Anharmonic Group Elements as Generated by Machine

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March 2011

1 Representing using Creation and Annihilation Operators

The quartic perturbed harmonic oscillator in quantum mechanics can be represented by creation an annihilation operators, like so:

$$
\begin{array}{rcl}
H_0 = \frac{(p^2+x^2)}{2} & \rightarrow & H_0 = BA + \frac{1}{2} \\
H_4 = H_0 + \frac{\lambda}{4}\cdot(x^4) & \rightarrow & H_4 = H_0 + \frac{\lambda}{4}\cdot((B+A)^4) \\
\\
[A, B] = 1 & &
\end{array}
$$

Table 1: Here, we have defined $B = a^\dagger$ and $A = a$, a more handy notation for our needs.

H_4 can be normal ordered to the following result:

$$
\begin{aligned}
H_4 &= H_0 + \frac{\lambda}{4}\cdot((B+A)^4) && (1) \\
&= H_0 + \lambda\cdot(0.25)\cdot(B^4 + A^4) + \lambda\cdot(B^3A + BA^3) \\
&\quad +\lambda\cdot(1.5)\cdot(B^2 + A^2) + \lambda\cdot(1.5)\cdot B^2A^2 \\
&\quad +\lambda\cdot(3)\cdot BA + \lambda\cdot(0.75) && (2)
\end{aligned}
$$

2 Solution to first order in λ

2.1 Step 1: Identify All Elements of the Lie Algebra

Elements of the Lie Algebra at first order ($L_m^{(k)}$ where $k = 1$) are determined by performing commutations with H_0 and H_4, as identified in Table 1. At first order, terms of order $O(\lambda^2)$ are ignored, so only one commutation is required.

The first commutator:

$$
\begin{aligned}
[H_0, H_4] &= [B{\cdot}A, \lambda{\cdot}(\tfrac{A+B}{\sqrt{2}})^4] \\
&= \lambda{\cdot}(B^4 - A^4) + \lambda{\cdot}(2){\cdot}(B^3A - BA^3) + \lambda{\cdot}(3){\cdot}(B^2 - A^2)
\end{aligned}
$$

At this point, we can identify all the terms of the Lie Algebra to first order.

		Nonperturbative Terms (λ^k where $k = 0$)
$L_0^{(1)}$	=	$I = 1$
$L_1^{(1)}$	=	BA
		First Order Terms (λ^k where $k = 1$)
$L_2^{(1)}$	=	$\lambda{\cdot}I = \lambda$
$L_3^{(1)}$	=	$\lambda{\cdot}BA$
$L_4^{(1)}$	=	$\lambda{\cdot}B^2A^2$
$L_5^{(1)}$	=	$\lambda{\cdot}(B^4 + A^4)$
$L_6^{(1)}$	=	$\lambda{\cdot}(B^3A + BA^3)$
$L_7^{(1)}$	=	$\lambda{\cdot}(B^2 + A^2)$
$L_8^{(1)}$	=	$\lambda{\cdot}(B^4 - A^4)$
$L_9^{(1)}$	=	$\lambda{\cdot}(B^3A - BA^3)$
$L_{10}^{(1)}$	=	$\lambda{\cdot}(B^2 - A^2)$

In this representation, we see the following:

$$
\begin{aligned}
H_0 &= L_1^{(1)} + \tfrac{1}{2}L_0^{(1)} \\
H_4 &= H_0 + (0.25)L_5^{(1)} + L_6^{(1)} \\
&\quad +(1.5)L_7^{(1)} + (1.5)L_4^{(1)} \\
&\quad +(3)L_3^{(1)} + (0.75)L_2^{(1)}
\end{aligned}
$$

This representation is complete for our purposes because it satisfies two conditions:

1. H_4 can be completely represented by terms in the algebra.
2. No two terms can be commuted to create a third non-trivial term not shown in the group. (Remember, $\lambda^2 = 0$).

2.2 Step 2: Construct a General Lie Group Element

In principle, the Lie group element could be constructed from all terms in the Lie algebra, like so:

$$U = \exp(\sum_{k=0}^{10} \alpha_k \cdot L_k) \tag{3}$$

But, by nature of the Hammard lemma (see Section 2.3.1), we can choose to exclude all terms that commute with H_0. So we construct U as follows:

$$U = \exp(\alpha_5 L_5 + \alpha_6 L_6 + \alpha_7 L_7 + \alpha_8 L_8 + \alpha_9 L_9 + \alpha_{10} L_{10}) \tag{4}$$

This gives us 6 constants we tune in order make this Lie group element a transformation of basis between perturbed and unperturbed eigenstates.

2.3 Step 3: Use the Hammard Lemma to Compute our Lie Group Element

It is our goal to choose a U such that the following is true:

$$H_4 - U^\dagger H_0 U = \Lambda_4 \tag{5}$$

where $[U, \Lambda_4] = 0 + O(\lambda^2)$

2.3.1 Step 3.1: Expand $U^\dagger H_0 U$ by the Hammard Lemma

$$U^\dagger H_0 U = H_0 + [-X, H_0] + \frac{1}{2!}([-X, [-X, H_0]]) + \cdots \tag{6}$$

where $X = \alpha_5 L_5 + \alpha_6 L_6 + \alpha_7 L_7 + \alpha_8 L_8 + \alpha_9 L_9 + \alpha_{10} L_{10}$.

