Assignment 4

Computer Graphics

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1 Exercise 1

1.1 Task 1

The two matrices R_{90} and T in homogeneous coordinates are as follows.

$$R_{90} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} \cos 90 & -\sin 90 & 0 \\ \sin 90 & \cos 90 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1.2 Task 2

The following are the computations for the rotated and translated points and vector.

$$\begin{aligned} p_1' &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \cdot 1 + (-1) \cdot 1 + 0 \cdot 1 \\ 1 \cdot 1 + 0 \cdot 1 + 0 \cdot 1 \\ 0 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot (-1) + 0 \cdot 1 + 1 \cdot 1 \\ 0 \cdot (-1) + 1 \cdot 1 + (-2) \cdot 1 \\ 0 \cdot (-1) + 0 \cdot 1 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ -1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} p_2' &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \cdot 3 + (-1) \cdot 2 + 0 \cdot 1 \\ 1 \cdot 3 + 0 \cdot 2 + 0 \cdot 1 \\ 0 \cdot 3 + 0 \cdot 2 + 1 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot (-2) + 0 \cdot 3 + 1 \cdot 1 \\ 0 \cdot (-2) + 1 \cdot 3 + (-2) \cdot 1 \\ 0 \cdot (-2) + 0 \cdot 3 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} p_2' &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \cdot 1 + (-1) \cdot 4 + 0 \cdot 1 \\ 1 \cdot 1 + 0 \cdot 4 + 0 \cdot 1 \\ 0 \cdot 1 + 0 \cdot 4 + 1 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot (-4) + 0 \cdot 1 + 1 \cdot 1 \\ 0 \cdot (-4) + 1 \cdot 1 + (-2) \cdot 1 \\ 0 \cdot (-4) + 0 \cdot 1 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -3 \\ -1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} p' &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.5 \\ 2.5 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \cdot 1.5 + (-1) \cdot 2.5 + 0 \cdot 1 \\ 1 \cdot 1.5 + 0 \cdot 2.5 + 0 \cdot 1 \\ 0 \cdot 1.5 + 0 \cdot 2.5 + 1 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2.5 \\ 1.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot (-2.5) + 0 \cdot 1.5 + 1 \cdot 1 \\ 0 \cdot (-2.5) + 1 \cdot 1.5 + (-2) \cdot 1 \\ 0 \cdot (-2.5) + 0 \cdot 1.5 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} -1.5 \\ -0.5 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -1.5 \\ -0.5 \end{bmatrix} \end{aligned}$$

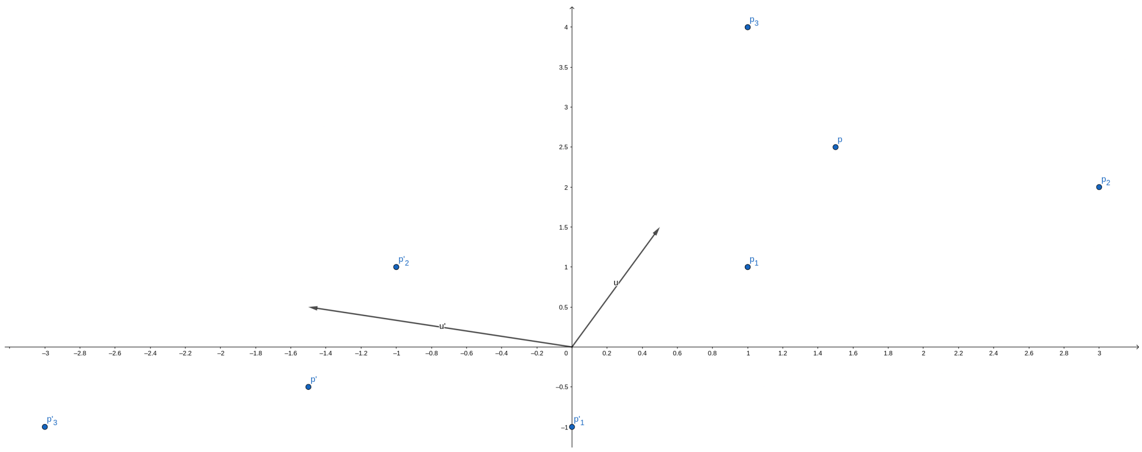
$$
\begin{aligned}
u' &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1.5 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \cdot 0.5 + (-1) \cdot 1.5 + 0 \cdot 0 \\ 1 \cdot 0.5 + 0 \cdot 1.5 + 0 \cdot 0 \\ 0 \cdot 0.5 + 0 \cdot 1.5 + 1 \cdot 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1.5 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot (-1.5) + 0 \cdot 0.5 + 1 \cdot 0 \\ 0 \cdot (-1.5) + 1 \cdot 0.5 + (-2) \cdot 0 \\ 0 \cdot (-1.5) + 0 \cdot 0.5 + 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} -1.5 \\ 0.5 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} -1.5 \\ 0.5 \end{bmatrix}
\end{aligned}
$$

