Probability, Expected Payoffs and Expected Utility

- In thinking about mixed strategies, we will need to make use of *probabilities*. We will therefore review the basic rules of probability and then derive the notion of expected value.
- We will also develop the notion of *expected utility* as an alternative to *expected payoffs*.
- Probabilistic analysis arises when we face uncertainty.
- In situations where outcomes (or "states of the world") are uncertain, a **probability** measures the likelihood that a particular outcome (or set of outcomes) occurs.
  - e.g. The probability that the roll of a die comes up 6. (1/6).
  - The probability that two randomly chosen cards in a 52 card deck add up to 21 (Natural Blackjack) (<5%).

Sample Space or Universe

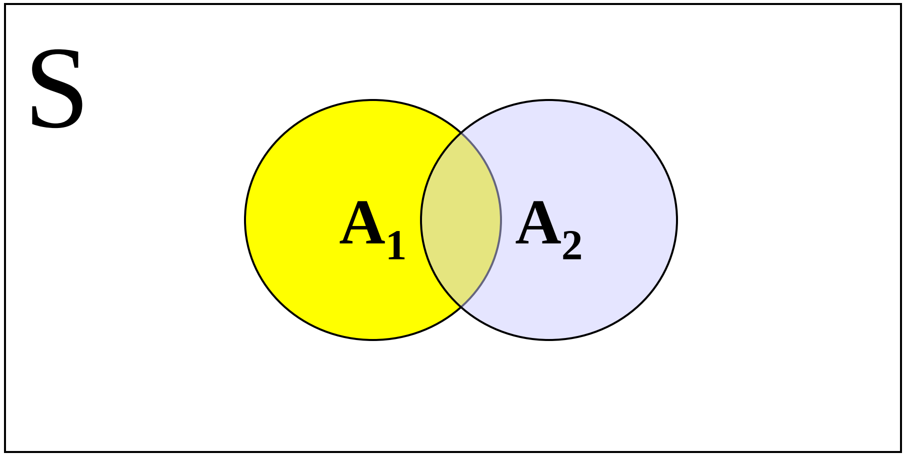
- Let S denote a set (collection or listing) of all possible outcomes or "states" of the environment known as the sample space or universe; a typical state is denoted as s. For example:
- $S=\{s_1, s_2\}$; success/failure, or low/high price.
- $S=\{s_1, s_2,...,s_{n-1},s_n\}$; number of n units sold or n offers received.
- $S=[0, \infty)$; stock price or salary offer. (continuous positive sample space).

States and Events

- A state is the result of an experiment or other situation involving uncertainty.
- An event is a collection of those states, s, that result in the occurrence of the event.
- An event can be that state s occurs or that multiple states occur, or that one of several states occurs (there are other possibilities).
- Event A is a subset of S, denoted as $A \subset S$.
- Event A occurs if the true state s is an element of the set A, written as $s \in A$.

Venn Diagrams

- Illustrates the sample space and events.



- **S** is the sample space and $\mathbf{A_1}$ and $\mathbf{A_2}$ are events within S.
- "Event A_1 does not occur." Denoted A_1^c (Complement of A_1)
- "Event A_1 *or* A_2 occurs." Denoted $A_1 \cup A_2$ (For probability use Addition Rules)
- "Event A_1 *and* A_2 both occur", denoted $A_1 \cap A_2$ (For probability use Multiplication Rules).

Probability

- To each uncertain event A, or set of events, e.g., A_1 *or* A_2, we would like to assign weights which measure the likelihood or importance of the events in a proportionate manner.
- Let $P(A_i)$ be the probability of A_i.
- We further assume that:

$$\bigcup_{\text{all } i} A_i = S$$

$$P\Big(\bigcup_{\text{all } i} A_i\Big) = 1$$

$$P(A_i) \geq 0.$$

Addition Rules

- The probability of event A *or* event B: $P(A \cup B)$
- If the events do not overlap, i.e. the events are disjoint subsets of S, so that $A \cap B = \emptyset$, then the probability of A or B is simply the sum of the two probabilities.

$$P(A \cup B) = P(A) + P(B).$$

- If the events overlap, (are not disjoint) $A \cap B \neq \emptyset$ use the *modified addition rule*:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example Using the Addition Rule

- Suppose you throw two dice. There are 6x6=36 possible ways in which both can land.
- Event A: What is the probability that both dice show the same number?