To first order in λ this simplifies to:

$$U^\dagger H_0 U = H_0 + [-X, H_0] \tag{7}$$

Performing the commutator of Equation 7 and normal ordering, we get the following:

$$
\begin{aligned}
[-X, H_0] &= [H_0, X] \\
&= [BA, X] \\
&= \lambda\cdot[BA, \alpha_5(B^4 + A^4) + \alpha_6(B^3A + BA^3) \\
&\quad +\alpha_7(B^2 + A^2) + \alpha_8(B^4 - A^4) \\
&\quad +\alpha_9(B^3A - BA^3) + \alpha_{10}(B^2 - A^2)]
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
&= \lambda\cdot((4\alpha_8)\cdot(B^4 + A^4) + (4\alpha_5)\cdot(B^4 - A^4) \\
&\quad +(2\alpha_9)\cdot(B^3A + BA^3) + (2\alpha_6)\cdot(B^3A - BA^3) \\
&\quad +(2\alpha_{10})\cdot(B^2 + A^2) + (2\alpha_7)\cdot(B^2 - A^2))
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
&= (4\alpha_8)\cdot L_5 + (4\alpha_5)\cdot L_8 \\
&\quad +(2\alpha_9)\cdot L_6 + (2\alpha_6)\cdot L_9 \\
&\quad +(2\alpha_{10})\cdot L_7 + (2\alpha_7)\cdot L_{10}
\end{aligned} \tag{10}
$$

2.3.2 Step 3.2: Tune α_k so Λ_4 is a Number Operator

From Equations 5 and 7,

$$
\Lambda_4 = H_4 - U^\dagger H_0 U
$$

$$
\begin{aligned}
&= (0.25 - 4\alpha_8)L_5 + (1 - 2\alpha_9)L_6 \\
&\quad +(1.5 - 2\alpha_{10})L_7 + (1.5)L_4 \\
&\quad +(3)L_3 + (0.75)L_2 \\
&\quad +(-4\alpha_5)\cdot L_8 + (-2\alpha_6)\cdot L_9 \\
&\quad +(-2\alpha_7)\cdot L_{10}
\end{aligned} \tag{11}
$$

Now, using our knowledge that Λ_4 must commute with U, we know that Λ_4 cannot have terms involving L_5, L_6, L_7, L_8, L_9 or L_{10}. Thus, the alphas must be tuned such that:

$$
\begin{aligned}
(-4\alpha_5) &= 0 &\rightarrow \quad \alpha_5 &= 0 \\
(-2\alpha_6) &= 0 &\rightarrow \quad \alpha_6 &= 0 \\
(-2\alpha_7) &= 0 &\rightarrow \quad \alpha_7 &= 0 \\
(0.25 - 4\alpha_8) &= 0 &\rightarrow \quad \alpha_8 &= \tfrac{1}{16} \\
(1 - 2\alpha_9) &= 0 &\rightarrow \quad \alpha_9 &= \tfrac{1}{2} \\
(1.5 - 2\alpha_{10}) &= 0 &\rightarrow \quad \alpha_{10} &= \tfrac{3}{4}
\end{aligned}
$$

Which leaves:

$$\Lambda_4 = \frac{3}{2}L_4 + 3L_3 + \frac{3}{4}L_2 \tag{12}$$
$$= \frac{3}{2}\lambda(B^2A^2) + 3\lambda(BA) + \frac{3}{4}\lambda \tag{13}$$

We have now completed the computation of the quartic oscillator to first order in λ for all states. Moving on...

3 Solution to second order in λ

3.1 Step 1: Identify All Elements of the Lie Algebra

We retain all of the previous terms in the Lie algebra, L_0 through L_{10}, but need to consider that $\lambda^2 \neq 0$. Thus, the commutators of all of these terms with one another are fair game. The non-trivial commutators are shown in Table 2.

By inspection, we can see that our first order Lie Algebra must be extended. The complete list is shown in Table 3.

The terms in Table 3 form a complete representation, as it satisfies the two conditions from 2.1, where $\lambda^2 \neq 0, \lambda^3 = 0$.

