Applied Time Series

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Theoretical Exercises

Exercise 1: Stationarity

Let $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ be a weak white noise with variance σ^2:

$$\forall t, \quad \mathbb{E}(\varepsilon_t) = 0, \qquad \mathrm{Var}(\varepsilon_t) = \mathbb{E}(\varepsilon_t^2) = \sigma^2 < \infty, \qquad \mathrm{Cov}(\varepsilon_t, \varepsilon_{t-h}) = \mathbb{E}(\varepsilon_t \varepsilon_{t-h}) = 0 \ (h \neq 0).$$

A process $\{X_t\}$ is weakly stationary if:

$$\begin{aligned} \mathbb{E}(X_t) &\text{ is constant,} \\ \mathrm{Var}(X_t) &\text{ is constant,} \\ \mathrm{Cov}(X_t, X_{t-h}) &\text{ depends only on } h \text{ (not on } t). \end{aligned}$$

1) $\boldsymbol{y_t = \varepsilon_t - \varepsilon_{t-1}}$

$\mathbb{E}(y_t) = 0.$

$$\begin{aligned} \mathrm{Var}(y_t) &= \mathrm{Var}(\varepsilon_t - \varepsilon_{t-1}) \\ &= \mathrm{Var}(\varepsilon_t) + \mathrm{Var}(\varepsilon_{t-1}) - 2\,\mathrm{Cov}(\varepsilon_t, \varepsilon_{t-1}) \\ &= \sigma^2 + \sigma^2 - 0 = 2\sigma^2. \end{aligned}$$

For the autocovariance function, for $h \in \mathbb{Z}$,

$$\gamma_y(h) := \mathrm{Cov}(y_t, y_{t-h}) = \mathrm{Cov}(\varepsilon_t - \varepsilon_{t-1},\, \varepsilon_{t-h} - \varepsilon_{t-h-1}).$$

Using that ε_t is weak white noise (uncorrelated over time), we get

$$\gamma_y(0) = 2\sigma^2, \qquad \gamma_y(1) = \mathrm{Cov}(y_t, y_{t-1}) = -\sigma^2, \qquad \gamma_y(h) = 0 \text{ for } |h| > 1.$$

Hence $\gamma_y(h)$ depends only on h and y_t is weakly stationary (it is an MA(1)).

2) $\boldsymbol{z_t = a\varepsilon_t + b\varepsilon_{t-1}}$

$$\mathbb{E}(z_t) = 0, \qquad \mathrm{Var}(z_t) = a^2\sigma^2 + b^2\sigma^2 = \sigma^2(a^2 + b^2).$$

Moreover, for $h \in \mathbb{Z}$,

$$\gamma_z(h) := \mathrm{Cov}(z_t, z_{t-h}) = \mathrm{Cov}(a\varepsilon_t + b\varepsilon_{t-1},\, a\varepsilon_{t-h} + b\varepsilon_{t-h-1}).$$

Thus,

$$\gamma_z(0) = \sigma^2(a^2 + b^2), \qquad \gamma_z(1) = ab\,\sigma^2, \qquad \gamma_z(h) = 0 \text{ for } |h| > 1,$$

so $\gamma_z(h)$ depends only on h and z_t is weakly stationary (MA(1)).

3) Random walk: $\boldsymbol{w_0 = 0}$, $\boldsymbol{w_t = w_{t-1} + \varepsilon_t}$ for $\boldsymbol{t > 0}$

$$w_t = \sum_{j=1}^{t} \varepsilon_j \quad \Rightarrow \quad \mathbb{E}(w_t) = 0, \qquad \mathrm{Var}(w_t) = \sum_{j=1}^{t} \mathrm{Var}(\varepsilon_j) = t\sigma^2.$$

The variance depends on t, so w_t is not weakly stationary.

Exercise 2: Transformation of a stationary process

Let $U_t = \mu + \sum_{k=0}^{\infty} \theta_k \, \varepsilon_{t-k}$, with $\sum_{k=0}^{\infty} |\theta_k| < \infty$.

1) Mean

$$
\begin{aligned}
\mathbb{E}(U_t) = \mathbb{E}\left(\mu + \sum_{k=0}^{\infty} \theta_k \, \varepsilon_{t-k}\right) \\
&= \mathbb{E}(\mu) + \mathbb{E}\left(\sum_{k=0}^{\infty} \theta_k \, \varepsilon_{t-k}\right) \\
&= \mu + \sum_{k=0}^{\infty} \theta_k \, \mathbb{E}(\varepsilon_{t-k}) \\
&= \mu + \sum_{k=0}^{\infty} \theta_k \cdot 0 \\
&= \mu.
\end{aligned}
$$

