SIAM J. SCI. STAT. COMPUT.
Vol. 13, No. 2, pp. 573–595, March 1992

009

GENERALIZED SCHWARZ SPLITTINGS*

WEI PAI TANG†

Abstract. A classic mathematical technique, the Schwarz Alternating Method (SAM), has recently attracted much attention from researchers in the field of parallel computations, as well as theoreticians. Its advantages in parallelism, wide applicability and great flexibility in implementation make SAM a competitive choice in parallel computations. However, the computational performance of the classical SAM and its modern extensions strongly depend on the amount of overlap between the neighboring subregions. Introducing a large overlap has changed the image of SAM from an impractical theoretical technique to a rewarding numerical approach. However, the duplication of work in these overlapped regions is undesirable. Reducing the amount of overlap without affecting the speed of convergence has become an important performance issue.

Schwarz Splitting (SS) has been proposed as an extension of SAM in numerical linear algebra, and a generalized SS is presented in this paper. The new approach allows utilization of the flexibility of the splitting to further improve convergence speed and complexity. A fast convergence is obtained by choosing a good splitting instead of increasing the overlap. The best performance of our generalized SS is much better than that of a previously recommended SS, in which a large overlap is used. Both convergence analysis and numerical results are presented here.

Key words. Schwarz Alternating Method (SAM), Schwarz Splitting (SS), generalized Schwarz splitting, domain decomposition, parallel computation, overlap

AMS(MOS) subject classifications. 65F10, 65N10

1. Introduction.

Experience with the new generation of parallel computers has promoted efforts to search for *truly* parallel algorithms rather than parallelizing the existing sequential algorithms. For coarse grain parallelism, domain decomposition has become an increasingly important focus of research for the numerical solution of partial differential equations.

A classic mathematical approach, the Schwarz Alternating Method (SAM) (1869), [16] appears to offer promise for the parallel solution of the very large systems of linear or nonlinear algebraic equations that arise when elliptic problems in elasticity, fluid dynamics, or other important areas are discretized by finite elements or finite differences. With this approach, a large problem is decomposed into several coupled subproblems. If a proper ordering is used, these subproblems can be solved independently. Starting from a given initial guess, the solution is iterated in each subregion and new values are exchanged on these coupled artificial boundaries. This process will converge to the solution for the entire region. Flexibility in mapping these subproblems into different parallel computer topologies and the advantageous ratio between communication and computation make SAM a tempting choice in parallel processing. It is also crucial for some complex fluid flow calculations that different modelings or grids be applied to different subdomains of the flow. For example, in many applications we need to merge Euler's equation, the Navier–Stokes equations, potential flow, and other models in suitable subregions for a single large problem. There are also applications where composite meshes in regions with complicated boundaries are needed. SAM can provide a natural framework within which all these requirements are met.

The recognition of SAM's potential in numerical computations was a rather recent event [2], [14], [11], [12], [5], [17], [4], [10], [3]. This delay may have been caused by

* Received by the editors January 30, 1989; accepted for publication (in revised form) January 14, 1991. This research was supported by the Natural Sciences and Engineering Research Council of Canada.

† Department of Computer Science, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1.

some disappointing experiments with earlier implementations of SAM [15]. During the past several years, understanding of SAM's computational behavior has become clearer. If the area of the overlap regions is a constant fraction of the subregions, it has been shown that the convergence of SAM is independent of the mesh [12], [3].

Several *modern extensions* of SAM have been proposed. For example, a generalization of SAM in linear algebra—Schwarz splittings—was introduced in [14], [17], and an additive version of SAM is being investigated [4], [3]. It is now known that the convergence of the classical SAM and most of its extensions strongly depend on the amount of overlap between subregions. Introducing a larger overlap does considerably improve the performance of SAM [12], [8]. In conjunction with other acceleration techniques, such as multilevel techniques, preconditioning or SOR accelerations, SAM has proven to be a useful method in large scale scientific computations. However, one undesirable feature of SAM is the duplication of work on the overlapped regions at each iteration. Reducing the amount of overlap without affecting speed of convergence has become an important performance issue.

Schwarz originally proposed a coupling between subregions which requires only the continuity of the unknown. Kantorovich and Krylov ([9, pp. 617–626]) presented a rather general convergence result of SAM for a second order partial differential equation of the form

$$L(u) = F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}\right) = 0,$$

with Dirichlet boundary condition. They also used Dirichlet boundary condition on the artificial boundaries. It is not difficult to observe that we can replace the continuity of the unknown on these artificial boundaries by other couplings, for example, the continuity of the unknown's derivative. Several attempts to use different boundary conditions on these overlapped regions were not very successful [17], [13], leading us to conjecture a few years ago that "For different problems the best choice of the type of boundary conditions may vary. It is a very interesting open problem for future research." For the model problem, it was shown that the Dirichlet boundary condition has a better convergence rate than the Neumann boundary condition [17].

In this paper, a new coupling between the overlap subregions is identified. If a successful coupling is chosen, a fast convergence of the alternating process can be achieved without a large overlap.

Before proceeding, let us introduce a generalized version of SAM. We consider the Dirichlet problem for a second order elliptic operator L

$$\begin{aligned} L(u(\mathbf{x})) &= 0, && \mathbf{x} \in \Omega, \\ u(\mathbf{x})\,|_{\Gamma_\Omega} &= \psi(\mathbf{x}), && \mathbf{x} \in \Gamma_\Omega, \end{aligned} \tag{1}$$

where Ω is a bounded region in k-dimensional space, Γ_Ω is the boundary of Ω, and $\mathbf{x} = \{x_1, x_2, \cdots, x_k\}$ is the independent variable (Fig. 1 shows an example of the two-dimensional case). To simplify the discussion, we consider a case for two subregions, although direct generalization can be made to more subregions. We also assume that the solution to this problem exists and is unique.

Split the solution domain Ω into two overlapping subdomains Ω_1 and Ω_2 (see Fig. 1), provided $\Omega_{12} = \Omega_1 \cap \Omega_2 \neq \emptyset$. Denote Γ_{Ω_1}, Γ_{Ω_2}, $\Gamma_{\Omega_{12}}$ the boundaries of Ω_1, Ω_2, and Ω_{12}, respectively. We denote Γ_i, $i = 1, 2$, the part of boundary which belongs to

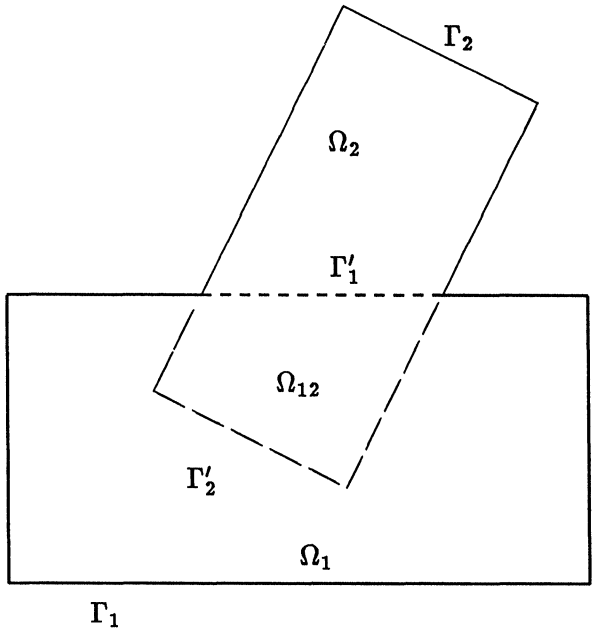

FIG. 1. *Two overlapping subregions.*

Γ_{Ω_i}. Let Γ_i', $i = 1, 2$, be the part of the artificial boundary in Ω_i. We have

$$\Gamma_{\Omega_1} = \Gamma_1 \cup \Gamma_1',$$
$$\Gamma_{\Omega_2} = \Gamma_2 \cup \Gamma_2'.$$

Denote u_1 and u_2 as the solutions on subdomain Ω_1 and Ω_2, respectively. Then, the following couplings

$$g_1(u_1)\,|_{\Gamma_1'} = g_1(u_2)\,|_{\Gamma_1'}, \tag{2}$$

$$g_2(u_2)\,|_{\Gamma_2'} = g_2(u_1)\,|_{\Gamma_2'}, \tag{3}$$

are true on the artificial boundaries Γ_1' and Γ_2', where

$$g_i(u) = \omega_i u + (1 - \omega_i)\frac{\partial u}{\partial n}, \qquad i = 1, 2. \tag{4}$$

With these new couplings we can formulate two coupled subproblems

$$\begin{aligned} L(u_1(\mathbf{x})) &= 0, \qquad \mathbf{x} \in \Omega_1, \\ u_1(\mathbf{x})\,|_{\Gamma_1} &= \psi(\mathbf{x}), \\ g_1(u_1(\mathbf{x}))\,|_{\Gamma_1'} &= g_1(u_2(\mathbf{x}))\,|_{\Gamma_1'}, \end{aligned} \tag{5}$$

$$\begin{aligned} L(u_2(\mathbf{x})) &= 0, \qquad \mathbf{x} \in \Omega_2, \\ u_2(\mathbf{x})\,|_{\Gamma_2} &= \psi(\mathbf{x}), \\ g_2(u_2(\mathbf{x}))\,|_{\Gamma_2'} &= g_2(u_1(\mathbf{x}))\,|_{\Gamma_2'}. \end{aligned} \tag{6}$$

We have the following result.

