Mixed Strategies

Keep ‘em guessing

Mixed Strategy Nash Equilibrium

- A mixed strategy is one in which a player plays his available pure strategies with certain probabilities.
- Mixed strategies are best understood in the context of repeated games, where each player's aim is to keep the other player(s) guessing, for example: Rock, Scissors Paper.
- If each player in an n-player game has a finite number of pure strategies, then there exists at least one equilibrium in (possibly) mixed strategies. (Nash proved this).
- If there are no pure strategy equilibria, there must be a unique mixed strategy equilibrium.
- However, it is possible for pure strategy and mixed strategy Nash equilibria to coexist, as in the Stag Hunt and Chicken games.

Example 1: Tennis

- Let p be the probability that Serena chooses DL, so that 1-p is the probability she chooses CC.
- Let q be the probability that Venus positions herself for DL, so that 1-q is the probability she positions herself for CC.
- To find mixed strategies, we add the p-mix and q-mix strategies to the payoff matrix.

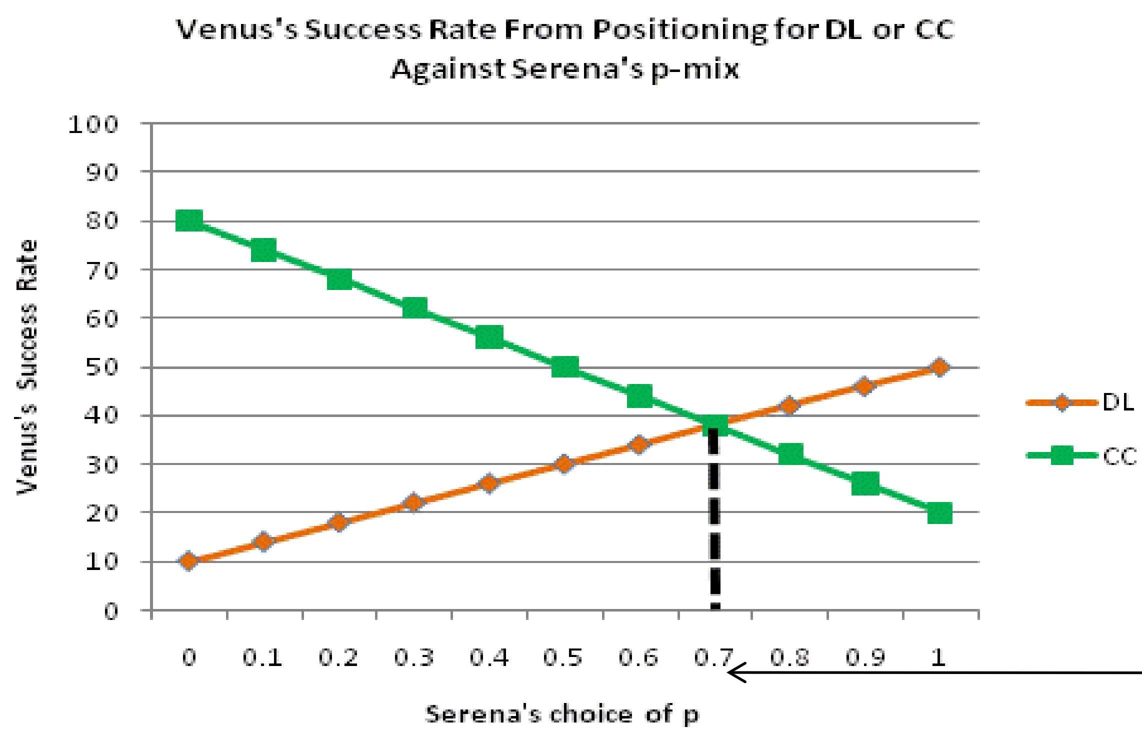
Venus Williams

Serena Williams		DL	CC	q-mix
	DL	50, 50	80,20	50q+80(1-q) 50q+20(1-q)
	CC	90, 10	20,80	90q+20(1-q) 10q+80(1-q)
	p-mix	50p+90(1-p) 50p+10(1-p)	80p+20(1-p) 20p+80(1-p)	