Now we can see that $u' = p' - p'_1$, as shown below:

$$
\begin{bmatrix} -1.5 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} -1.5 \\ -0.5 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1.5 \\ 0.5 \\ 0 \end{bmatrix}
$$

The translation only works with points, while the rotation works with both points and vectors. For this reason, the vector u' is only affected by the rotation matrix, and not by the translation matrix.

Finally, here is a visual representation of all the points and vector before and after the transformations.



1.3 Task 3

Here we have the scaling matrix which scales both x and y by a factor of 2.

$$
S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

The following are the computations for the scaled points and vector.

$$
p''_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 0 + 0 \cdot (-1) + 0 \cdot 1 \\ 0 \cdot 0 + 2\,(-1) + 0 \cdot 1 \\ 0 \cdot 0 + 0 \cdot (-1) + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ -2 \end{bmatrix}
$$

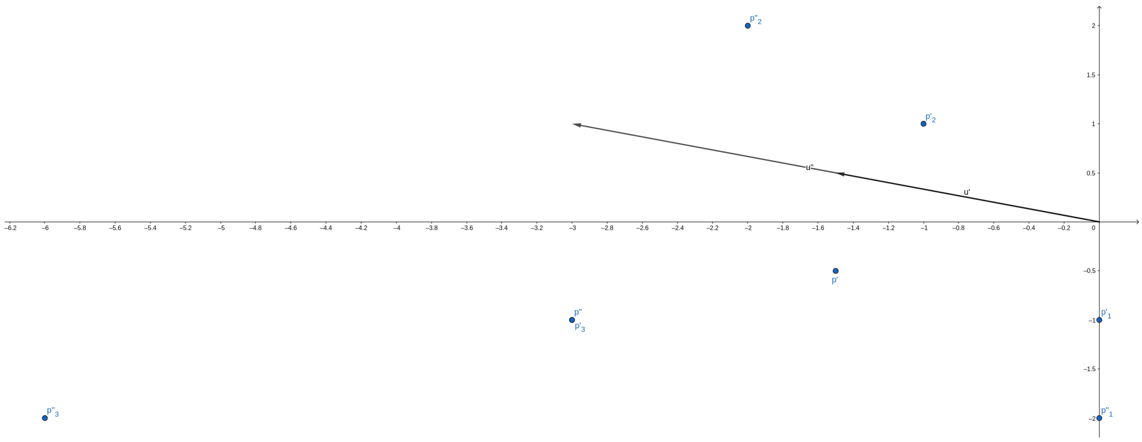
$$
p''_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\,(-1) + 0 \cdot 1 + 0 \cdot 1 \\ 0 \cdot (-1) + 2 \cdot 1 + 0 \cdot 1 \\ 0 \cdot (-1) + 0 \cdot 1 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -2 \\ 2 \end{bmatrix}
$$

