Assignment 5

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1 Market equilibrium

Consider an economy with 3 risky assets with expected returns

$$\mu = \begin{bmatrix} 0.06 \\ 0.08 \\ 0.10 \end{bmatrix}$$

The variance-covariance matrix of returns is given by

$$\Sigma = \begin{bmatrix} 0.0144 & 0.0015 & 0.002 \\ 0.0015 & 0.0225 & 0.003 \\ 0.002 & 0.003 & 0.04 \end{bmatrix}$$

The risk-free rate is $R_0 = 3\%$.

1. What is the optimal portfolio for a mean-variance investor (call him X) with a risk aversion of $a_X = 4$? Does he borrow or lend?

2. Assuming that the economy is populated only with mean-variance investors, what is the composition of the market portfolio? What is its expected return and standard deviation?

3. How much is the economy-wide aggregate risk aversion implicit in the market portfolio? Interpret this value.

4. Consider now a second mean-variance investor (call her Y) who has the same initial wealth as the investor X. Let's suppose that there are only two investors in this market and that the risk-free asset is in zero net supply.

(a) What is the position of investor Y in the risk-free asset? Interpret your result.

(b) What is the optimal portfolio of investor Y?

(c) Compute the risk aversion of investor Y (a_Y).

2 Empirics: Leverage CAPM & Portfolio Construction

In this question, we test the empirical implications of the leverage CAPM and construct an optimal portfolio based on its predictions. To this end, we use the same data as in Problem Set 4.

The leverage CAPM is the version of the CAPM where investors can't borrow more than a fraction l^i of their wealth at the risk-free rate. In this context, the security market line is given by

$$\mu_n = R_f + \psi + \beta_n(\mu_M - R_f - \psi) \tag{1}$$

where $\psi = \sum_i \omega^i \frac{\lambda^i}{1+l^i}$ is the weighted tightness of the funding constraints, with $\omega^i = \frac{W^i}{W}\frac{a}{a^i}$, W^i and a^i are respectively the wealth and the risk-aversion of investor i, W is the aggregate wealth, a is the weighted average risk aversion in the economy defined as $a = \left(\sum_i \frac{W^i}{W}\frac{1}{a^i}\right)^{-1}$, and λ^i is the Lagrange multiplier in the leverage-constrained optimization.

(a) **Estimating Rolling Betas.** Compute the time-varying $\beta_{t,n}$ for each stock by running monthly rolling 5-year regressions of stock-specific excess returns on the excess market return. Require at least 36 months of observations for each stock. Appendix provides some example code on how to efficiently compute the betas in python.

(b) **Portfolio Sorts.** At every month t, sort all stocks into deciles based on their beta. Then compute monthly returns for 10 decile portfolios that equal weight all stocks in each decile. Plot the average annualized portfolio returns across the 10 deciles in a barplot. Repeat for value-weighted returns. Are the results in line with the prediction of the leverage CAPM (1)?

(c) **Constructing the BAB factor.** Now we construct the betting-against-beta factor as in Frazzini & Pedersen (2014). At every month t, we construct two portfolios, a high-beta portfolio w_H and a low-beta portfolio w_L. (See Appendix for the construction of the portfolios) Construct the BAB factor return as

$$R_{t+1}^{BAB} = \frac{R_{t+1}^{L} - R_f}{\beta_L} - \frac{R_{t+1}^{H} - R_f}{\beta_H} \tag{2}$$

where $R_{t+1}^{H} = w_H^\top R_{t+1}$ is the return on the high-beta portfolio and $R_{t+1}^{L} = w_L^\top R_{t+1}$ is the return on the low-beta portfolio. $\beta_H = w_H^\top \beta_t$ and $\beta_L = w_L^\top \beta_t$ are the corresponding betas of the portfolios. Report the sharpe ratio and CAPM alpha of the BAB factor.

(d) **Long-Short Portfolios** R_{t+1}^{BAB} is the return to a self-financing (i.e. dollar neutral) long short portfolio. Is R_{t+1}^{BAB} also dollar neutral in terms of risky securities only? Intuitively, describe the difference to the portfolio $R_{t+1}^{H} - R_{t+1}^{L}$.

(e) **Long-Short Portfolios** Given the CAPM alpha and idiosyncratic volatility of the BAB factor R_{t+1}^{BAB}, what is the optimal allocation to the risk free rate, the market porfolio and the BAB factor that maximizes the Sharpe ratio of the portfolio?

(f) **BAB Return and Leverage Constraints** Plot the rolling 10-year average of the BAB factor return R_{t+1}^{BAB}. What is the relationship to the aggregate tightness of the funding constraint ψ?

Appendix

- **Construct Rolling Betas**:

    1. The instead of running a *groupby* OLS regression, it will be more efficient to use the built in *cov()* function.
    2. Use *groupby.rolling(60, min_periods=36).cov()* to get the rolling covariance matrix between excess stock returns and excess market returns. Then divide the estimated covariance by the market variance

- **Constructing w_H and w_L**:
  $w_H = k(z - \bar{z})^+$ and $w_L = k(z - \bar{z})^-$ where z is a vector of cross-sectional beta ranks $z_n = rank(\beta_n)$. $\bar{z}$ is the cross-sectional average rank, and $k = \frac{2}{\mathbf{1}^\top |z-\bar{z}|}$ is a normalizing factor. $()^+$ and $()^-$ denote the positive and negative elements of a vector respectively.