Row Player's Optimal Choice of *p*

- Chose *p* so as to equalize the payoff your opponent receives from playing either of her pure strategies.
- This requires understanding how your opponent's payoff varies with your own choice of *p*.
- Graphically, in the Tennis example:

For Serena's choice of p, Venus's *expected payoff* from playing DL is: 50p+10(1-p) and from playing CC is: 20p+80(1-p)



Venus is made indifferent if Serena chooses p=.70

Algebraically

- Serena solves for the value of p that equates Venus's payoff from positioning herself for DL or CC:

$$50p+10(1-p) = 20p+80(1-p), \text{ or}$$

$$50p+10-10p = 20p+80-80p, \text{ or}$$

$$40p+10 = 80-60p, \text{ or}$$

$$100p = 70, \text{ so}$$

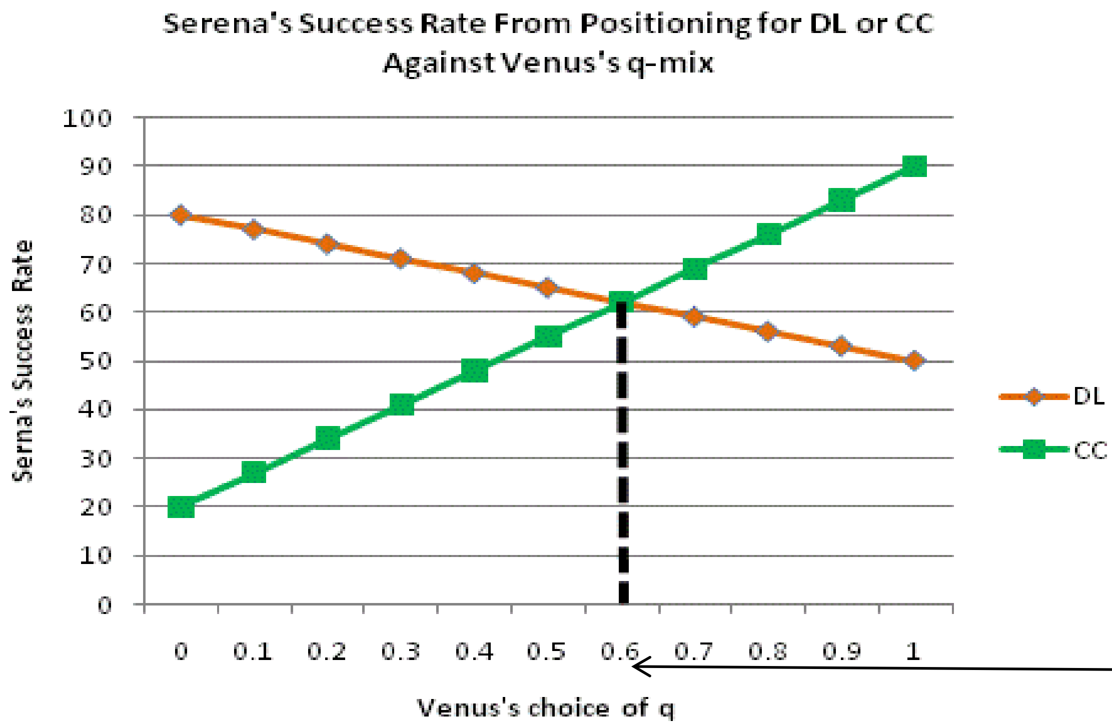
$$p = 70/100 = .70.$$

- If Serena plays DL with probability p=.70 and CC with probability 1-p=.30, then Venus's success rate from DL=50(.70)+10(.30)=38%=Venus's success rate from CC=20(.70)+80(.30)=38%.
- Since this is a constant sum game, Serena's success rate is 100%-Venus's success rate = 100-38=62%.