THEOREM 1. *If the boundary value problem*

$$
\begin{aligned}
L(w(\mathbf{x})) &= 0, \qquad \mathbf{x} \in \Omega_{12}, \\
g_1(w(\mathbf{x}))\,|_{\Gamma_1'} &= 0, \\
g_2(w(\mathbf{x}))\,|_{\Gamma_2'} &= 0,
\end{aligned} \tag{7}
$$

has only trivial solution and the solutions u_1, u_2 *of* (5) *and* (6) *exist, then*

1. $u_1(\mathbf{x}) = u_2(\mathbf{x}), \quad \mathbf{x} \in \Omega_{12}$.
2. $u(\mathbf{x}) = u_1(\mathbf{x}), \quad \mathbf{x} \in \Omega_1$ *and* $u(\mathbf{x}) = u_2(\mathbf{x}), \quad \mathbf{x} \in \Omega_2$,

where u, *and* u_1, u_2 *are the solutions of* (1) *and, respectively,* (5) *and* (6).

The proof of this theorem is straightforward and a direct generalization of this result to a finite number of overlapping subregions can be made. However, the same proof can work only for the case in which no three subregions have a common overlap region. We say that problem (1) is *equivalent* to (5) and (6), if 1 and 2 in Theorem 1 are true. A version of this result in linear algebra will be shown in the next section. Following Theorem 1, we can replace problem (1) by (5) and (6).

Since there are unknowns which are coupled in the boundary conditions of (5) and (6), we cannot solve the two problems independently. Given an initial guess $u_2^{(0)}\,|_{\Gamma_1'} = \psi_0$, we will then be able to construct a sequence $\{u_1^{(i)}, u_2^{(i)}\}$ as follows:

$$
\begin{aligned}
L(u_1^{(i)}) &= 0, \qquad \mathbf{x} \in \Omega_1, \\
u_1^{(i)}\,|_{\Gamma_1} &= \psi, \\
g_1(u_1^{(i)})\,|_{\Gamma_1'} &= g_1(u_2^{(i-1)})\,|_{\Gamma_1'},
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
L(u_2^{(i)}) &= 0, \qquad \mathbf{x} \in \Omega_2, \\
u_2^{(i)}\,|_{\Gamma_2} &= \psi, \\
g_2(u_2^{(i)})\,|_{\Gamma_2'} &= g_2(u_1^{(i)})\,|_{\Gamma_2'}, \qquad i = 1, 2, \cdots.
\end{aligned} \tag{9}
$$

A key question to ask is "Under what conditions will the sequence $\{u_1^{(i)}, u_2^{(i)}\}$ converge to the solutions $\{u_1, u_2\}$ of (5) and (6)?" If it converges, then the solution of (1) can be constructed from the solution of (5) and (6). An analysis for the model problem will be given in §3.

The generalized SAM provides us with a general framework. Several questions of implementation which affect efficiency are left open. We can effectively tailor this approach to different problems or a different computer architecture. In particular, the following issues have an important impact on performance in real applications.

- The choice of the couplings $g_i(u)$. A better choice of g_i can yield substantial improvement in the convergence rate (see numerical results in §3).
- The decomposition of the solution domain. The flexibility in decomposition makes it possible to choose the geometry of most of the subregions to meet the requirements imposed by fast solvers or by grids. A fast biharmonic solver on irregular domains using generalized SAM is studied in [1].
- The selection of the individual solution technique for each subdomain. We are able to use different solution techniques for different subproblems. It is also

possible to use different ways to obtain the solution of the same subproblem in the different stages of the computation, allowing us to use an optimal approach at any particular moment and in any particular location. Hierarchical grid and inexact solution strategies are typical examples here [18], [7].

- Numerical model for each subproblem. Special boundary shapes or local behavior of the solution may require different models in different subregions. The decoupled subproblems allow us to localize the special treatment to the place where it is needed. Composite grids are a good example of this case.

A particularly important application of SAM is for parallel computations. In the previous description of generalized SAM, the parallelism is not very obvious. When the number of the subregions is greater than or equal to the number of processors, we can color the subregions such that subregions with the same color can be solved independently. There are many issues which need to be considered in practical parallel implementation such as load balancing, communication, synchronization, and ordering of the solution of the subproblems. These are very important in terms of parallel efficiency; however, we shall not study them in depth here.

In the next section, a generalized Schwarz Splitting (generalized SS) and an equivalence theorem are presented. This generalization is an analogy of the generalization from SAM to Schwarz Splitting. Then, an application of this generalized SS to the solution of elliptic equations is shown in §3. The convergence analysis of the strip case and our numerical results indicate that the performance of a proposed generalized SS depends mostly on a coupling parameter α. A fast convergence rate based on a proper choice of α can be obtained with very little overlap, thus the concern about the duplication of computation in the traditional SAM can be alleviated.

2. Generalized Schwarz Splittings. In this section we present an extension of the generalized SAM to numerical linear algebra. For a matrix equation $Ax = f$, we first introduce a Schwarz enhanced equation $\widetilde{A}\widetilde{x} = \widetilde{f}$. The corresponding matrix $\widetilde{A}$ is called a Schwarz enhanced matrix. A necessary and sufficient condition for the equivalence of the original equation and the Schwarz enhanced equation is shown. The analogy of applying generalized SAM to the matrix equation is equivalent to applying a particular block Gauss–Seidel scheme to the Schwarz enhanced matrix. The corresponding splitting of the Schwarz enhanced matrix is called generalized Schwarz Splitting (generalized SS). With this extension, many classical results in numerical linear algebra can be applied to this problem.

2.1. Definitions. As we mentioned in the introduction, the approach of the generalized SAM to a problem is to create an equivalent problem which consists of several loosely coupled subproblems, then to solve the subproblems iteratively. It is not necessary to view SAM only as a way of solving elliptic partial differential equations, as it can also be viewed as a general method for problem solving. Here the generalized SAM is discussed in terms of matrix theory. This approach provides new opportunities for generalizing and improving the original SAM.

Consider a matrix problem:

$$Ax = f, \tag{10}$$

where A is an $N \times N$ nonsingular matrix, f and x are N vectors. A partitioned form of (10) will be used in the rest of this paper. A partition is defined by the integers $n_1, n_2, \cdots, n_{2k+1}$ such that

$$n_1 + n_2 + \cdots + n_{2k+1} = N, \tag{11}$$

$$n_{2i} > 0, \qquad n_{2i+1} \geq 0, \qquad i = 1, \cdots, k. \tag{12}$$

Given a set $\{n_i\}_{i=1}^{2k+1}$ which satisfies (11) and (12), the $(2k+1) \times (2k+1)$ partitioned form of the matrix A is then given by

$$A = \begin{bmatrix} A_{1\,1} & A_{1\,2} & \cdots & A_{1\,2k+1} \\ A_{2\,1} & A_{2\,2} & \cdots & A_{2\,2k+1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{2k+1\,1} & A_{2k+1\,2} & \cdots & A_{2k+1\,2k+1} \end{bmatrix}, \tag{13}$$

where $A_{i\,j}$ is an $n_i \times n_j$ submatrix. We always assume that the unknown vector x and the known vector f in the matrix equation $Ax = f$ are partitioned in a form consistent with A. Thus, if A is given by (13), then x is assumed to be partitioned as

$$x = [x_1, x_2, \cdots, x_{2k+1}]^T, \tag{14}$$

where x_i is an $n_i \times 1$ matrix (column vector). An *augmented vector* of x

$$\tilde{x} = [x_1, x_2, x_2, x_3, x_4, x_4, x_5, \cdots, x_{2k}, x_{2k}, x_{2k+1}]^T \tag{15}$$

is defined such that all even subvectors x_{2i}, $i = 1, \cdots, k$ are duplicated once in their places, and all odd subvectors remain the same.

We present the cases for $N = 3$ and 5 here for readability. The generalization to a large N is direct. A dense 3×3 partitioned matrix can be written as

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}.$$

If the operator $L(u)$ in (1) is a linear second order elliptic operator, then the discretized problem can be written as a matrix equation:

$$Ax = \begin{bmatrix} \boxed{\begin{matrix} A_{11} & A_{12} \\ A_{21} & \end{matrix}} & \boxed{\begin{matrix} A_{13} \\ A_{22} \end{matrix}} & \\ & \boxed{\begin{matrix} A_{23} \\ A_{32} \end{matrix}} & \boxed{A_{33}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = f. \tag{16}$$

The order of the unknowns is arranged so that $[x_1, x_2]$ corresponds to the unknowns in Ω_1, $[x_2, x_3]$ corresponds to the unknowns in Ω_2, and $[x_2]$ corresponds to the unknowns in Ω_{12}, which is the overlapped region. The numerical generalized SAM for the above problem solves the following subproblems alternately:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & B_2 \end{bmatrix} \begin{bmatrix} x_1^{(i)} \\ x_2^{(i-1/2)} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} 0 & A_{13} \\ C_2 & A_{23} \end{bmatrix} \begin{bmatrix} x_2^{(i-1)} \\ x_3^{(i-1)} \end{bmatrix},$$

$$\begin{bmatrix} B_2' & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_2^{(i)} \\ x_3^{(i)} \end{bmatrix} = \begin{bmatrix} f_2 \\ f_3 \end{bmatrix} + \begin{bmatrix} A_{21} & C_2' \\ A_{31} & 0 \end{bmatrix} \begin{bmatrix} x_1^{(i)} \\ x_2^{(i-1/2)} \end{bmatrix}, \tag{17}$$

where

$$A_{22} = B_2 + C_2 = B_2' + C_2'. \tag{18}$$

The splittings in (18) correspond to the couplings in (2) and (3). In the next section we will show that a good choice of the splitting of A_{22} can significantly affect the convergence of SAM. It is therefore a very interesting research problem for further improvement of SAM.