  A={{1,1}, {2,2}, {3,3}, {4,4}, {5,5}, {6,6}} so P(A)=6/36
- Event B: What is the probability that the two die add up to eight?

  B={{2,6}, {3,5}, {4,4}, {5,3}, {6,2}} so P(B)=5/36.
- Event C: What is the probability that A *or* B happens, i.e. $P(A \cup B)$? First, note that $A \cap B$ = {{4,4}} so $P(A \cap B)$ =1/36.

  $P(A \cup B) = P(A)+P(B)-P(A \cap B) = 6/36+5/36-1/36=10/36$ (5/18).

Multiplication Rules

- The probability of event A *and* event B: $P(A \cap B)$
- Multiplication rule applies if A and B are independent events.
- A and B are independent events if P(A) does not depend on whether B occurs or not, and P(B) does not depend on whether A occurs or not.

$$P(A \cap B) = P(A) \times P(B) = P(AB)$$

- Conditional probability for *non independent events*. The probability of A given that B has occurred is $P(A|B) = P(AB)/P(B)$.

Examples Using Multiplication Rules

- An unbiased coin is flipped 5 times. What is the probability of the sequence: TTTTT?

  P(T)=.5, 5 independent flips, so .5x.5x.5x.5x.5=.03125.

- Suppose a card is drawn from a standard 52 card deck. Let B be the event: the card is a Queen, P(B)=4/52. Let A be the event: the card is the Queen of Hearts, P(A)=1/52. *Conditional* on having drawn a Queen, what is the probability that it is the Queen of Hearts, P(A|B)?
  - First note that $P(AB)=P(A\cap B)=1/52$. (Probability the card drawn is the Queen of Hearts *and* the card is a Queen).
  - $P(A|B)=P(AB)/P(B) = (1/52)/(4/52)=1/4$.

Bayes Rule

- Used for making inferences: given a particular outcome, event A, can we infer the unobserved cause of that outcome, some event B_1, B_2,...B_n?
- Suppose we know the *prior* probabilities, $P(B_i)$ and the conditional probabilities $P(A|B_i)$
- Suppose that B_1, B_2,...,B_n form a *complete partition* of the sample space S, so that $\cup_i B_i = S$ and $B_i \cap B_j = \emptyset$ for any $i \neq j$. In this case we have that:

$$P(A) = \sum_{i=1}^{n} P[A \mid B_i] P[B_i] \quad (1)$$

- Bayes rule is a formula for computing the *posterior* probabilities e.g. the probability that event B_k was the cause of outcome A, denoted $P(B_k|A)$:

$$P(B_k \mid A) = P(B_k \cap A) / P(A)$$

Using the conditional probability rule

$$= P(A \mid B_k) P(B_k) / P(A)$$

Using expression (1) above.

$$= \frac{P(A \mid B_k) P(B_k)}{\sum_{i=1}^{n} P(A \mid B_i) P(B_i)}$$

This is Bayes Rule

Bayes Rule-Special Case

- Suppose S consists of just B and *not* B, i.e. B^c
- Then Bayes rule can be stated as:

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A \mid B)P(B) + P(A \mid B^c)P(B^c)}$$

- Example: Suppose a drug test is 95% effective: the test will be positive on a drug user 95% of the time, and will be negative on a non-drug user 95% of the time. Assume 5% of the population are drug users. Suppose an individual tests positive. What is the probability he is a drug user?

Bayes Rule Example

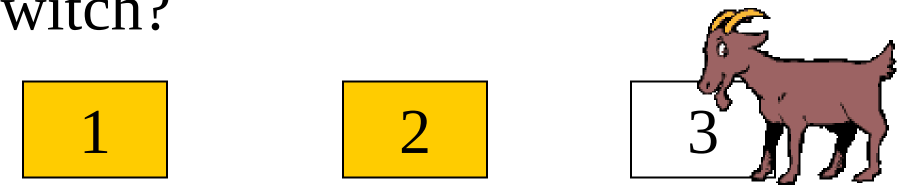
- Let A be the event that the individual tests positive. Let B be the event individual is a drug user. Let B^c be the complementary event, that the individual is *not* a drug user. Find $P(B|A)$.
- $P(A|B)=.95$. $P(A|B^c)=.05$, $P(B)=.05$, $P(B^c)=.95$