$$
p''_3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\,(-3) + 0 \cdot (-1) + 0 \cdot 1 \\ 0 \cdot (-3) + 2\,(-1) + 0 \cdot 1 \\ 0 \cdot (-3) + 0 \cdot (-1) + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -6 \\ -2 \end{bmatrix}
$$

$$p'' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1.5 \\ -0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\,(-1.5) + 0 \cdot (-0.5) + 0 \cdot 1 \\ 0 \cdot (-1.5) + 2\,(-0.5) + 0 \cdot 1 \\ 0 \cdot (-1.5) + 0 \cdot (-0.5) + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

$$u'' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1.5 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\,(-1.5) + 0 \cdot 0.5 + 0 \cdot 0 \\ 0 \cdot (-1.5) + 2 \cdot 0.5 + 0 \cdot 0 \\ 0 \cdot (-1.5) + 0 \cdot 0.5 + 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Finally, here is a visual representation of the points and the vector before and after the scaling.



1.4 Task 4

In order to apply the inverse transformations, we need to compute the inverse matrices for rotation, translation and scaling.

$$R_{90}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And we compute the transformation matrix M.

$$\begin{aligned}
M = R_{90}^{-1} T^{-1} S^{-1} &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 0 & 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 & 0 \cdot (-1) + 1 \cdot 2 + 0 \cdot 1 \\ (-1) \cdot 1 + 0 \cdot 0 + 0 \cdot 0 & (-1) \cdot 0 + 0 \cdot 1 + 0 \cdot 0 & (-1)\,(-1) + 0 \cdot 2 + 0 \cdot 1 \\ 0 \cdot 1 + 0 \cdot 0 + 1 \cdot 0 & 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 & 0 \cdot (-1) + 0 \cdot 2 + 1 \cdot 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \cdot 0.5 + 1 \cdot 0 + 2 \cdot 0 & 0 \cdot 0 + 1 \cdot 0.5 + 2 \cdot 0 & 0 \cdot 0 + 1 \cdot 0 + 2 \cdot 1 \\ (-1) \cdot 0.5 + 0 \cdot 0 + 1 \cdot 0 & (-1) \cdot 0 + 0 \cdot 0.5 + 1 \cdot 0 & (-1) \cdot 0 + 0 \cdot 0 + 1 \cdot 1 \\ 0 \cdot 0.5 + 0 \cdot 0 + 1 \cdot 0 & 0 \cdot 0 + 0 \cdot 0.5 + 1 \cdot 0 & 0 \cdot 0 + 0 \cdot 0 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 & 0.5 & 2 \\ -0.5 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Now that we have the transformation matrix M, we can finally compute the transformation of points p_1'', p_2'', p_3'', p'' and of vector u''.

$$p_1 = \begin{bmatrix} 0 & 0.5 & 2 \\ -0.5 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \cdot 0 + 0.5\,(-2) + 2 \cdot 1 \\ (-0.5) \cdot 0 + 0 \cdot (-2) + 1 \cdot 1 \\ 0 \cdot 0 + 0 \cdot (-2) + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$p_2 = \begin{bmatrix} 0 & 0.5 & 2 \\ -0.5 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \cdot (-2) + 0.5 \cdot 2 + 2 \cdot 1 \\ (-0.5)\,(-2) + 0 \cdot 2 + 1 \cdot 1 \\ 0 \cdot (-2) + 0 \cdot 2 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$p_2 = \begin{bmatrix} 0 & 0.5 & 2 \\ -0.5 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -6 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \cdot (-6) + 0.5\,(-2) + 2 \cdot 1 \\ (-0.5)\,(-6) + 0 \cdot (-2) + 1 \cdot 1 \\ 0 \cdot (-6) + 0 \cdot (-2) + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

$$p = \begin{bmatrix} 0 & 0.5 & 2 \\ -0.5 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \cdot (-3) + 0.5\,(-1) + 2 \cdot 1 \\ (-0.5)\,(-3) + 0 \cdot (-1) + 1 \cdot 1 \\ 0 \cdot (-3) + 0 \cdot (-1) + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 2.5 \\ 1 \end{bmatrix}$$

$$u = \begin{bmatrix} 0 & 0.5 & 2 \\ -0.5 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \cdot (-3) + 0.5 \cdot 1 + 2 \cdot 0 \\ (-0.5)\,(-3) + 0 \cdot 1 + 1 \cdot 0 \\ 0 \cdot (-3) + 0 \cdot 1 + 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.5 \\ 0 \end{bmatrix}$$