2) Variance

$$
\mathrm{Var}(U_t) = \mathrm{Var}\left(\sum_{k=0}^{\infty} \theta_k \varepsilon_{t-k}\right) = \sum_{k=0}^{\infty} \theta_k^2 \, \mathrm{Var}(\varepsilon_{t-k}) = \sigma^2 \sum_{k=0}^{\infty} \theta_k^2.
$$

3) Autocovariance $\gamma_h = \mathrm{Cov}(U_t, U_{t-h})$, $h > 0$

Write

$$
U_t - \mu = \sum_{k=0}^{\infty} \theta_k \varepsilon_{t-k}, \qquad U_{t-h} - \mu = \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-h-j}.
$$

Then

$$
\begin{aligned}
\gamma_h &= \mathrm{Cov}(U_t, U_{t-h}) = \mathbb{E}\Big[(U_t - \mu)(U_{t-h} - \mu)\Big] \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \theta_k \theta_j \, \mathbb{E}(\varepsilon_{t-k} \varepsilon_{t-h-j}) \, .
\end{aligned}
$$

Since $\{\varepsilon_t\}$ is a weak white noise,

$$
\mathbb{E}(\varepsilon_{t-k} \varepsilon_{t-h-j}) = \begin{cases} \sigma^2, & \text{if } t-k = t-h-j, \\ 0, & \text{otherwise,} \end{cases}
$$

i.e. it is non-zero only when $t - k = t - h - j \iff j = k - h$. Because $j \geq 0$, this requires $k \geq h$. Therefore,

$$
\gamma_h = \sigma^2 \sum_{k=h}^{\infty} \theta_k \theta_{k-h}.
$$

Now perform the index shift $m = k - h$ (so $k = m + h$):

$$
\gamma_h = \sigma^2 \sum_{m=0}^{\infty} \theta_{m+h} \theta_m.
$$

Hence γ_h depends only on h.

Exercise 3: Forecast error

$X_t = 0.5 + 0.8X_{t-1} + \varepsilon_t, \qquad (\phi = 0.8 < 1,\ c = 0.5).$

1) Best linear forecast

We consider the AR(1):

$$X_t = c + \phi X_{t-1} + \varepsilon_t, \qquad |\phi| < 1, \quad \mathbb{E}(\varepsilon_t \mid \Omega_{t-1}) = 0.$$

Let $\mu = \mathbb{E}(X_t)$. Taking expectations:

$$\mathbb{E}(X_t) = c + \phi\,\mathbb{E}(X_{t-1}) + \mathbb{E}(\varepsilon_t).$$

Since $\mathbb{E}(X_t) = \mathbb{E}(X_{t-1}) = \mu$ and $\mathbb{E}(\varepsilon_t) = 0$,

$$\mu = c + \phi\mu \quad \Rightarrow \quad (1-\phi)\mu = c \quad \Rightarrow \quad \mu = \frac{c}{1-\phi}.$$

Here $c = 0.5$ and $\phi = 0.8$, hence $\mu = 2.5$.

h-step-ahead forecast

Rewrite the process in deviations from the mean:

$$X_t - \mu = c + \phi X_{t-1} + \varepsilon_t - \mu.$$

Using $c = (1-\phi)\mu$,

$$X_t - \mu = \phi(X_{t-1} - \mu) + \varepsilon_t.$$

Let $Y_t := X_t - \mu$. Then $Y_t = \phi Y_{t-1} + \varepsilon_t$.

1-step-ahead forecast

Taking conditional expectation given Ω_t:

$$\begin{aligned}
\mathbb{E}(Y_{t+1} \mid \Omega_t) &= \mathbb{E}(\phi Y_t + \varepsilon_{t+1} \mid \Omega_t) \\
&= \phi Y_t + \underbrace{\mathbb{E}(\varepsilon_{t+1} \mid \Omega_t)}_{=0} \\
&= \phi Y_t.
\end{aligned}$$

Thus

$$\mathbb{E}(X_{t+1} \mid \Omega_t) - \mu = \phi(X_t - \mu) \quad \Rightarrow \quad \mathbb{E}(X_{t+1} \mid \Omega_t) = \mu + \phi(X_t - \mu).$$

h-step-ahead forecast (iteration)

By iteration,

$$\mathbb{E}(Y_{t+h} \mid \Omega_t) = \phi^h Y_t \quad \Rightarrow \quad X_t^*(h) := \mathbb{E}(X_{t+h} \mid \Omega_t) = \mu + \phi^h (X_t - \mu).$$

Here,

$$X_t^*(h) = 2.5 + 0.8^h (X_t - 2.5), \qquad h \to \infty :\ X_t^*(h) \to 2.5.$$

2) Forecast error

By definition,

$$e_t(h) = X_{t+h} - X_t^*(h).$$

Iterating the deviation equation gives

$$Y_{t+h} = \phi^h Y_t + \sum_{j=0}^{h-1} \phi^j \, \varepsilon_{t+h-j}.$$

Therefore,

$$e_t(h) = X_{t+h} - X_t^*(h) = \sum_{j=0}^{h-1} \phi^j \, \varepsilon_{t+h-j}.$$

