Power Spectral Density and LTI Filters

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Introduction

This document wants to fully show the mathematical elements and derivations necessary to compute the power spectral density at the output of an LTI filter when a stationary random process is at its input. Beware that, despite the huge effort to produce an error-free document, some might still be present.
Please, if you find something that is not correct, contact *edoardo@veredo.net*

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1 Probability concepts

In order to achieve the objective of this document it is useful to recall some probability concepts.

1.1 Covariance

Given any given two given variables, X and Y, we can define the *covariance* $C_{ov}[X,Y]$ as:

$$C_{ov}[X,Y] = E\left[(X - E[X])\,(Y - E[Y])\right] \tag{1}$$

The covariance is used to describe how the two variables behave with respect to their average value:

- if $C_{ov}[X,Y] > 0$ then the two variables move in the same direction i.e. if X is getting larger than E[X], then Y will do the same. This holds also if X (or Y) evolves to values smaller than the respective average.
- if $C_{ov}[X,Y] < 0$ then the two variables evolve in opposite directions i.e. if Y (or X) increases above its average, then X (or Y) will decrease.
- if $C_{ov}[X,Y] = 0$ then the two variables do not evolve consistently together.

We can expand the previous definition into a more compact form:

$$\begin{align} C_{ov}[X,Y] &= E\left[XY + E[X]E[Y] - XE[Y] - E[X]Y\right] \tag{2} \\ &= E[XY] - E[X]E[Y] \tag{3} \end{align}$$

Now, recall the definition of the *variance* of a random variable $Var(X)$:

$$\begin{align} Var(X) &= E[(X - E(X))^2] \tag{4} \\ &= E[X^2 - 2XE[X] + E^2[X]] \tag{5} \\ &= E[X^2] - E^2[X] \tag{6} \end{align}$$

Therefore, we can relate *variance* and *covariance* as follows:

$$Var(X) = C_{ov}[X,X] \tag{7}$$

1.2 Autocovariance function

The autocovariance function of a random process $X(t)$ is a function of two times t and $s = t+\tau$. This function specifies the covariance between the value of the process at time t and the value at time s.

$$C_X(s,t) = C_{ov}[X(t), X(s)] \tag{8}$$

As showed before, we can obtain the variance of the random process X(t) using the covariance function, infact:

$$Var(X(t)) = C_{ov}[X(t), X(t)] = C_X(t,t) \tag{9}$$

Example: consider a (discrete) white noise process {W[n]} consisting of independent identically distributed (i.i.d) random variables with variance $\sigma^2 = Var(W[n])$. We can now compute the covariance between $W[n_1]$ and $W[n_2]$ using the previous definition:

$$C_{ov}[W(n_1), W(n_2)] = E[W(n_1)W(n_2)] - E[W(n_1)]E[W(n_2)] \tag{10}$$

Remembering that $W[n_1]$ and $W[n_2]$ are i.i.d random processes, the above relation yields:

$$C_{ov}[W(n_1), W(n_2)] = \begin{cases} 0 & n_1 \neq n_2 \\ \sigma^2 & n_1 = n_2 \end{cases} \tag{11}$$

Rewriting this result with only one expression, we would get:

$$C_{ov}[W(n_1), W(n_2)] = \sigma^2 \delta(n_1 - n_2) \tag{12}$$

1.3 Wide-Sense Stationary process

A random process X(t) is said to be wide-sense stationary (WSS) if its mean and its autocovariance function are invariant under time shifts, that is:

- the mean value $\mu_X(t) \equiv \mu_X$ is a constant
- $C_X(s,t)$, the autocovariance function, only depends on $\tau = s - t$, where s and t are two different time instants.

1.4 Autocorrelation function

The autocorrelation function $R_x(s,t)$ of a random process X(t) is a function of two times s and t:

$$\boxed{R_x(s,t) = E[X(s)X(t)]} \tag{13}$$

If we recall the autocovariance function in its compact form:

$$C_{ov}[X(s), X(t)] = E[X(s)X(t)] - E[X(s)]E[X(t)] \tag{14}$$

Then it is immediate to write:

$$R_x(s,t) = E[X(s)X(t)] = C_{ov}[X(s), X(t)] + E[X(s)]E[X(t)] \tag{15}$$

$$= C_X[s,t] + \mu_x^2 \tag{16}$$

For a WSS process, the autocorrelation only depends on the time difference $\tau = s - t$:

$$R_X(\tau) = C_X[\tau] + \mu_x^2 \tag{17}$$

1.5 Power of a stationary process

The instantaneous power of any given signal X(t) is defined as:

$$P_X(t) = X(t)^2 \tag{18}$$

If the signal X(t) is a random process, it is useful to define its expected power:

$$E[X(t)^2] \tag{19}$$

The expected power of a random process can be linked to the autocorrelation function, infact:

$$E[X(t)^2] = E[X(t)X(t)] = R_X(t,t) \tag{20}$$

Note that the autocorrelation function written as $R_X(t,t)$ is equivalent to $R_X(0)$, since the time difference $\tau = 0$.

