Sequential Move Games

Using Backward Induction (Rollback) to Find Equilibrium

Sequential Move Class Game:
Century Mark

- Played by fixed pairs of players taking turns.
- At each turn, each player chooses a number (an integer) between 1 and 10 inclusive.
- This choice is added to sum of all previous choices (the initial sum is 0).
- The first player to take the cumulative sum greater 100 loses the game.
- No talking! Who are my first two volunteers?

cheesy game show music

Analysis of the Game

- What is the winning strategy?
  - Broadly speaking, bring the total to 89. Then, your opponent cannot possibly win and you can win for certain.
- The first mover can guarantee a win!
  - How to do this: to get to 89, need to get to 78, which can be done by getting to 67, 56, 45, 34, 23, 12, etc.
- Winning strategy: choose 11 minus the number chosen by the other player – a *complete plan of action*, or strategy.

Sequential Move Games with Perfect Information

- Models of strategic situations where there is a strict order of play.

- Perfect information implies that players know everything that has happened prior to making a decision.

- Sequential move games are most easily represented in extensive form, that is, using a *game tree*.

- The investment game we played in class was an example.

Constructing a sequential move game

- **Who** are the players?
- **What** are the action choices/strategies available to each player.
- **When** does each player get to move?
- **How much** do they stand to gain/lose?

Example 1: The merger game. Suppose an industry has six large firms (think airlines). Denote the largest firm as firm 1 and the smallest firm as firm 6. Suppose firm 1 proposes a merger with firm 6. Firm 2 must then decide whether to merge with firm 5.

The Merger Game Tree

Since Firm 1 moves first, they are placed at the **root node** of the game tree.
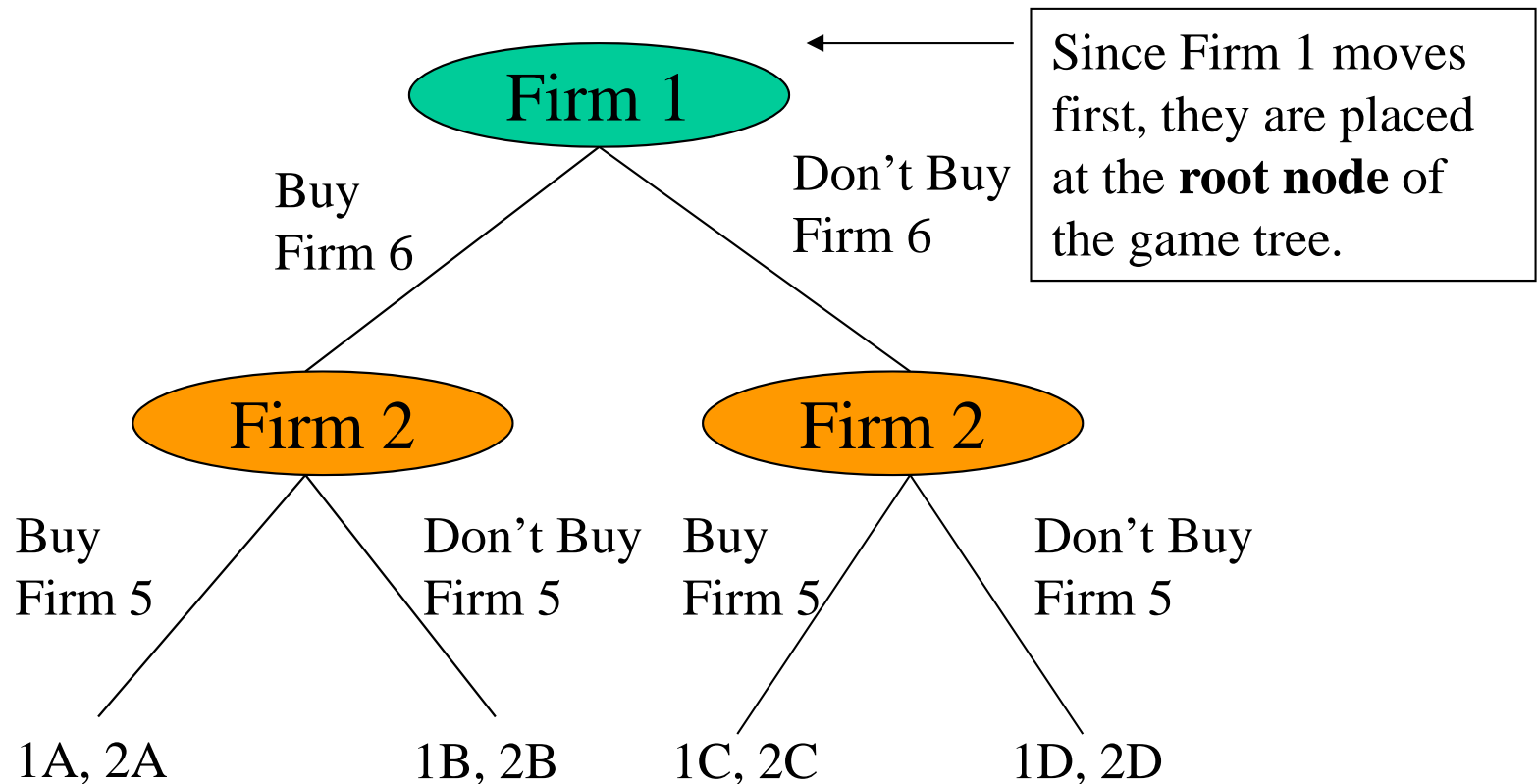
- What payoff values do you assign to firm 1's payoffs 1A, 1B, 1C, 1D? To firm 2's payoffs 2A, 2B, 2C, 2D? Think about the relative profitability of the two firms in the four possible outcomes, or **terminal nodes** of the tree. Use your economic intuition to rank the outcomes for each firm.