It is not difficult to observe that this procedure is equivalent to a 2×2 block Gauss–Seidel iteration for the following matrix equation:

$$\widetilde{A}\widetilde{x} = \begin{bmatrix} A_{11} & A_{12} & 0 & A_{13} \\ A_{21} & B_2 & C_2 & A_{23} \\ A_{21} & C_2' & B_2' & A_{23} \\ A_{31} & 0 & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \widetilde{x}_2' \\ \widetilde{x}_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_2 \\ f_3 \end{bmatrix} = \widetilde{f}. \tag{19}$$

Under certain conditions [14], [17], we know that the procedure (17) will converge, the solution of (19) satisfies $\widetilde{x}_2 = \widetilde{x}_2'$, and $[\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3]^T$ is a solution of (16). This is to say, the augmented vector of the solution of (16) is the solution of (19) and vice versa. Later we will prove that this conclusion can be true only when $(B_2 - C_2')^{-1}$ exists. For most approximations of an elliptic partial differential equation, this restriction is not very difficult to satisfy. We shall call (19) the generalized Schwarz enhanced equation of (16), and the corresponding matrix $\widetilde{A}$ in (19) the generalized Schwarz enhanced matrix of the matrix A.

It may be observed that the second equation in (16) becomes a pair of dual equations in (19):

$$A_{21}\widetilde{x}_1 + B_2\widetilde{x}_2 + C_2\widetilde{x}_2' + A_{23}\widetilde{x}_3 = f_2,$$
$$A_{21}\widetilde{x}_1 + C_2'\widetilde{x}_2 + B_2'\widetilde{x}_2' + A_{23}\widetilde{x}_3 = f_2.$$

They are almost identical, except the term $A_{22}x_2$ of the second equation in (16) is split in two different ways:

$$A_{22}x_2 \implies B_2\widetilde{x}_2 + C_2\widetilde{x}_2',$$
$$A_{22}x_2 \implies C_2'\widetilde{x}_2 + B_2'\widetilde{x}_2'.$$

Here is another example of a 5×5 block matrix equation and its generalized Schwarz enhanced equation:

$$Ax = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} = f,$$

$$\widetilde{A}\widetilde{x} = \begin{bmatrix} A_{11} & A_{12} & 0 & A_{13} & A_{14} & 0 & A_{15} \\ A_{21} & B_2 & C_2 & A_{23} & A_{24} & 0 & A_{25} \\ A_{21} & C_2' & B_2' & A_{23} & A_{24} & 0 & A_{25} \\ A_{31} & 0 & A_{32} & A_{33} & A_{34} & 0 & A_{35} \\ A_{41} & 0 & A_{42} & A_{43} & B_4 & C_4 & A_{45} \\ A_{41} & 0 & A_{42} & A_{43} & C_4' & B_4' & A_{45} \\ A_{51} & 0 & A_{52} & A_{53} & 0 & A_{54} & A_{55} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2' \\ x_3 \\ x_4 \\ x_4' \\ x_5 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_2 \\ f_3 \\ f_4 \\ f_4 \\ f_5 \end{bmatrix} = \widetilde{f},$$

where

$$A_{22} = B_2 + C_2 = B_2' + C_2',$$
$$A_{44} = B_4 + C_4 = B_4' + C_4'.$$

For a general partitioned matrix (13), the splittings of the submatrices $A_{2i\ 2i}$ are

$$A_{2i\ 2i} = B_{2i} + C_{2i} = B_{2i}' + C_{2i}'.$$

From these examples, we can summarize the following rules for constructing the Schwarz enhanced equation. The odd-numbered equations in $Ax = f$ are changed to

$$\sum_{j=1}^{2k+1} A_{1\ j}x_j = f_1,$$

$$\sum_{j=1}^{i} A_{2i-1\ 2j-1}x_{2j-1} + \sum_{j=1}^{i-1} A_{2i-1\ 2j}x_{2j}' + \sum_{j=2i}^{2k+1} A_{2i-1\ j}x_j = f_{2i-1}, \quad 1 < i \le k, \quad k \ne 1,$$

$$\sum_{j=1}^{k+1} A_{2k+1\ 2j-1}x_{2j-1} + \sum_{j=1}^{k} A_{2k+1\ 2j}x_{2j}' = f_{2k+1},$$

while the even-numbered equations become a pair of dual equations in the generalized Schwarz enhanced equation:

$$\sum_{j=1}^{i} A_{2i\ 2j-1}x_{2j-1} + \sum_{j=1}^{i-1} A_{2i\ 2j}x_{2j}' + B_{2i}x_{2i} + C_{2i}x_{2i}' + \sum_{j=2i+1}^{2k+1} A_{2i\ j}x_j = f_{2i},$$

(20)

$$\sum_{j=1}^{i} A_{2i\ 2j-1}x_{2j-1} + \sum_{j=1}^{i-1} A_{2i\ 2j}x_{2j}' + C_{2i}'x_{2i} + B_{2i}'x_{2i}' + \sum_{j=2i+1}^{2k+1} A_{2i\ j}x_j = f_{2i}, \quad i = 1, \cdots, k.$$

Only two terms are different in the two dual equations. We will not describe the details of how to form the generalized Schwarz enhanced matrix in general cases, as it is similar to the Schwarz enhanced matrix described in [17]. From the construction of the generalized Schwarz enhanced equation, it is easy to see the following result.

LEMMA 1. *If the vector* $x = (x_1, x_2, \cdots, x_{2k+1})^T$ *is the solution of* (10), *then its augmented vector* $\tilde{x}$ *is the solution of the generalized Schwarz enhanced equation* $\tilde{A}\tilde{x} = \tilde{f}$, *where* $\tilde{f}$ *is the augmented vector of* f.

The matrices $A_{2i\ 2i}$, $i = 1, \cdots, k$, are also called overlapped blocks. Let two matrices $\tilde{B}$ and $\tilde{C}$ be the Schwarz enhanced matrices of the same matrix A and their overlapped blocks are $B_{2i\ 2i}$ and $C_{2i\ 2i}, i = 1, \cdots, k$, respectively. If $B_{2i\ 2i}$ and $C_{2i\ 2i}$ have a relationship such that each $B_{2i\ 2i}$ is a submatrix of the corresponding $C_{2i\ 2i}$, we then say $\tilde{C}$ has *more overlap* than $\tilde{B}$. This overlap is closely related to the overlap area of the solution regions for the subregions mentioned in the introduction. As we have shown in [17], for the continuous model problem, if the amount of overlap increases, then the convergence rate will increase if a traditional SAM is applied. For the matrix model we have a similar result [12].

2.2. Equivalence theorem. A necessary and sufficient condition for the equivalence of (10) and its Schwarz enhanced equation (19) is given in this section. Let A be the same partitioned matrix in (13) and $\widetilde{A}$ be its Schwarz enhanced matrix.

THEOREM 2. *Let $\lambda(A)$, $\lambda(\widetilde{A})$, and $\lambda(B_{2i} - C'_{2i})$, $i = 1, \cdots, k$, be the sets of eigenvalues of $A, \widetilde{A}$, and $(B_{2i} - C'_{2i})$, $i = 1, \cdots, k$, respectively. Then $\lambda(\widetilde{A}) \subset \lambda(A) \cup (\bigcup_{i=1}^{k} \lambda(B_{2i} - C'_{2i}))$.*

Proof. Let λ be an eigenvalue of $\widetilde{A}$ and

$$\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}'_2, \cdots, \widetilde{x}_{2k+1})$$

be the corresponding eigenvector. Substituting $\widetilde{x}$ into the equation $2i$ and its dual equation, we have

$$\sum_{j=1}^{i} A_{2i\ 2j-1}\widetilde{x}_{2j-1} + \sum_{j=1}^{i-1} A_{2i\ 2j}\widetilde{x}'_{2j} + B_{2i}\widetilde{x}_{2i} + C_{2i}\widetilde{x}'_{2i} + \sum_{j=2i+1}^{2k+1} A_{2i\ j}\widetilde{x}_j = \lambda\widetilde{x}_{2i},$$

$$\sum_{j=1}^{i} A_{2i\ 2j-1}\widetilde{x}_{2j-1} + \sum_{j=1}^{i-1} A_{2i\ 2j}\widetilde{x}'_{2j} + C'_{2i}\widetilde{x}_{2i} + B'_{2i}\widetilde{x}'_{2i} + \sum_{j=2i+1}^{2k+1} A_{2i\ j}\widetilde{x}_j = \lambda\widetilde{x}'_{2i}.$$

As we mentioned in the last section, only two terms are different in the left-hand sides of the two equations. Subtracting the first equation from the second, we have

$$(B_{2i} - C'_{2i})(\widetilde{x}_{2i} - \widetilde{x}'_{2i}) = \lambda(\widetilde{x}_{2i} - \widetilde{x}'_{2i}), \qquad i = 1, \cdots, k.$$

If $\widetilde{x}_{2i} - \widetilde{x}'_{2i} \neq 0$ for some i, then we have $\lambda \in \bigcup_{i=1}^{k} \lambda(B_{2i} - C'_{2i})$. If $\lambda \notin \bigcup_{i=1}^{k} \lambda(B_{2i} - C'_{2i})$, then $\widetilde{x}_{2i}$ has to be equal to $\widetilde{x}'_{2i}$ for $i = 1, \cdots, k$. Therefore, $\widetilde{x}$ is an augmented vector of $x = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \cdots, \widetilde{x}_{2k+1})^T$, which will satisfy equation

$$Ax = \lambda x.$$

Thus $\lambda \in \lambda(A)$, which concludes the proof. □

Define the Schwarz enhanced equation (19) as *equivalent* to (10) if $\widetilde{A}^{-1}$ exists and the solution vector $\widetilde{x}$ is an augmented vector of the solution x of (10). Similarly, we say that the Schwarz enhanced matrix $\widetilde{A}$ is *equivalent* to matrix A if $\widetilde{A}^{-1}$ exists. With this definition and the result from Theorem 2 we have the following theorem.