Column Player's Optimal Choice of *q*

- Choose *q* so as to equalize the payoff your opponent receives from playing either pure strategy.
- This requires understanding how your opponent's payoff varies with your choice of *q*.
- Graphically, in our example:

For Venus's choice of q, Serena's *expected payoff* from playing DL is: 50q+80(1-q) and from playing CC is: 90q+20(1-q)



Serena is made indifferent if Venus chooses q=.60

Algebraically

- Venus solves for the value of q that equates Serena's payoff from playing DL or CC:

$$50q+80(1-q) = 90q+20(1-q), \text{ or}$$

$$50q+80-80q = 90q+20-20q, \text{ or}$$

$$80-30q = 70q+20, \text{ or}$$

$$60 = 100q, \text{ so}$$

$$q = 60/100 = .60.$$

- If Venus positions herself for DL with probability q=.60 and CC with probability 1-q=.40, then Serena's success rate from DL=50(.60)+80(.40)=62%=Serena's success rate from CC=90(.60)+20(.40)=62%.
- Since this is a constant sum game, Venus's success rate is 100%-Serena's success rate = 100-62=38%.

The Mixed Strategy Equilibrium

- A strictly mixed strategy Nash equilibrium in a 2 player, 2 choice (2x2) game is a $p > 0$ and a $q > 0$ such that p is a best response by the row player to column player's choices, and q is a best response by the column player to the row player's choices.
- In our example, p=.70, q=.60. The row player's payoff (Serena) was 62 and the column player's payoff (Venus) was 38. (Serena wins 62%, Venus 38%).
- **Pure strategies** can now be understood as a ***special case of mixed strategies***, where p is chosen from the set {0, 1} and q is chosen from the set {0, 1}. For example, if $p=0$ and $q=1$, then row player always plays CC and column player always plays DL.

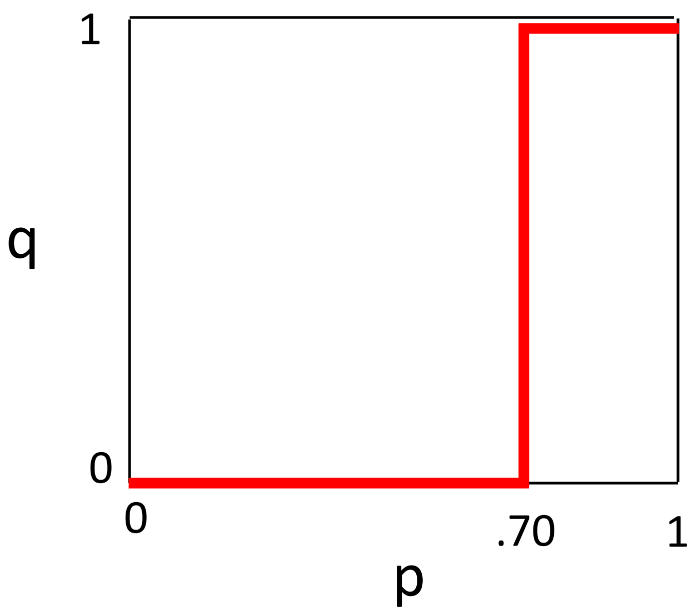
Keeping the Opponent Indifferent

- Why is this a useful objective for determining the mixing probability?
  - In constant sum games, such as the tennis example, making your opponent indifferent in expected payoff terms is equivalent to minimizing your opponents' ability to recognize and exploit systematic patterns of behavior in your own choice.
  - In constant sum games, keeping your opponent indifferent is equivalent to keeping yourself indifferent
  - The same objective works for finding mixed strategy equilibria in *non-constant sum games as well, where players* interests are not totally opposed to one another.
- Necessarily suggests that the game is played repeatedly.