Assigning Payoffs
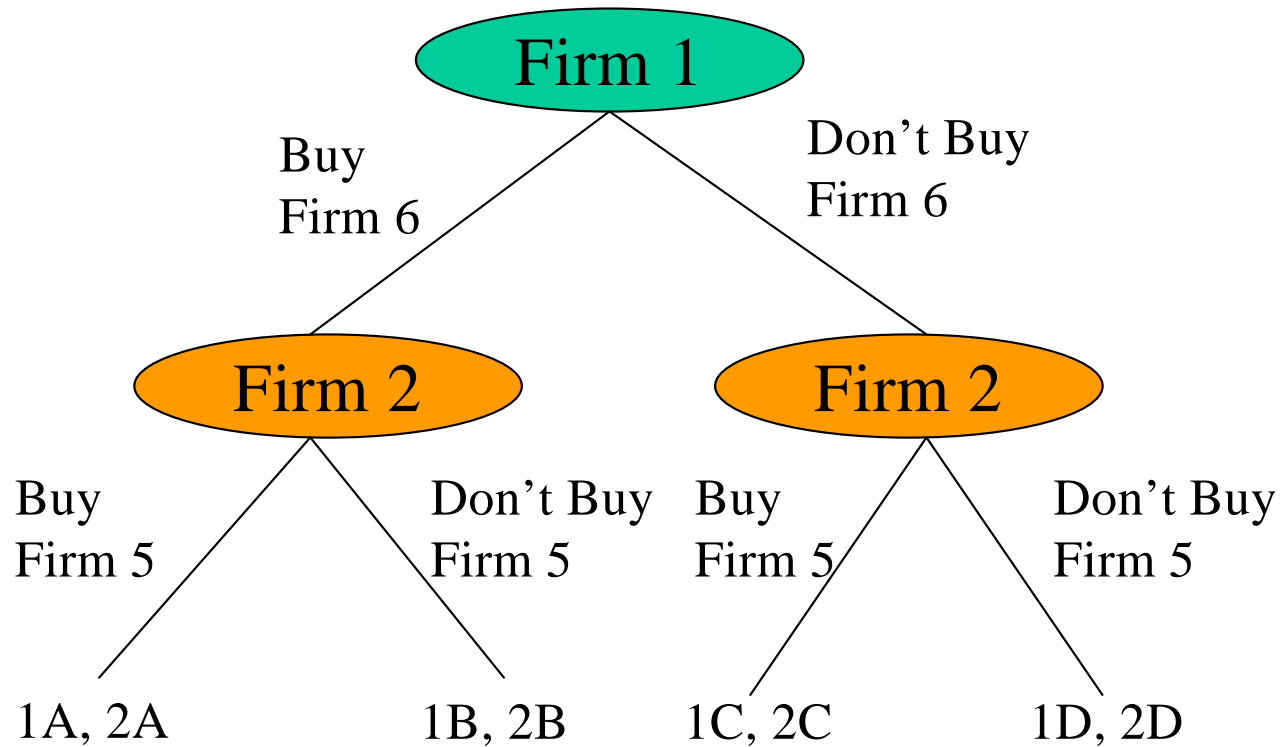
- Firm 1's Ranking: 1B > 1A > 1D > 1C. Use 4, 3, 2, 1
- Firm 2's Ranking: 2C > 2A > 2D > 2B. Use 4, 3, 2, 1

The Completed Game Tree
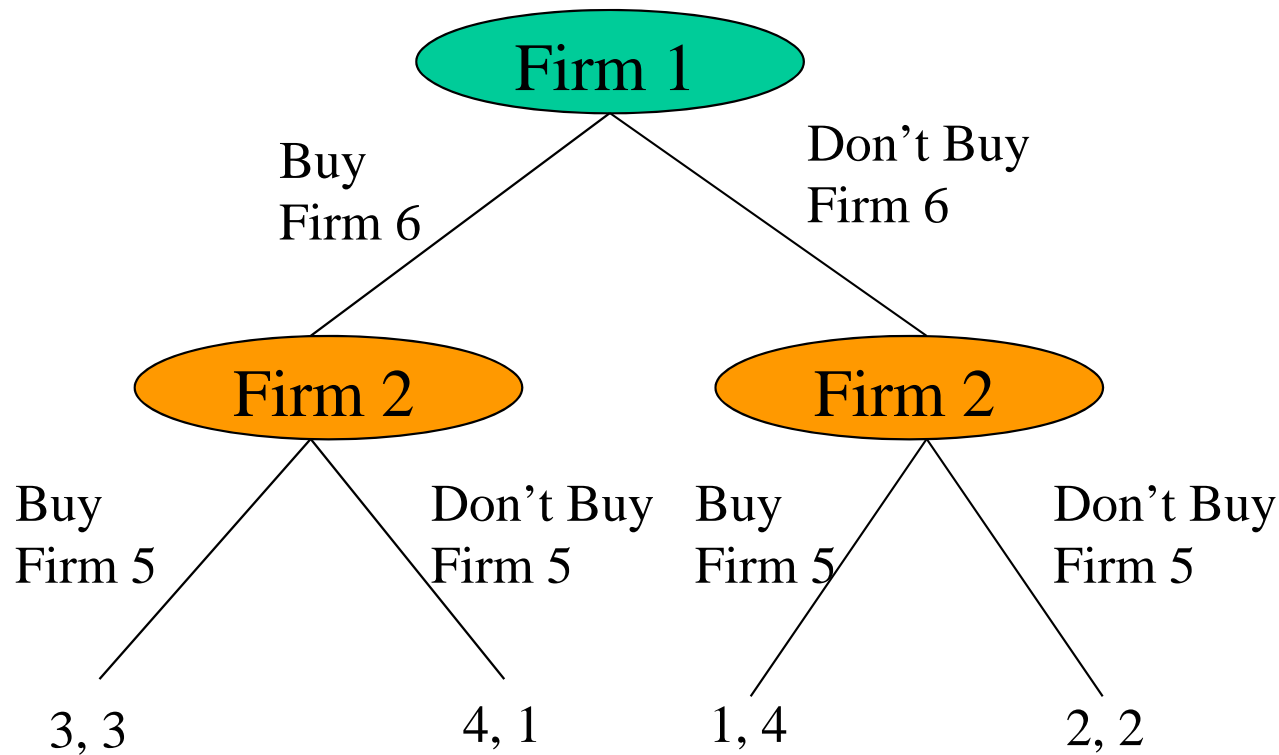
- What is the equilibrium? Why?

