STAT 150 Notes

September 4, 2025

1 Markov Chains

Consider a Markov chain (MC) with three states $\{0, 1, 2\}$. Suppose 0 and 2 are absorbing states.

$$\alpha + \beta + \delta = 1$$

Definition 1.1 (Stopping Time τ).

$$\tau = \min\{n \geq 0 : X_n = 0 \text{ or } X_n = 2\}.$$

$$\nu = \Pr(X_\tau = 0) = \frac{\alpha}{1 - \beta} = \frac{\alpha}{\alpha + \gamma}.$$

$$V_i = \mathbb{E}[\tau \mid X_0 = i].$$

Conditioning on X_1

$$V = \mathbb{E}[\tau] = \sum_{i=0}^{2} \Pr(X_1 = i)\, \mathbb{E}[\tau \mid X_1 = i].$$

Explicitly:

$$V = \alpha \cdot 1 + \beta(V + 1) + \gamma \cdot 1 = (\alpha + \beta + \gamma) + \beta V = 1 + \beta V.$$

Theorem 1.1 (Tail-Sum Formula). *For a nonnegative random variable τ,*

$$\mathbb{E}[\tau] = \sum_{n=0}^{\infty} \Pr(\tau > n).$$

Proof. Apply monotone convergence to the representation $\tau = \sum_{n=0}^{\infty} \mathbf{1}_{\{\tau>n\}}$. $\square$

$$\Pr(\tau > n) = \beta^n \quad \Longrightarrow \quad \mathbb{E}[\tau] = \sum_{n=0}^{\infty} \beta^n = \frac{1}{1-\beta}.$$

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2 Classification of States

Definition 2.1 (Accessibility)**.** *State j is **accessible** from i if*

$$\exists n \geq 0 \quad s.t.\ P_{ij}^{(n)} > 0.$$

Definition 2.2 (Communication)**.** *We say i and j **communicate**, denoted $i \leftrightarrow j$, if i is accessible from j and j is accessible from i.*

Theorem 2.1. *Communication is an equivalence relation:*

1. *Reflexive: $i \leftrightarrow i$.*
2. *Symmetric: $i \leftrightarrow j \iff j \leftrightarrow i$.*
3. *Transitive: If $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$.*

Proof. Suppose $P_{ij}^{(m)} > 0$ and $P_{jk}^{(n)} > 0$. Then

$$P_{ik}^{(m+n)} \;\geq\; P_{ij}^{(m)} \cdot P_{jk}^{(n)} \;>\; 0.$$

Thus $i \leftrightarrow k$. $\square$

Example: Transition Matrix

$$P = \begin{bmatrix} a & b & c & 0 & 0 \\ d & e & f & 0 & 0 \\ g & h & i & 0 & 0 \\ 0 & 0 & 0 & u & v \\ 0 & 0 & 0 & w & x \end{bmatrix}.$$

Equivalence classes: $\{1,2,3\}$ and $\{4,5\}$.

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3 Recurrence and Hitting Times

Definition 3.1 (First Hitting Time)**.** *For state y,*

$$T_y = \min\{n \geq 1 : X_n = y\}.$$

$$f_{yy} = \Pr_y(T_y < \infty) = \sum_{n=1}^{\infty} \Pr(T_y = n).$$

Interpretation: probability that the chain, starting from y, eventually returns to y.

By the Markov property, the probability of returning to y at least k times is

$$(f_{yy})^k.$$

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4 Stopping Times

Definition 4.1 (Stopping Time)**.** *A random time T is a **stopping time** if $\{T = n\}$ depends only on $(X_0, \ldots, X_n)$.*

Example 4.1.

$$\{T_y = n\} = \{X_1 \neq y, X_2 \neq y, \ldots, X_{n-1} \neq y, X_n = y\}.$$

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5 Strong Markov Property

Theorem 5.1 (Strong Markov Property (Durrett, Thm 1.2))**.** *If τ is a stopping time and E is an event measurable with respect to $\{X_0, \ldots, X_{\tau-1}\}$, then*

$$\Pr\big(X_{\tau+k} = j \mid X_\tau = i, E\big) = \Pr\big(X_{\tau+k} = j \mid X_\tau = i\big).$$

Remark 5.1. *Equivalently, defining $Y_n := X_{\tau+n}$, we have that (Y_n) is itself a Markov chain.*