Portfolio Construction Through Mixed-Integer Programming

PORTFOLIO SELECTION*

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ɪE PROCESS OF SELECTING a portfolio may be divided into two stage
.e first stage starts with observation and experience and ends wi
.iefs about the future performances of available securities. Tl
:ond stage starts with the relevant beliefs about future performanc
d ends with the choice of portfolio. This paper is concerned with tl
:ond stage. We first consider the rule that the investor does (or shoul
ıximize discounted expected, or anticipated, returns. This rule is r
:ted both as a hypothesis to explain, and as a max
stment behavior. We next consider the rule that th
ɔuld) consider expected return a desirable thing *a*

Portfolio Construction Through Mixed-Integer Programming at Grantham, Mayo, Van Otterloo and Company

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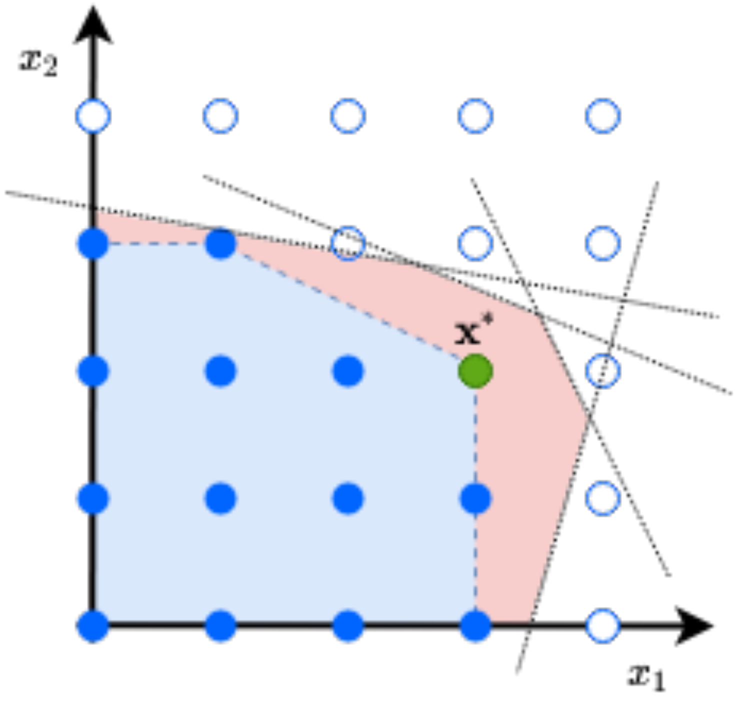
Implementing portfolio construction

Complications that the classical theory does not address

MIP APPROACH

OUR OBJECTIVE FUNCTION

- Parameters to consider:
  - number of different stocks
  - number of transactions
  - transaction fees



MIP APPROACH

OUR OBJECTIVE FUNCTION

COMPLETE PROBLEM FORMULATION

- We set a target portfolio **wT** monthly:
  - w0 is the current portfolio
  - wF will be the desirable portfolio to own after rebalancing monthly

Our objective function is formulated according to:

1. closeness to target portfolio **wT**
2. exposure to different economic sectors close to that of the **wT**
3. small # of names
4. small # of transactions
5. high return
6. high liquidity
7. low transaction costs

OUR OBJECTIVE FUNCTION

COMPLETE PROBLEM FORMULATION

CLOSENESS TO MARKET PORTFOLIO

Minimize

$$
\begin{aligned}
&\sum_{i=1}^{N} y(i) + \sum_{i=1}^{K} \lambda_{\text{sec}}(s)x(s) \\
&\quad + \lambda_{\text{names}} \sum_{i=1}^{N} y_{\text{names}}(i) \\
&+ \lambda_{\text{tickets}} \sum_{i=1}^{N} y_{\text{tickets}}(i) - \lambda_{\alpha} \sum_{i=1}^{N} \alpha(i)w_f(i) \\
&+ \lambda_{\text{illiquidity}} \sum_{i=1}^{N} (ls(1,\, i)x_1(i) + ls(2,\, i)x_2(i) \\
&+ ls(3,\, i)x_3(i)) \\
&+ \lambda_{\text{tc}} \sum_{i=1}^{N} (cs(1,\, i)z_1(i) + cs(2,\, i)z_2(i) \\
&+ cs(3,\, i)z_3(i))
\end{aligned}
$$

subject to

$$
\begin{aligned}
&\sum_{i=1}^{N} w_f(i) = 1, \\
&w_f(i) - w_t(i) \le y(i), \quad \forall i \\
&-(w_f(i) - w_t(i)) \le y(i), \quad \forall i \\
&w_f(i) - w_0(i) \le f(i), \quad \forall i \\
&-(w_f(i) - w_0(i)) \le f(i), \quad \forall i \\
&x(s) \geqslant \sum_{i=1}^{N} M_s(i)(w_f(i) - w_t(i)), \quad \forall s \\
&x(s) \geqslant -\sum_{i=1}^{N} M_s(i)(w_f(i) - w_t(i)), \quad \forall S \\
&w_f(i) \le y_{\text{names}}(i), \quad \forall i \\
&f(i) \le y_{\text{tickets}}(i), \quad \forall i \\
&w_f(i) = x_1(i) + x_2(i) + x_3(i), \quad \forall i \\
&f(i) = z_1(i) + z_2(i) + z_3(i), \quad \forall i \\
&0 \le x_1(i) \le \text{lq}(i), \quad \forall i \\
&0 \le x_2(i) \le \text{lq}(i), \quad \forall i \\
&0 \le x_3(i) \le 2\cdot\text{lq}(i), \quad \forall i \\
&0 \le z_1(i) \le 0.1\cdot\text{vol}(i), \quad \forall i \\
&0 \le z_2(i) \le 0.2\cdot\text{vol}(i), \quad \forall i \\
&0 \le z_3(i) \le 0.2\cdot\text{vol}(i), \quad \forall i \\
&y(i), w_f(i), f(i) \geqslant 0, \quad \forall i \\
&x(s) \geqslant 0, \quad \forall s \\
&y_{\text{names}}(i), y_{\text{tickets}}(i) \in \{0, 1\}, \quad \forall i.
\end{aligned}
$$

COMPLETE PROBLEM FORMULATION

CLOSENESS TO MARKET PORTFOLIO

EXPOSURE TO DIFFERENT SECTORS

It is desirable for the portfolio wF to be as close as possible to wT:

$$\sum_{i=1}^{N} | w_f(i) - w_t(i) |$$

CLOSENESS TO MARKET PORTFOLIO

EXPOSURE TO DIFFERENT SECTORS

NUMBER OF NAMES

It is desirable to have exposure to different sectors between wT and wF to be as close as possible.

$$\sum_{s=1}^{K} \lambda_{sec}(s) \left| \sum_{i=1}^{N} M_s(i)(w_f(i) - w_t(i)) \right|$$

- Ms(i) binary index denoting membership of stock i in sector s
- Lambda_sec(s) is a user-specified penalty for sector s.