Example 2: The Senate Race Game

- Incumbent Senator Gray will run for reelection. The challenger is Congresswoman Green.

- Senator Gray moves first, and must decide whether or not to run advertisements early on.

- The challenger Green moves second and must decide whether or not to enter the race.

- Issues to think about in modeling the game:
    - Players are Gray and Green. Gray moves first.
    - Strategies for Gray are Ads, No Ads; for Green: In or Out.
    - Ads are costly, so Gray would prefer not to run ads.
    - Green will find it easier to win if Gray does not run ads.

Computer Screen View
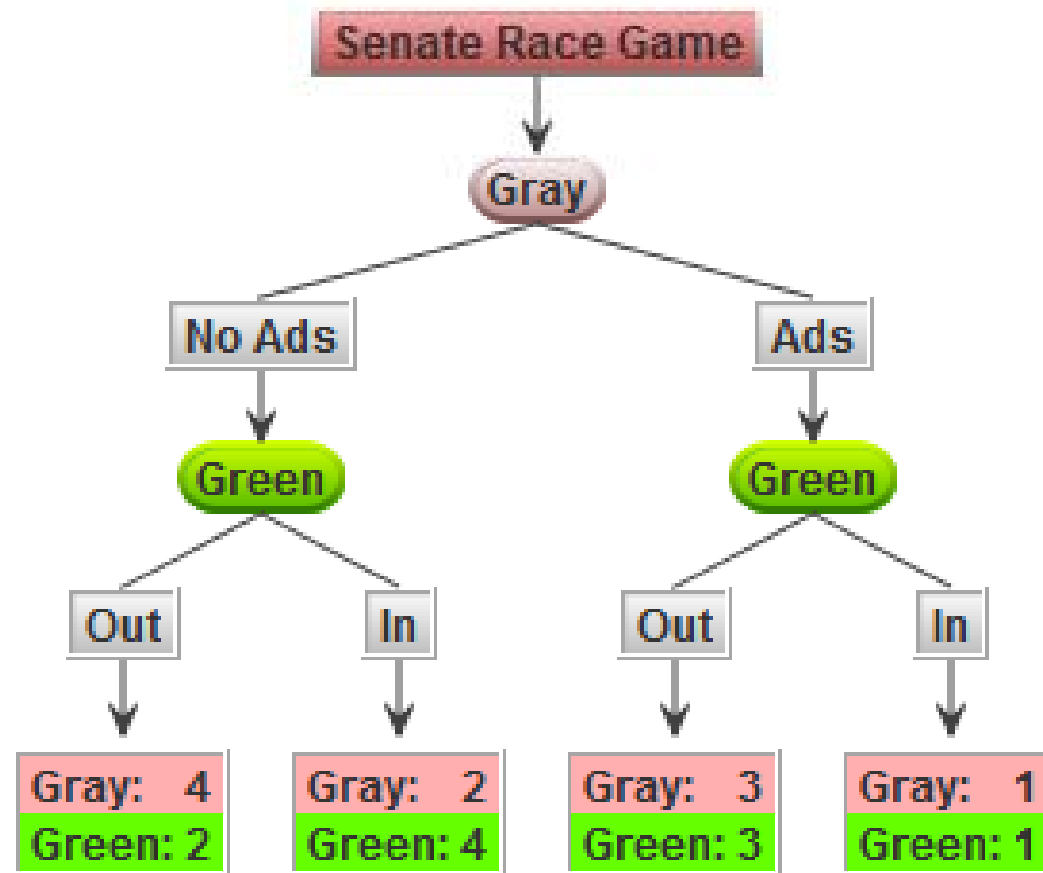
What are the strategies?

- A *pure strategy* for a player is a complete plan of action that specifies the choice to be made at each decision node.

- Gray has two pure strategies: Ads or No Ads.

- Green has four pure strategies:

  1. If Gray chooses Ads, choose In and if Gray chooses No Ads choose In.
  2. If Gray chooses Ads, choose Out and if Gray chooses No Ads choose In.
  3. If Gray chooses Ads, choose In and if Gray chooses No Ads choose Out.
  4. If Gray chooses Ads, choose Out and if Gray chooses No Ads choose Out.