THEOREM 3. *If a matrix $\widetilde{A}$ is a Schwarz enhanced matrix of the nonsingular matrix A, then the following conditions are equivalent:*

1. *Matrix $\widetilde{A}$ is equivalent to matrix A.*
2. $0 \notin \bigcup_{i=1}^{k} \lambda(B_{2i} - C'_{2i})$.

Proof. If $0 \notin \bigcup_{i=1}^{k} \lambda(B_{2i} - C'_{2i})$, then from Theorem 2 we know $\widetilde{A}^{-1}$ exists. Applying the same strategy used in the previous proof, we can show that the solution $\widetilde{x}$ of $\widetilde{A}\widetilde{x} = \widetilde{f}$ is an augmented vector of the solution x of $Ax = f$.

Now we show that $0 \notin \bigcup_{i=1}^{k} \lambda(B_{2i} - C'_{2i})$ is also a necessary condition. Suppose there is a j such that $0 \in \lambda(B_{2j} - C'_{2j})$. We know that $(B_{2j} - C'_{2j})$ is singular, hence so is $(B'_{2j} - C_{2j})$, since $(B_{2j} - C'_{2j}) = (B'_{2j} - C_{2j})$. Now, if we subtract row $2j$ from $2j'$ in matrix $\widetilde{A}$, we will have

$$0, \cdots, 0, (B_{2j} - C'_{2j}), -(B'_{2j} - C_{2j}), 0, \cdots, 0. \tag{21}$$

This means that $\widetilde{A}$ is singular. The proof is complete. □

If a matrix is a positive definite matrix or an M-matrix,[1] any principal minor of this matrix is also a positive definite matrix or an M-matrix, respectively. Thus, if we choose $C_{2i} = 0$ and $C'_{2i} = 0$,[2] we immediately have the following corollaries.

COROLLARY 1. *Any Schwarz enhanced matrix of a positive definite matrix A is equivalent to A if $C_{2i} = 0$ and $C'_{2i} = 0, i = 1, \cdots, k$.*

COROLLARY 2. *Any Schwarz enhanced matrix of an M-matrix A is equivalent to A if $C_{2i} = 0$ and $C'_{2i} = 0, i = 1, \cdots, k$.*

3. A parameterized generalized Schwarz Splitting. The general framework of a generalized SS is given in the last section. Here, the convergence behavior of a particular generalized SS for the elliptic equation, namely, parameterized generalized SS, is studied. In a traditional approach to SS, we choose $C_{2i} \equiv 0$. In this case, it is well understood that the amount of overlap is a key factor which affects the convergence rate. Even though a larger overlap means more duplication of work on these overlapping regions, the overall complexity is still better than with a smaller overlap. However, a natural question is raised: is a larger overlap the ultimate choice? The generalized Schwarz Splitting discussed in the previous section provides a way to explore possibilities of further improving the performance of SAM. In particular, we will examine the importance of splitting

$$A_{2i\ 2i} = B_{2i} + C_{2i} = B'_{2i} + C'_{2i}.$$

First, an application of generalized SS to a two-point boundary value problem is investigated. A similar approach can also be applied to two-dimensional problems.

Consider a two-point boundary value problem

$$U''(x) + qU(x) = f(x), \qquad x \in (0,1),$$
$$U(0) = a_0, \qquad U(1) = a_1,$$

where $q \leq 0$. After discretization using a centered finite difference, the resulting linear system is

$$T_n(\beta)x = b, \tag{22}$$

where

$$T_n(\beta) = Tridiagonal\{-1, \quad \beta, \quad -1\}_{n\times n}$$

and $\beta \geq 2$. If there is no ambiguity, it will be abbreviated as T_n. Denote

$$T_n(x_1, x_2, x_3)$$

as the same $n \times n$ tridiagonal matrix $T_n(x_2)$, except the first diagonal element is x_1 and the last is x_3.

The generalized SAM for solving this problem divides the region into k overlapping subregions Ω_i $i = 1, \cdots, k$ as shown in Fig. 2. (To simplify the analysis we assume the overlap pattern is uniform). Let h be the grid size, ℓ the length of the overlap and η

[1] Any $n \times n$ matrix $A = (a_{ij})$ with $a_{ij} \leq 0$ for all $i \neq j$ is an M-matrix if A is nonsingular, and $A^{-1} \geq 0$.

[2] In the traditional approach of SAM, we always choose $C_{2i} = 0$ and $C'_{2i} = 0$.

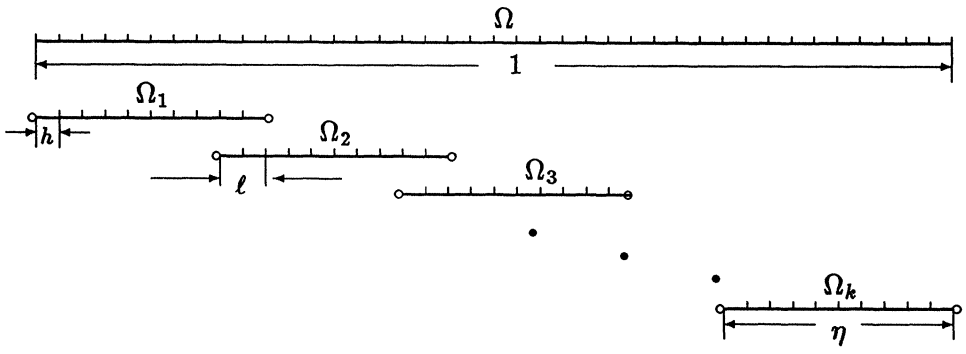

FIG. 2. *One-dimensional overlapping grid.*

the length of every subregion. Then let $n+1 = 1/h$, $l = \ell/h$ and $m+1 = \eta/h$. Here we assume

$$l < m/2,$$

which means no three subregions have a common overlap part. The open circle points in Fig. 2 are the boundaries of the subregions.

We display the case of $k = 3$ for readability. The general case can be easily extended from this case. The partitioned form of (22) is now

$$T_n x = \begin{bmatrix} T_{m-l} & -F_1 & 0 & 0 & 0 \\ -E_2 & T_l & -F_2 & 0 & 0 \\ 0 & -E_3 & T_{m-2l} & -F_3 & 0 \\ 0 & 0 & -E_4 & T_l & -F_4 \\ 0 & 0 & 0 & -E_5 & T_{m-l} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} = b.$$

Its corresponding Schwarz enhanced equation is

$$\widetilde{T}_n \widetilde{x} = \begin{bmatrix} T_{m-l} & -F_1 & 0 & 0 & 0 & 0 & 0 \\ -E_2 & B_2 & C_2 & -F_2 & 0 & 0 & 0 \\ -E_2 & C_2' & B_2' & -F_2 & 0 & 0 & 0 \\ 0 & 0 & -E_3 & T_{m-2l} & -F_3 & 0 & 0 \\ 0 & 0 & 0 & -E_4 & B_4 & C_4 & -F_4 \\ 0 & 0 & 0 & -E_4 & C_4' & B_4' & -F_4 \\ 0 & 0 & 0 & 0 & 0 & -E_5 & T_{m-l} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2' \\ x_3 \\ x_4 \\ x_4' \\ x_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_2 \\ b_3 \\ b_4 \\ b_4 \\ b_5 \end{bmatrix} = \widetilde{b}.$$

The quantities above are defined as

- F_1: an $(m-l) \times l$ matrix with zero elements everywhere except for a 1 in position $(m-l, 1)$.
- F_2: an $l \times (m-2l)$ matrix with zero elements everywhere except for a 1 in position $(m-2l, 1)$.
- F_3: an $(m-2l) \times l$ matrix with zero elements everywhere except for a 1 in position $(m-2l, 1)$.

- F_4: an $l \times (m-l)$ matrix with zero elements everywhere except for a 1 in position $(m-l, 1)$.
- E_2: an $l \times (m-l)$ matrix with zero elements everywhere except for a 1 in position $(1, m-l)$.
- E_3: an $(m-2l) \times l$ matrix with zero elements everywhere except for a 1 in position $(1, l)$.
- E_4: an $l \times (m-2l)$ matrix with zero elements everywhere except for a 1 in position $(1, m-2l)$.
- E_5: an $(m-l) \times l$ matrix with zero elements everywhere except for a 1 in position $(1, l)$.