EXPOSURE TO DIFFERENT SECTORS

NUMBER OF NAMES

NUMBER OF TRANSACTIO-NS

We define: $$y_{\text{names}}(i) = \begin{cases} 1, & \text{if } w_f(i) > 0, \\ 0, & \text{if } w_f(i) = 0. \end{cases}$$

Requirement captured by: $$\lambda_{\text{names}} \sum_{i=1}^{N} y_{\text{names}}(i)$$

- We want to have a small amount of Names (stocks) in the portfolio to minimize:
  - custodial fees
  - transaction costs

NUMBER OF NAMES

NUMBER OF TRANSACTIO-NS

HIGH RETURN

We define: $$y_{\text{tickets}}(i) = \begin{cases} 1, & \text{if } |w_f(i) - w_0(i)| > 0, \\ 0, & \text{if } |w_f(i) - w_0(i)| = 0. \end{cases}$$

Requirement captured by: $$\lambda_{\text{tickets}} \sum_{i=1}^{N} y_{\text{tickets}}(i)$$

Number of transactions needs to be kept low,

to minimize custodial fees and transaction costs

NUMBER OF TRANSACTIO- NS

HIGH RETURN

HIGH LIQUIDITY

- Requirement captured by: $-\lambda_{\alpha} \sum_{i=1}^{N} \alpha(i) w_f(i),$
- $\alpha(i)$ is the expected return of stock i (*)
- The sign is negative is because of the overall objective of minimization

(*) We estimated $\alpha(i)$ by taking the mean return of stock i for each month

HIGH RETURN

HIGH LIQUIDITY

LOW TRANSACTION COST

- Illiquidity term is captured by:

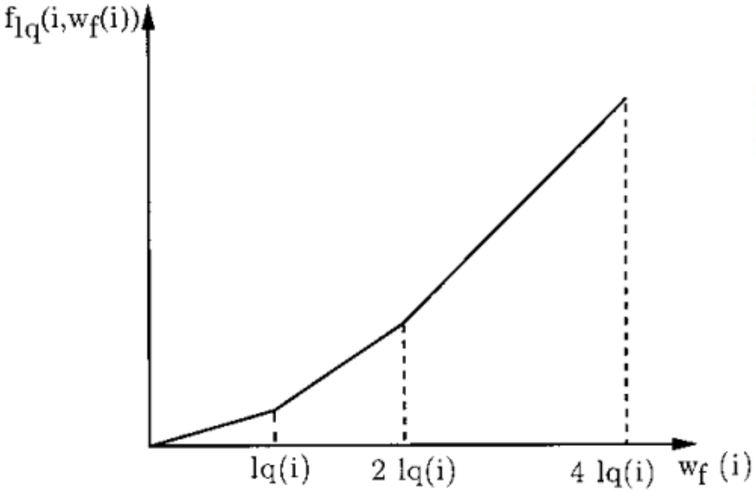
$$\lambda_{\text{illiquidity}} \sum_{i=1}^{N} f_{\text{lq}}\ (i,\ w_f(i)),$$

HIGH RETURN

HIGH LIQUIDITY

LOW TRANSACTION COST

- We want high liquidity aka low illiquidity. In particular
- As the position in a stock increases, it is harder and thus more expensive to trade it.
- A piece-wise linear convex function captures illiquidity of stock.
- For every stock i there is an illiquidity inxed lq(i) s.t. the illiquidity function: $f_{\mathrm{lq}}(i, w_f(i))$



$$f_{\mathrm{lq}}(i, x) = \begin{cases} \mathrm{ls}(1, i)x, & 0 \le x \le \mathrm{lq}(i), \\ \mathrm{ls}(2, i)(x - \mathrm{lq}(i)) & \\ \quad + \mathrm{ls}(1, i)\mathrm{lq}(i), & \mathrm{lq}(i) \le x \le 2\cdot\mathrm{lq}(i), \\ \mathrm{ls}(3, i)(x - 2\cdot\mathrm{lq}(i)) & \\ + \mathrm{ls}(2, i)2\cdot\mathrm{lq}(i), & 2\cdot\mathrm{lq}(i) \le x \le 4\cdot\mathrm{lq}(i). \end{cases}$$