$[L1, L5]$	$=$	$\lambda\cdot(4)\cdot(B^4 - A^4)$
$[L1, L6]$	$=$	$\lambda\cdot(2)\cdot(B^3A - BA^3)$
$[L1, L7]$	$=$	$\lambda\cdot(2)\cdot(B^2 - A^2)$
$[L1, L8]$	$=$	$\lambda\cdot(4)\cdot(B^4 + A^4)$
$[L1, L9]$	$=$	$\lambda\cdot(2)\cdot(B^3A + BA^3)$
$[L1, L10]$	$=$	$\lambda\cdot(2)\cdot(B^2 + A^2)$
$[L3, L5]$	$=$	$\lambda^2\cdot(4)\cdot(B^4 - A^4)$
$[L3, L6]$	$=$	$\lambda^2\cdot(2)\cdot(B^3A - BA^3)$
$[L3, L7]$	$=$	$\lambda^2\cdot(2)\cdot(B^2 - A^2)$
$[L3, L8]$	$=$	$\lambda^2\cdot(4)\cdot(B^4 + A^4)$
$[L3, L9]$	$=$	$\lambda^2\cdot(2)\cdot(B^3A + BA^3)$
$[L3, L10]$	$=$	$\lambda^2\cdot(2)\cdot(B^2 + A^2)$
$[L4, L5]$	$=$	$\lambda^2\cdot(8)\cdot(B^5A - BA^5) + \lambda^2\cdot(12)\cdot(B^4 - A^4)$
$[L4, L6]$	$=$	$\lambda^2\cdot(4)\cdot(B^4A^2 - B^2A^4) + \lambda^2\cdot(6)\cdot(B^3A - BA^3)$
$[L4, L7]$	$=$	$\lambda^2\cdot(4)\cdot(B^3A - BA^3) + \lambda^2\cdot(2)\cdot(B^2 - A^2)$
$[L4, L8]$	$=$	$\lambda^2\cdot(8)\cdot(B^5A + BA^5) + \lambda^2\cdot(12)\cdot(B^4 + A^4)$
$[L4, L9]$	$=$	$\lambda^2\cdot(4)\cdot(B^4A^2 + B^2A^4) + \lambda^2\cdot(6)\cdot(B^3A + BA^3)$
$[L4, L10]$	$=$	$\lambda^2\cdot(4)\cdot(B^3A + BA^3) + \lambda^2\cdot(2)\cdot(B^2 + A^2)$
$[L5, L6]$	$=$	$\lambda^2\cdot(-4)\cdot(B^6 - A^6) + \lambda^2\cdot(-12)\cdot(B^4A^2 - B^2A^4)$ $+\lambda^2\cdot(-36)\cdot(B^3A - BA^3) + \lambda^2\cdot(-24)\cdot(B^2 - A^2)$
$[L5, L7]$	$=$	$\lambda^2\cdot(-8)\cdot(B^3A - BA^3) + \lambda^2\cdot(-12)\cdot(B^2 - A^2)$
$[L5, L8]$	$=$	$\lambda^2\cdot(32)\cdot B^3A^3 + \lambda^2\cdot(144)\cdot B^2A^2$ $+\lambda^2\cdot(192)\cdot BA + \lambda^2\cdot(48)$
$[L5, L9]$	$=$	$\lambda^2\cdot(-4)\cdot(B^6 + A^6) + \lambda^2\cdot(12)\cdot(B^4A^2 + B^2A^4)$ $+\lambda^2\cdot(36)\cdot(B^3A + BA^3) + \lambda^2\cdot(24)\cdot(B^2 + A^2)$
$[L5, L10]$	$=$	$\lambda^2\cdot(8)\cdot(B^3A + BA^3) + \lambda^2\cdot(12)\cdot(B^2 + A^2)$
$[L6, L7]$	$=$	$\lambda^2\cdot(2)\cdot(B^4 - A^4)$
$[L6, L8]$	$=$	$\lambda^2\cdot(4)\cdot(B^6 + A^6) + \lambda^2\cdot(12)\cdot(B^4A^2 + B^2A^4)$ $+\lambda^2\cdot(36)\cdot(B^3A + BA^3) + \lambda^2\cdot(24)\cdot(B^2 + A^2)$
$[L6, L9]$	$=$	$\lambda^2\cdot(16)\cdot B^3A^3 + \lambda^2\cdot(36)\cdot B^2A^2$ $+\lambda^2\cdot(12)\cdot BA$
$[L6, L10]$	$=$	$\lambda^2\cdot(2)\cdot(B^4 + A^4) + \lambda^2\cdot(12)\cdot B^2A^2$ $+\lambda^2\cdot(12)\cdot BA$
$[L7, L8]$	$=$	$\lambda^2\cdot(8)\cdot(B^3A + BA^3) + \lambda^2\cdot(12)\cdot(B^2 + A^2)$
$[L7, L9]$	$=$	$\lambda^2\cdot(-2)\cdot(B^4 + A^4) + \lambda^2\cdot(12)\cdot B^2A^2$ $+\lambda^2\cdot(12)\cdot BA$
$[L7, L10]$	$=$	$\lambda^2\cdot(8)\cdot BA + \lambda^2\cdot(4)$
$[L8, L9]$	$=$	$\lambda^2\cdot(-4)\cdot(B^6 - A^6) + \lambda^2\cdot(12)\cdot(B^4A^2 - B^2A^4)$ $+\lambda^2\cdot(36)\cdot(B^3A - BA^3) + \lambda^2\cdot(24)\cdot(B^2 - A^2)$
$[L8, L10]$	$=$	$\lambda^2\cdot(8)\cdot(B^3A - BA^3) + \lambda^2\cdot(12)\cdot(B^2 - A^2)$
$[L9, L10]$	$=$	$\lambda^2\cdot(2)\cdot(B^4 - A^4)$

Table 2: All non-trivial commutators of terms in the first order Lie Algebra – *i.e.* a computed list of terms associated with the second order Lie Algebra.

