COMP270 Worksheet B solutions

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1 Calculating the shot speed

Let u and θ be defined as in the worksheet. Let g be the acceleration due to gravity, and x be the distance between the two tanks. Assume the position of the player tank is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so the position of the enemy tank is $\begin{pmatrix} x \\ 0 \end{pmatrix}$.

The initial velocity of the projectile is

$$\mathbf{u} = \begin{pmatrix} u \cos \theta \\ u \sin \theta \end{pmatrix}$$

When the projectile hits the enemy tank, its position relative to the origin is

$$\mathbf{s} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

The acceleration on the projectile is

$$\mathbf{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

Substituting these into the equation of motion $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$ gives

$$\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} u \cos \theta \\ u \sin \theta \end{pmatrix} t + \frac{1}{2} \begin{pmatrix} 0 \\ -g \end{pmatrix} t^2$$

Considering the x and y components separately gives two simultaneous equations:

$$x = u \cos(\theta) t \tag{1}$$

$$0 = u \sin(\theta) t - \frac{1}{2} g t^2 \tag{2}$$

We have two equations in two unknowns (u and t), so we should be able to solve this. We want to find u, so let's begin by eliminating t.

Rearranging (1) (dividing both sides by $u \cos \theta$ gives

$$t = \frac{x}{u \cos \theta}$$

Substituting this into (2) gives

$$0 = \frac{u\sin(\theta)x}{u\cos\theta} - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

Cancelling u, squaring $\frac{x}{u\cos\theta}$ and simplifying gives

$$0 = \frac{x\sin\theta}{\cos\theta} - \frac{gx^2}{2u^2\cos^2\theta}$$

Now let's rearrange this to get a formula for u.

Moving the second term from the right hand side to the left gives

$$\frac{gx^2}{2u^2\cos^2\theta} = \frac{x\sin\theta}{\cos\theta}$$

Moving u^2 to the right and everything else to the left gives

$$\frac{gx^2\cos\theta}{2x\sin\theta\cos^2\theta} = u^2$$

Taking the square root and cancelling $x\cos\theta$ gives

$$u = \sqrt{\frac{gx}{2\sin\theta\cos\theta}}$$

Finally, applying the double angle formula for sin gives

$$u = \sqrt{\frac{gx}{\sin 2\theta}}$$

2 Accounting for height difference

Assume the position of the player tank is $\begin{pmatrix}0\\0\end{pmatrix}$, and the position of the enemy tank is $\begin{pmatrix}x\\y\end{pmatrix}$. Now $\mathbf{u}$ and $\mathbf{a}$ are as in Task 1, but we now have

$$\mathbf{s} = \begin{pmatrix}x\\0\end{pmatrix}$$

Again substituting into $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$ gives

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}u\cos\theta\\u\sin\theta\end{pmatrix}t + \frac{1}{2}\begin{pmatrix}0\\-g\end{pmatrix}t^2$$

which gives simultaneous equations

$$x = u\cos(\theta)t \qquad (1)$$

$$y = u\sin(\theta)t - \frac{1}{2}gt^2 \qquad (2)$$

Just as in Task 1, rearranging (1) gives

$$t = \frac{x}{u\cos\theta}$$

Substituting this into (2) and simplifying as before gives

$$y = \frac{x\sin\theta}{\cos\theta} - \frac{gx^2}{2u^2\cos^2\theta}$$

Rearranging terms gives

$$\frac{gx^2}{2u^2\cos^2\theta} = \frac{x\sin\theta}{\cos\theta} - y$$

Combining the right hand side into a single fraction gives

$$\frac{gx^2}{2u^2\cos^2\theta} = \frac{x\sin\theta - y\cos\theta}{\cos\theta}$$

Moving u^2 to the right and everything else to the left gives

$$\frac{gx^2\cos\theta}{2\cos^2\theta(x\sin\theta - y\cos\theta)} = u^2$$

Cancelling $\cos\theta$ and multiplying out the denominator gives

$$u^2 = \frac{gx^2}{2x\sin\theta\cos\theta - 2y\cos^2\theta}$$

Applying double angle formulae gives

$$u^2 = \frac{gx^2}{x\sin 2\theta - y(\cos 2\theta + 1)}$$

Finally, taking the square root gives

$$u = \sqrt{\frac{gx^2}{x\sin 2\theta - y(\cos 2\theta + 1)}}$$

Note that this working is done in a coordinate system where the y-axis points upwards, however in the program the y-axis points downwards. We compensate for this by flipping the sign of y, so we have