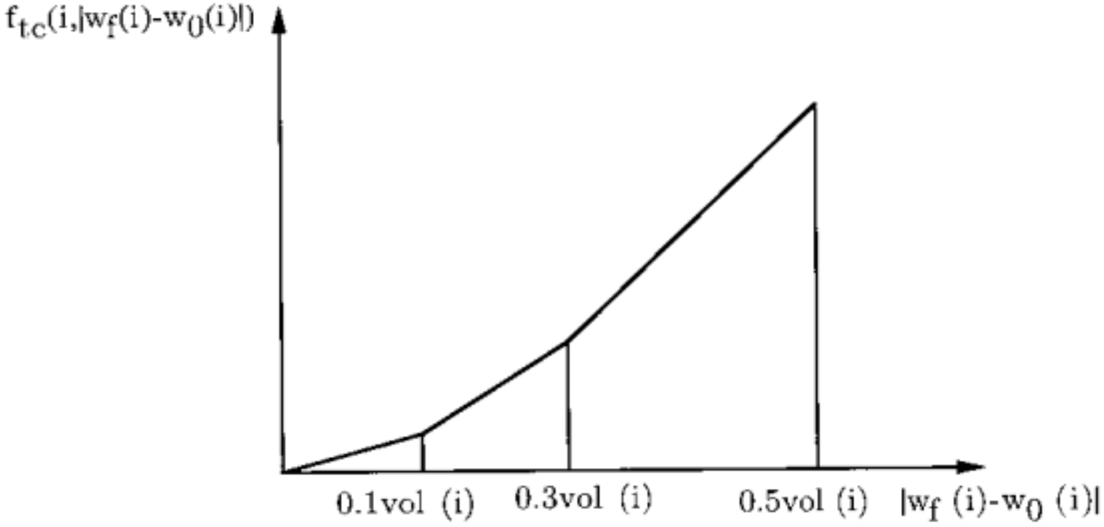
HIGH LIQUIDITY

LOW TRANSACTION COST

SUMMARY

- It is desirable to minimize total transaction costs.
- Higher the daily volume of a stock traded, the more transaction costs increase because portfolio trades have an impact on the market.
- The transaction cost from trading stock i is captured by a piecewise linear convex function:

$$f_{tc}(i, |w_f(i) - w_0(i)|)$$



HIGH LIQUIDITY

LOW TRANSACTION COST

SUMMARY

Function defined by:

$$f_{\mathrm{tc}}(i, x) = \begin{cases} \mathrm{cs}(1, i)x, & 0 \le x \le 0.1\cdot\mathrm{vol}(i), \\ \mathrm{cs}(2, i)(x - 0.1\cdot\mathrm{vol}(i)) \\ \quad + \mathrm{cs}(1, i)0.1\cdot\mathrm{vol}(i), & 0.1\cdot\mathrm{vol}(i) \le x \le 0.3\cdot\mathrm{vol}(i), \\ \mathrm{cs}(3, i)(x - 0.3\cdot\mathrm{vol}(i)) \\ \quad + \mathrm{cs}(3, i)0.3\cdot\mathrm{vol}(i), & 0.3\cdot\mathrm{vol}(i) \le x \le 0.5\cdot\mathrm{vol}(i). \end{cases}$$

The transaction cost consideration is captured by:

$$\lambda_{\mathrm{tc}} \sum_{i=1}^{N} f_{\mathrm{tc}}(i, | w_f(i) - w_0(i) |),$$

LOW TRANSACTION COST

SUMMARY

BASELINE STRATEGY

- The model uses various penalties, denoted by lambda; they capture the relative importance of the various objectives.
- We choose penalties trying to focus our attention on the different characteristics considered.
- If we consider a characteristic of the portfolio unsatisfactory, we increase its corresponding penalty.

SUMMARY

BASELINE STRATEGY

THE MARKET PORTFOLIO

- We simulated the market portfolio
- **Market portfolio** weights:

$$\tilde{w}_{M,i} = \frac{\text{capital in asset } i}{\text{total capital}} = \frac{N_i P_i}{\sum_{i=1}^{n} N_i P_i}, i = 1, \ldots, n$$

- According to Markowitz's theory, every efficient portfolio lies on the CML and therefore can be built as a combination of a risk-free asset and the market portfolio.
- Therefore, every profile of risk can be satisfied

BASELINE STRATEGY

THE MARKET PORTFOLIO

IMPLEMEN-TATION

- The market portfolio evolves constantly but our portfolio changes monthly.
- At the end of every month, our portfolio tries to replicate the current market portfolio
- Room for further improvements: which is the best target every month?
  - We chose the market portfolio at the moment of recalibration but one can choose the (weighted) average during the last month

THE MARKET PORTFOLIO

IMPLEMEN-TATION

LQ and VOL

- We have simulated a market of 20 stocks (based on the CSVs) and we computed the weights (wT) of the market portfolio daily (multiplying the price of a stock by the number of shares).
- Every month we evaluate: the weights target the portfolio, the volume of a stock, and the return of every stock (as the average return in the month).

```
for i in range(48):
    w_t_loop = w_t[:,i]
    vol_loop = vol[:,i]

    alpha_loop = np.transpose(alpha[i][:])

    #I suppose that both are following the law f(x) = x^2 for ease of use
    ls = np.concatenate([np.power(lq,2), np.power(2*lq,2), np.power(4*lq,2)], axis=1)

    cs = np.zeros([3,20])
    cs[:][0] = np.transpose(np.power(0.1*vol_loop,2))
    cs[:][1] = np.transpose(np.power(0.3*vol_loop,2))
    cs[:][2] = np.transpose(np.power(0.5*vol_loop,2))

    cs = np.transpose(cs)

    print(cs)

    w_f[:,i+1] = MIP(N, S, L, w_0, w_t_loop, alpha_loop, lq, vol_loop, M, ls, cs).reshape(-1)
    w_0 = w_f[:,i]
```

IMPLEMEN-
TATION

LQ and VOL

CHOICE OF λ

- How did we make those assumptions?
  - Lq = ask-bid (doesn't change over time in our model)
  - Vol = averageVolLast10Days/averageVolWholePeriod (changes every month)
- What does it mean to choose a more convex function than another?
  - The bigger the weight of a stock the more illiquid it becomes, this pushes for more diverse portfolios
  - We chose x^2 as a convex function
- How does this really affect the model?
  - It doesn't affect the model as it is, because we are considering only medium to large-cap stocks, if we included other more illiquid assets it would affect the model

LQ and VOL

CHOICE OF λ

$\lambda = 0$

Everyone has is own choice of lamba according to his situation.