		Nonperturbative Terms (λ^k where $k=0$)
$L_0^{(2)}$	$=$	$I = 1$
$L_1^{(2)}$	$=$	BA
		First Order Terms (λ^k where $k=1$)
$L_2^{(2)}$	$=$	$\lambda \cdot I = \lambda$
$L_3^{(2)}$	$=$	$\lambda \cdot BA$
$L_4^{(2)}$	$=$	$\lambda \cdot B^2A^2$
$L_5^{(2)}$	$=$	$\lambda \cdot (B^4 + A^4)$
$L_6^{(2)}$	$=$	$\lambda \cdot (B^3A + BA^3)$
$L_7^{(2)}$	$=$	$\lambda \cdot (B^2 + A^2)$
$L_8^{(2)}$	$=$	$\lambda \cdot (B^4 - A^4)$
$L_9^{(2)}$	$=$	$\lambda \cdot (B^3A - BA^3)$
$L_{10}^{(2)}$	$=$	$\lambda \cdot (B^2 - A^2)$
		Second Order Terms (λ^k where $k=2$)
$L_{11}^{(2)}$	$=$	$\lambda^2 \cdot I = \lambda^2$
$L_{12}^{(2)}$	$=$	$\lambda^2 \cdot BA$
$L_{13}^{(2)}$	$=$	$\lambda^2 \cdot B^2A^2$
$L_{14}^{(2)}$	$=$	$\lambda^2 \cdot B^3A^3$
$L_{15}^{(2)}$	$=$	$\lambda^2 \cdot (B^6 + A^6)$
$L_{16}^{(2)}$	$=$	$\lambda^2 \cdot (B^5A + BA^5)$
$L_{17}^{(2)}$	$=$	$\lambda^2 \cdot (B^4A^2 + B^2A^4)$
$L_{18}^{(2)}$	$=$	$\lambda^2 \cdot (B^4 + A^4)$
$L_{19}^{(2)}$	$=$	$\lambda^2 \cdot (B^3A + BA^3)$
$L_{20}^{(2)}$	$=$	$\lambda^2 \cdot (B^2 + A^2)$
$L_{21}^{(2)}$	$=$	$\lambda^2 \cdot (B^6 - A^6)$
$L_{22}^{(2)}$	$=$	$\lambda^2 \cdot (B^5A - BA^5)$
$L_{23}^{(2)}$	$=$	$\lambda^2 \cdot (B^4A^2 - B^2A^4)$
$L_{24}^{(2)}$	$=$	$\lambda^2 \cdot (B^4 - A^4)$
$L_{25}^{(2)}$	$=$	$\lambda^2 \cdot (B^3A - BA^3)$
$L_{26}^{(2)}$	$=$	$\lambda^2 \cdot (B^2 - A^2)$

Table 3: The list of all terms in the second order Lie algebra, derived (by inspection) from Table 2.

3.2 Step 2: Construct a General Lie Group Element

Once again, we may discard terms that commute with H_0, (the number operator terms, L_0 through L_4 and L_{11} through L_{14}) when constructing the general Lie group element. We may also discard terms that are eliminated by the first-order coefficients as determined in Table 2.3.2. Thus, L_5 through L_7 are not included.

$$\begin{aligned} U \quad = \exp(\quad & \beta_8 L_8 + \beta_9 L_9 + \beta_{10} L_{10} \\ & + \beta_{15} L_{15} + \beta_{16} L_{16} + \beta_{17} L_{17} + \beta_{18} L_{18} + \beta_{19} L_{19} \\ & + \beta_{20} L_{20} + \beta_{21} L_{21} + \beta_{22} L_{22} + \beta_{23} L_{23} + \beta_{24} L_{24} \\ & + \beta_{25} L_{25} + \beta_{26} L_{26}) \end{aligned} \tag{14}$$

3.3 Step 3: Use the Hammard Lemma to Compute our Lie Group Element

There are 18 coefficients to compute in this group element. We proceed as before.

3.3.1 Step 3.1: Expand $U^\dagger H_0 U$ by the Hammard Lemma

$$U^\dagger H_0 U = H_0 + [-X, H_0] + \frac{1}{2!}([-X, [-X, H_0]]) \tag{15}$$

where $X = \beta_8 L_8 + \beta_9 L_9 + \beta_{10} L_{10} + \beta_{15} L_{15} + \beta_{16} L_{16} + \beta_{17} L_{17} + \beta_{18} L_{18} + \beta_{19} L_{19} + \beta_{20} L_{20} + \beta_{21} L_{21} + \beta_{22} L_{22} + \beta_{23} L_{23} + \beta_{24} L_{24} + \beta_{25} L_{25} + \beta_{26} L_{26}$.