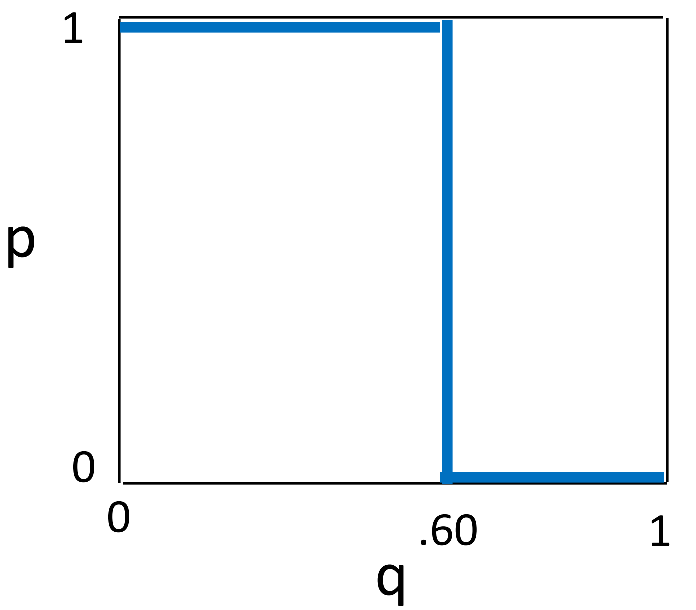
Best Response Functions

- Another way to depict each player's choice of the mixing probability. (Recall *p*=Pr(DL) by Serena, *q*=Pr(DL) by Venus).
- Shows strategic best response of *q*=f(*p*) and *p*=g(*q*).
- *p*, *q* =0 is always play CC, *p*, *q =1* is always play DL.