Some choices are too big and make the objective function unbalanced

L_SEC	100	UTILITIES
	100	INDUSTRIALS
	0	ENERGY
	100	UTILITIES
	100	CONSUMER CYCLICAL
	100	CONSUMER DEFENSIVE
	0	FINANCIAL SERVICES
	100	REAL ESTATE
	0	HEALTHCARE
	0	COMMUNICATION SERVICES
	100	BASIC MATERIALS
L_names	0,1	
L_tickets	1E-05	
L_alpha	1	
L_illiquidity	1E-04	
L_tc	1E-04	

We modified the lambdas from the excel file to experiment different objective functions
and different solutions

CHOICE OF λ

λ = 0

λ = 0 WITHOUT CONSTRAIN-TS

- Is the only purpose of our portfolio to replicate the target?
- Yes but still there are some active constraints related to volatility and transaction cost
- If we eliminate such constraints our final portfolio shadows the target portfolio (with a one-month delay)

$$w_f(i) = x_1(i) + x_2(i) + x_3(i), \quad \forall i$$
$$f(i) = z_1(i) + z_2(i) + z_3(i), \quad \forall i$$

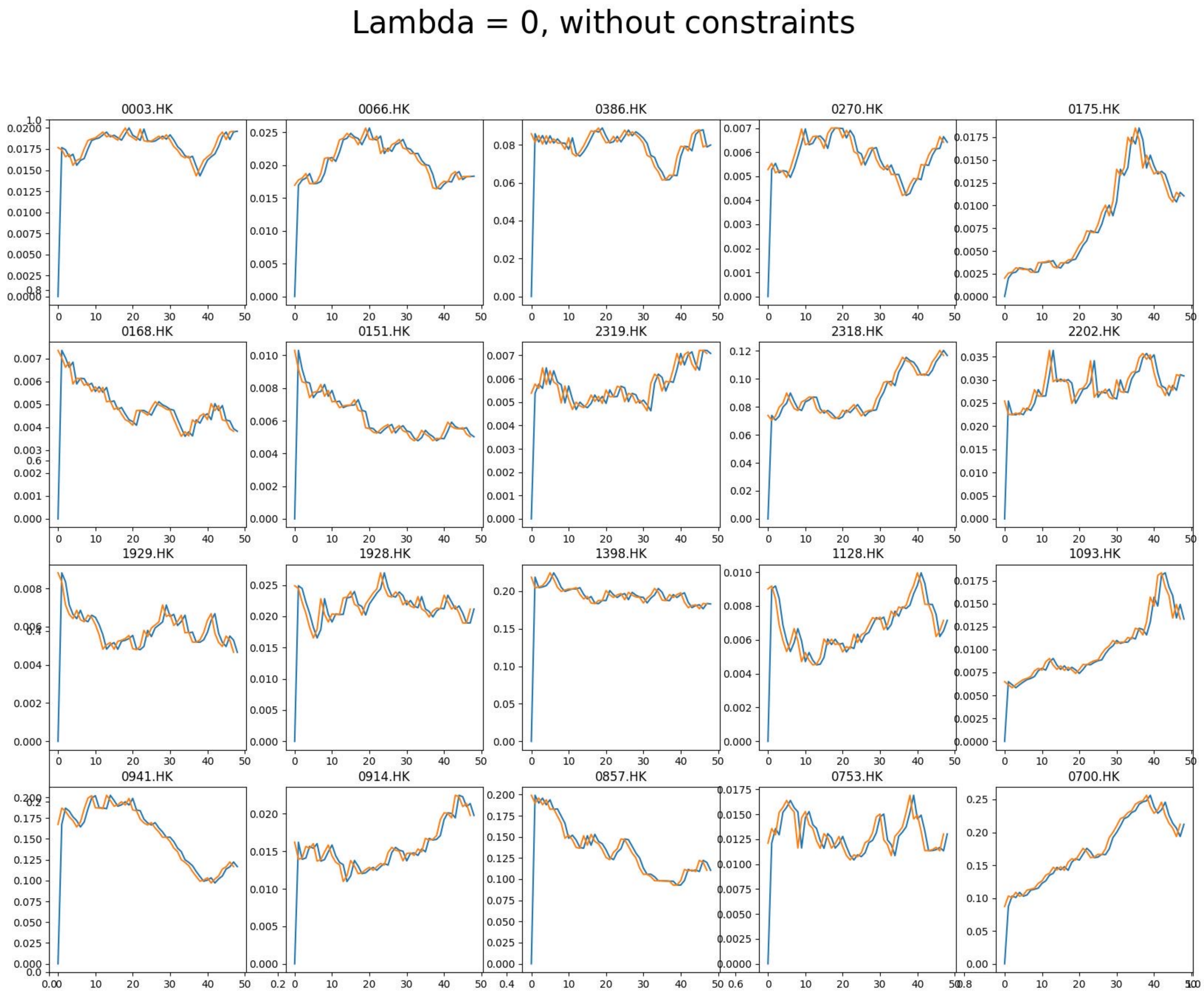
```
#Constraints for Volatility
m.addConstrs((F[i] == z_1[i] + z_2[i] + z_3[i] for i in range(N)), name='F=z1+z2+z3')
```

```
#Constraints for Liquidity
m.addConstrs((Weights[i] == x_1[i] + x_2[i] + x_3[i] for i in range(N)),name='W=x1+x2+x3')
```