There are many ways to split the matrix T_l. We will introduce a parameterized generalized SS for this problem below. Let

- F be an $l \times l$ matrix with zero elements everywhere except for a 1 in position (l, l) and

$$C = C_2 = C_4 = \alpha F, \qquad B = B_2 = B_4 = T_l - C.$$

- E be an $l \times l$ matrix with zero elements everywhere except for a 1 in position $(1, 1)$ and

$$C' = C_2' = C_4' = \alpha E, \qquad B' = B_2' = B_4' = T_l - C',$$

where $0 \le \alpha < 1$. It is not difficult to show that

$$\det(B - C') = \det(T_l(\beta - \alpha, \beta, \beta - \alpha)) \neq 0,$$

provided $\beta \ge 2$. The resulting Schwarz enhanced equation is equivalent to (22). Then the parameterized generalized SS of (22) is defined as

$$\begin{aligned} \widetilde{T}_n &= M(\alpha) - N(\alpha) \\ &= \begin{bmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{bmatrix} - \begin{bmatrix} 0 & U_1 & 0 \\ L_2 & 0 & U_2 \\ 0 & L_3 & 0 \end{bmatrix}, \end{aligned}$$

where

$$T_1 = T_m(\beta, \beta, \beta - \alpha), \quad T_2 = T_m(\beta - \alpha, \beta, \beta - \alpha), \quad T_3 = T_m(\beta - \alpha, \beta, \beta),$$

$$L_2 = \begin{bmatrix} E_2 & \alpha E \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad L_3 = \begin{bmatrix} 0 & E_4 & \alpha E \\ 0 & 0 & 0 \end{bmatrix},$$

$$U_1 = \begin{bmatrix} 0 & 0 & 0 \\ \alpha F & F_2 & 0 \end{bmatrix}, \qquad U_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \alpha F & F_4 \end{bmatrix}.$$

A simple calculation can show that the relationship between α and ω in (4) is

$$\omega = \frac{1-\alpha}{1-\alpha+h\alpha}.$$

When $\alpha = 0$, we have $\omega = 1$. Thus, this parameterized generalized SS reduced to the traditional SS, namely, a Dirichlet boundary condition, is used on these artificial

boundaries. If $\beta > 2$ and $\alpha = 1$, we have $\omega = 0$. It is equivalent to using a Neumann condition on the artificial boundaries. (If $\beta = 2$, we can use a Neumann condition on only one of the boundaries for interior subregions. Otherwise, the resulting Schwarz enhanced matrix is singular. For two-dimensional problems, if a strip decomposition is employed, then a Neumann boundary condition can be used for both artificial boundaries.) For $0 < \alpha < 1$, this generalized SS corresponds to

$$g_1(u) = g_2(u) = \omega u + (1-\omega)\frac{\partial u}{\partial n}.$$

The convergence analysis of this parameterized generalized SS is therefore reduced to calculating the spectral radius of the block Jacobi matrix $J = M^{-1}N$. Notice that the matrix $N(\alpha)$ has only eight nonzero elements. So the matrix $J = M^{-1}N$ has only eight nonzero columns, provided $l < m/2$. They are related only to the elements in the last or first columns of the matrices T_1^{-1}, T_2^{-1}, and T_3^{-1}. Let $t_{i,j}$ be the elements of the matrix $T_n^{-1}(\beta)$ and $D_i(\beta) = \det T_i(\beta)$. We have the following results (see [6]):

$$D_k(\beta) = \begin{cases} \sinh(n+1)\theta/\sinh\theta, & \beta > 2, \quad 2\cosh\theta = \beta, \\ n+1, & \beta = 2, \\ \sin(n+1)\theta/\sinh\theta, & \beta < 2, \quad 2\cos\theta = \beta, \end{cases}$$

$$t_{i,j} = \begin{cases} D_{j-1}(\beta)D_{n-i}(\beta)/D_n(\beta), & i \geq j, \\ D_{i-1}(\beta)D_{n-j}(\beta)/D_n(\beta), & i < j. \end{cases}$$

Based on this result, the elements of T_1^{-1} and T_2^{-1} can be easily derived from the Sherman–Morrison formula. We will not elaborate on the detailed derivation here.

Denote the last columns of the matrices T_1^{-1} and T_2^{-1} by $t^{(1)}$ and $t^{(2)}$, respectively:

$$t^{(1)} = (t_1^{(1)}, t_2^{(1)}, \cdots, t_m^{(1)}),$$
$$t^{(2)} = (t_1^{(2)}, t_2^{(2)}, \cdots, t_m^{(2)}).$$

Note that elements $t_i^{(j)}$ are functions of α. Since matrix T_3^{-1} is a permuted matrix of T_1^{-1}, the first column of the inverse T_3^{-1} can be derived from the last column of T_1^{-1} by a simple permutation. Let P^T be a permutation matrix; permute columns $m-l$, $m-l+1$; $m+l$, $m+l+1$; $2m-l$, $2m-l+1$; $2m+l$, $2m+l+1$ to $3m-k+1$, $k = 8, 7, \cdots, 1$, respectively. J can be similarly transformed to $\tilde{J}$:

$$\tilde{J} = PJP^T = \begin{bmatrix} 0 & K \\ 0 & G \end{bmatrix},$$

where

$$G = \begin{pmatrix} 0 & 0 & -\alpha t_{m-l}^{(1)} & t_{m-l}^{(1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha t_{m-l+1}^{(1)} & t_{m-l+1}^{(1)} & 0 & 0 & 0 & 0 \\ t_{m-l+1}^{(2)} & -\alpha t_{m-l+1}^{(2)} & 0 & 0 & 0 & 0 & -\alpha t_{l-1}^{(2)} & t_{l-1}^{(2)} \\ t_{m-l}^{(2)} & -\alpha t_{m-l}^{(2)} & 0 & 0 & 0 & 0 & -\alpha t_l^{(2)} & t_l^{(2)} \\ t_l^{(2)} & -\alpha t_l^{(2)} & 0 & 0 & 0 & 0 & -\alpha t_{m-l}^{(2)} & t_{m-l}^{(2)} \\ t_{l-1}^{(2)} & -\alpha t_{l-1}^{(2)} & 0 & 0 & 0 & 0 & -\alpha t_{m-l+1}^{(2)} & t_{m-l+1}^{(2)} \\ 0 & 0 & 0 & 0 & t_{m-l+1}^{(1)} & -\alpha t_{m-l+1}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & t_{m-l}^{(1)} & -\alpha t_{m-l}^{(1)} & 0 & 0 \end{pmatrix}.$$

Note that matrix G has only four independent columns. After a simple reduction, we know the following matrix includes four nonzero eigenvalues of G:

$$G' = \begin{bmatrix} 0 & t^{(1)}_{m-l}-\alpha t^{(1)}_{m-l+1} & 0 & 0 \\ t^{(2)}_{m-l}-\alpha t^{(2)}_{m-l+1} & 0 & 0 & t^{(2)}_{l}-\alpha t^{(2)}_{l-1} \\ t^{(2)}_{l}-\alpha t^{(2)}_{l-1} & 0 & 0 & t^{(2)}_{m-l}-\alpha t^{(2)}_{m-l+1} \\ 0 & 0 & t^{(1)}_{m-l}-\alpha t^{(1)}_{m-l+1} & \end{bmatrix}$$

$$= \begin{bmatrix} 0 & g_1 & 0 & 0 \\ g_2 & 0 & 0 & g_3 \\ g_3 & 0 & 0 & g_2 \\ 0 & 0 & g_1 & 0 \end{bmatrix}.$$

Let

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

We have

$$HG'H = \begin{bmatrix} 0 & g_1 & 0 & 0 \\ g_2+g_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_2-g_3 \\ 0 & 0 & g_1 & 0 \end{bmatrix}.$$

Thus, the four eigenvalues of G' are:

$$\lambda_{1,2} = \pm\sqrt{g_1(g_2+g_3)},$$
$$\lambda_{3,4} = \pm\sqrt{g_1(g_2-g_3)}.$$

Here we present two figures to show the relationship between the spectral radius and the parameter α. For both cases, the size of each subproblem is $m = 10$ and $\beta = 2$. When $\beta > 2$, the generalized SS has a faster convergence rate. Figures 6 and 7 will show the results for the latter case. The x-axis is the parameter α, while the y-axis is the spectral radius of the Jacobi matrix for the generalized SS. The top figure shows the case of one overlapping node while the lower one demonstrates the case of overlapping half the subregion. In Fig. 3, we can see that the traditional SS (when $\alpha = 0$) has a very poor convergence rate, since the overlap is so small. The convergence factor is 0.9. To reduce l_2 norm of the residual by a factor of 10^6 requires more than 60 iterations. When the parameter α approaches 1, an amazing improvement of the convergence rate appears. For $\alpha = 0.9$, the convergence factor is less than 10^{-4}. That is to say, only very few iterations are needed for any particular computation. From the second picture, we can observe when $\alpha = 0.85$ the convergence rate of the generalized SS approaches the optimum. But the optimal convergence rate in this case is even worse than having minimum overlap, and the only positive point here is the sensitivity of the convergence rate with the parameter α.

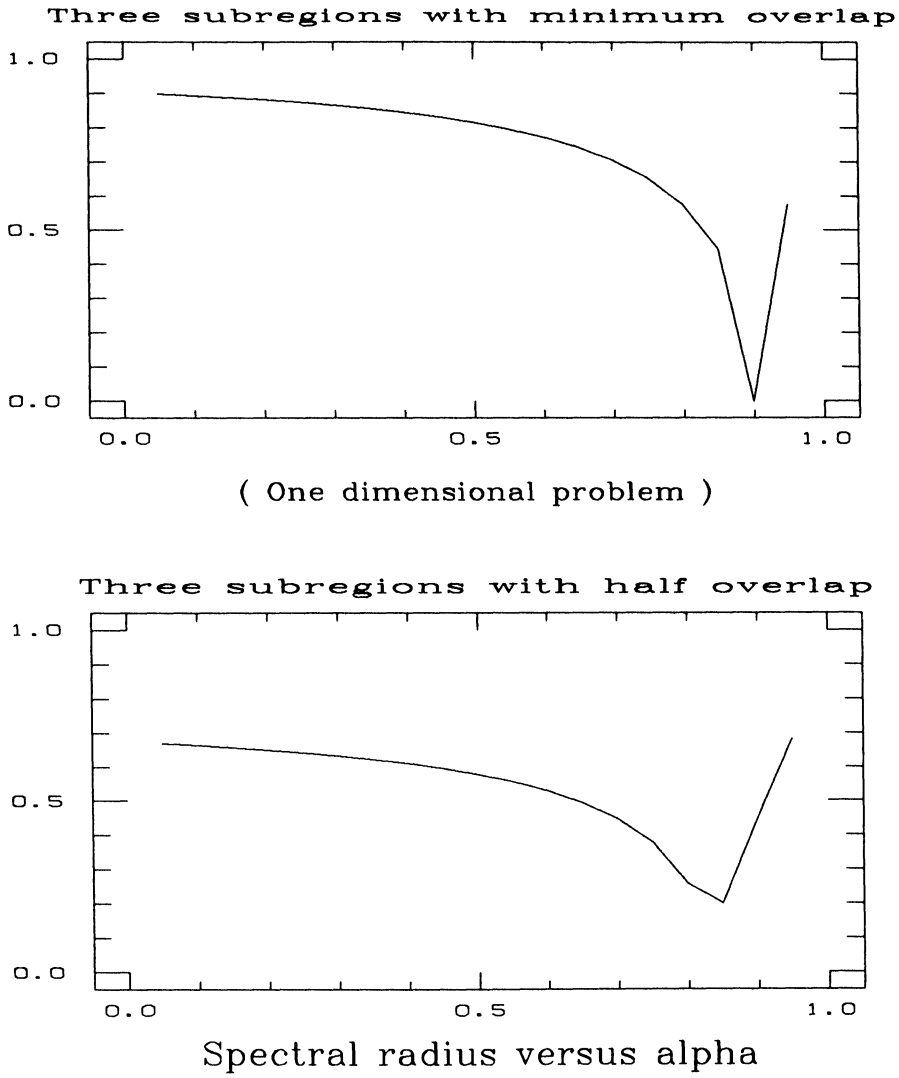

FIG. 3.