As we can see, by multiplying the transformation matrix with p_1'', p_2'', p_3'', p'' and u'', we will get back p_1, p_2, p_3, p and u.

1.5 Task 5

First we need to compute the barycentric coordinates of point p with respect to the triangle made up by the points p_1, p_2 and p_3. To do so we use the following system of linear equations.

$$\begin{cases} x = x_1 + \lambda_2(x_2 - x_1) + \lambda_3(x_3 - x_1) \\ y = y_1 + \lambda_2(y_2 - y_1) + \lambda_3(y_3 - y_1) \end{cases}$$

And compute λ_1 as follows.

$$\lambda_1 = 1 - \lambda_2 - \lambda_3$$

Now we compute the actual values.

$$\begin{cases} 1.5 = 1 + \lambda_2(3-1) + \lambda_3(1-1) \\ 2.5 = 1 + \lambda_2(2-1) + \lambda_3(4-1) \end{cases}$$

$$\begin{cases} 1.5 = 1 + 2\lambda_2 \\ 2.5 = 1 + \lambda_2 + 3\lambda_3 \end{cases}$$

$$\begin{cases} -2\lambda_2 = -1.5 + 1 \\ -3\lambda_3 = -2.5 + 1 + \lambda_2 \end{cases}$$

$$\begin{cases} -2\lambda_2 = -0.5 \\ -3\lambda_3 = -2.5 + 1 + \lambda_2 \end{cases}$$

$$\begin{cases} \lambda_2 = \dfrac{0.5}{2} \\ 3\lambda_3 = 2.5 - 1 - \lambda_2 \end{cases}$$

$$\begin{cases} \lambda_2 = 0.25 \\ \lambda_3 = \dfrac{2.5 - 1 - 0.25}{3} \end{cases}$$

$$\begin{cases} \lambda_2 = 0.25 \\ \lambda_3 = 0.42 \end{cases}$$

$$\lambda_1 = 1 - 0.25 - 0.42 = 0.33$$

Thus we have the following values for the lambdas.

$$\lambda_1 = \frac{1}{3}$$

$$\lambda_2 = \frac{1}{4}$$

$$\lambda_3 = \frac{5}{12}$$

Now we compute the barycentric coordinates of point p'' with respect to the triangle made up by the points p_1'', p_2'' and p_3''. To do so we use the following system of linear equations.

$$\begin{cases} x = x_1 + \lambda_2(x_2 - x_1) + \lambda_3(x_3 - x_1) \\ y = y_1 + \lambda_2(y_2 - y_1) + \lambda_3(y_3 - y_1) \end{cases}$$

And compute λ_1 as follows.

$$\lambda_1 = 1 - \lambda_2 - \lambda_3$$

Now we compute the actual values.

$$\begin{cases} -3 = \lambda_2(-2 - 0) + \lambda_3(-6 - 0) \\ -1 = -2 + \lambda_2(2 + 2) + \lambda_3(-2 + 2) \end{cases}$$

$$\begin{cases} -3 = -2\lambda_2 - 6\lambda_3 \\ -1 = -2 + 4\lambda_2 \end{cases}$$

$$\begin{cases} 6\lambda_3 = 3 - 2\lambda_2 \\ -4\lambda_2 = 1 - 2 \end{cases}$$

$$\begin{cases} 6\lambda_3 = 3 - 2\lambda_2 \\ 4\lambda_2 = 1 \end{cases}$$

$$\begin{cases} 6\lambda_3 = 2.5 \\ \lambda_2 = \frac{1}{4} \end{cases}$$

$$\begin{cases} \lambda_3 = \frac{2.5}{6} \\ \lambda_2 = 0.25 \end{cases}$$

$$\begin{cases} \lambda_3 = 0.42 \\ \lambda_2 = 0.25 \end{cases}$$

$$\lambda_1 = 1 - 0.25 - 0.42 = 0.33$$

Thus we have the following values for the lambdas.