λ = 0

λ = 0 WITHOUT CONSTRAIN-TS

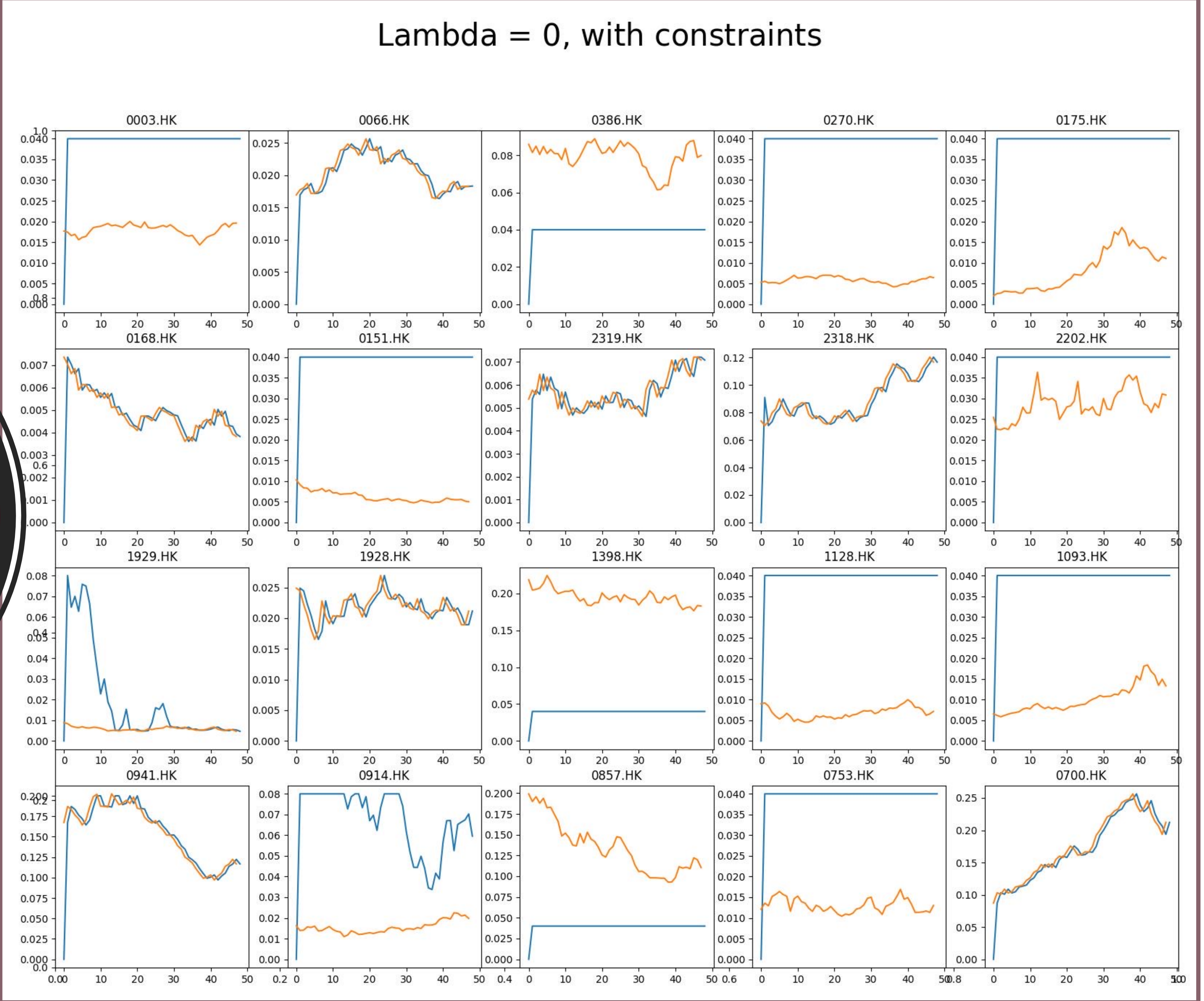
λ = 0 WITH CONSTRAI-NTS



λ = 0
WITHOUT CONSTRAIN-TS

λ = 0
WITH CONSTRAI-NTS

λα = 1

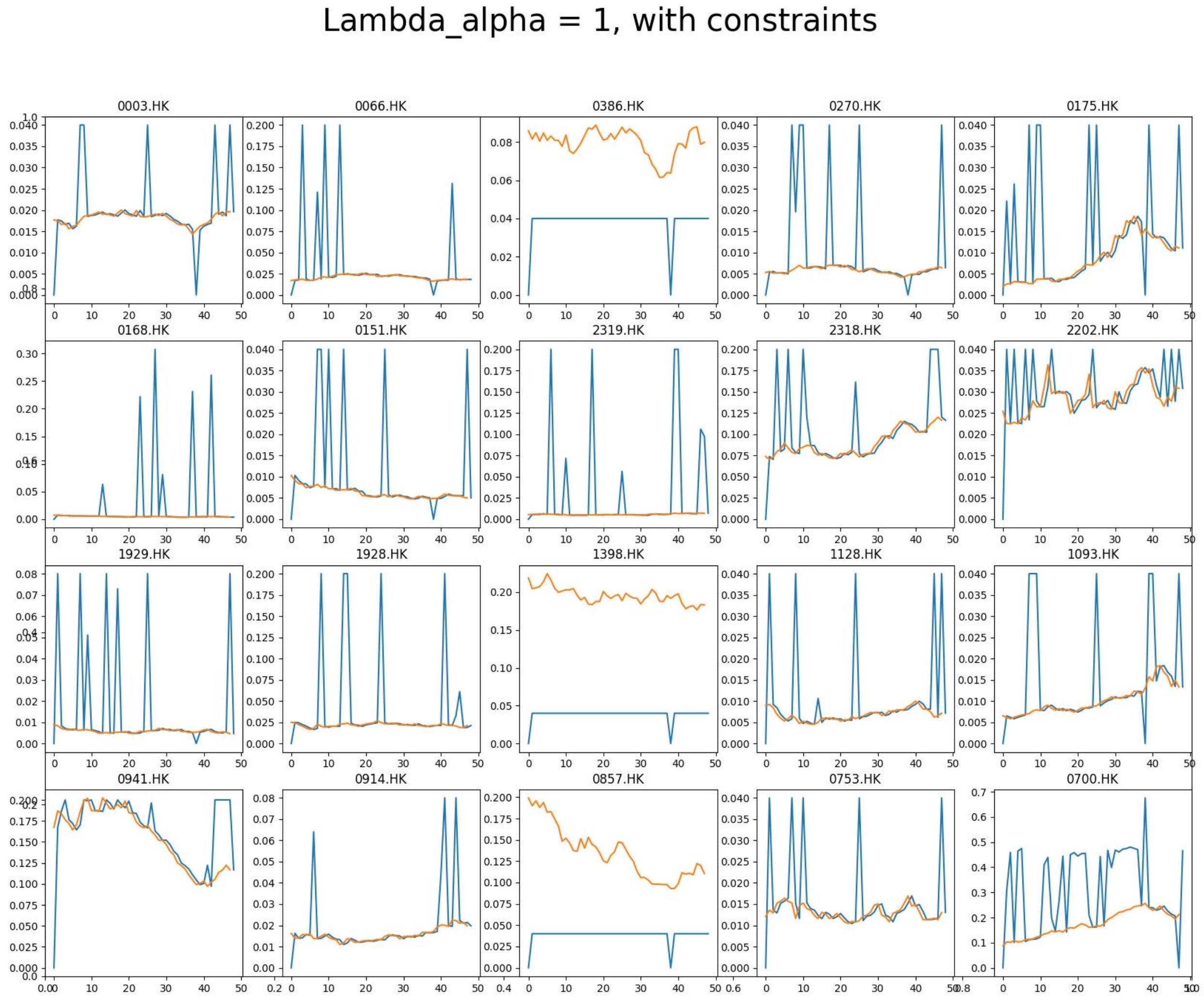


$\lambda = 0$
WITH CONSTRAI-NTS

$\lambda\alpha = 1$

$\lambda_{names} = 0.1$

After testing we found that $\lambda\alpha = 1$ is a reasonable size for the parameter