Venus's Best
Response Function

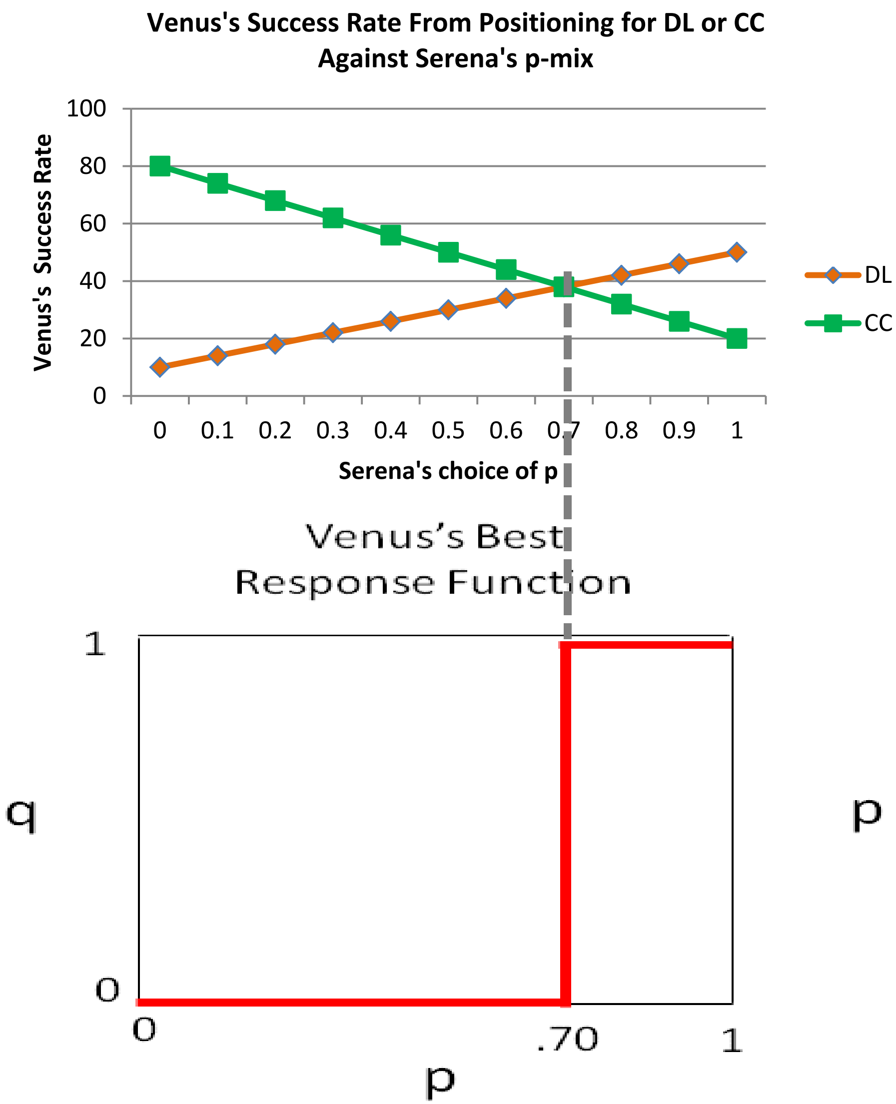


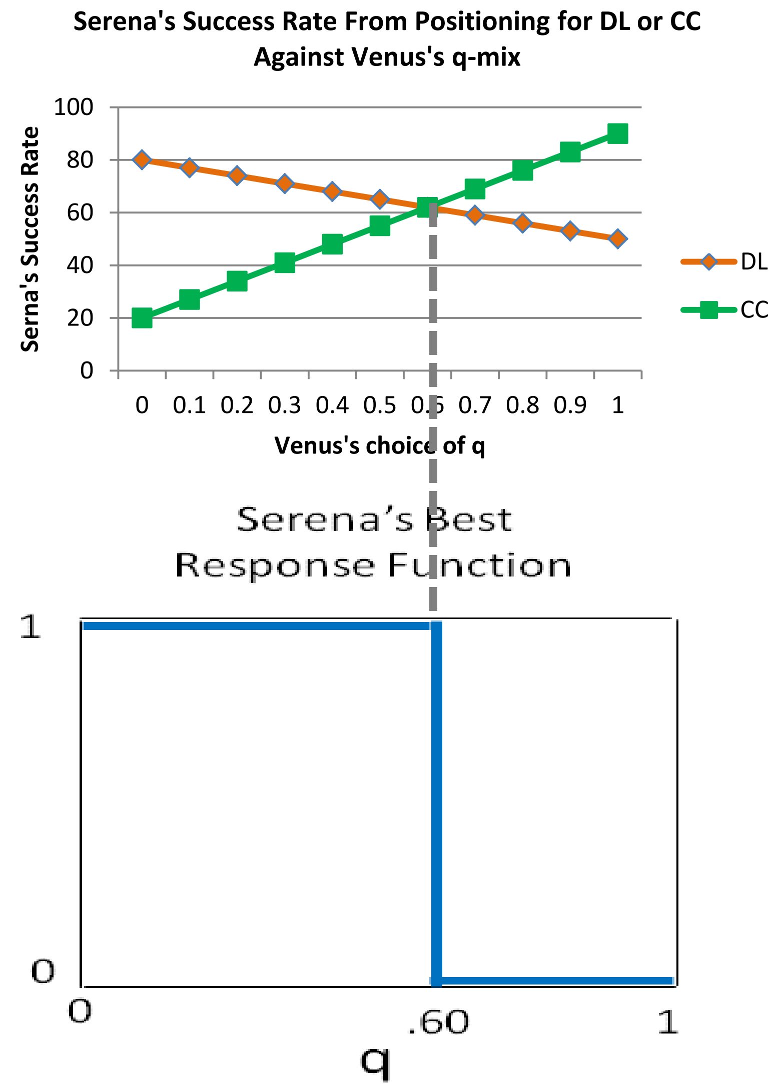
Serena's Best
Response Function



Construction of Best Response Functions

- Use the graphs of the optimal choices of q=f(p) and p=g(q):

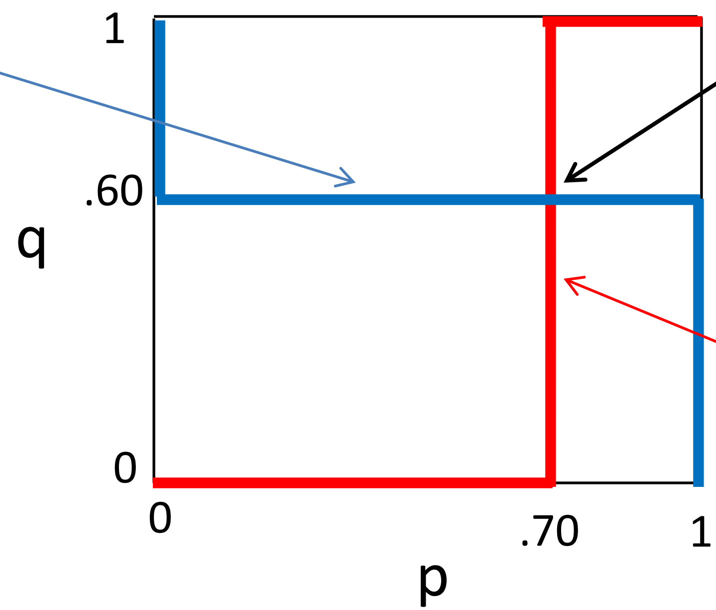




Combining the Best Response Functions Reveals the Mixed Strategy Nash Equilibrium

Combined Best
Response Functions
Nash equilibria are mutual best responses

Serena's Best Response Function



Mixed Strategy Nash Equilibrium Occurs at p=.70, q=.60.