- Summary: Gray's pure strategies, Ads, No Ads.

- Greens' pure strategies: (In, In), (Out, In), (In, Out), (Out, Out).

Using *Rollback* or *Backward Induction* to find the Equilibrium of a Game

- Suppose there are two players A and B. A moves first and B moves second.
- Start at each of the terminal nodes of the game tree. What action will the last player to move, player B choose starting from the immediate prior decision node of the tree?
- Compare the payoffs player B receives at the terminal nodes, and assume player B always chooses the action giving him the maximal payoff.
- Place an arrow on these branches of the tree. Branches without arrows are "pruned" away.
- Now treat the next-to-last decision node of the tree as the terminal node. Given player B's choices, what action will player A choose? Again assume that player A always chooses the action giving her the maximal payoff. Place an arrow on these branches of the tree.
- Continue rolling back in this same manner until you reach the root node of the tree. The path indicated by your arrows is the equilibrium path.

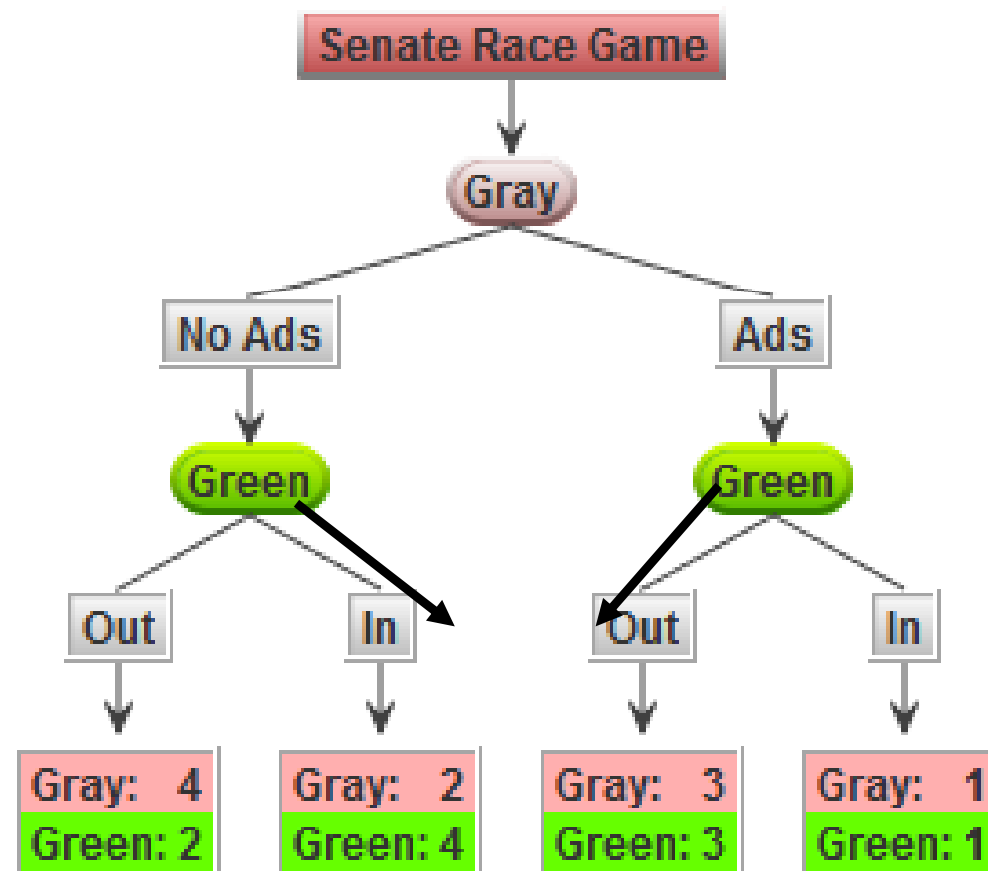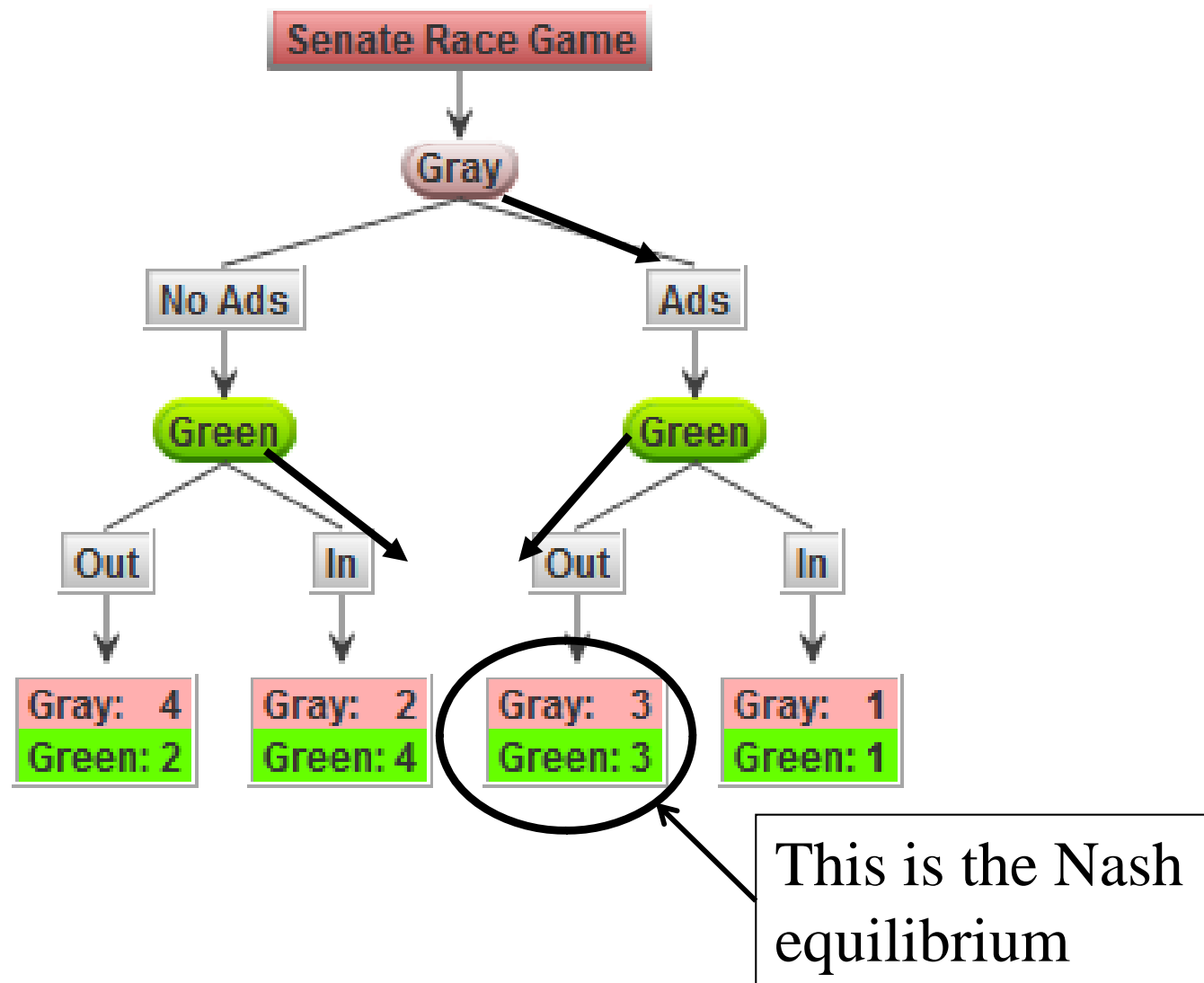# Illustration of Backward Induction in Senate Race Game: Green's Best Response