A numerical test has verified this analysis. The problem we are testing is

$$y''(x) = 2e^x \cos x, \qquad x \in (0,1),$$
$$y(0) = 0, \qquad y(1) = e\sin(1),$$

which has a solution $y(x) = e^x \sin(x)$. The solution region is covered by three overlap subregions with m unknowns each. Two neighboring subregions have one overlapping grid node. A random initial guess is used when the iteration starts and the l_2 norm is used to measure convergence. For $\alpha = 0.948, m = 20$, and $l = 2$, the residual is reduced by a factor of 10^{14} after three iterations. The results are the same for different mesh sizes. For $\alpha = 0.9, m = 10$, and $l = 2$, the residual is reduced by a factor of 10^{15} after three iterations. By comparison, the traditional SS with the same overlap will take 60 iterations to reduce the residual by only a factor of 10^5.

We apply the traditional SS to the same problem with an overlap of half of the subregion. Numerical testing shows that 11 iterations are needed to achieve a reduction of the initial residual by a factor of 10^5. If an optimal α is used, 4 iterations are needed. This again verifies the analysis shown in Fig. 3. The above results are summarized in Table 1.

TABLE 1

	Minimum overlap		Half overlap	
	Convergence factor	Number of iterations	Convergence factor	Number of iterations
Classical SAM	0.91	60	0.68	11
Generalized SAM	10^{-4}	3	0.06	4

In general, for k overlapping subregions, the nonzero eigenvalues of the Jacobi matrix are included in those of the following $(k-1)\times(k-1)$ block matrix:

$$G_k = \begin{bmatrix} D_1 & L & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ U & D_2 & L & 0 & \cdot & \cdot & 0 & 0 \\ & & \ddots & \ddots & & & & \\ & & & \ddots & \ddots & & & \\ 0 & 0 & \cdot & \cdot & \cdot & U & D_2 & L \\ 0 & 0 & \cdot & \cdot & & 0 & U & D_3 \end{bmatrix}.$$

There will not be a closed form for the eigenvalues of G_k when $k > 4$. In Fig. 4 we present a numerical result for the spectral radius of J where five overlapping subregions exist. The first picture is for the minimum overlap case, while the second is for the overlapping half of the subregions. Again, a significant improvement can be obtained by choosing a good parameter α.

For the two-dimensional problem

$$\triangle U(x,y) - qU(x,y) = f(x,y), \qquad (x,y) \in (0,1)\times(0,1),$$
$$U(x,y)\,|_\Gamma = g(x,y),$$

where $q < 0$, a matrix equation

$$Ax = b$$

can be derived using the centered finite difference. Given grid size $h = 1/(n+1)$, A can be written as

$$A = T_n(\beta)\otimes I_n + I_n \otimes T_n(2),$$

where $\beta > 2$. Decompose the solution region into three overlapping subregions (in strip). The overlap pattern in the x-direction is exactly the same as in the one-dimensional case. The corresponding Schwarz enhanced matrix is

$$\widetilde{A} = \begin{bmatrix} A_1 & -F_1 & 0 \\ -E_1 & A_2 & -F_1 \\ 0 & -E_1 & A_3 \end{bmatrix},$$

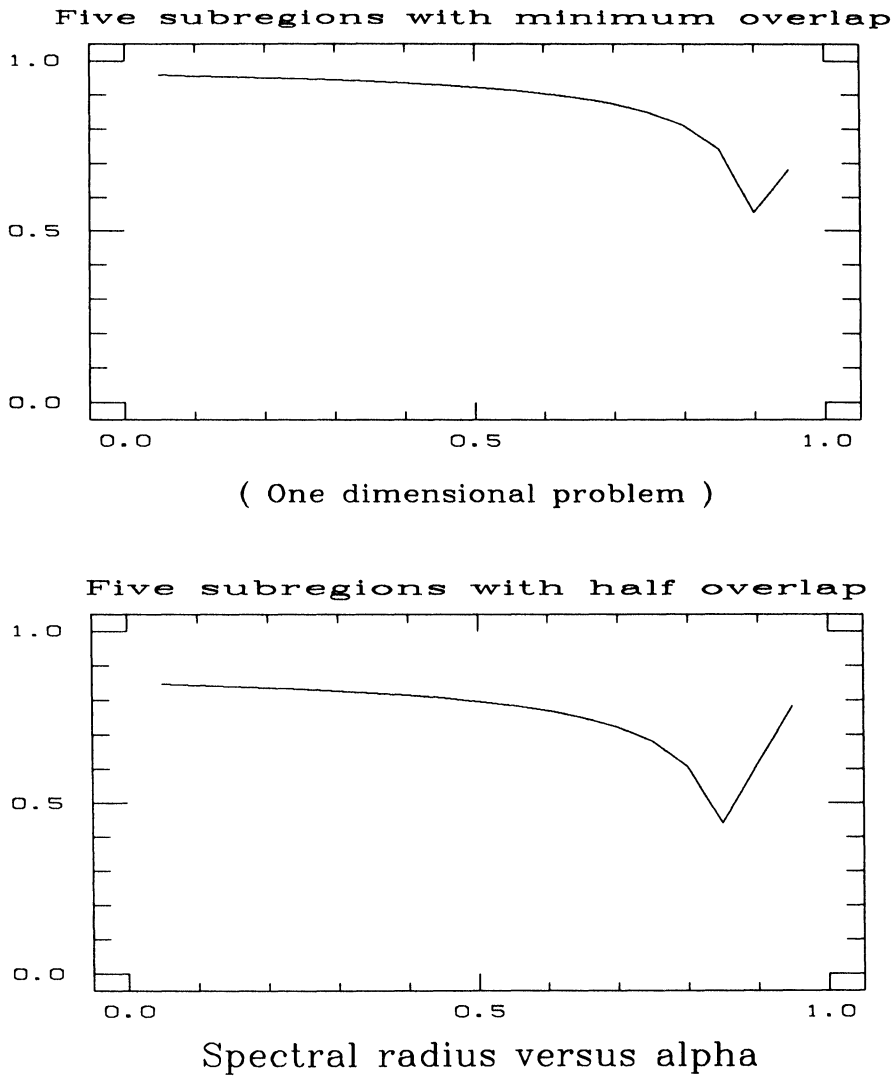

FIG. 4.

where

$$A_1 = T_1 \otimes I_n + I_m \otimes T_n(2),$$
$$A_2 = T_2 \otimes I_n + I_m \otimes T_n(2),$$
$$A_3 = T_3 \otimes I_n + I_m \otimes T_n(2),$$
$$E_1 = E_m \otimes I_n,$$
$$F_1 = F_m \otimes I_n,$$

and

- E_m is an $m \times m$ matrix with zero elements everywhere except for a 1 in position $(1, m-l)$ and α in position $(1, m-l+1)$.

- F_m is an $m \times m$ matrix with zero elements everywhere except for an α in position $(m, m-l)$ and 1 in position $(m, l-1)$.

The Jacobi iterative matrix for the generalized Schwarz splitting is

$$J = \begin{bmatrix} A_1^{-1} & 0 & 0 \\ 0 & A_2^{-1} & 0 \\ 0 & 0 & A_3^{-1} \end{bmatrix} \begin{bmatrix} 0 & F_1 & 0 \\ E_1 & 0 & F_1 \\ 0 & E_1 & 0 \end{bmatrix}$$
$$= M^{-1}N.$$

Let

$$U = \begin{bmatrix} I_m \otimes X_n & 0 & 0 \\ 0 & I_m \otimes X_n & 0 \\ 0 & 0 & I_m \otimes X_n \end{bmatrix},$$

where X_n is an orthogonal matrix. Each column in X_n corresponds to an eigenvector of matrix $T_n(2)$ and $X_n T_n(2) X_n = D_n = \text{diag}\,\{d_i\}$, $d_i = 2 + 2\cos(i\pi/(n+1))$, $i = 1, \cdots, n$. Note that U is orthogonal and $UNU = N$. So

$$J' = UJU^T$$
$$= (UMU^T)^{-1}N$$
$$= \widetilde{M}N$$

where

$$\widetilde{M} = \begin{bmatrix} \widetilde{A}_1 & 0 & 0 \\ 0 & \widetilde{A}_2 & 0 \\ 0 & 0 & \widetilde{A}_3 \end{bmatrix},$$
$$\widetilde{A}_i = (I_m \otimes X_n) A_i (I_m \otimes X_n)^T$$
$$= T_i \otimes I_n + I_m \otimes D_n.$$

Let P be the permutation matrix which permutes row $(k-1)n+i$ to $(i-1)3m+k$, $k = 1, \cdots, 3m$, $i = 1, \cdots, n$. Then

$$PJ'P^T = \begin{bmatrix} J(d_1) & & & \\ & J(d_2) & & \\ & & \ddots & \\ & & & J(d_n) \end{bmatrix},$$

where each $J(d_i)$ is the Jacobi iterative matrix of the generalized SAM for matrix $T_n(d_i)$ in the one-dimensional case. Similar to the traditional SAM, we found that the convergence of the lower frequency components are slower than that of higher frequencies. We present two pictures in Fig. 5 to show how $\rho_{J(d_i)}$ changes when d_i changes. The first represents three subregions with minimum overlap, while the second shows the same number of subregions with half overlap.