Venus's Best Response Function

Equilibria, pure or mixed, obtain wherever the best response functions intersect.

- Note that there is no equilibrium in pure strategies in this game.
- Intersections in the middle of the box are *mixed* strategy equilibria; intersections at the four corners of the box correspond to *pure* strategy equilibria.

Example 2: Rock, Paper, Scissors

- A two-player game with *three* strategies – how to find the mixed strategy equilibrium in that case?
- Graphically it is more difficult, but we use the same principle: make the opponent indifferent over his three strategy choices.
- An example of a three strategy game with a unique mixed strategy equilibrium is Rock, Paper Scissors.
- Rules are: Two players simultaneously choose Rock, Paper or Scissors with the understanding that Rock beats Scissors; Paper beats Rock; Scissors beats paper. All other outcomes, e.g. Rock, Rock result in ties.
- Denote a win by a 1, a loss by -1 and a tie by 0

Rock, Paper, Scissors: Solved!

- The game is symmetric, so it suffices to find one player's mixed strategy, say the Column player.
- Let r=Pr(Rock), p=Pr(Paper) and 1-r-p=Pr(Scissors)

	Rock	Paper	Scissors	Mixture
Rock	0, 0	-1, 1	1, -1	1-r-2p
Paper	1, -1	0, 0	-1, 1	2r+p-1
Scissors	-1, 1	1, -1	0, 0	-r+p

- Column chooses r and p the make row indifferent:
- $2r+p-1=-r+p \Rightarrow 3r=1, r=1/3$;
- $1-r-2p=2r+p-1 \Rightarrow 1-1/3 -2p=2/3+p-1 \Rightarrow 1=3p, p=1/3$;
- It follows that the unique, mixed strategy equilibrium is p=r=1/3, i.e., play all three strategies with equal probability.

Example 3: A Market Entry Game

- Two Firms, MD and BK must decide whether to open one of their restaurants in a new shopping mall.
- The strategies are to “Enter” or “Don’t Enter”.
- If either firm plays Don’t Enter, it earns 0 profits.
- If one firm plays Enter and the other plays Don’t Enter, the Firm that plays Enter earns $300,000 per year in profits (Don’t enter always yields 0 profits).
- If both firms choose to play Enter, both lose $100,000 per year as there is not enough demand for two restaurants to make positive profits.

The Market Entry Game is Non-Zero Sum

- Computer Screen View

Payoffs are in units of $100,000	Enter	Don't Enter	BK MD
Enter	1 -1 -1	3 0 3	
Don't Enter	2 3 0	4 0 0	

- What are the Nash equilibria of this game?

Cell-by-Cell Inspection Reveals 2 Pure Strategy Equilibria

Payoffs are in units of $100,000	Enter	Don't Enter
Enter	1: -1, -1	3: 3, 0
Don't Enter	2: 0, 3	4: 0, 0

