Assignment 3

Prof: Pierre Collin-Dufresne
TA: Florian Perusset

1. **Efficient portfolios**

   Consider an investor who wants to invest in N risky assets with return $R_i \ \forall i = 1, \dots, N$ with expected return $E[R_i] = \mu_i$ and variance $V[R_i] = \sigma_i^2$, and in a risk-free asset with return R_f. The investor seeks a N-risky asset portfolio weight vector w (and a weight $1 - w^\top \mathbf{1}$ in the risk-free asset), such that her portfolio return $R_p = R_f + w^\top (R - R_f \mathbf{1})$ maximizes her mean-variance objective function $U(w) = E[R_p] - \frac{\gamma}{2} V[R_p]$.

   (a) Show that an optimal portfolio weight vector w is such that for the corresponding mean-variance efficient portfolio return R_p we have

   $$\mu_i - R_f = \gamma \, cov[R_i, R_p] \ \ \forall i = 1, \dots N$$

   (b) Show that for any such mean-variance efficient portfolio, we have

   $$\mu_i - R_f = \beta_{i,P}(\mu_P - R_f)$$

   where $\beta_{i,P} = \frac{cov(R_i, R_P)}{\sigma_P^2}$ is the linear regression coefficient of return R_i on the mean-variance efficient portfolio return R_P.

   (c) In turn, show that this implies that, if R_p is the return to a mean-variance efficient portfolio, then for any return i we have

   $$R_i = R_f + \beta_i (R_P - R_f) + \epsilon_i$$

   where $cov(R_P, \epsilon_i) = 0$.

   Hint: use the definition of a linear regression

(d) Show that all mean-variance efficient portfolios have the same Sharpe ratio where we define its Sharpe ratio as $SR_p = \frac{\mu_p - R_f}{\sigma_p}$.

2. **Portfolio Math**

   (a) Show that any risky-asset only minimum variance frontier portfolio w can be rewritten as a convex combination of any two arbitrary minimum variance frontier portfolios w_a, w_b in the sense that $w = \alpha w_a + (1 - \alpha) w_b$.

   (b) Let R_{min} denote the return on the global minimum-variance portfolio of risky assets. Let R be the return on any risky asset or portfolio of risky assets, efficient or not. Show that $Cov(R_{min}, R) = Var(R_{min})$. Hint: Consider a portfolio consisting of a fraction w in this risky asset and a fraction $(1 - w)$ in the global minimum-variance portfolio. Compute the variance of the return on this portfolio and observe that the variance has to be minimized for $w = 0$.

3. **Risk Parity** The optimal mean-variance portfolio is a complex function of estimated means, volatilities, and correlations of asset returns. There are many parameters to estimate. Optimized mean-variance portfolios can give poor results when these inputs are imperfectly estimated. In their paper "leverage aversion and risk- parity" (posted on moodle) Asness, Frazzini, and Pedersen (AFP) suggest that **risk-parity allocation**, which has become widely popular and ignores information in sample-means, dominates the standard mean-variance portfolio because it exploits leverage aversion of investors. Here we will try to replicate some of their findings.

   (a) Download from CRSP the monthly value-weighted CRSP MSCI index, the 1-month risk-free rate and the value-weighted Bond index from 1960 to 2023. (The respective SQL requests are reported in the appendix below). Compute the mean, standard deviation and Sharpe ratio of a portfolio that invests 60% in stocks and 40% in bonds (the 60/40 portfolio). Find the tangency portfolio and give its mean, standard deviation and Sharpe ratio.

   (b) The risk-parity portfolio holds stocks and bonds in proportion to the inverse of their (full-sample) volatility. The levered risk-parity portfolio is levered at the risk-free T-Bill rate such that the portfolio's (full sample) volatility is equal to the volatility of the 60/40 portfolio. Following AFP construct the two portfolios and compute their mean, standard deviation and Sharpe ratio. Plot the RP and

RP-unlevered portfolios on the efficient frontier along with the Tangency portfolio and the 60/40 portfolio. What explains the difference between the RP and RP-unlevered portfolio performance?

(c) Why does the RP strategy in Figure 1 of AFP not lie on the efficient frontier?

(d) Split the sample into two periods from 1960-1990 and 1990-2023. For both subsamples, find the optimal portfolio for a mean-variance investor with the same risk as a 60/40 portfolio. Also, find the levered RP portfolio with the same standard deviation. Discuss your findings.

SQL Requests

- `stocks = db.raw sql("select date,vwretd from crsp.msi where date>='1960-01-01and date<='2023-12-31", date cols=['date'])`
- `bonds = db.raw sql("select caldt, b2ret from crsp.mcti where caldt>='1960-01-01' and caldt<='2023-12-31'", date cols=['caldt'])`
- `Rf=db.raw sql("select mcaldt,tmytm from crsp.tfz mth rf where kytreasnox = 2000001 and mcaldt>='1960-01-01'and mcaldt<='2023-12-31'", date cols=['mcaldt'])`