LOCATE SUBMARINE WITH ACOUSTIC DATA USING SIGNAL FILTERING AND FOURIER TRANSFORM

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Abstract. We successfully find the location of the submarine using four-dimensional acoustic pressure data through the application of fast-fourier transformation (FFT) and Gaussian filtering. Multiple Gaussian filters are tried and compared for robustness. Planar plots and spacial plot of the location of the submarine are shown.

1. Introduction and Overview

We are to locate a submarine in the Puget Sound using acoustic signal obtained over 24 hours in half-hour increments was provided as a 262144×49 array with entries of complex numbers, where the real parts of the data denotes acoustic pressure and the imaginary part of the data denotes phase of the signal. In order to trace the trail of the submarine, we assume that the maximum signal captured in the acoustic data corresponds to the acoustic energy emitted from the submarine.

The above-mentioned task can therefore be modeled into a frequency transform problem. We first gather the spacial-temporal data in the signal domain and then use Fourier Transform to reconstruct the signal into frequency domain, where we can further manipulate the frequency through truncating, filtering etc. Such manipulation techniques are largely driven by the fact that signal collected usually are not ideal. When a Gaussian noise – i.e. mean-zero white noise – affects all frequencies the same time, we can preliminarily remove such noise by averaging all samples data ([1]).

After removing the white noise, a signature frequency (or center frequency) can be observed through all 49 temporal samples we reconstructed from the signal domain. Since the maximum frequencies were identified, further denoising techniques can be applied to obtain cleaner signals in the frequency domain before an inverse Fourier Transform (IFFT) to be applied to each sample. A three-dimensional Gaussian filtering technique was implemented while different values of σ (standard deviation) were tried to see robustness of the trace. The inverse Fourier Transform aims to regain the time and location information lost in the frequency domain back into the signal domain, where the locations of the submarine correspond to its signature frequencies we just identified in each time sample. We observe the planar trace of the submarine by plotting an x-y planar; we also plotted a spacial trace in three-dimensional coordinates.

2. Theoretical Background

A given function $f(x)$ can be represented into Fourier space with sin and cos as its new orthogonal basis where Fourier coefficients can be interpreted as the length of the new basis.

Moreover, Fourier Series can be further extended from $\mathbb{R}$ into $\mathbb{C}$ using Euler formula:

$$e^{ikx} = \cos kx + i \sin kx.$$

where coefficient index k now fills from $-\infty$ to $+\infty$, and the coefficients $a_k, b_k \in \mathbb{R}$ are now united into $c_k \in \mathbb{C}$.

The Fourier Series in the complex space give rise to prosperous applications in engineering problems, e.g. signal analysis and processing. In order to analyze the measured signals, we will need to transform our data into frequency spectrograms. Mathematically, Fourier Transform enables us to expand the periodic signals to real line:

$$\mathcal{F}[f(x)] = \widehat{f}_k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx. \tag{1}$$

We can expand (1) into three-dimensional Fourier Transform which will be used in our case:

$$\mathcal{F}[f(x,y,z)] = \widehat{f_{k,l,m}} = \int\int\int f(x,y,z) e^{-i2\pi(kx+ly+mz)} dx\ dy\ dz. \tag{2}$$

where Fourier Transform of the original function is a function of k, l, m. As we can see, the spacial-temporal information contained in x, y, z is now "lost", and the reconstructed function $\widehat{f_{k,l,m}}$ showcases the spectrum of all frequencies construed in k, l, m.

The beauty of Fourier Transform is further embodied through its inverse property, where information lost in x (2) can now be recaptured through:

$$\mathcal{F}^{-1}[\widehat{f_{k,l,m}}] = f(x,y,z) = \int\int\int \widehat{f_{k,l,m}} e^{i2\pi(kx+ly+mz)} dk\ dl\ dm. \tag{3}$$

Presumably, the recorded acoustic pressure contained noises which we need to filter out. One usual technique to accomplish this denoising task is to convolute the signal with a Gaussian filter using the property from Convolutional Theorem of Fourier Transform:

$$f * g = \mathcal{F}^{-1}[\widehat{f * g}] = \mathcal{F}^{-1}[2\pi \cdot \widehat{f}_k \cdot \widehat{g}_k]. \tag{4}$$

where function g denotes our Gaussian filter, and $f * g$ is defined as $\int_{\mathbb{R}^3} f(\mathbf{x}-\mathbf{y}) g(\mathbf{y}) d\mathbf{y}$. In our three-dimensional case, the Gaussian filter g is further defined as the following:

$$g(k_x, k_y, k_z; \mu_x, \mu_y, \mu_z, \sigma) = e^{-\frac{(k_x-\mu_x)^2+(k_y-\mu_y)^2+(k_z-\mu_z)^2}{2\sigma^2}} \tag{5}$$

where μ_x, μ_y, μ_z are the mean of a sample, which in our case correspond to the signature frequency of the submarine.