3) Variance of forecast error

$$\begin{aligned}
\mathrm{Var}(e_t(h)) &= \mathrm{Var}\left(\sum_{j=0}^{h-1} \phi^j \varepsilon_{t+h-j} \right) = \sum_{j=0}^{h-1} \phi^{2j} \, \mathrm{Var}(\varepsilon_{t+h-j}) \\
&= \sigma_\varepsilon^2 \sum_{j=0}^{h-1} \phi^{2j} = \sigma_\varepsilon^2 \, \frac{1-\phi^{2h}}{1-\phi^2}.
\end{aligned}$$

With $\phi = 0.8$,

$$\mathrm{Var}(e_t(h)) = \sigma_\varepsilon^2 \, \frac{1-0.8^{2h}}{0.36}, \qquad h \to \infty : \ \mathrm{Var}(e_t(h)) \to \frac{\sigma_\varepsilon^2}{0.36}.$$

4) Prediction interval

Under normality, $e_t(h) \sim \mathcal{N}(0, \sigma_h^2)$ with $\sigma_h^2 = \mathrm{Var}(e_t(h))$. Thus a $(1-\alpha)$ prediction interval centered at $X_t^*(h)$ is

$$\mathrm{CI}_\alpha\big(X_t^*(h)\big) = X_t^*(h) \pm z_{\alpha/2} \, \sigma_h.$$

Using $\sigma_h^2 = \sigma_\varepsilon^2 \frac{1-\phi^{2h}}{1-\phi^2}$,

$$\mathrm{CI}_\alpha\big(X_t^*(h)\big) = X_t^*(h) \pm z_{\alpha/2} \sqrt{\sigma_\varepsilon^2 \, \frac{1-\phi^{2h}}{1-\phi^2}}.$$

As $h \to \infty$,

$$\mathrm{CI}_\alpha\big(X_t^*(h)\big) \to \mu \pm z_{\alpha/2} \sqrt{\frac{\sigma_\varepsilon^2}{1-\phi^2}}.$$

Exercise 4: The MA(1)-GARCH(1) model

$$y_t = \theta_0 + \varepsilon_t + \theta_1 \varepsilon_{t-1}, \qquad h_t = \mathrm{Var}_{t-1}(\varepsilon_t) = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 h_{t-1}.$$

Let $\mathcal{F}_{t-1}$ be the information set at time $t-1$ and assume $\mathbb{E}(\varepsilon_t \mid \mathcal{F}_{t-1}) = 0$.

1) Conditional and unconditional mean

$$\mathbb{E}(y_t \mid \mathcal{F}_{t-1}) = \mathbb{E}\big(\theta_0 + \varepsilon_t + \theta_1 \varepsilon_{t-1} \mid \mathcal{F}_{t-1}\big) = \theta_0 + \theta_1 \varepsilon_{t-1}, \qquad \mathbb{E}(y_t) = \theta_0.$$

2) Conditional and unconditional variance

Since $\mathbb{E}(y_t \mid \mathcal{F}_{t-1}) = \theta_0 + \theta_1 \varepsilon_{t-1}$,

$$y_t - \mathbb{E}(y_t \mid \mathcal{F}_{t-1}) = \varepsilon_t \quad \Rightarrow \quad \mathrm{Var}_{t-1}(y_t) = \mathrm{Var}_{t-1}(\varepsilon_t) = h_t.$$

For the unconditional variance, use $y_t - \mathbb{E}(y_t) = \varepsilon_t + \theta_1 \varepsilon_{t-1}$, hence

$$\mathrm{Var}(y_t) = \mathrm{Var}(\varepsilon_t) + \theta_1^2 \mathrm{Var}(\varepsilon_{t-1}) + 2\theta_1 \, \mathrm{Cov}(\varepsilon_t, \varepsilon_{t-1}).$$

In a (zero-mean) GARCH model, the innovations ε_t are uncorrelated over time, so $\mathrm{Cov}(\varepsilon_t, \varepsilon_{t-1}) = 0$, and therefore

$$\mathrm{Var}(y_t) = (1 + \theta_1^2)\, v,$$

where $v = \mathrm{Var}(\varepsilon_t) = \mathbb{E}(h_t)$. Taking expectation in the GARCH recursion:

$$v = \alpha_0 + \alpha_1 v + \alpha_2 v \quad \Rightarrow \quad v = \frac{\alpha_0}{1 - \alpha_1 - \alpha_2} \qquad (\alpha_1 + \alpha_2 < 1).$$

Hence

$$\mathrm{Var}(y_t) = (1 + \theta_1^2)\, \frac{\alpha_0}{1 - \alpha_1 - \alpha_2}.$$