Taking the first term:

$$
\begin{aligned}
[-X, H_0] \quad = \quad & \lambda^2\cdot(6\cdot\beta_{21})\cdot(B^6 + A^6) + \lambda^2\cdot(6\cdot\beta_{15})\cdot(B^6 - A^6) \\
& +\lambda^2\cdot(4\cdot\beta_{22})\cdot(B^5A + BA^5) + \lambda^2\cdot(4\cdot\beta_{16})\cdot(B^5A - BA^5) \\
& +\lambda^2\cdot(2\cdot\beta_{23})\cdot(B^4A^2 + B^2A^4) + \lambda^2\cdot(2\cdot\beta_{17})\cdot(B^4A^2 - B^2A^4) \\
& +\lambda\cdot(4\cdot\beta_8)\cdot(B^4 + A^4) \\
& +\lambda^2\cdot(4\cdot\beta_{24})\cdot(B^4 + A^4) + \lambda^2\cdot(4\cdot\beta_{18})\cdot(B^4 - A^4) \\
& +\lambda\cdot(2\cdot\beta_9)\cdot(B^3A + BA^3) \\
& +\lambda^2\cdot(2\cdot\beta_{25})\cdot(B^3A + BA^3) + \lambda^2\cdot(2\cdot\beta_{19})\cdot(B^3A - BA^3) \\
& +\lambda\cdot(2\cdot\beta_{10})\cdot(B^2 + A^2) \\
& +\lambda^2\cdot(2\cdot\beta_{26})\cdot(B^2 + A^2) + \lambda^2\cdot(2\cdot\beta_{20})\cdot(B^2 - A^2)
\end{aligned}
$$

And now, the second term:
$\frac{1}{2!}[-X,[-X,H_0]]$

$$\begin{aligned}
\tfrac{1}{2!}[-X,[-X,H_0]] \quad = \quad & \lambda^2\cdot(-4\cdot\beta_8\cdot\beta_9)\cdot(B^6+A^6) \\
& +\lambda^2\cdot(36\cdot\beta_8\cdot\beta_9)\cdot(B^4A^2+B^2A^4) \\
& +\lambda^2\cdot(108\cdot\beta_8\cdot\beta_9+24\cdot\beta_8\cdot\beta_{10})\cdot(B^3A+BA^3) \\
& +\lambda^2\cdot(72\cdot\beta_8\cdot\beta_9+36\cdot\beta_8\cdot\beta_{10})\cdot(B^2+A^2) \\
& +\lambda^2\cdot(64\cdot\beta_8^2+16\cdot\beta_9^2)\cdot B^3A^3 \\
& +\lambda^2\cdot(288\cdot\beta_8^2+36\cdot\beta_9^2+24\cdot\beta_9\cdot\beta_{10})\cdot B^2A^2 \\
& +\lambda^2\cdot(384\cdot\beta_8^2+12\cdot\beta_9^2+24\cdot\beta_9\cdot\beta_{10}+8\cdot\beta_{10}^2)\cdot BA \\
& +\lambda^2\cdot(96\cdot\beta_8^2+4\cdot\beta_{10}^2)
\end{aligned}$$

And finally, the Λ_4

$$
\begin{aligned}
H_4 - U^\dagger H_0 U &= \Lambda_4 \\
&= \tfrac{\lambda}{4}(A+B)^4 - \left([-X, H_0] + \tfrac{1}{2!}[-X,[-X,H_0]]\right) \\
&= \lambda^2\cdot(-6\cdot\beta_{21} + 4\cdot\beta_8\cdot\beta_9)\cdot(B^6 + A^6) + \lambda^2\cdot(-6\cdot\beta_{15})\cdot(B^6 - A^6) \\
&\quad +\lambda^2\cdot(-4\cdot\beta_{22})\cdot(B^5A + BA^5) + \lambda^2\cdot(-4\cdot\beta_{16})\cdot(B^5A - BA^5) \\
&\quad +\lambda^2\cdot(-2\cdot\beta_{23} - 36\cdot\beta_8\cdot\beta_9)\cdot(B^4A^2 + B^2A^4) + \lambda^2\cdot(-2\cdot\beta_{17})\cdot(B^4A^2 - B^2A^4) \\
&\quad +\lambda\cdot(0.25 - 4\cdot\beta_8)\cdot(B^4 + A^4) \\
&\quad +\lambda^2\cdot(-4\cdot\beta_{24})\cdot(B^4 + A^4) + \lambda^2\cdot(-4\cdot\beta_{18})\cdot(B^4 - A^4) \\
&\quad +\lambda\cdot(1 - 2\cdot\beta_9)\cdot(B^3A + BA^3) \\
&\quad +\lambda^2\cdot(-2\cdot\beta_{25} - 108\cdot\beta_8\cdot\beta_9 - 24\cdot\beta_8\cdot\beta_{10})\cdot(B^3A + BA^3) + \lambda^2\cdot(-2\cdot\beta_{19})\cdot(B^3A - BA^3) \\
&\quad +\lambda\cdot(1.5 - 2\cdot\beta_{10})\cdot(B^2 + A^2) \\
&\quad +\lambda^2\cdot(-2\cdot\beta_{26} - 72\cdot\beta_8\cdot\beta_9 - 36\cdot\beta_8\cdot\beta_{10})\cdot(B^2 + A^2) + \lambda^2\cdot(-2\cdot\beta_{20})\cdot(B^2 - A^2) \\
&\quad +\lambda^2\cdot(-64\cdot\beta_8^2 - 16\cdot\beta_9^2)\cdot B^3A^3 \\
&\quad +\lambda\cdot(1.5)\cdot B^2A^2 \\
&\quad +\lambda^2\cdot(-288\cdot\beta_8^2 - 36\cdot\beta_9^2 - 24\cdot\beta_9\cdot\beta_{10})\cdot B^2A^2 \\
&\quad +\lambda\cdot(3)\cdot BA \\
&\quad +\lambda^2\cdot(-384\cdot\beta_8^2 - 12\cdot\beta_9^2 - 24\cdot\beta_9\cdot\beta_{10} - 8\cdot\beta_{10}^2)\cdot BA \\
&\quad +\lambda\cdot(0.75) \\
&\quad +\lambda^2\cdot(-96\cdot\beta_8^2 - 4\cdot\beta_{10}^2)
\end{aligned}
$$

