Assignment 4

Computer Graphics

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1 Exercise 1

1.1 Task 1

The two matrices R_{90} and T in homogeneous coordinates are as follows.

$$R_{90} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} \cos 90 & -\sin 90 & 0 \\ \sin 90 & \cos 90 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1.2 Task 2

The following are the computations for the rotated and translated points and vector.

$$\begin{aligned} p_1' &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \cdot 1 + (-1) \cdot 1 + 0 \cdot 1 \\ 1 \cdot 1 + 0 \cdot 1 + 0 \cdot 1 \\ 0 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot (-1) + 0 \cdot 1 + 1 \cdot 1 \\ 0 \cdot (-1) + 1 \cdot 1 + (-2) \cdot 1 \\ 0 \cdot (-1) + 0 \cdot 1 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ -1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} p_2' &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \cdot 3 + (-1) \cdot 2 + 0 \cdot 1 \\ 1 \cdot 3 + 0 \cdot 2 + 0 \cdot 1 \\ 0 \cdot 3 + 0 \cdot 2 + 1 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot (-2) + 0 \cdot 3 + 1 \cdot 1 \\ 0 \cdot (-2) + 1 \cdot 3 + (-2) \cdot 1 \\ 0 \cdot (-2) + 0 \cdot 3 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} p_2' &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \cdot 1 + (-1) \cdot 4 + 0 \cdot 1 \\ 1 \cdot 1 + 0 \cdot 4 + 0 \cdot 1 \\ 0 \cdot 1 + 0 \cdot 4 + 1 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot (-4) + 0 \cdot 1 + 1 \cdot 1 \\ 0 \cdot (-4) + 1 \cdot 1 + (-2) \cdot 1 \\ 0 \cdot (-4) + 0 \cdot 1 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -3 \\ -1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} p' &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.5 \\ 2.5 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \cdot 1.5 + (-1) \cdot 2.5 + 0 \cdot 1 \\ 1 \cdot 1.5 + 0 \cdot 2.5 + 0 \cdot 1 \\ 0 \cdot 1.5 + 0 \cdot 2.5 + 1 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2.5 \\ 1.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot (-2.5) + 0 \cdot 1.5 + 1 \cdot 1 \\ 0 \cdot (-2.5) + 1 \cdot 1.5 + (-2) \cdot 1 \\ 0 \cdot (-2.5) + 0 \cdot 1.5 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} -1.5 \\ -0.5 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -1.5 \\ -0.5 \end{bmatrix} \end{aligned}$$

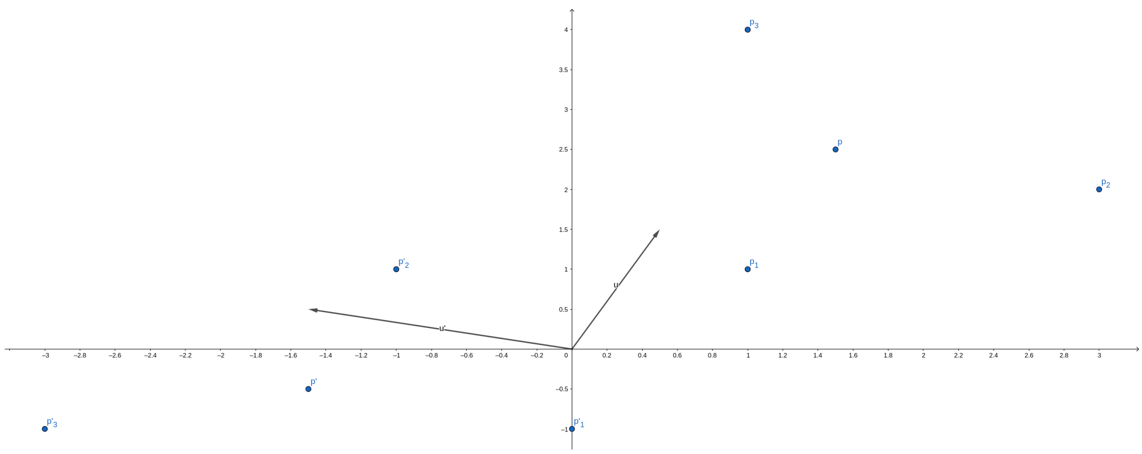
$$u' = \begin{bmatrix}1 & 0 & 1\\0 & 1 & -2\\0 & 0 & 1\end{bmatrix}\left(\begin{bmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}0.5\\1.5\\0\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 1\\0 & 1 & -2\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\cdot 0.5 + (-1)\cdot 1.5 + 0\cdot 0\\1\cdot 0.5 + 0\cdot 1.5 + 0\cdot 0\\0\cdot 0.5 + 0\cdot 1.5 + 1\cdot 0\end{bmatrix}$$
$$= \begin{bmatrix}1 & 0 & 1\\0 & 1 & -2\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}-1.5\\0.5\\0\end{bmatrix} = \begin{bmatrix}1\cdot(-1.5) + 0\cdot 0.5 + 1\cdot 0\\0\cdot(-1.5) + 1\cdot 0.5 + (-2)\cdot 0\\0\cdot(-1.5) + 0\cdot 0.5 + 1\cdot 0\end{bmatrix} = \begin{bmatrix}-1.5\\0.5\\0\end{bmatrix} \mapsto \begin{bmatrix}-1.5\\0.5\end{bmatrix}$$