Illustration of Backward Induction in Senate Race Game: Gray's Best Response

Is There a First Mover Advantage?

- Suppose the sequence of play in the Senate Race Game is changed so that Green gets to move first. The payoffs for the four possible outcomes are exactly the same as before, except now, Green's payoff is listed first.
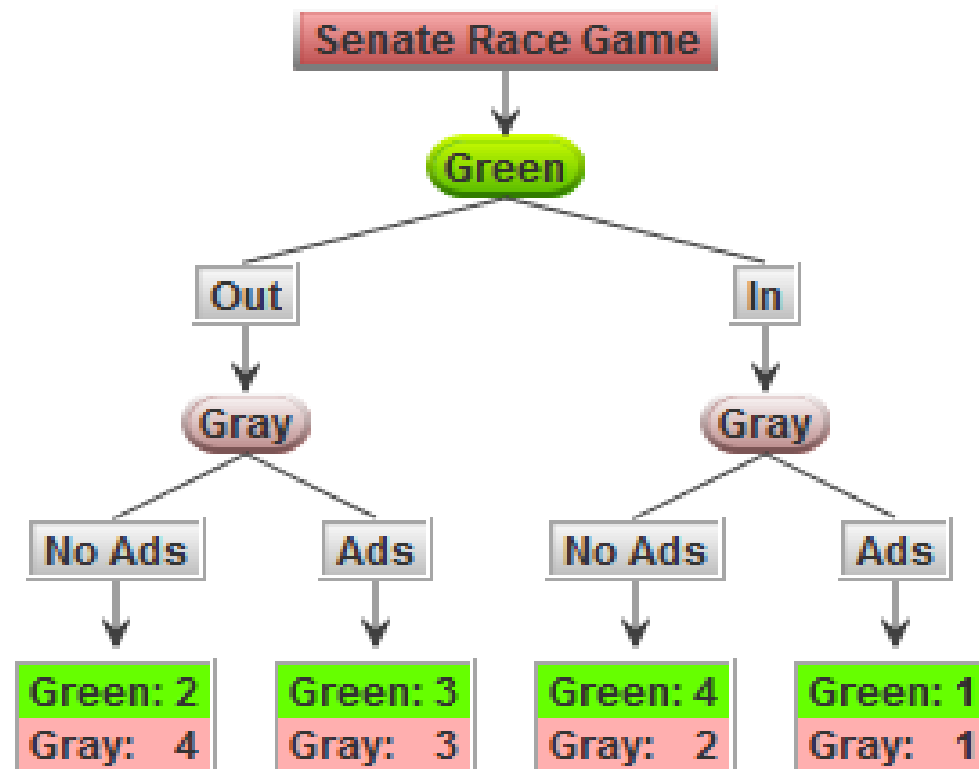
Whether there is a first mover advantage depends on the game.