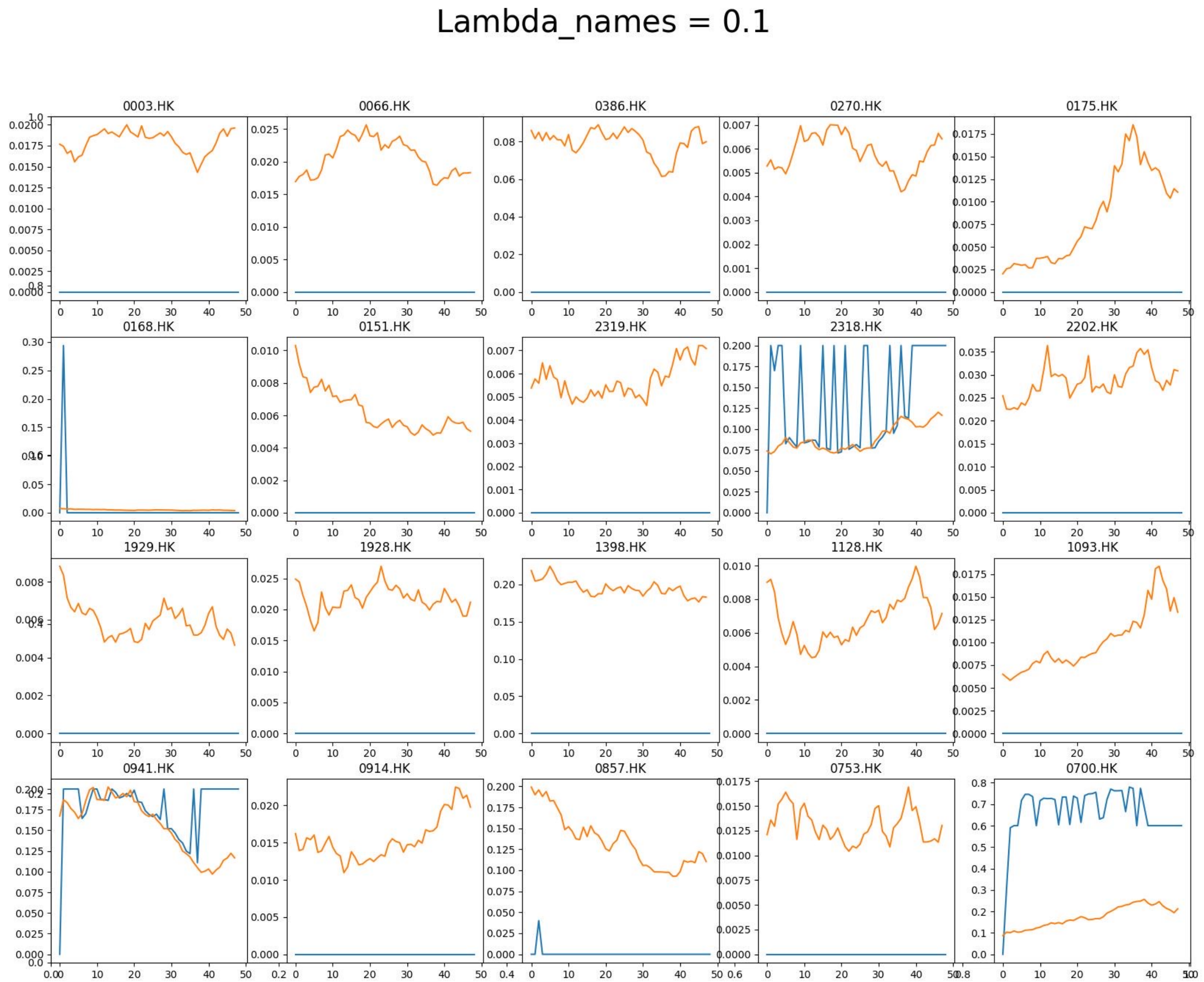


$\lambda_\alpha = 1$

$\lambda_{names} = 0.1$

λ_{illy} and λ_{tc}

Choosing $\lambda_{names} = 0.1$ gives too much importance to **having fewer stocks.**