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A \mid B)P(B) + P(A \mid B^c)P(B^c)}$$

$$= \frac{(.95)(.05)}{(.95)(.05) + (.05)(.95)} = .50$$

Monty Hall's 3 Door Problem

- There are three closed doors. Behind one of the doors is a brand new sports car. Behind each of the other two doors is a smelly goat. You can't see the car or smell the goats.
- You win the prize behind the door you choose.
- The sequence of play of the game is as follows:
  - You choose a door and announce your choice.
  - The host, Monty Hall, who knows where the car is, always selects one of the two doors that you did not choose, which he knows has a goat behind it.
  - Monty then asks if you want to switch your choice to the unopened door that you did not choose.
- Should you switch?



You Should *Always* Switch

- Let C_i be the event "car is behind door i" and let G be the event: "Monty chooses a door with a goat behind it."
- Suppose without loss of generality, the contestant chooses door 1. Then Monty shows a goat behind door number 3
- According to the rules, P(G)=1, and so $P(G|C_1)=1$;
- Initially, $P(C_1)=P(C_2)=P(C_3)=1/3$. By the addition rule, we also know that $P(C_2 \cup C_3)=2/3$.
- After Monty's move, $P(C_3)=0$. $P(C_1)$ remains 1/3, but $P(C_2)$ now becomes 2/3!
- According to Bayes Rule:

$$P(C_1 | G) = \frac{P(G | C_1)P(C_1)}{P(G)} = \frac{1 \times 1/3}{1} = 1/3.$$

- It follows that $P(C_2|G)=2/3$, so the contestant always does better by *switching*; the probability is 2/3 he wins the car.

Here is Another Proof

- Let (w,x,y,z) describe the game.
  - w=your initial door choice, x=the door Monty opens, y=the door you finally decide upon and z=W/L (whether you win or lose).
  - Without loss of generality, **assume the car is behind door number 1**, and that there are goats behind door numbers 2 and 3.
  - Suppose you adopt the *never* switch strategy. The sample space under this strategy is: S=[(1,2,1,W),(1,3,1,W),(2,3,2,L),(3,2,3,L)].
  - If you choose door 2 or 3 you always lose with this strategy.
  - But, if you initially choose one of the three doors randomly, it must be that the outcome (2,3,2,L) and (3,2,3,L) each occur with probability 1/3. That means the two outcomes (1,2,1,W) and (1,3,1,W) have the remaining 1/3 probability $\Rightarrow$ you win with probability 1/3.
  - Suppose you adopt the *always* switch strategy. The sample space under this strategy is: S=[(1,2,3,L),(1,3,2,L),(2,3,1,W),(3,2,1,W)].
  - Since you initially choose door 2 with probability 1/3 and door 3 with probability 1/3, the probability you win with the switching strategy is 1/3+1/3=2/3 $\Rightarrow$ you should always switch.

Expected Value (or Payoff)

- One use of probabilities to calculate expected values (or payoffs) for uncertain outcomes.
- Suppose that an outcome, e.g. a money payoff is uncertain. There are n possible values, X_1, X_2,...,$X_{N.}$ Moreover, we know the probability of obtaining each value.
- The **expected value (or expected payoff)** of the uncertain outcome is then given by:

$$P(X_1)X_1+P(X_2)X_2+\ldots P(X_N)X_{N.}$$

An Example

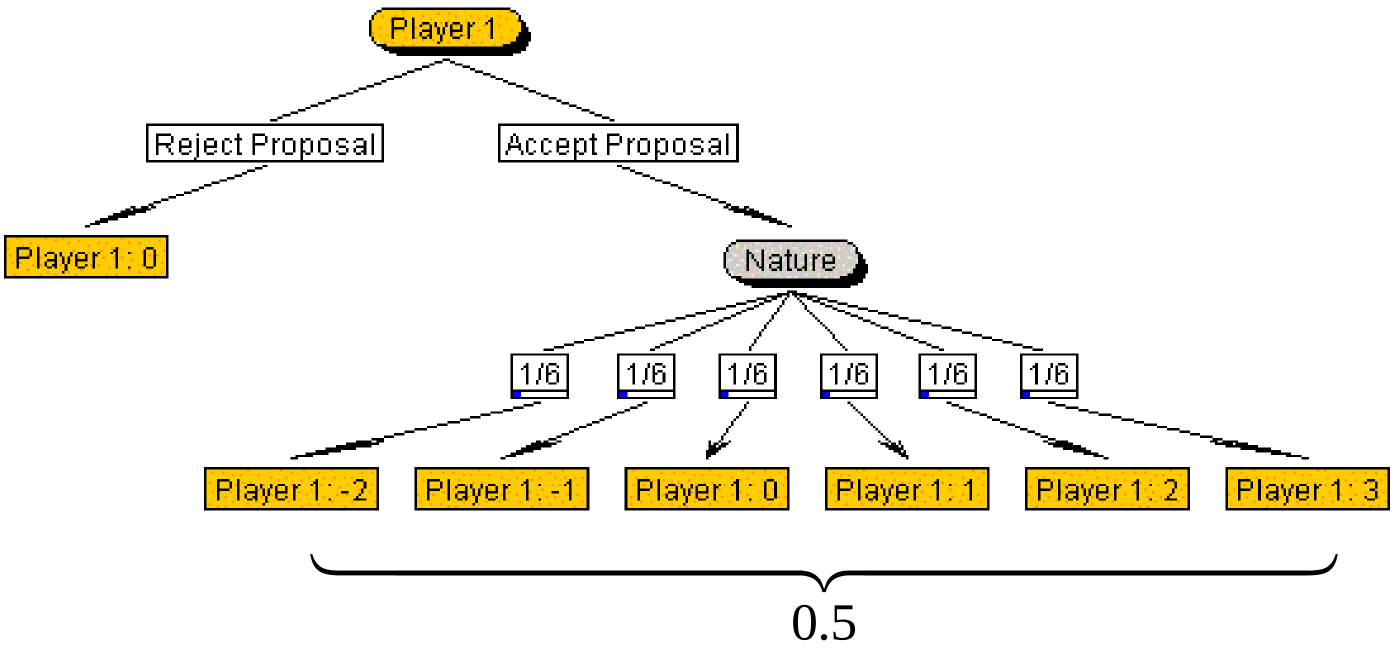
- You are made the following proposal: You pay \$3 for the right to roll a die once. You then roll the die and are paid the number of dollars shown on the die. Should you accept the proposal?
- The *expected payoff* of the uncertain die throw is:

$$\frac{1}{6}\times\$1+\frac{1}{6}\times\$2+\frac{1}{6}\times\$3+\frac{1}{6}\times\$4+\frac{1}{6}\times\$5+\frac{1}{6}\times\$6=\$3.50$$

- The expected payoff from the die throw is greater than the \$3 price, so a (risk neutral) player accepts the proposal.

Extensive Form Illustration: *Nature* as a Player

- Payoffs are in *net* terms: $ winnings-$3.



Another Example: Deal or No Deal

http://www.youtube.com/watch?v=GfDaplU9zuw

- Two cases remain:
  - One has $1,000,0000 in it.
  - The other has $200,000.
- Deal is for $561,000.
- Deal or No Deal?

Accounting for Risk Aversion

- The assumption that individuals treat *expected payoffs* the same as *certain payoffs* (i.e. that they are *risk neutral*) may not hold in practice.
- Recall our earlier examples:
  - A risk neutral person is indifferent between \$25 for certain or a 25% chance of earning \$100 and a 75% chance of earning 0.
  - A risk neutral person agrees to pay \$3 to roll a die once and receive as payment the number of dollars shown on the die.
- Many people are *risk averse* and prefer \$25 with certainty to the uncertain *gamble*, or might be unwilling to pay \$3 for the right to roll the die once, so imagining that people base their decisions on expected payoffs alone *may yield misleading results*.
- What can we do to account for the fact that many people are risk averse? We can use the concept of *expected utility*.