- To see if the order matters, rearrange the sequence of moves as in the senate race game.
- Other examples in which order may matter:
  - Adoption of new technology. Better to be first or last?
  - Class presentation of a project. Better to be first or last?
- Sometimes order does not matter. For example, is there a first mover advantage in the merger game as we have modeled it? Why or why not?
- Is there such a thing as a second mover advantage?
  - Sometimes, for example:
    - Sequential biding by two contractors.
    - Cake-cutting: One person cuts, the other gets to decide how the two pieces are allocated.
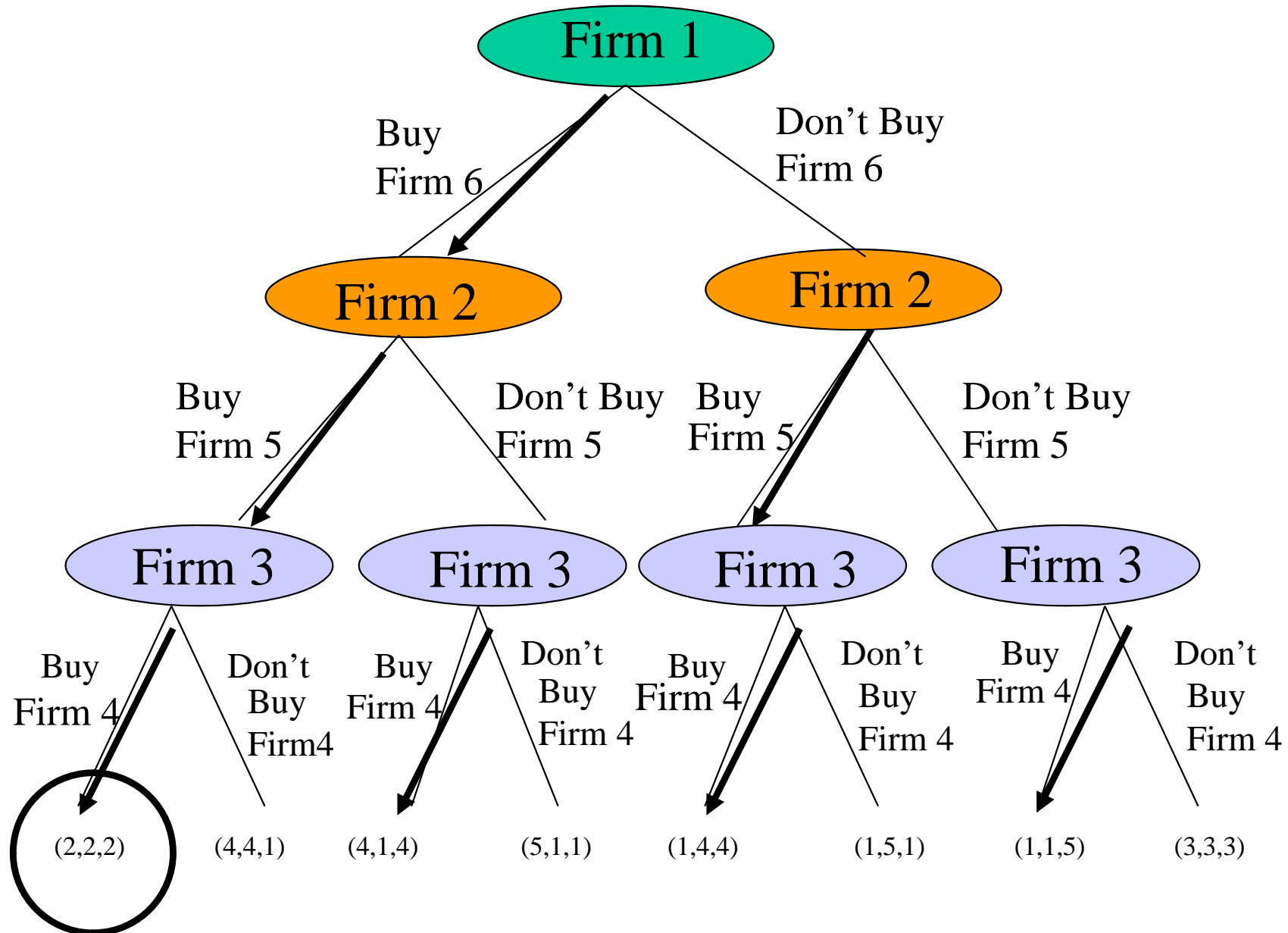
Adding more players

- Game becomes more complex.
- Backward induction, rollback can still be used to determine the equilibrium.
- Example: The merger game. There are 6 firms.
  - If firms 1 and 2 make offers to merge with firms 5 and 6, what should firm 3 do?
  - Make an offer to merge with firm 4?
 - Depends on the payoffs.