Example: consider a (discrete) white noise process {W[n]} consisting of i.i.d random variables such that $\mu_W = E[W(n)]$ and $Var(W[n]) = \sigma^2$. We can write that:

$$\begin{aligned} R_W(k) &= C_W[k] + \mu_W^2 && (21)\\ &= \sigma^2\delta(k) + \mu_W^2 \quad \text{where } k = n_1 - n_2 && (22) \end{aligned}$$

Therefore

$$E[W(n)^2] = R_W(0) = C_W[0] + \mu_W^2 = \sigma^2\delta(0) + \mu_W^2 = \sigma^2 + \mu_W^2 \tag{23}$$

1.6 Power spectral density

The power spectral density (PSD) $S_X(f)$ of a stationary random process {X(t)} is the Fourier transform of the autocorrelation function $R(\tau)$.

$$R_X(\tau) \xRightarrow{\mathscr{F}} S_X(f) \tag{24}$$

An equivalent definition is the following:

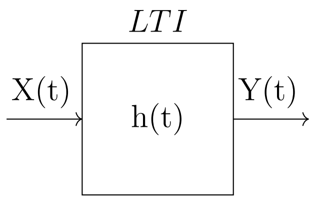
$$E[X(t)^2] = R_X(0) = \int_{-\infty}^{+\infty} S_X(f)df \tag{25}$$

Which means that the integral of the power spectral density in the frequency interval $(-\infty, \infty)$ must return the signal expected power.

2 Linear filtering of stochastic processes

Consider a stochastic process X(t) is filtered through an LTI filter with unit-impulse response $h(t)$. If we define Y(t) as the output of the filter, then we have that:

$$Y(t) = \int_{-\infty}^{+\infty} h(u)X(t-u)du = h(t) * X(t) \tag{26}$$



Compute the *expected value of the output*:

$$E[Y(t)] = E\left[\int_{-\infty}^{+\infty} h(u)X(t-u)du\right] = \int_{-\infty}^{+\infty} h(u)E[X(t-u)]du = h(t) * E[X(t)] \tag{27}$$

if X(t) is a WSS process, then E[X(t)] is a constant, therefore:

$$E[Y(t)] = \mu_x \int_{-\infty}^{+\infty} h(u)du \tag{28}$$

Compute the *crosscorrelation* between input (WSS) and output:

$$\begin{aligned}
R_{XY}(\tau) &= E[X(t)Y(t+\tau)] && (29)\\
&= E[X(t)\int_{-\infty}^{+\infty} h(u)X(t+\tau-u)du] && (30)\\
&= \int_{-\infty}^{+\infty} h(u)E[X(t)X(t+\tau-u)]du && (31)\\
&= \int_{-\infty}^{+\infty} h(u)R_X(\tau-u)du && (32)\\
&= h(\tau) * R_X(\tau) && (33)
\end{aligned}$$

Compute the *autocorrelation of the output*

$$\begin{aligned}
R_Y(\tau) &= E[Y(t)Y(t+\tau)] && (34)\\
&= E\left[\int_{-\infty}^{+\infty} h(u)X(t-u)du \int_{-\infty}^{+\infty} h(v)X(t+\tau-v)dv\right] && (35)\\
&= \int_{-\infty}^{+\infty} h(u)\int_{-\infty}^{+\infty} h(v)E[X(t-u)X(t+\tau-v)]dvdu && (36)\\
&= \int_{-\infty}^{+\infty} h(u)\int_{-\infty}^{+\infty} h(v)R_X(\tau-v+u)dvdu && (37)\\
&= \int_{-\infty}^{+\infty} h(u)R_{XY}(\tau+u)du && (38)\\
&= h(-\tau) * R_{XY}(\tau) && (39)\\
&= h(-\tau) * h(\tau) * R_X(\tau) && (40)
\end{aligned}$$

2.1 Filter Theorem

Let X(t) be a stationary random process with power spectral density $S_X(f)$. If X(t) is passed through an LTI filter with impulse response h(t), then the output process Y(t) has PSD:

$$S_Y(f) = |H(f)|^2 S_X(f) \tag{41}$$

Proof: From the definition of autocorrelation, we know that:

$$R_X(\tau) = C_X(\tau) + \mu_X^2 \tag{42}$$

but also, we previously saw that, after passing though an LTI filter, the autocorrelation of the output $R_Y(\tau)$ can be written as:

$$\begin{aligned}
R_Y(\tau) &= h(-\tau) * h(\tau) * R_X(\tau) && (43)\\
&= h(-\tau) * h(\tau) * \left(C_X(\tau) + \mu_X^2\right) && (44)
\end{aligned}$$

The PSD of Y(t) is defined as the Fourier transform of its autocorrelation function, therefore:

$$S_Y(f) = \mathscr{F}[R_Y(\tau)] \tag{45}$$

$$S_Y(f) = H(-f) \cdot H(f) \cdot S_X(f) = H^*(f)H(f)S_X(f) \tag{46}$$

$$\boxed{S_Y(f) = |H(f)|^2 S_X(f)} \tag{47}$$

3 Useful Fourier properties

In the following some useful properties of Fourier transforms are reported:

$$h(t) \xRightarrow{\mathscr{F}} \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt = H(f) \tag{48}$$

$$h(-t) \xRightarrow{\mathscr{F}} \int_{-\infty}^{+\infty} h(-t)e^{-j\omega t}dt = \int_{-\infty}^{+\infty} h(\tau)e^{j\omega\tau}d\tau = H(-f) \tag{49}$$

$$h^*(t) \xRightarrow{\mathscr{F}} \int_{-\infty}^{+\infty} h^*(t)e^{-j\omega t}dt = H^*(-f) \tag{50}$$

$$H(f) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt \tag{51}$$

$$H(-f) = \int_{-\infty}^{+\infty} h(t)e^{j\omega t}dt \tag{52}$$

$$H^*(f) = \int_{-\infty}^{+\infty} h^*(t)e^{j\omega t}dt \tag{53}$$

Only if h(t) is real then $H(-f) = H^*(f)$