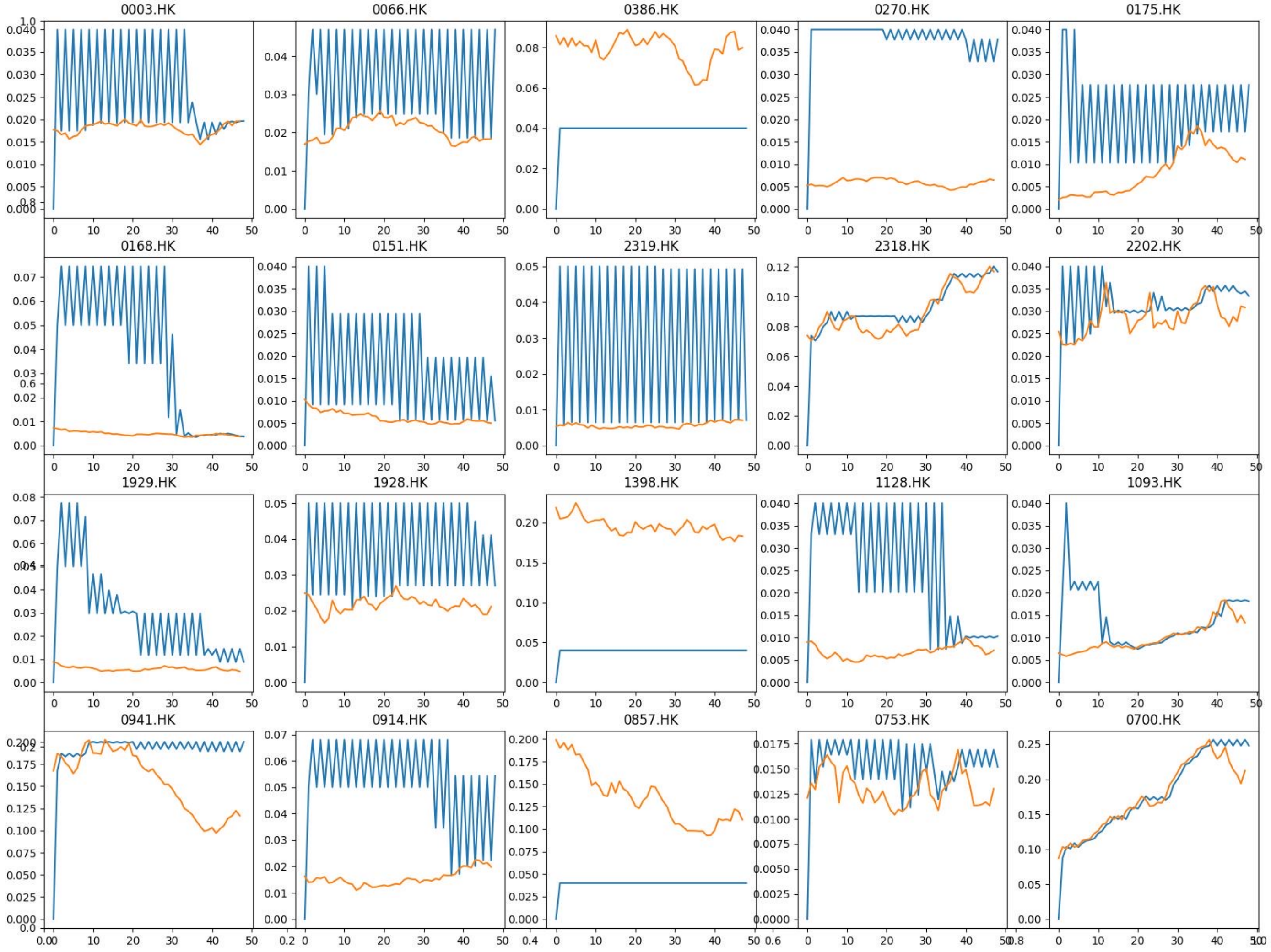
$\lambda_{\text{names}} = 0.1$

λ_{illy} and λ_{tc}

λ_{sec}

With λ_{illy}=1e-4, we focus on the **stocks that have high liquidity**, so the weights of these stocks are heavier than the others.

Choosing λ_{tc}=0.1, we don’t give too much importance to doing few transactions, and it can be seen by the **high frequency of changes on portfolio weights**



λ_{illy} and λ_{tc}

λ_{sec}

Example Portfolio

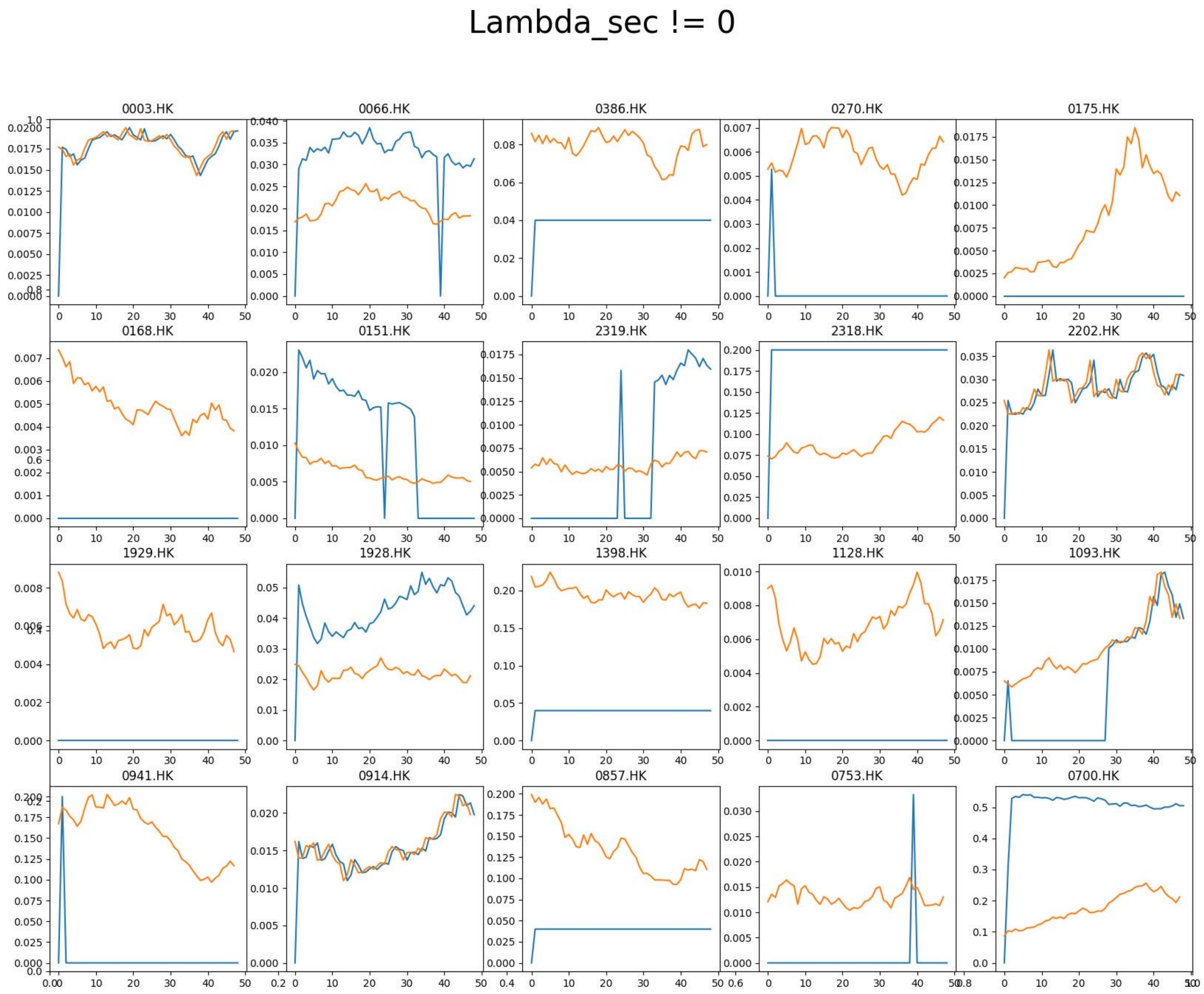
Now we chose to have a reduced number of names and to prioritize stokes from **“Communication services”**: as a result, stocks like **“0700.HK”** make up to 50% of the portfolio.
This is against the idea of shadowing the target portfolio but shows the goodness of the algorithm