3 Player Merger Game
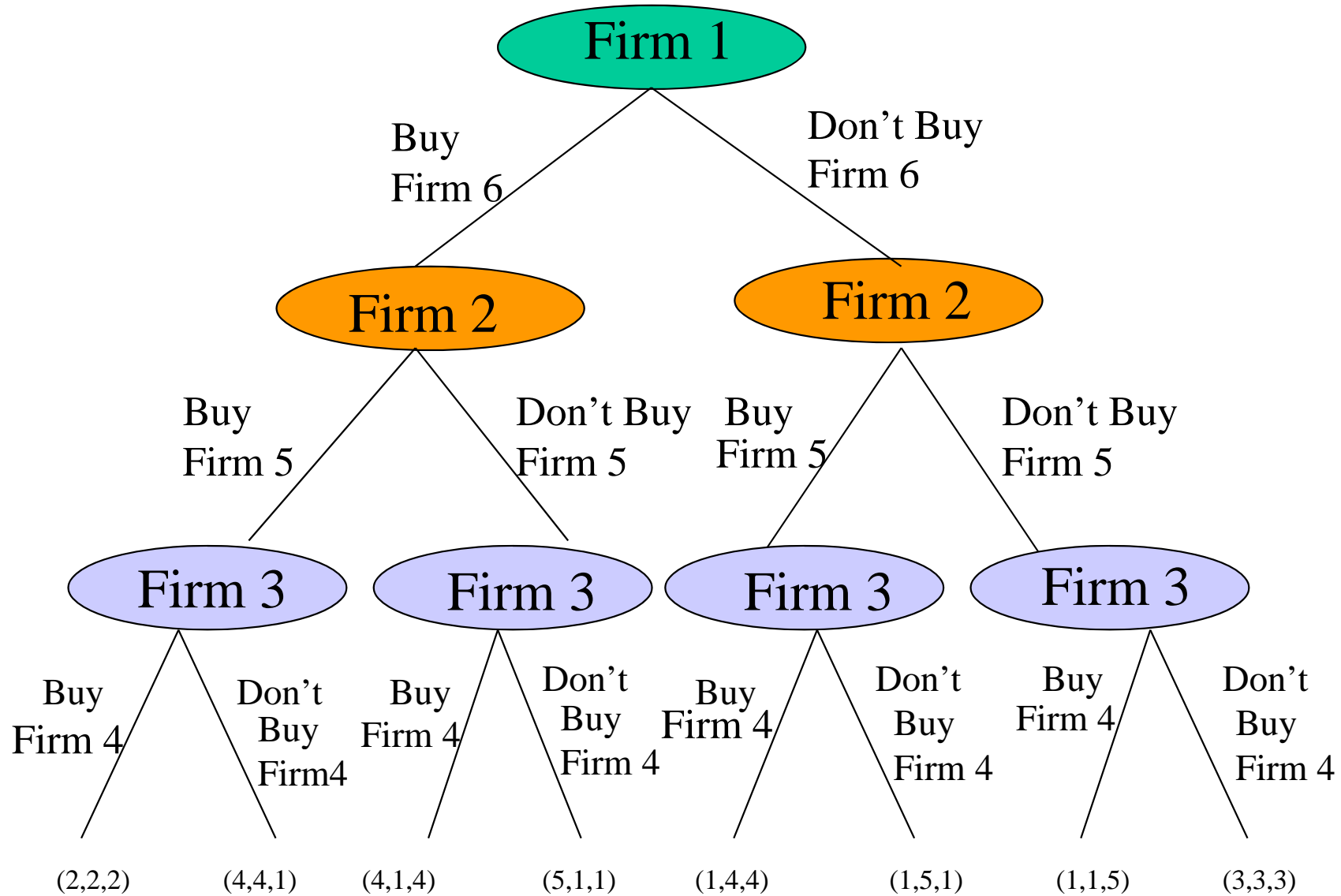
Solving the 3 Player Game

Adding More Moves

- Again, the game becomes more complex.
- Consider, as an illustration, the Game of Nim
- Two players, move sequentially.
- Initially there are two piles of matches with a certain number of matches in each pile.
- Players take turns removing any number of matches from a single pile.
- The winner is the player who removes the *last* match from either pile.
- Suppose, for simplicity that there are 2 matches in the first pile and 1 match in the second pile. We will summarize the initial state of the piles as (2,1), and call the game Nib (2,1)
- What does the game look like in extensive form?

Nib (2,1) in Extensive Form
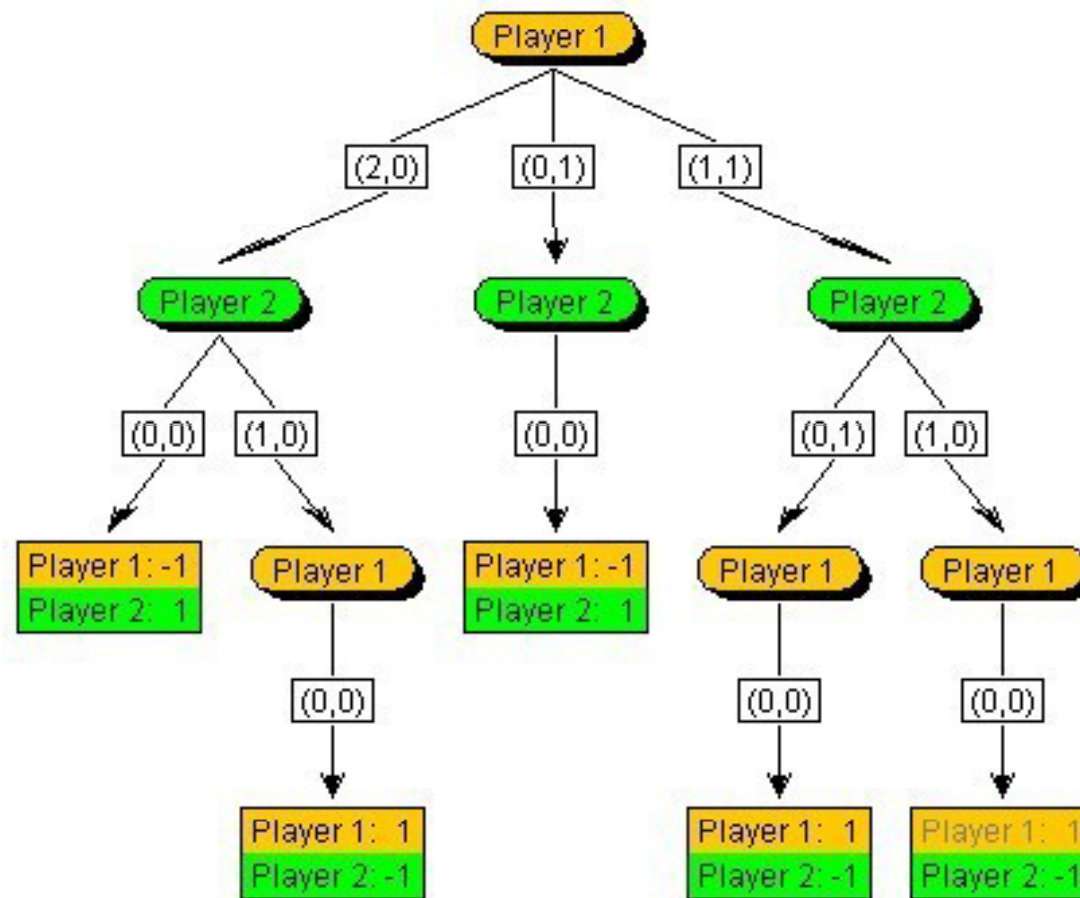
How reasonable is rollback/backward induction as a behavioral principle?