3.3.2 Step 3.2: Tune β_k so Λ_4 is a Number Operator

Tuning the β values is easily done by setting each term that is not a number operator in Λ_4 equal to zero, and solving for β_k. The results are shown in Table 4.

$\beta_8 = \frac{1}{16}$	$\beta_{15} = 0$	$\beta_{21} = \frac{1}{48}$
$\beta_9 = \frac{1}{2}$	$\beta_{16} = 0$	$\beta_{22} = 0$
$\beta_{10} = \frac{3}{4}$	$\beta_{17} = 0$	$\beta_{23} = -\frac{9}{16}$
	$\beta_{18} = 0$	$\beta_{24} = 0$
	$\beta_{19} = 0$	$\beta_{25} = -\frac{9}{4}$
	$\beta_{20} = 0$	$\beta_{26} = -\frac{63}{32}$

Table 4: Values for β_k.

The resulting form for Λ_4 is

$$\Lambda_4 = \lambda(\frac{3}{2}B^2A^2 + 3BA + \frac{3}{4}) + \lambda^2(\frac{-17}{4}B^3A^3 + \frac{-153}{8}B^2A^2 + (-18)BA + \frac{-21}{8}) \tag{16}$$

Not surprisingly, setting λ^2 equal to zero will return the first order perturbation result.

4 Solution to third order in λ

4.1 Step 1: Identify All Elements of the Lie Algebra

We retain all of the previous terms in the Lie algebra, L_0 through L_{26}, but need to consider that $\lambda^3 \neq 0$. Thus, commutators from our second order Lie Algebra should be taken. Many of these are shown in Table 5.

$$
\begin{array}{rcl}
[L_3, L_{11}] & = & 0 \\
[L_3, L_{12}] & = & 0 \\
[L_3, L_{13}] & = & 0 \\
[L_3, L_{14}] & = & 0 \\
[L_3, L_{15}] & = & \lambda^3 \cdot (6) \cdot (B^6 - A^6) \\
[L_3, L_{16}] & = & \lambda^3 \cdot (4) \cdot (B^5 A - BA^5) \\
[L_3, L_{17}] & = & \lambda^3 \cdot (2) \cdot (B^4 A^2 - B^2 A^4) \\
[L_3, L_{18}] & = & \lambda^3 \cdot (4) \cdot (B^4 - A^4) \\
[L_3, L_{19}] & = & \lambda^3 \cdot (2) \cdot (B^3 A - BA^3) \\
[L_3, L_{20}] & = & \lambda^3 \cdot (2) \cdot (B^2 - A^2) \\
[L_3, L_{21}] & = & \lambda^3 \cdot (6) \cdot (B^6 + A^6) \\
[L_3, L_{22}] & = & \lambda^3 \cdot (4) \cdot (B^5 A + BA^5) \\
[L_3, L_{23}] & = & \lambda^3 \cdot (2) \cdot (B^4 A^2 + B^2 A^4) \\
[L_3, L_{24}] & = & \lambda^3 \cdot (4) \cdot (B^4 + A^4) \\
[L_3, L_{25}] & = & \lambda^3 \cdot (2) \cdot (B^3 A + BA^3) \\
[L_3, L_{26}] & = & \lambda^3 \cdot (2) \cdot (B^2 + A^2) \\
[L_4, L_{11}] & = & 0 \\
[L_4, L_{12}] & = & 0 \\
[L_4, L_{13}] & = & 0 \\
[L_4, L_{14}] & = & 0 \\
[L_4, L_{15}] & = & \lambda^3 \cdot (12) \cdot (B^7 A - BA^7) \\
 & & +\lambda^3 \cdot (30) \cdot (B^6 - A^6) \\
[L_4, L_{16}] & = & \lambda^3 \cdot (8) \cdot (B^6 A^2 - B^2 A^6) \\
 & & +\lambda^3 \cdot (20) \cdot (B^5 A - BA^5) \\
[L_4, L_{17}] & = & \lambda^3 \cdot (4) \cdot (B^5 A^3 - B^3 A^5) \\
 & & +\lambda^3 \cdot (10) \cdot (B^4 A^2 - B^2 A^4) \\
[L_4, L_{18}] & = & \lambda^3 \cdot (8) \cdot (B^5 A - BA^5) \\
 & & +\lambda^3 \cdot (12) \cdot (B^4 - A^4) \\
[L_4, L_{19}] & = & \lambda^3 \cdot (4) \cdot (B^4 A^2 - B^2 A^4) \\
 & & +\lambda^3 \cdot (6) \cdot (B^3 A - BA^3) \\
[L_4, L_{20}] & = & \lambda^3 \cdot (4) \cdot (B^3 A - BA^3) \\
 & & +\lambda^3 \cdot (2) \cdot (B^2 - A^2) \\
[L_4, L_{21}] & = & \lambda^3 \cdot (12) \cdot (B^7 A + BA^7) \\
 & & +\lambda^3 \cdot (30) \cdot (B^6 + A^6) \\
[L_4, L_{22}] & = & \lambda^3 \cdot (8) \cdot (B^6 A^2 + B^2 A^6) \\
 & & +\lambda^3 \cdot (20) \cdot (B^5 A + BA^5) \\
[L_4, L_{23}] & = & \lambda^3 \cdot (4) \cdot (B^5 A^3 + B^3 A^5) \\
 & & +\lambda^3 \cdot (10) \cdot (B^4 A^2 + B^2 A^4)
\end{array}
$$