L_SEC	100	UTILITIES
	100	INDUSTRIALS
	100	ENERGY
	100	UTILITIES
	100	CONSUMER CYCLICAL
	100	CONSUMER DEFENSIVE
	100	FINANCIAL SERVICES
	100	REAL ESTATE
	100	HEALTHCARE
	0	COMMUNICATION SERVICES
	100	BASIC MATERIALS
L_names	1	
L_tickets	0	
L_alpha	0	
L_illiquidity	0	
L_tc	0	

λilly and λ_{tc}

λ_{sec}

Example Portfolio



λ_{sec}

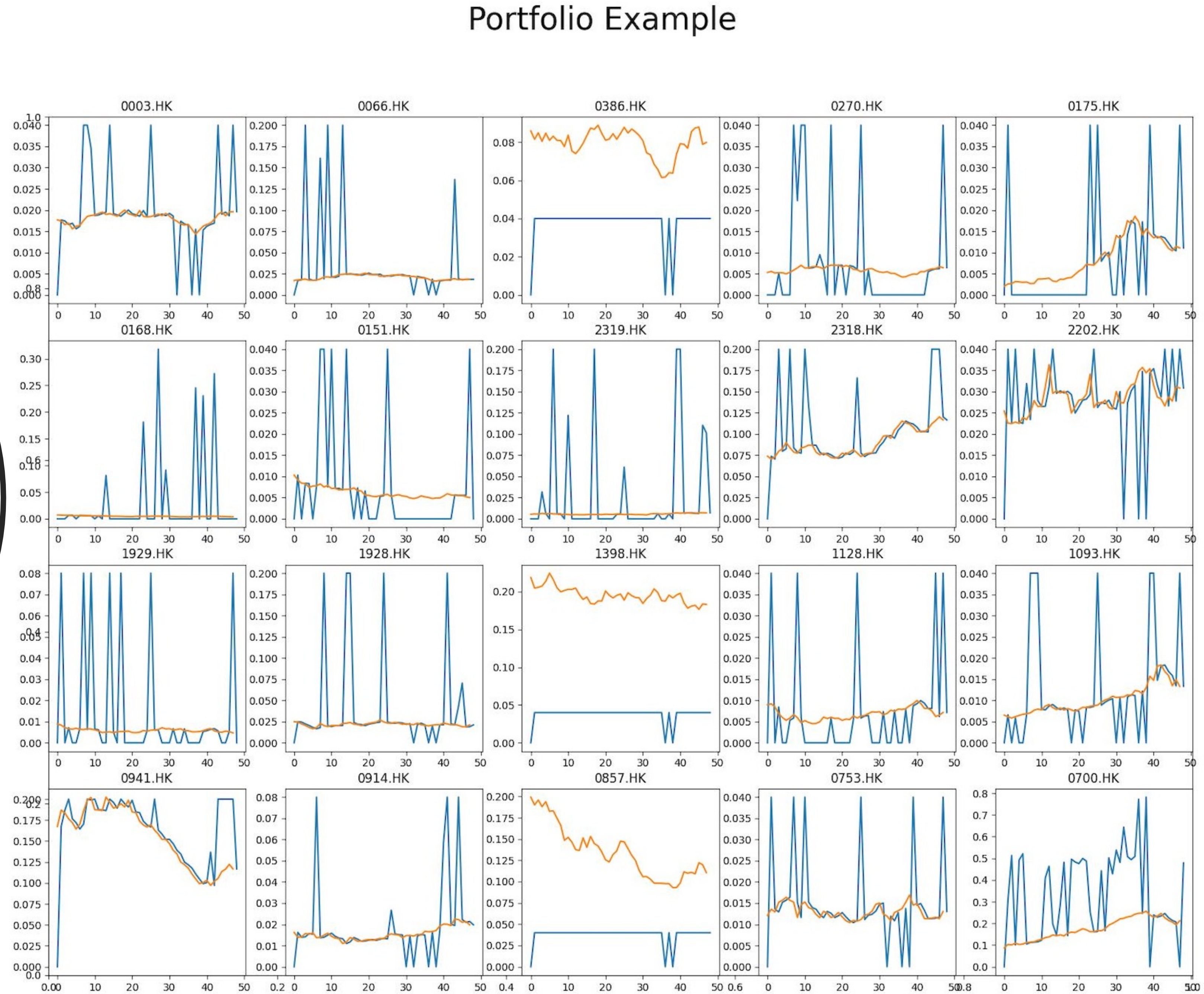
Example Portfolio

Idea of possible lambdas for a portfolio: we shadow the market portfolio and try to reduce the number of names and of transactions and their size.

L_SEC	100	UTILITIES
	100	INDUSTRIALS
	0	ENERGY
	100	UTILITIES
	100	CONSUMER CYCLICAL
	100	CONSUMER DEFENSIVE
	0	FINANCIAL SERVICES
	100	REAL ESTATE
	0	HEALTHCARE
	0	COMMUNICATION SERVICES
	100	BASIC MATERIALS
L_names	0.1	
L_tickets	1E-05	
L_alpha	1	
L_illiquidity	1E-04	
L_tc	1E-04	

λsec

Example Portfolio



Example Portfolio

Thank You