- May work to explain actual outcomes in simple games, with few players and moves.
- More difficult to use in complex sequential move games such as Chess.
  - We can't draw out the game tree because there are too many possible moves, estimated to be on the order of 10^{120}.
  - Need a rule for assigning payoffs to non-terminal nodes – a intermediate valuation function.
- May not always predict behavior if players are unduly concerned with "fair" behavior by other players and do not act so as to maximize their own payoff, e.g., they choose to punish "unfair" behavior.

Existence of a Solution to Perfect Information Games

Games of perfect information are ones where every information set consists of a single node in the tree.

Kuhn's Theorem: Every game of perfect information with a finite number of nodes, *n, has a solution to backward* induction.

 Corollary: If the payoffs to players at all terminal nodes are unequal, (no ties) then the backward induction solution is unique.

Sketch of Proof: Consider a game with a maximum number of *n nodes. Assume the game with just n-1 steps has a* backward induction solution. (Think e.g. *n=2). Figure out* what the best response of the last player to move at step *n,* taking into account the terminal payoffs. Then prune the tree, and assign the appropriate terminal payoffs to the *n-1 node.* Since the game with just *n-1 steps has a solution, by* induction, so does the entire *n-step game.*

"Nature" as a Player

- Sometimes we allow for special type of player, - nature- to make random decisions. Why?

- Often there are uncertainties that are inherent to the game, that do not arise from the behavior of other players.
  - e.g., whether you can find a parking place or not.

- A simple example: 1 player game against Nature. With probability ½ Nature chooses G and with probability ½ Nature chooses B.
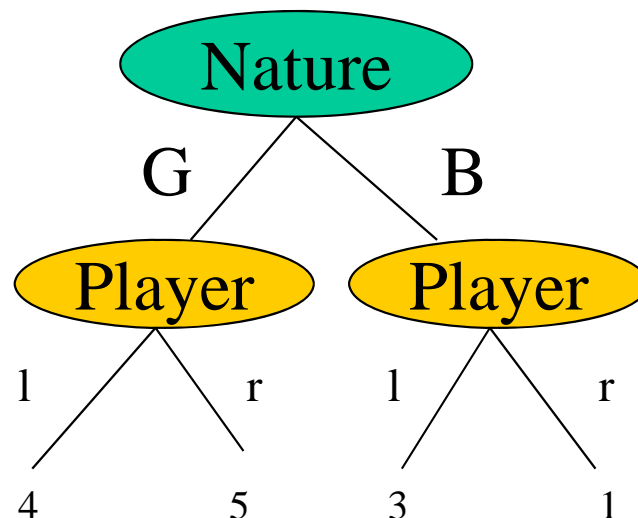
- In this sequential move game, nature moves first. Equilibria are G,r and B,l

Playing Against Nature, Cont'd

- Suppose the game is changed to one of *simultaneous* moves:
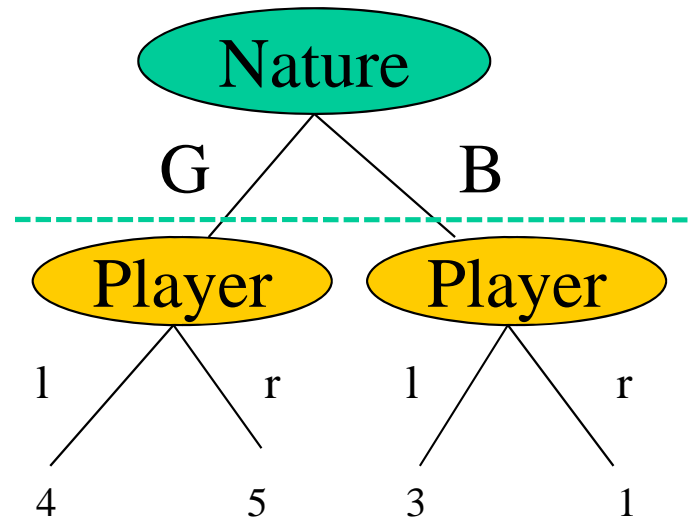
- Player doesn't know what nature will do, as symbolized by the ---'d line.

- What is your strategy for playing this game if you are the player?

- A *risk neutral* player treats expected payoffs the same as certain payoffs: Expected payoff from left=½*4+½*3=7/2; Expected payoff from right = ½*5+½*1=3: Choose left (l).