Computer Graphics Formulae

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Contents

1 Vectors

1.1 Dot Product

$$\langle a, b \rangle = \sum_{i=1}^{3} a_i b_i$$

1.2 Cross Product

$$a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

1.3 Magnitude

$$||a|| = \sqrt{\sum_{i=1}^{3} a_i^2}$$

2 Matrices

2.1 Inverse (Diagonal Matrix)

$$A^{-1} = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 & 0 \\ 0 & \frac{1}{b} & 0 & 0 \\ 0 & 0 & \frac{1}{c} & 0 \\ 0 & 0 & 0 & \frac{1}{d} \end{bmatrix}$$

2.2 Transpose

$$A^T = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}^T = \begin{bmatrix} a & e & i & m \\ b & f & j & n \\ c & g & k & o \\ d & h & l & p \end{bmatrix}$$

3 Rays and Objects

3.1 Ray

$$\gamma(t) = o + dt$$

3.2 Ray-Sphere Intersection

$$a = \langle c, d \rangle$$
$$D = \sqrt{\|\ c\ \|^2 - \langle c, d \rangle^2}$$
$$t_{1,2} = \langle c, d \rangle \pm \sqrt{r^2 - D^2}$$

Where c is the vector from the eye to the center of the sphere, d is the viewing direction, and r is the ray of the circle.

3.3 Ray-Cone Intersection

$$a = d_x^2 + d_z^2 - d_y^2$$
$$b = 2(d_x o_x - d_z o_z - d_y o_y)$$
$$c = o_x^2 + o_z^2 - o_y^2$$
$$t_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where d is the view direction, and o is the position of the eye.

3.4 Ray-Triangle Intersection

$$p = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$p_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}$$
$$n = (p_2 - p_1) \times (p_3 - p1)$$
$$n_i = (p_{i+1} - p) \times (p_{i-1} - p)$$
$$W = \frac{||n||}{2}$$
$$w_i = \frac{||n_i||}{2} \cdot \operatorname{sign}(\langle n_i, n \rangle)$$
$$\lambda_1(p) = \frac{w_1}{W} \qquad \lambda_2(p) = \frac{w_2}{W} \qquad \lambda_3(p) = \frac{w_3}{W}$$

Where $\lambda_1(p)$, $\lambda_2(p)$ and $\lambda_3(p)$ are the three barycentric coordinates.

4 Illumination

4.1 Diffuse Reflection

$$I_d = \rho_d \cdot \langle n, l \rangle \cdot I$$

Where ρ_d is the diffuse coefficient, n is the normal vector of the point, l is the vector pointing to the light source, and I is the intensity of the light.

4.2 Ambient Illumination

$$I_a = \rho_a \cdot I$$

Where ρ_a is the ambient coefficient, and I is the ambient light intensity.

4.3 Specular Reflection

$$r = 2n \cdot \langle n, l \rangle - l$$
$$I_s = \rho_s \cdot \langle r, v \rangle^k \cdot I$$

Where n is the normal vector of the point, l is the vector pointing to the light source, ρ_s is the reflection coefficient, r is the reflection ray, v is the viewing direction, k is the shininess of the object, and I is the intensity of the light source.

4.4 Blinn-Phong Specular Reflection

$$h = \frac{1}{2}(l + v)$$
$$I_s = \rho_s \cdot \langle n, h \rangle^{4k} \cdot I$$

Where l is the vector pointing to the light source, v is the viewing direction, ρ_s is the reflection coefficient, n is the normal of the point, h is the bisection vector, k is the shininess of the object, and I is the intensity of the light source.

4.5 Distance Attenuation

$$att(r) = \frac{1}{a_1 + a_2 r + a_3 r^2}$$

Where r is the ray of light, and a_1, a_2 and a_3 are constant values.

4.6 Phong Lighting Model

$$I = I_e + \rho_a \cdot I_a + \sum_{j=1}^{n} (\rho_d \cdot \langle n, l_j \rangle + \rho_s \cdot \langle n, h_j \rangle^{4k}) \cdot I_j \cdot att(||l_j||) \cdot s_j(p)$$

Where I_e is the self-emitting intensity, ρ_a is the diffuse coefficient, I_a is the ambient intensity, ρ_d is the diffuse coefficient, n is the normal vector, l is the vector pointing to the light source, ρ_s is the specular coefficient, h is the bisection vector, v is the direction vector, k is the shininess, I_j is the intensity of the j-th light source, and $s_j(p)$ indicates if the point p indicates whether the point is in shadow or not $-$ 1 p is not in shadow, 0 p is in shadow.