$$u = \sqrt{\frac{gx^2}{x\sin 2\theta + y(\cos 2\theta + 1)}}$$

3 Accounting for wind

We now have $\mathbf{u}$ and $\mathbf{s}$ as in the previous task, but

$$\mathbf{a} = \begin{pmatrix} w \\ -g \end{pmatrix}$$

Again substituting into $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$ gives

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u\cos\theta \\ u\sin\theta \end{pmatrix} t + \frac{1}{2} \begin{pmatrix} w \\ -g \end{pmatrix} t^2$$

which gives simultaneous equations

$$x = u\cos(\theta)t + \frac{1}{2}wt^2 \tag{1}$$
$$y = u\sin(\theta)t - \frac{1}{2}gt^2 \tag{2}$$

Multiplying (1) through by g and (2) through by w gives

$$gx = gu\cos(\theta)t + \frac{1}{2}gwt^2 \tag{1}$$
$$wy = wu\sin(\theta)t - \frac{1}{2}gwt^2 \tag{2}$$

Adding both of these equations together gives

$$gx + wy = gu\cos(\theta)t + wu\sin(\theta)t$$

(The $+\frac{1}{2}gwt^2$ and $-\frac{1}{2}gwt^2$ cancel out.) Rearranging gives

$$t = \frac{gx + wy}{u(g\cos\theta + w\sin\theta)}$$

To make life easier, define

$$\alpha = \frac{gx + wy}{g\cos\theta + w\sin\theta}$$

So now

$$t = \frac{\alpha}{u}$$

Substituting this into (2) gives

$$y = \frac{u\alpha\sin\theta}{u} - \frac{g\alpha^2}{2u^2}$$

Cancelling u and rearranging gives

$$\frac{g\alpha^2}{2u^2} = \alpha\sin\theta - y$$

Further rearranging gives

$$\frac{g\alpha^2}{2(\alpha \sin\theta - y)} = u^2$$

Taking the square root gives

$$u = \sqrt{\frac{g\alpha^2}{2(\alpha \sin\theta - y)}}$$

At this point we could substitute α back in and simplify, but this form is good enough for doing the calculation.

4 Calculating the shot angle

Let's look at the case with no wind (as in Task 2). From there we have

$$y = \frac{x \sin\theta}{\cos\theta} - \frac{gx^2}{2u^2 \cos^2\theta}$$

From definition of sec we have

$$y = \frac{x \sin\theta}{\cos\theta} - \frac{gx^2 \sec^2\theta}{2u^2}$$

Applying identities $\tan = \frac{\sin}{\cos}$ and $1 + \tan^2 = \sec^2$ gives

$$y = x \tan\theta - \frac{gx^2(1 + \tan^2\theta)}{2u^2}$$

Rearranging gives

$$\frac{gx^2}{2u^2} \tan^2\theta - x \tan\theta + \frac{gx^2}{2u^2} + y = 0$$

This is a quadratic of the form

$$a\phi^2 + b\phi + c = 0$$

with

$$\begin{aligned}
\phi &= \tan\theta \quad \text{(to be found)} \\
a &= \frac{gx^2}{2u^2} \\
b &= x \\
c &= \frac{gx^2}{2u^2} + y
\end{aligned}$$

Hence its solutions can be found using the quadratic formula

$$\phi = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So the shot angle is

$$\theta = \tan^{-1}\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$$

Note that there are two possible solutions (due to the $\pm$) — hence there are two trajectories that hit the target.