Another two sets of figures (Figs. 6 and 7) present the relations between the spectral radius $\rho_{J(d_i)}$ and the parameter α. The first set is for $J(2.01)$ and the other is

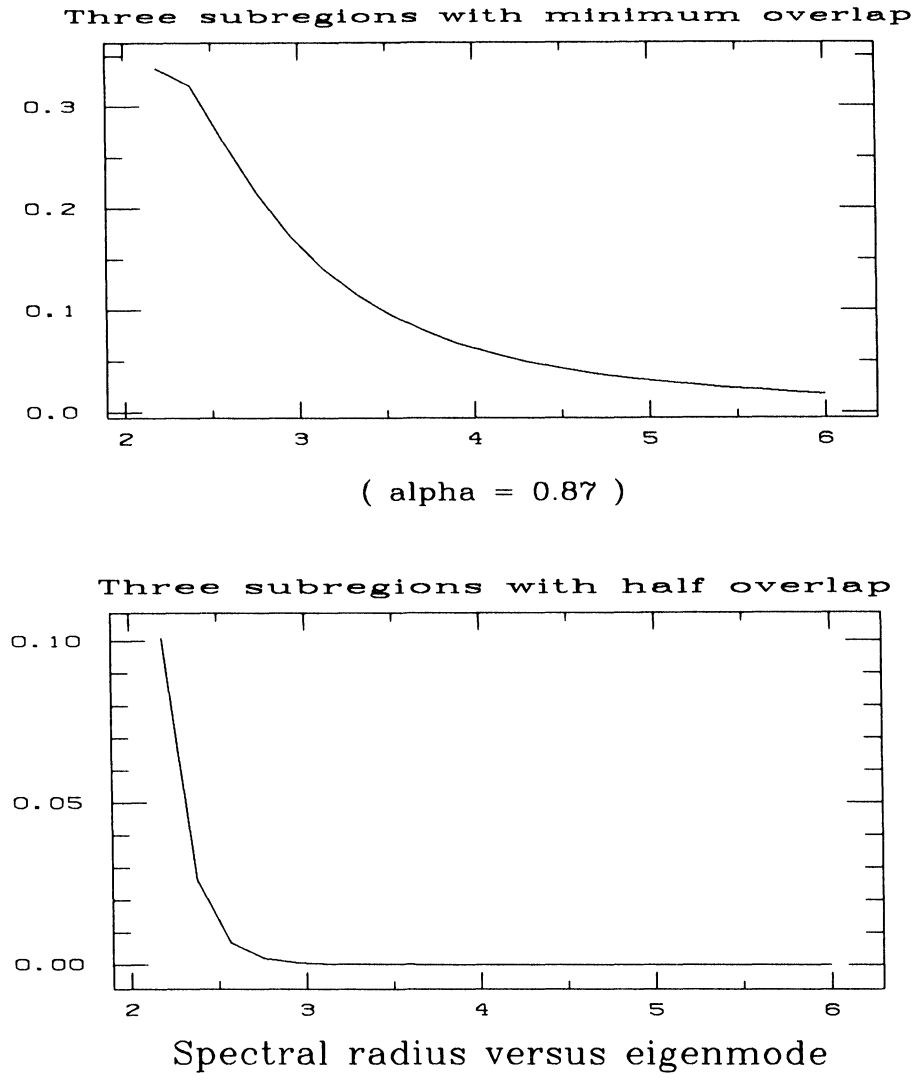

FIG. 5.

for $J(5.99)$, which represent the lowest and the highest frequencies of the eigenmodes, respectively. Both sets have one figure for minimum overlapping (one grid line) and another for overlapping half of the subregion. We can see that the sensitivity of the convergence rate with the parameter α is better in a two-dimensional problem. It is also noticeable that the convergence of the higher frequency mode is very fast for all α.

Numerical testing results for the model problem

$$\triangle U(x,y) = -2x(1-x) - 2y(1-y), \qquad (x,y) \in (0,1) \times (0,1),$$
$$U(x,y)\,|_{\Gamma} = 0$$

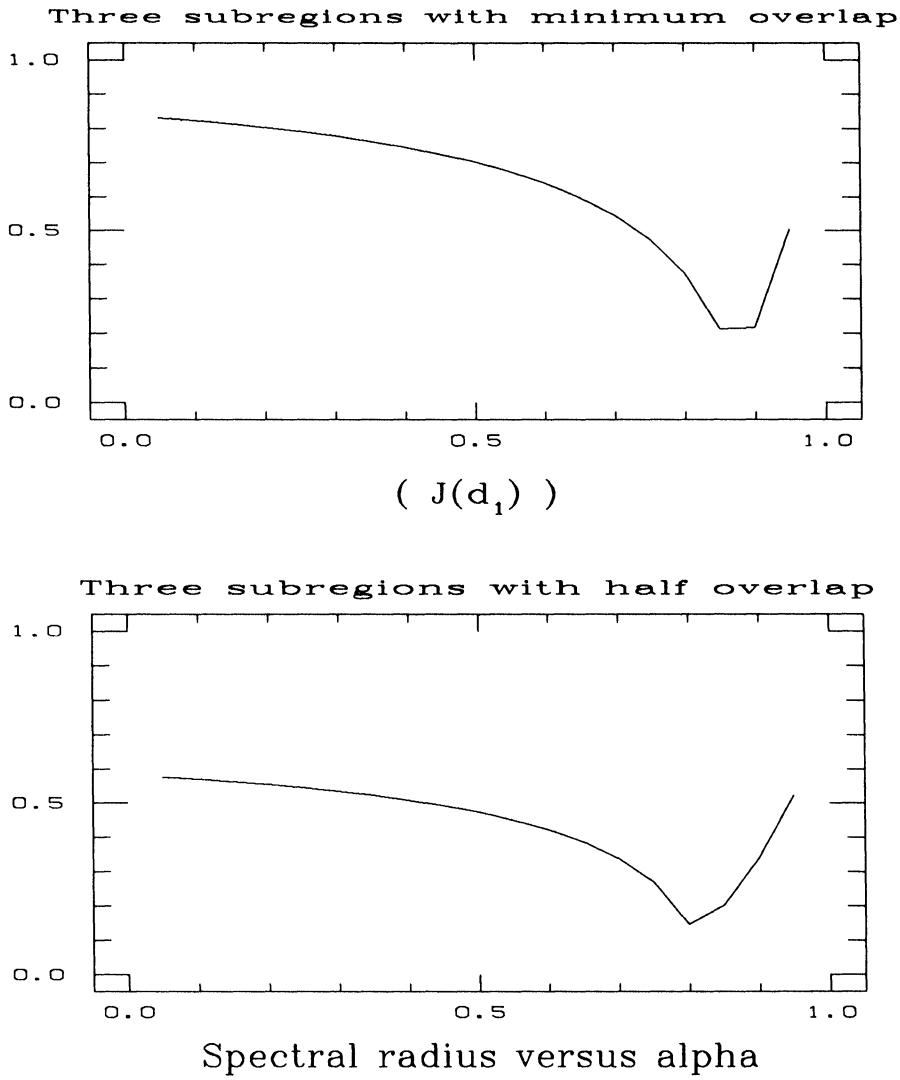

FIG. 6.

are given in Fig. 8. We present the relations between the number of iterations and the parameter α in these figures. Testing is carried out for three and five subregion cases, and for each decomposition, both minimum overlap and half overlap are tested. The initial guess is randomly generated. To make the programming easier, the grid size is slightly different for each case. h is between 1/40 and 1/50. The results plainly verify our analysis. The x-axis is the parameter α while the y-axis is the number of iterations needed for reducing the initial error by a factor of 10^5.