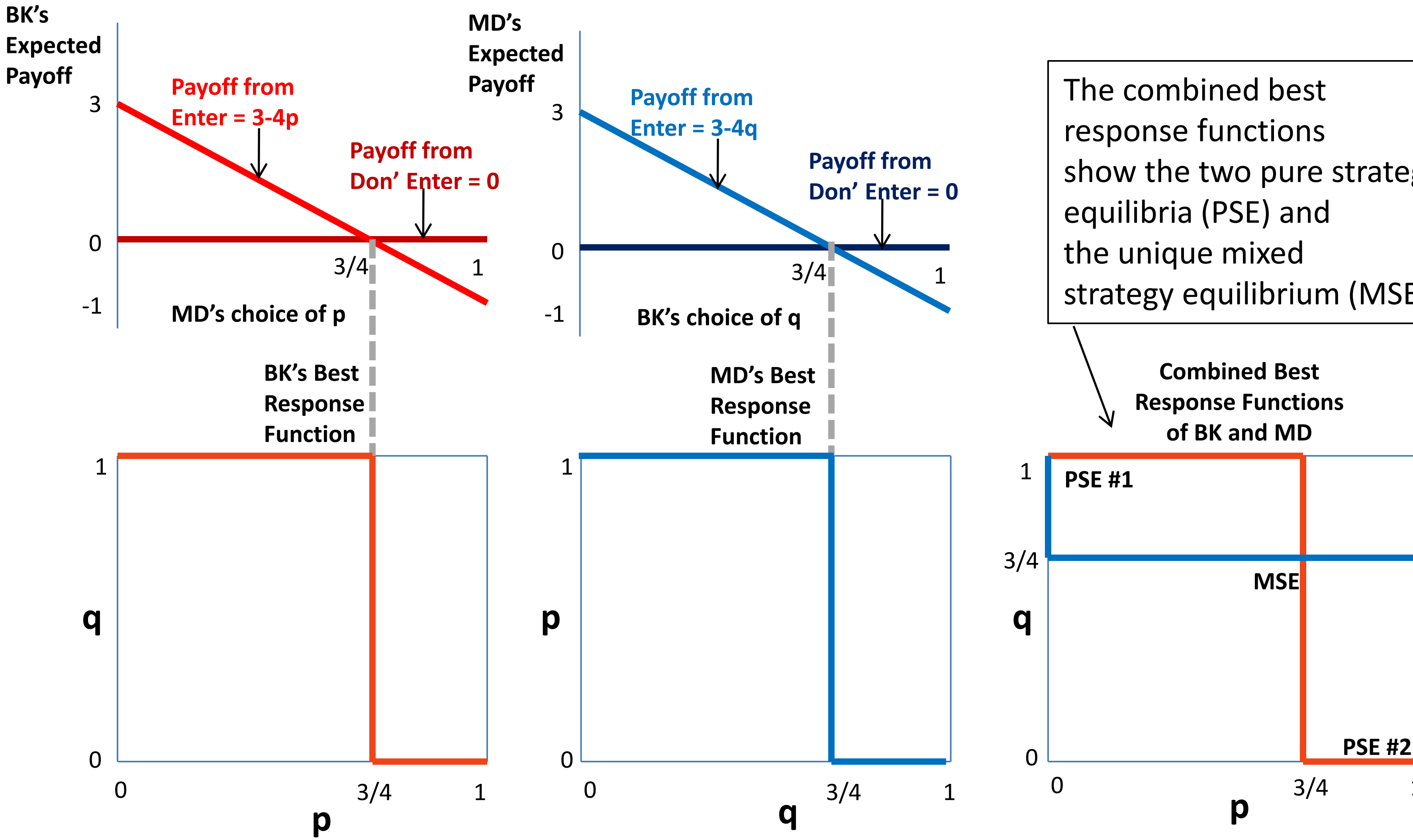
BK
MD

- BUT....are there any mixed strategy equilibria? Let's check

Calculation of the Mixed Strategy Probabilities

- MD (BK) wants to choose a p(q) to make BK (MD) indifferent between playing Enter or Don't Enter.
- **MD**: Choose p such that: BK's payoff from Enter: $p(-1)+(1-p)(3)=p(0)+(1-p)(0)=0$, BK's payoff from Don't Enter. So $-p+3-3p=0$, or $3=4p$, so $p=3/4$.
- **BK**: Choose q such that: MD's payoff from Enter: $q(-1)+(1-q)3=q(0)+(1-q)(0)=0$, MD's payoff from Don't Enter. So $–q+3-3q=0$, or $3=4q$, so $q=3/4$.
- In the mixed strategy Nash equilibrium both firms choose enter with probability ¾, and Don't Enter with probability ¼.