Utility Function Transformation

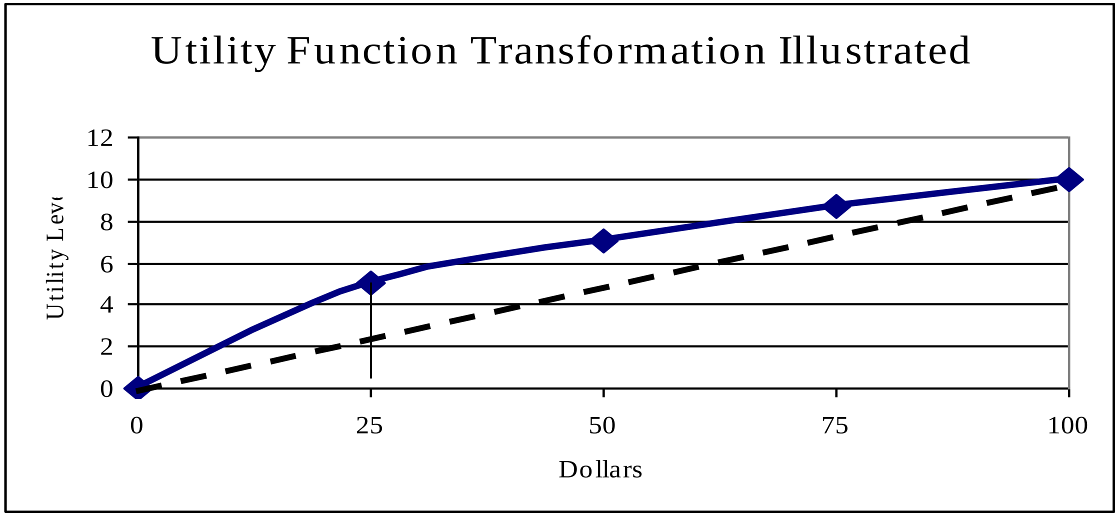
- Let *x* be the payoff amount in dollars, and let *U(x)* be a *continuous*, *increasing* function of x.
- The function U(x) gives an individual's level of satisfaction in fictional "utils" from receiving payoff amount x, and is known as a *utility function*.
- If the certain payoff of $25 is preferred to the gamble, (due to risk aversion) then we want a utility function that satisfies:

  U($25) > .25 U($100) +.75 U($0).
- The left hand side is the utility of the certain payoff and the right hand side is the *expected utility* from the gamble.
- In this case, *any concave* function *U(x)* will work, e.g. $U(X)=\sqrt{X}$

$$\sqrt{25} > .25\sqrt{100} + .75\sqrt{0}, \Leftrightarrow 5 > 2.5$$

Graphical Illustration

- The blue line shows the utility of any *certain monetary payoff* between \$0 and \$100, assuming $U(X) = \sqrt{X}$



- Utility diminishes with increases in monetary payoff – this is just the principle of diminishing marginal utility (requires risk aversion).
- Black (dashed) line shows the *expected utility* of risky payoff
- At \$25, the certain payoff yields higher utility than the risky payoff.

Another Example

- If keeping \$3 were preferred to rolling a die and getting paid the number of dollars that turns up (expected payoff \$3.5) we need a utility function that satisfied:

$$U(\$3) > \frac{1}{6}U(\$1) + \frac{1}{6}U(\$2) + \frac{1}{6}U(\$3) + \frac{1}{6}U(\$4) + \frac{1}{6}U(\$5) + \frac{1}{6}U(\$6)$$

- In this case, where the expected payoff \$3.5 is strictly higher than the certain amount – the \$3 price – the utility function must be *sufficiently concave* for the above relation to hold.
  - If we used $U(x) = \sqrt{x} = x^{1/2},$ we would find that the left-hand-side of the expression above was $\sqrt{3} = 1.732,$ while the right-hand-side equals 1.805, so we need a *more concave* function.
- We would need a utility function transformation of

$$U(x) = x^{1/100}$$

for the inequality above to hold, (50 times more risk aversion)!

Summing up

- The notions of probability and expected payoff are frequently encountered in game theory.
- We mainly assume that players are *risk neutral*, so they seek to maximized *expected payoff*.
- We are aware that expected monetary payoff might not be the relevant consideration – that aversion to risk may play a role.
- We have seen how to transform the objective from payoff to utility maximization so as to capture the possibility of risk aversion – the trick is to assume some concave utility function transformation.
- Now that we know how to deal with risk aversion, we are going to largely ignore it, and assume risk neutral behavior ☺