Now we can see that $u' = p' - p'_1$, as shown below:

$$\begin{bmatrix}-1.5\\0.5\\0\end{bmatrix} = \begin{bmatrix}-1.5\\-0.5\\1\end{bmatrix} - \begin{bmatrix}0\\-1\\1\end{bmatrix} = \begin{bmatrix}-1.5\\0.5\\0\end{bmatrix}$$

The translation only works with points, while the rotation works with both points and vectors. For this reason, the vector u' is only affected by the rotation matrix, and not by the translation matrix.

Finally, here is a visual representation of all the points and vector before and after the transformations.



1.3 Task 3

Here we have the scaling matrix which scales both x and y by a factor of 2.

$$S = \begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 1\end{bmatrix}$$

The following are the computations for the scaled points and vector.

$$p''_1 = \begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\-1\\1\end{bmatrix} = \begin{bmatrix}2\cdot 0 + 0\cdot(-1) + 0\cdot 1\\0\cdot 0 + 2\,(-1) + 0\cdot 1\\0\cdot 0 + 0\cdot(-1) + 1\cdot 1\end{bmatrix} = \begin{bmatrix}0\\-2\\1\end{bmatrix} \mapsto \begin{bmatrix}0\\-2\end{bmatrix}$$

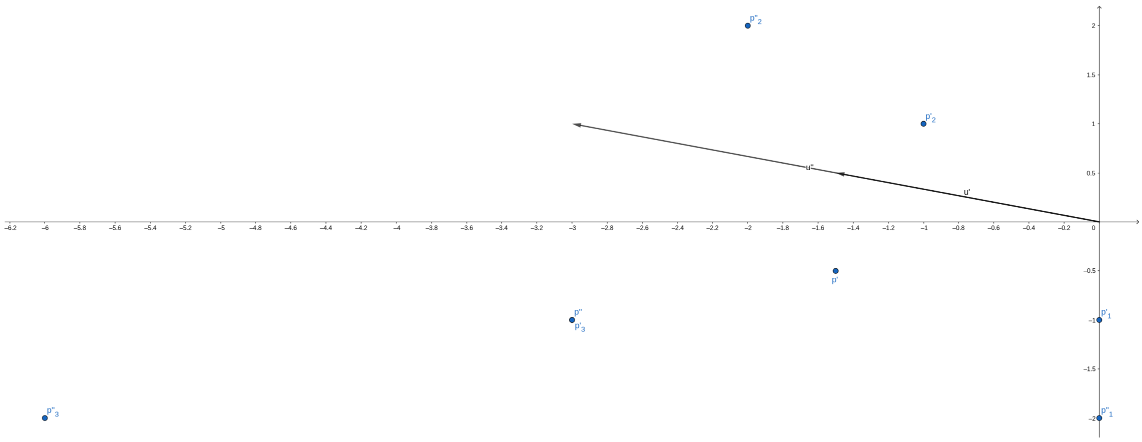
$$p''_2 = \begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}-1\\1\\1\end{bmatrix} = \begin{bmatrix}2\,(-1) + 0\cdot 1 + 0\cdot 1\\0\cdot(-1) + 2\cdot 1 + 0\cdot 1\\0\cdot(-1) + 0\cdot 1 + 1\cdot 1\end{bmatrix} = \begin{bmatrix}-2\\2\\1\end{bmatrix} \mapsto \begin{bmatrix}-2\\2\end{bmatrix}$$

$$p''_3 = \begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}-3\\-1\\1\end{bmatrix} = \begin{bmatrix}2\,(-3) + 0\cdot(-1) + 0\cdot 1\\0\cdot(-3) + 2\,(-1) + 0\cdot 1\\0\cdot(-3) + 0\cdot(-1) + 1\cdot 1\end{bmatrix} = \begin{bmatrix}-6\\-2\\1\end{bmatrix} \mapsto \begin{bmatrix}-6\\-2\end{bmatrix}$$