$$
\begin{aligned}
[L_4, L_{24}] &= \lambda^3 \cdot (8) \cdot (B^5 A + B A^5) \\
&\quad + \lambda^3 \cdot (12) \cdot (B^4 + A^4) \\
[L_4, L_{25}] &= \lambda^3 \cdot (4) \cdot (B^4 A^2 + B^2 A^4) \\
&\quad + \lambda^3 \cdot (6) \cdot (B^3 A + B A^3) \\
[L_4, L_{26}] &= \lambda^3 \cdot (4) \cdot (B^3 A + B A^3) \\
&\quad + \lambda^3 \cdot (2) \cdot (B^2 + A^2) \\
[L_5, L_{11}] &= 0 \\
[L_5, L_{12}] &= \lambda^3 \cdot (-4) \cdot (B^4 - A^4) \\
[L_5, L_{13}] &= \lambda^3 \cdot (-8) \cdot (B^5 A - B A^5) \\
&\quad + \lambda^3 \cdot (-12) \cdot (B^4 - A^4) \\
[L_5, L_{14}] &= \lambda^3 \cdot (-12) \cdot (B^6 A^2 - B^2 A^6) \\
&\quad + \lambda^3 \cdot (-36) \cdot (B^5 A - B A^5) \\
&\quad + \lambda^3 \cdot (-24) \cdot (B^4 - A^4) \\
[L_5, L_{15}] &= \lambda^3 \cdot (24) \cdot (B^5 A^3 - B^3 A^5) \\
&\quad + \lambda^3 \cdot (180) \cdot (B^4 A^2 - B^2 A^4) \\
&\quad + \lambda^3 \cdot (480) \cdot (B^3 A - B A^3) \\
&\quad + \lambda^3 \cdot (360) \cdot (B^2 - A^2) \\
[L_5, L_{16}] &= \lambda^3 \cdot (-4) \cdot (B^8 - A^8) \\
[L_5, L_{17}] &= \lambda^3 \cdot (-8) \cdot (B^7 A - B A^7) \\
&\quad + \lambda^3 \cdot (-16) \cdot (B^5 A^3 - B^3 A^5) \\
&\quad + \lambda^3 \cdot (-12) \cdot (B^6 - A^6) \\
&\quad + \lambda^3 \cdot (-72) \cdot (B^4 A^2 - B^2 A^4) \\
&\quad + \lambda^3 \cdot (-96) \cdot (B^3 A - B A^3) \\
&\quad + \lambda^3 \cdot (-24) \cdot (B^2 - A^2) \\
[L_5, L_{18}] &= 0 \\
[L_5, L_{19}] &= \lambda^3 \cdot (-4) \cdot (B^6 - A^6) \\
&\quad + \lambda^3 \cdot (-12) \cdot (B^4 A^2 - B^2 A^4) \\
&\quad + \lambda^3 \cdot (-36) \cdot (B^3 A - B A^3) \\
&\quad + \lambda^3 \cdot (-24) \cdot (B^2 - A^2) \\
[L_5, L_{20}] &= \lambda^3 \cdot (-8) \cdot (B^3 A - B A^3) \\
&\quad + \lambda^3 \cdot (-12) \cdot (B^2 - A^2) \\
[L_5, L_{21}] &= \lambda^3 \cdot (24) \cdot (B^5 A^3 + B^3 A^5) \\
&\quad + \lambda^3 \cdot (180) \cdot (B^4 A^2 + B^2 A^4) \\
&\quad + \lambda^3 \cdot (480) \cdot (B^3 A + B A^3) \\
&\quad + \lambda^3 \cdot (360) \cdot (B^2 + A^2) \\
[L_5, L_{22}] &= \lambda^3 \cdot (-4) \cdot (B^8 + A^8) \\
&\quad + \lambda^3 \cdot (40) \cdot B^4 A^4 \\
&\quad + \lambda^3 \cdot (240) \cdot B^3 A^3 \\
&\quad + \lambda^3 \cdot (480) \cdot B^2 A^2 \\
&\quad + \lambda^3 \cdot (240) \cdot B A \\
[L_5, L_{23}] &= \lambda^3 \cdot (-8) \cdot (B^7 A + B A^7) \\
&\quad + \lambda^3 \cdot (16) \cdot (B^5 A^3 + B^3 A^5) \\
&\quad + \lambda^3 \cdot (-12) \cdot (B^6 + A^6) \\
&\quad + \lambda^3 \cdot (72) \cdot (B^4 A^2 + B^2 A^4) \\
&\quad + \lambda^3 \cdot (96) \cdot (B^3 A + B A^3) \\
&\quad + \lambda^3 \cdot (24) \cdot (B^2 + A^2)
\end{aligned}
$$

