Assignment 4

Prof: Pierre Collin-Dufresne

TA: Florian Perusset

1 The Capital Asset Pricing Model: Theory

Consider an economy with $n = 1, ..., N$ risky assets with returns R_n and a risk-free asset with return R_f. $E[R] = \mu$ is the N-vector of asset expected returns and Σ is the $N \times N$ variance-covariance matrix of returns. There are $i = 1, ..., I$ investors with risk aversion γ^i that choose an N-risky asset portfolio weight vector w^i, such that their portfolio return $R_p^i = R_f + (w^i)^\top (R - R_f \mathbf{1})$ maximizes their mean-variance objective function $U^i(w^i) = E[R_p^i] - \frac{\gamma^i}{2} V[R_p^i]$. The investors' dollar assets are given by W^i. The market capitalization of risky assets is given by the vector M with elements M_n. The vector of market portfolio weights is given by $w_M = \frac{M}{\mathbf{1}^\top M}$.

(a) **Optimal Portfolios.** Show that every investor i invests in only two funds, the risk free asset and the risky tangency portfolio $w_{MV} = \frac{\Sigma^{-1}(\mu - R_f \mathbf{1})}{\mathbf{1}^\top \Sigma^{-1}(\mu - R_f \mathbf{1})}$, where the allocation between the two funds is driven by the investor's risk aversion γ^i. Therefore, each investor's assets $W^i = W_0^i + W_{MV}^i$ where W_0^i are the dollars held in the risk free asset and W_{MV}^i the dollars in the tangency portfolio.

(b) **Equilibrium.** The risk free asset is in zero net supply. Therefore $\sum_{i=1}^I W_0^i = 0$. The risky asset supply is given by their market capitalization M. In equilibrium, the sum of all investors must hold the market, i.e. $\sum_{i=1}^I W_{MV}^i w_{MV} = M$. Impose these equilibrium conditions to show that in equilibrium, the market portfolio and the tangency portfolio coincide $w_M = w_{MV}$.

(c) **Market Risk Premium.** Show that the expected return on the market portfolio is given by $\mu_M = R_f + \sigma_M^2 \gamma_M$ where $\mu_M = w_M^\top \mu$, $\sigma_M^2 = w_M^\top \Sigma w_M$ and $\gamma_M = \left(\sum_{i=1}^I \frac{1}{\gamma^i} \frac{W^i}{\sum_{i=1}^I W^i} \right)^{-1}$. Provide a brief intuition on the economic meaning of γ_M.

(d) **Security Market Line.** Show that for any risky asset n the equilibrium expected return is given by

$$\mu_n = R_f + \beta_n(\mu_M - R_f) \tag{1}$$

where $\beta_n = \frac{cov(R_n, R_m)}{\sigma_M^2}$. How would you empirically test this equation?

2 The Capital Asset Pricing Model: Empirics

The CAPM (1) implies that each stock's expected excess return or the risk premium is proportional the stock's market beta. In this question we test the empirical relevance of the CAPM. We will try to replicate Figure 2 in Fama & French (2004). To this end, download monthly stock returns from CRSP 1928 to 2004 for all common stocks traded on the New York Stock Exchange (NYSE). Also download the value-weighted CRSP market return and 1-month T-bill returns as a risk-free rate. The relevant SQL requests are in the appendix.

(a) **Compute stock-specific betas.** Compute β_n for each stock in the sample via a simple time-series OLS regression on the market return $R_{M,t}$

$$R_{n,t} - R_{f,t} = \alpha_n + \beta_n(R_{M,t} - R_{f,t}) + \epsilon_{n,t}. \tag{2}$$

Plot a histogram of the estimated betas and alphas and the fraction of variance explained by systematic risk. Interpret your result. Also plot the shrinkage estimator that shrinks the betas towards 1 with a shrinkage weight of $\frac{2}{3}$ as used by Bloomberg.

(b) **Beta Portfolios.** For each month, sort all stocks by their beta into 10 decile portfolios. For each of the 10 deciles portfolios, compute the market-cap weighted average excess return (see appendix for details). Then compute the beta of the portfolio excess returns based with respect to the market excess return for the full sample. Plot the 10 average portfolio returns for the full sample versus the portfolios' betas. Repeat your analysis for equal-weighted portfolio returns. Intuitively, why are we sorting stocks into beta portfolios instead of directly testing (1) over the entire cross-section of stocks?

(c) **Testing the CAPM.** Fit a line through the estimated betas and mean returns. How does the slope of that line compare to the average market excess return for the sample? Are these findings consistent with the CAPM? Explain verbally, how you would construct a trading strategy with positive alpha based on your findings above.

Appendix

SQL Requests

- Market Return:

  ```
  db.raw sql("select date,vwretd from crsp.msi where date>='1928-01-01and date<='2004
  date cols=['date'])
  ```

- Stock Returns (common stocks traded on the NYSE):

  ```
  db.raw_sql(" select a.permno, a.date, b.shrcd, b.exchcd, a.ret, a.shrout,
  a.prc from crsp.msf as a left join crsp.msenames as b on a.permno=b.permno
  and b.namedt<=a.date and a.date<=b.nameendt where a.date between '01/01/1928'
  and '12/31/2004' and b.exchcd between 1 and 2 and b.shrcd between 10 and
  11", date_cols=['date'])
  ```

- Risk Free Rate (1 month T-bill:

  ```
  db.raw_sql("select mcaldt, tmytm from crsp.tfz_mth_rf where kytreasnox
  = 2000001 and mcaldt>='1928-01-01' and <='2004-12-31'")
  ```

Market Cap Weighting

- Compute the market cap of each stock as $M_{n,t} = P_{n,t} * Shrout_{n,t}$ where $P_{n,t}$ is the share price of stock n and $Shrout_{n,t}$ are the shares outstanding

- Let $S \subset N$ denote a subset of stocks. Value-weights (aka market cap weights) over this subset of stocks are then given by

$$vw_{n,t} = \frac{M_{n,t}}{\sum_{n \in S} M_{n,t}}$$

- Note that value-weighted returns are computed using lagged weights. i.e. $R_t^{vw} = \sum_{n \in S} R_{n,t} vw_{n,t-1} = \sum_{n \in S} R_{t,n} \frac{M_{n,t-1}}{\sum_{n \in S} M_{n,t-1}}$