$$p'' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1.5 \\ -0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\,(-1.5) + 0 \cdot (-0.5) + 0 \cdot 1 \\ 0 \cdot (-1.5) + 2\,(-0.5) + 0 \cdot 1 \\ 0 \cdot (-1.5) + 0 \cdot (-0.5) + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

$$u'' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1.5 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\,(-1.5) + 0 \cdot 0.5 + 0 \cdot 0 \\ 0 \cdot (-1.5) + 2 \cdot 0.5 + 0 \cdot 0 \\ 0 \cdot (-1.5) + 0 \cdot 0.5 + 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Finally, here is a visual representation of the points and the vector before and after the scaling.



1.4 Task 4

In order to apply the inverse transformations, we need to compute the inverse matrices for rotation, translation and scaling.

$$R_{90}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And we compute the transformation matrix M.

$$\begin{aligned} M = R_{90}^{-1} T^{-1} S^{-1} &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 0 & 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 & 0 \cdot (-1) + 1 \cdot 2 + 0 \cdot 1 \\ (-1) \cdot 1 + 0 \cdot 0 + 0 \cdot 0 & (-1) \cdot 0 + 0 \cdot 1 + 0 \cdot 0 & (-1)\,(-1) + 0 \cdot 2 + 0 \cdot 1 \\ 0 \cdot 1 + 0 \cdot 0 + 1 \cdot 0 & 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 & 0 \cdot (-1) + 0 \cdot 2 + 1 \cdot 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \cdot 0.5 + 1 \cdot 0 + 2 \cdot 0 & 0 \cdot 0 + 1 \cdot 0.5 + 2 \cdot 0 & 0 \cdot 0 + 1 \cdot 0 + 2 \cdot 1 \\ (-1) \cdot 0.5 + 0 \cdot 0 + 1 \cdot 0 & (-1) \cdot 0 + 0 \cdot 0.5 + 1 \cdot 0 & (-1) \cdot 0 + 0 \cdot 0 + 1 \cdot 1 \\ 0 \cdot 0.5 + 0 \cdot 0 + 1 \cdot 0 & 0 \cdot 0 + 0 \cdot 0.5 + 1 \cdot 0 & 0 \cdot 0 + 0 \cdot 0 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 & 0.5 & 2 \\ -0.5 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Now that we have the transformation matrix M, we can finally compute the transformation of points p_1'', p_2'', p_3'', p'' and of vector u''.

$$p_1 = \begin{bmatrix} 0 & 0.5 & 2 \\ -0.5 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \cdot 0 + 0.5\,(-2) + 2 \cdot 1 \\ (-0.5) \cdot 0 + 0 \cdot (-2) + 1 \cdot 1 \\ 0 \cdot 0 + 0 \cdot (-2) + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$p_2 = \begin{bmatrix} 0 & 0.5 & 2 \\ -0.5 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \cdot (-2) + 0.5 \cdot 2 + 2 \cdot 1 \\ (-0.5)\,(-2) + 0 \cdot 2 + 1 \cdot 1 \\ 0 \cdot (-2) + 0 \cdot 2 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$p_2 = \begin{bmatrix} 0 & 0.5 & 2 \\ -0.5 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -6 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \cdot (-6) + 0.5\,(-2) + 2 \cdot 1 \\ (-0.5)\,(-6) + 0 \cdot (-2) + 1 \cdot 1 \\ 0 \cdot (-6) + 0 \cdot (-2) + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

$$p = \begin{bmatrix} 0 & 0.5 & 2 \\ -0.5 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \cdot (-3) + 0.5\,(-1) + 2 \cdot 1 \\ (-0.5)\,(-3) + 0 \cdot (-1) + 1 \cdot 1 \\ 0 \cdot (-3) + 0 \cdot (-1) + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 2.5 \\ 1 \end{bmatrix}$$

$$u = \begin{bmatrix} 0 & 0.5 & 2 \\ -0.5 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \cdot (-3) + 0.5 \cdot 1 + 2 \cdot 0 \\ (-0.5)\,(-3) + 0 \cdot 1 + 1 \cdot 0 \\ 0 \cdot (-3) + 0 \cdot 1 + 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.5 \\ 0 \end{bmatrix}$$

As we can see, by multiplying the transformation matrix with p_1'', p_2'', p_3'', p'' and u'', we will get back p_1, p_2, p_3, p and u.