4.7 Gamma Correction and Tone Mapping

$$I_{in} = \max((\alpha \cdot I^{\beta})^{\frac{1}{\gamma}}, 1.0)$$

Where I_{in} is the input for our display, I is the intensity computed by the Phong model, γ is the gamma coefficient, and α and β are the tone mapping coefficients.

5 Transformations

5.1 Translation

$$T = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5.2 Shear

$$S_{yz} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ d_y & 1 & 0 & 0 \\ d_z & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad S_{xz} = \begin{bmatrix} 1 & d_x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & d_z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S_{xy} = \begin{bmatrix} 1 & 0 & d_x & 0 \\ 0 & 1 & d_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5.3 Scale

$$S = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5.4 Rotation

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad R_y = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5.5 Normal

$$n_{new} = Mn$$
$$n_{new} = (M^{-1})^T n$$

Where M is the transformation matrix. The first equation is used in the case that the transformation preserves angles, otherwise we use the second equations.

6 Advanced

6.1 Snell's Law

$$\frac{\sin\Theta_1}{\sin\Theta_2} = \frac{\delta_1}{\delta_2} = \frac{v_1}{v_2}$$

Where δ is the index of refraction of the medium, and v is the velocity of light in the medium.

6.2 Reflection

$$r = i - 2n \cdot \langle n, i \rangle$$

Where r is the reflection vector, i is the viewing direction, and n is the normal of the point.

6.3 Refraction

$$
\begin{aligned}
a &= n \cdot \langle n, i \rangle \\
b &= i - a \\
\beta &= \frac{\delta_1}{\delta_2} \\
\alpha &= \sqrt{1 + (1 - \beta^2)\frac{||b||^2}{||a||^2}} \\
r &= \alpha a + \beta b
\end{aligned}
$$

Where n is the normal of the vector, i is the viewing direction, and δ is the index of refraction of the medium.

6.4 Fresnel Effect

$$
F_{\mathrm{Rl}} = \frac{1}{2}\left(\left(\frac{\delta_2 \cos\Theta_1 - \delta_1 \cos\Theta_2}{\delta_2 \cos\Theta_1 + \delta_1 \cos\Theta_2}\right)^2 + \left(\frac{\delta_1 \cos\Theta_1 - \delta_2 \cos\Theta_2}{\delta_1 \cos\Theta_1 + \delta_2 \cos\Theta_2}\right)^2\right)
$$

$$
F_{\mathrm{Rf}} = 1 - F_{\mathrm{Rl}}
$$

7 Transformation Pipeline

7.1 Vectors

$$
\begin{aligned}
eye &= VP \\
z' &= \frac{VPN}{||VPN||} \\
x' &= \frac{VUP \times z'}{||VUP|| \times z'||} \\
y' &= z' \times x'
\end{aligned}
$$

Where VP is the camera position, VPN is the view plane normal, and VUP is the view up vector

7.2 Viewing Matrix

$$
VM = \begin{bmatrix} - & x^T & - & -x'^T \cdot eye \\ - & y^T & - & -y'^T \cdot eye \\ - & z^T & - & -z'^T \cdot eye \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

7.3 Projection Matrix

$$PM = \begin{bmatrix} \frac{1}{r} & 0 & 0 & 0 \\ 0 & \frac{1}{t} & 0 & 0 \\ 0 & 0 & \frac{-2}{f-n} & -\frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

8 Texture Pipeline

8.1 Normal Mapping

$$n_{new} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = 2 \begin{bmatrix} R \\ G \\ B \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

8.2 Tangent Space

$$p_i - p_j = \langle (u_i - u_j), T \rangle + \langle (v_i - v_j), B \rangle$$
$$T = T - \langle N, T \rangle \cdot N$$
$$B = N \times T$$

Where T is the tangent, B is the bitangent, and N is the normal to the point.