Best Response Functions for the Market Entry Game



The combined best response functions show the two pure strategy equilibria (PSE) and the unique mixed strategy equilibrium (MSE)

Economics of the Mixed Strategy Equilibrium

- Both firms choose to play Enter with probability ¾.
- Expected payoffs from Enter = 3-4(¾)=0 which is the same expected payoff from Don't Enter (always 0).
- Probability both Enter (multiplication rule) (¾) (¾)= 9/16, (about half the time).
- Probability MD enters and BK does not (¾)(¼)=3/16.
- Probability MD does not enter and BK does (¼)(¾)=3/16.
- Probability that neither firm Enters:(¼) (¼)=1/16.
- Note that 9/16+3/16+3/16+1/16=1.0; we have accounted for all outcomes.
- Payoff calculation for MD (same for BK):

$$(9/16)(-1)+(3/16)(3)+3/16(0)+(1/16)(0)=0.$$

Asymmetric Mixed Strategy Equilibrium

- Suppose we change the payoff matrix so that MD has a *competitive advantage over BK in one situation: if MD is the* sole entrant, it earns profits of 400,000.
- Otherwise the game is the same as before.

		BK	
		Enter	**Don't Enter**
MD	**Enter**	**-1, -1**	**4, 0**
	Don't Enter	**0, 3**	**0, 0**

- The pure strategy equilibria remain the same, (E,D) or (D, E). What happens to the mixed strategy probability?

Changes in the Mixed Strategy Probability?

- Since BK's payoffs have not changed, MD's mixture probability choice for p does not change. It chooses p so that $-1(p)+3(1-p)=0$, or $3=4p$, $p=3/4$.
- However, since MD's payoffs have changed, BK's mixture probability choice for q must change.
- BK chooses q so as to make MD just indifferent betweenentering and earning $(-1)q+4(1-q)$ and not entering and earning 0: $-q+4-4q=0$, or $4=5q$, so $q=4/5$
- Note that the probability that BK enters goes up from 3/4 to 4/5. If it did not, then MD would choose $p=1$. Why?
- MD's expected payoff remains zero: $-1(4/5)+4(1/5)=0$.

Example 4: An N-Player Market Entry Game

- Market entry games frequently involve more than 2 firms (players).
- Let there be *N>2 firms.*
- Let *m be the number of the N firms choosing* to Enter. (*N-m choose Don't Enter).*
- Suppose the payoff to each firm *i* is given by:

$$\text{Payoff to firm i} = \begin{cases} 10 & \text{if i chooses Don't Enter} \\ 10 + 2(7\text{-}m) & \text{if i chooses Enter} \end{cases}$$

Equilibria in the N-player Market Entry Game

- Pure strategy equilibrium: Each firm either plays Enter or Don't Enter.
- Firms compare the payoff from playing Don't Enter, with the payoff from playing Enter:
- When Does Don't Enter yield the same payoff as Enter?

$$10=10+2(7-m)$$

- Only when $m=7$. *So the pure strategy equilibrium* is for 7/N firms to play Enter, and (N-7)/N firms to play Don't Enter.

Mixed Strategy Equilibria in the N-Player Market Entry Game

- There is also a *symmetric mixed strategy* equilibrium in this game, where every firm mixes between Enter and Don't Enter with the *same* probability.
- Let p be the probability that each firm plays Enter.
- In the mixed strategy equilibrium,

$$p = \frac{7-1}{N-1} = \frac{6}{N-1}$$

- All players earn an expected payoff of 10.

Calculation of the Symmetric Mixed Strategy Probability

- The entry payoff is 10+2(7-m), and the no entry payoff is 10.
- First, for simplicity, subtract 10 from both payoffs.
- Let p be individual i's probability of entry. Assume all i players have the same p (mixed strategy is *symmetric).* p[2(7-m)]=(1-p)(0)=0.
- Recall that m is the total number of entrants *including player i.*
- *If player i chooses to enter*, his prediction for m=p(N-1)+1, the expected number of entrants among the N-1 players +1 more (himself). If player *i* chooses not enter he gets 0.
- So p[2(7-[p(N-1)+1])]=0, or p[2(7-1)-2p(N-1)]=0, or 2p(7-1)=2p2(N-1), or (7-1)=p(N-1), or p=(7-1)/(N-1).