3. Algorithm Implementation and Development

To analyze the acoustic data, we use `Python` in which we also import `numpy` as `np` and `matplotlib` as `plt` for computing.

After discretizing our space into cubical grids, we first reshape our raw data into spatial volume data in 49 time steps, i.e. four-dimensional matrix; then we normalize the volume data for each time step and apply a three-dimensional Fourier Transform (2) (in `Python`: `np.ftt.fttn`) to the spatial signal; we then preliminarily remove Gaussian white noise by averaging the transformed data, and therefore find the signature frequency; in order to further denoise the signal, we then apply a three-dimensional Gaussian filter (5) before we finally inverse back to signal domain (3) (in `Python`: `np.ftt.ifttn`); and finally we plot the trajectory of the submarine.

Here is the detailed algorithm:

```
for j = 0, ..., 48 do
    Reshape jth column into a three-dimensional matrix.
    Normalize the signal by dividing the maximum absolute value.
    Apply 3D Fourier Transform (3DFT) for jth column.
end for
Compute the average of the 3DFT across 49 time steps.
Find the location of the maxima of the averaged-3DFT in the matrix to get the signature
    frequency of the submarine.
Define a 3D Gaussian function in Fourier space.
for j = 0, ..., 49 do
    Multiply 3D Gaussian function to jth time step of the 3DFT.
    Apply a 3D Inverse Fourier Transform for jth time step.
    Find the location of the maxima value to get the location of the submarine in real space.
end for
Plot the trajectory.
```

4. Computational Results

4.1. **Task 1. Averaging Fourier Transform to Find Frequency Signature.** After removing the mean-zero Gaussian white noise, we find that the maxima of the averaged normalized Fourier Transform of the spacial signal is located at

$$(39, 49, 10) \tag{6}$$

in the grid of the frequency space (i.e. the k-grid). Since the starting index of the k-grid is $-\frac{N}{2}$ where N $= 64$, we need to add the starting index -32 from (6) and multiply by the step length constructed by `np.linspace`. That is to say, the signature frequency of the submarine is located at:

$$\frac{2\pi}{2L} \times (39 - 32, 49 - 32, 10 - 32) = \frac{\pi}{10} \times (7, 17, -22) \tag{7}$$

To show the signature frequency, we first flatten the three-dimensional frequency array into a single-dimensional array and then plot the frequency amplitudes as shown in Figure 1. As we

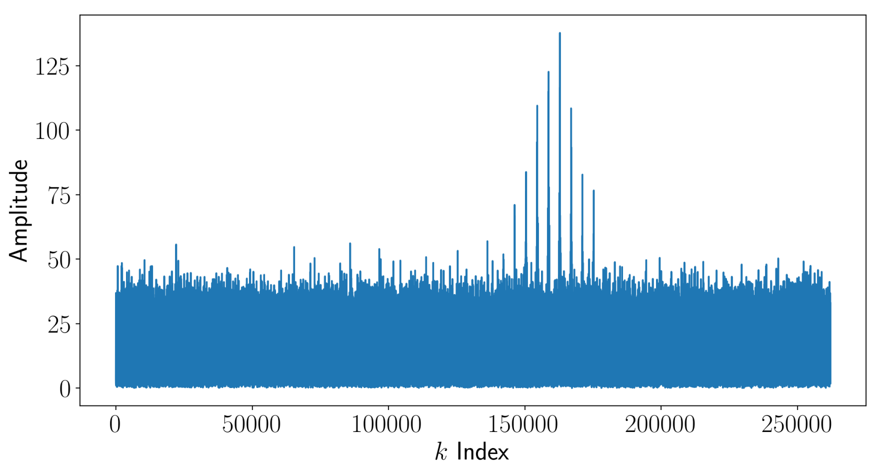
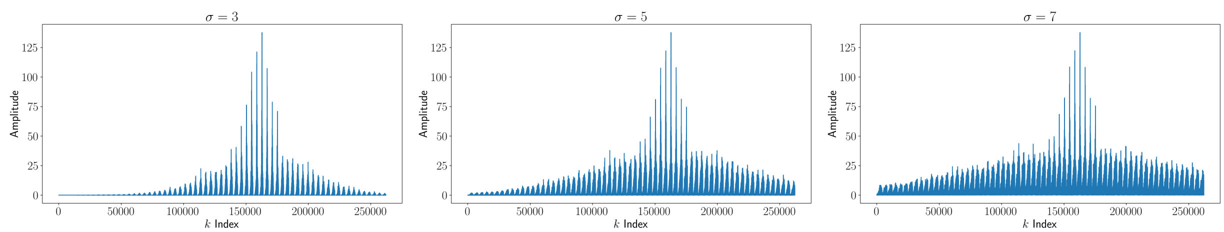


Figure 1. Amplitude of three-dimensional signal in the frequency domain, flattened into a single-dimensional array for visualization.

can see from Figure 1, there is an obvious spike on the near right of the plot, showing a peaked amplitude in the frequency domain; however, it is also easily observable that significant base noises exist ubiquitously across all frequencies. Therefore, we need to apply further denoising techniques to extract a cleaner data set. A Gaussian filter could be a good way to get us started.