$$\lambda_1 = \frac{1}{3}$$

$$\lambda_2 = \frac{1}{4}$$

$$\lambda_3 = \frac{5}{12}$$

As expected, the barycentric coordinates to not change. This is because together with points p_1, p_2 and p_3, also p undergoes the same kind of transformation. This means that p'' will be in the same position relative to p_1'', p_2'' and p_3'' as it was p for p_1, p_2 and p_3.

2 Exercise 2

In order to project the points p_i onto the line $x = 1$, we need to use the following matrix.

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

This is because if we multiply a random point of the form:

$$p = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

We will obtain the following.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ x \end{bmatrix}$$

And finally, in order to obtain the homogeneous coordinates of the point – and transform them into cartesian coordinates, we need to divide every term of the matrix by x. This is because the third term of the matrix is a x. The computation is as follows.

$$\begin{bmatrix} x/x \\ y/x \\ x/x \end{bmatrix} = \begin{bmatrix} 1 \\ y/x \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ y/x \end{bmatrix}$$

3 Exercise 3

In order to project any point p_i onto the general line $y = ax + b$, we need to use the following matrix.

$$M = \begin{bmatrix} b & 0 & 0 \\ 0 & b & 0 \\ -a & 1 & 0 \end{bmatrix}$$

This matrix comes from the solution of the system of linear equations composed by:

$$y' = \frac{y}{x} x'$$

And

$$y' = ax' + b$$

First of all we substitute the y' of the second equation in the first equation.

$$\begin{aligned} \frac{y}{x} x' &= ax' + b \\ b &= \frac{y}{x} x' - ax' \\ b &= x' \left(\frac{y}{x} - a \right) \\ x' &= \frac{b}{\frac{y}{x} - a} \end{aligned}$$

Next we substitute the x' we just found into the second equation.

$$\begin{aligned} y' &= a \left(\frac{b}{\frac{y}{x} - a} \right) + b \\ y' &= \frac{ab}{\frac{y}{x} - a} + b \\ y' &= \frac{ab + \frac{by}{x} - ba}{\frac{y}{x} - a} \\ y' &= \frac{\frac{by}{x}}{\frac{y}{x} - a} \end{aligned}$$

Now that we found the equations for both x' and y', we need to find the matrix that allows for this transformation.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{b}{\frac{y}{x}-a} \\ \frac{\frac{by}{x}}{\frac{y}{x}-a} \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ \frac{by}{x} \\ \frac{y}{x} - a \end{bmatrix} = \begin{bmatrix} b \\ by \\ y - ax \end{bmatrix}$$

In the above formula I've multiplied the three terms by $\frac{y}{x} - a$ and by x, since divisions cannot be translated in the matrix M. Finally we obtain the following.

$$\begin{bmatrix} b & 0 & 0 \\ 0 & b & 0 \\ -a & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ by \\ y - ax \end{bmatrix}$$

Thus making M the correct matrix used to compute the point p_i'. In order to obtain the cartesian coordinates of the point, we need to divide all of the three terms by $y - ax$, thus obtaining:

$$\begin{bmatrix} \frac{b}{y-ax} \\ \frac{by}{y-ax} \\ \frac{y-ax}{y-ax} \end{bmatrix} = \begin{bmatrix} \frac{b}{y-ax} \\ \frac{by}{y-ax} \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} \frac{b}{y-ax} \\ \frac{by}{y-ax} \end{bmatrix}$$