4. Conclusion. From the above analysis, a generalization of the traditional SAM is presented and the improvement of its performance is significant. The results of this study suggest that there may be other interesting splittings or couplings with good or even better performance characteristics. So far, our analysis has been

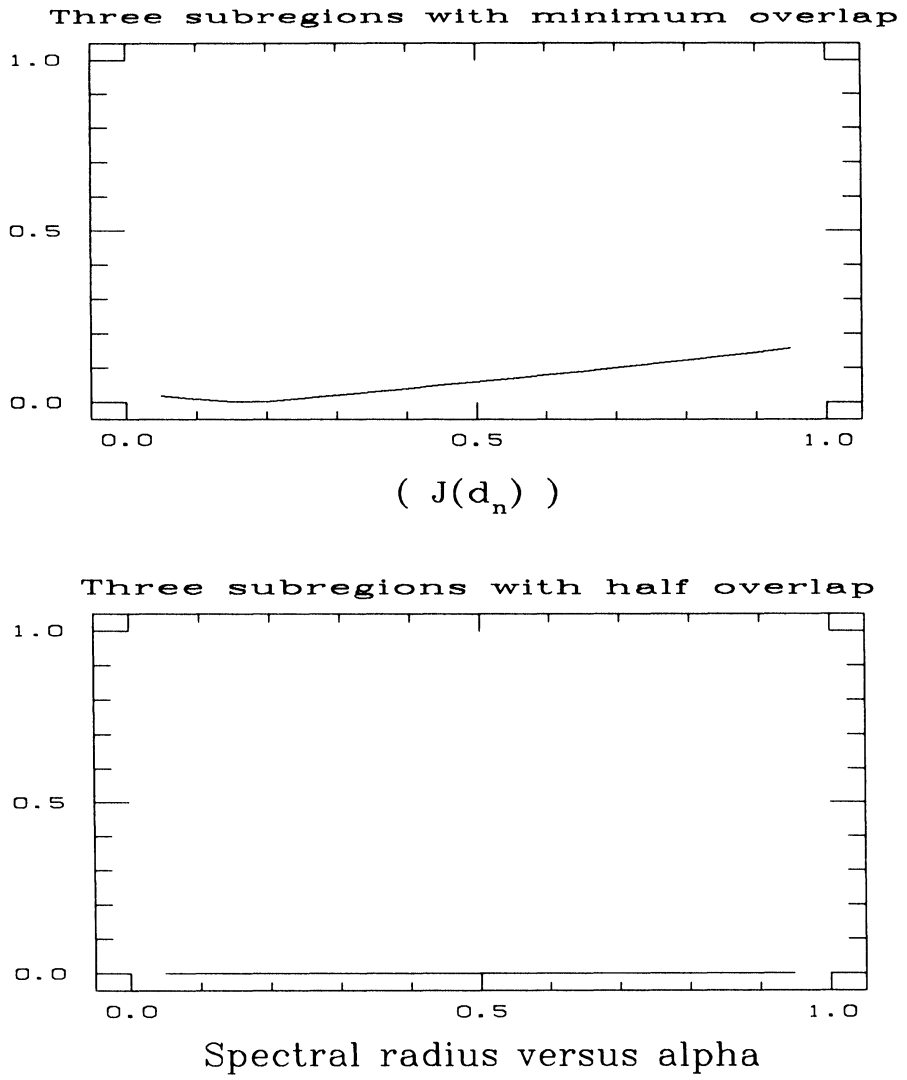

FIG. 7.

restricted to a simple case, namely, the strip decomposition. In particular, the close form of the spectral radius did not provide us with an insight of how the convergence is related to the decomposition and the parameter α. We do not have the same intuitive understanding we had for the classical SAM. More interesting problems remain to be studied. The effects of different couplings on the convergence of SAM have also been observed in a study for the fourth order equation [1].

We should also indicate that there is an extreme sensitivity between the parameter α and the convergence rate of generalized SS. A better understanding of this sensitivity is needed to make this generalized SS a practical technique.

Acknowledgment. The author would like to thank Gérard Meurant and referees for their valuable comments on this paper.

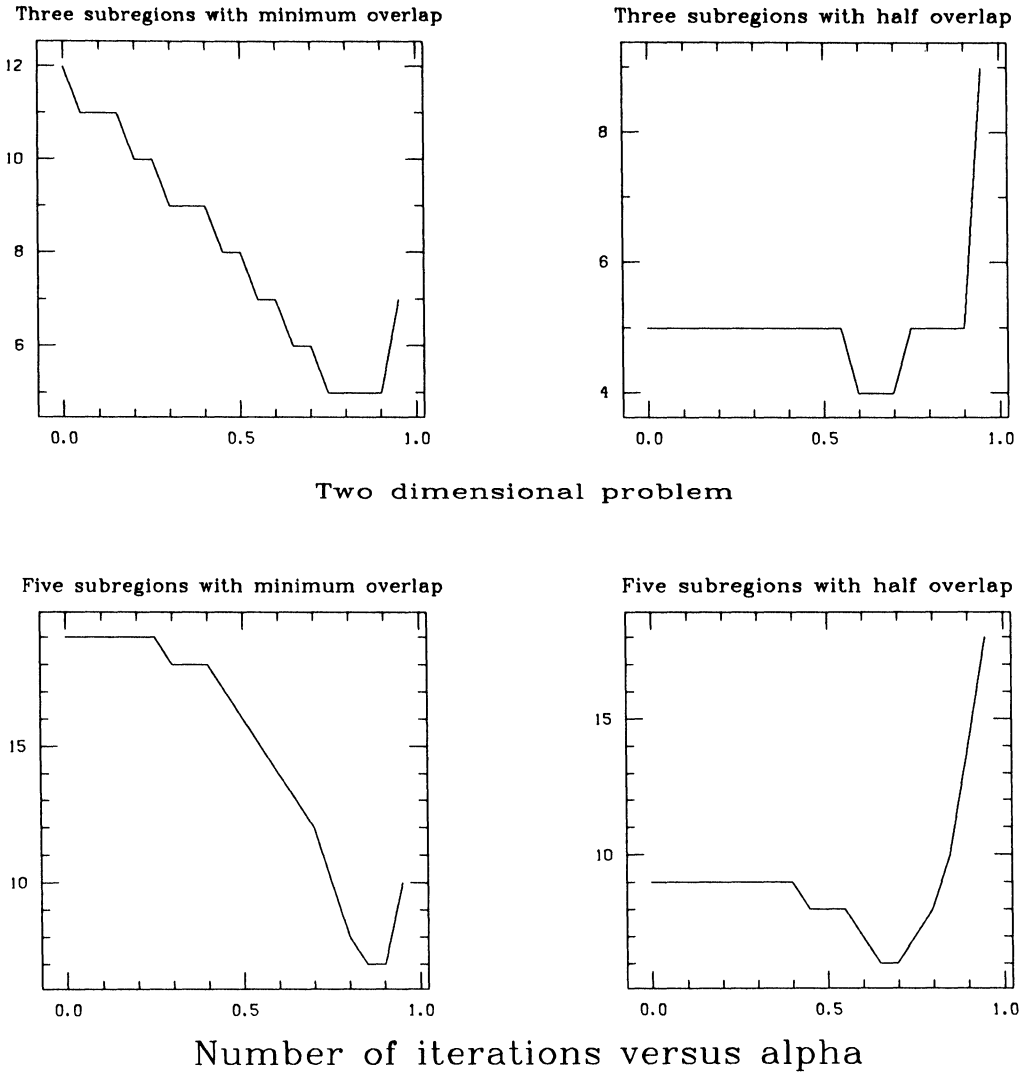

FIG. 8.

REFERENCES

[1] T. CHAN, D. GOOVAERTS, AND W. P. TANG, *A fast biharmonic solver on irregular domains using a generalized Schwarz alternating method*, Tech. Report 37, University of Waterloo, Waterloo, Ontario, Canada, 1989.

[2] R. V. DINH, R. GLOWINSKI, AND J. PERIAUX, *Applications of domain decomposition techniques to the numerical solution of the Navier–Stokes equations*, Tech. Report, INF-LAB, France, 1980.

[3] M. DRYJA, *An additive Schwarz algorithm for two- and three-dimensional finite element elliptic problems*, in Domain Decomposition Methods, T. Chan, R. Glowinski, J. Periaux, and O. Widlund, eds., Society for Industrial and Applied Mathematics, Philadelphia, PA, 1989, pp. 168–172.

[4] M. DRYJA AND O. WIDLUND, *An additive variant of Schwarz alternating method for the case of many subregions*, Tech. Report 339, New York University, New York, NY, 1987.

[5] L. W. EHRLICH, *The numerical Schwarz alternating procedure and SOR*, SIAM J. Sci. Statist. Comput., 7 (1986), pp. 989–993.

[6] C. F. FISCHER AND R. A. USMANI, *Properties of some tridiagonal matrices and their application to boundary value problems*, SIAM J. Numer. Anal., 6 (1969), pp. 127–142.

[7] G. H. GOLUB AND M. L. OVERTON, *The convergence of inexact Chebychev and Richardson iterative methods for solving linear systems*, Tech. Report, Stanford University, Stanford, CA, 1987.

[8] A. GREENBAUM, C. LI, AND H. Z. CHAO, *Parallelizing preconditioned conjugate gradient algorithms*, Tech. Report, New York University, New York, NY, 1988.

[9] L. V. KANTOROVICH AND V. I. KRYLOV, *Approximate methods of higher analysis*, Fourth Edition, P. Noordhoff LTD, Groningen, the Netherlands, 1958.

[10] P. L. LIONS, *On the Schwarz alternating method.* I, in First International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. Golub, G. Meurant, and J. Periaux, eds., Society for Industrial and Applied Mathematics, Philadelphia, 1988, pp. 1–42.

[11] U. MEIER, *Two parallel SOR variants of the Schwarz alternating procedure*, Tech. Report, Zentralinstitut für Angewandte Mathematik, 5170 Jülich, West Germany, 1986.

[12] J. OLIGER, W. SKAMAROCK, AND W. TANG, *Convergence analysis and acceleration of the Schwarz alternating method*, Tech. Report, Computer Science Dept., Stanford University, Stanford, CA, 1986.

[13] G. RODRIGUE, Personal communication, 1987.

[14] G. RODRIGUE AND J. SIMON, *A generalization of the numerical Schwarz algorithm*, in Computing Methods in Applied Sciences and Engineering VI, R. Glowinski and J. Lions, eds., North-Holland, Amsterdam, New York, Oxford, 1984, pp. 273–282.

[15] R. SCHREIBER, Personal communication, 1986.

[16] H. A. SCHWARZ, *Ueber einige abbildungsaufgaben*, J. Reine Angew. Math., 70 (1869), pp. 105–120.

[17] W. P. TANG, *Schwarz splitting and template operators*, Ph.D. thesis, Computer Science Dept., Stanford University, Stanford, CA, 1987.

[18] H. YSERENTANT, *Hierarchical bases give conjugate type method a multigrid speed of convergence*, Appl. Math. Comput., 19 (1986), pp. 147–158.