4.2. **Task 2. Filtering.** With trial and error, we apply Guassian filter as defined in (5) with $\sigma = 3, 5, 7$ respectively, with center locating at $(\mu_x, \mu_y, \mu_z) = \frac{\pi}{10} \times (7, 17, -22)$. We plot the filtered signal in Figure 2 with each subplot showing result of $\sigma = 3, 5, 7$ from left to right respectively.



(A) $\sigma = 3$ (B) $\sigma = 5$ (C) $\sigma = 7$

Figure 2. Amplitude of three-dimensional *filtered* signal in the frequency domain, flattened into a single-dimensional array for visualization. Subplot from left to right showing result of $\sigma = 3, 5, 7$ respectively.

From our visualization, it is clearly observed that $\sigma = 3$ Gaussian filter in Figure 2a provides us with the least background noise, thus may arguably be the empirical "best" Gaussian filter among its peers in Figure 2b and Figure 2c. In addition, we can observe a trend from the left to the right in Figure 2 that when σ increases, the base noise also increments. The observation fits into our expectation: the smaller value of σ, the narrower shape of the Gaussian filter, thus lowering all base frequencies; on the other hand, with larger values of σ, the wider Gaussian filter may result in low contrast of the frequency amplitude.

We also plot the three-dimensional spatial path of the submarine, which is achieved by applying an inverse Fourier Transform to the filtered signal. The maxima locations of the 49 time steps comprise the trajectory of the submarine, as is shown in Figure 3.

4.3. **Task 3. Plot the x, y coordinates of the submarine.** After we obtained the filtered data and captured a three-dimensional visualization of the trajectory of the submarine, it is straightforward to plot the x, y coordinates of the submarine during the 24 hour period. The planar plot is shown in Figure 4.

5. Summary and Conclusions

To find the location of the submarine, we analyzed the acoustic pressure signals by (1) using three-dimensional Fourier Transform, (2) preliminarily removing the Gaussian noises, (3) determining the signature frequency, (4) applying Gaussian filters to further denoise the data, and (5) applying the inverse Fourier Transform to extract trajectory. We discussed our three choices of parameter σ and observed a more robust trace denoting locations of maxima frequencies using three-dimensional inverse Fourier Transform of the filtered data. As a result, we successfully recovered the trajectory of the submarine.

Acknowledgements

The author is thankful to Prof. Eli for useful discussions about conceptualizing three-dimensional application of Fourier Transform and filtering techniques. Prof. Eli's inspirational advices on spacial-temporal discretization furthers the realization of modeling this mathematical problem in

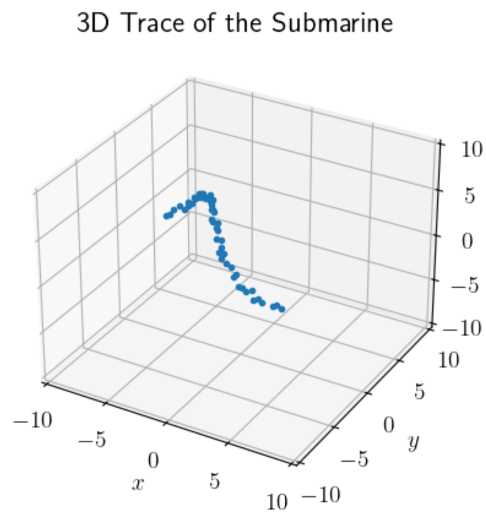


FIGURE 3. Three-dimensional spatial trajectory of the submarine in real space. The trajectory consists of maxima frequency locations of 49 time steps. We obtain the trajectory through inverse Fourier Transform the *filtered* data.

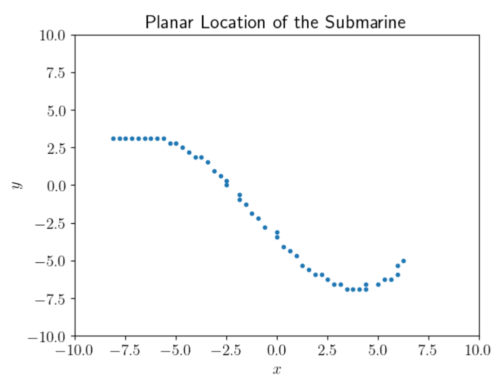


FIGURE 4. Two-dimensional (x, y coordinates) planar trajectory of the submarine during the 24 hour period. We know from the location information that the trajectory starts from the upper left side to the lower right side of the scatter plot.

Python. We are also thankful to TA Saba and Juan for advices regarding understanding data and clearing out our nuances during code realization.

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[1] J. Kutz. *Data-Driven Modeling & Scientific Computation: Methods for Complex Systems & Big Data*. Data-driven Modeling & Scientific Computation: Methods for Complex Systems & Big Data. OUP Oxford, 2013.