$$
\begin{array}{rcl}
[L_5, L_{24}] & = & \lambda^3 \cdot (32) \cdot B^3A^3 \\
 & & +\lambda^3 \cdot (144) \cdot B^2A^2 \\
 & & +\lambda^3 \cdot (192) \cdot BA \\
 & & +\lambda^3 \cdot (48) \\
[L_5, L_{25}] & = & \lambda^3 \cdot (-4) \cdot (B^6 + A^6) \\
 & & +\lambda^3 \cdot (12) \cdot (B^4A^2 + B^2A^4) \\
 & & +\lambda^3 \cdot (36) \cdot (B^3A + BA^3) \\
 & & +\lambda^3 \cdot (24) \cdot (B^2 + A^2) \\
[L_5, L_{26}] & = & \lambda^3 \cdot (8) \cdot (B^3A + BA^3) \\
 & & +\lambda^3 \cdot (12) \cdot (B^2 + A^2) \\
[L_6, L_{11}] & = & 0 \\
[L_6, L_{12}] & = & \lambda^3 \cdot (-2) \cdot (B^3A - BA^3) \\
[L_6, L_{13}] & = & \lambda^3 \cdot (-4) \cdot (B^4A^2 - B^2A^4) \\
 & & +\lambda^3 \cdot (-6) \cdot (B^3A - BA^3) \\
[L_6, L_{14}] & = & \lambda^3 \cdot (-6) \cdot (B^5A^3 - B^3A^5) \\
 & & +\lambda^3 \cdot (-18) \cdot (B^4A^2 - B^2A^4) \\
 & & +\lambda^3 \cdot (-6) \cdot (B^3A - BA^3) \\
[L_6, L_{15}] & = & \lambda^3 \cdot (6) \cdot (B^8 - A^8) \\
 & & +\lambda^3 \cdot (18) \cdot (B^6A^2 - B^2A^6) \\
 & & +\lambda^3 \cdot (90) \cdot (B^5A - BA^5) \\
 & & +\lambda^3 \cdot (120) \cdot (B^4 - A^4) \\
[L_6, L_{16}] & = & \lambda^3 \cdot (2) \cdot (B^7A - BA^7) \\
 & & +\lambda^3 \cdot (14) \cdot (B^5A^3 - B^3A^5) \\
 & & +\lambda^3 \cdot (60) \cdot (B^4A^2 - B^2A^4) \\
 & & +\lambda^3 \cdot (60) \cdot (B^3A - BA^3) \\
[L_6, L_{17}] & = & \lambda^3 \cdot (-2) \cdot (B^6A^2 - B^2A^6) \\
 & & +\lambda^3 \cdot (-6) \cdot (B^5A - BA^5) \\
[L_6, L_{18}] & = & \lambda^3 \cdot (4) \cdot (B^6 - A^6) \\
 & & +\lambda^3 \cdot (12) \cdot (B^4A^2 - B^2A^4) \\
 & & +\lambda^3 \cdot (36) \cdot (B^3A - BA^3) \\
 & & +\lambda^3 \cdot (24) \cdot (B^2 - A^2) \\
[L_6, L_{19}] & = & 0 \\
[L_6, L_{20}] & = & \lambda^3 \cdot (2) \cdot (B^4 - A^4) \\
[L_6, L_{21}] & = & \lambda^3 \cdot (6) \cdot (B^8 + A^8) \\
 & & +\lambda^3 \cdot (18) \cdot (B^6A^2 + B^2A^6) \\
 & & +\lambda^3 \cdot (90) \cdot (B^5A + BA^5) \\
 & & +\lambda^3 \cdot (120) \cdot (B^4 + A^4) \\
[L_6, L_{22}] & = & \lambda^3 \cdot (2) \cdot (B^7A + BA^7) \\
 & & +\lambda^3 \cdot (14) \cdot (B^5A^3 + B^3A^5) \\
 & & +\lambda^3 \cdot (60) \cdot (B^4A^2 + B^2A^4) \\
 & & +\lambda^3 \cdot (60) \cdot (B^3A + BA^3) \\
[L_6, L_{23}] & = & \lambda^3 \cdot (-2) \cdot (B^6A^2 + B^2A^6) \\
 & & +\lambda^3 \cdot (-6) \cdot (B^5A + BA^5) \\
 & & +\lambda^3 \cdot (20) \cdot B^4A^4 \\
 & & +\lambda^3 \cdot (72) \cdot B^3A^3 \\
 & & +\lambda^3 \cdot (48) \cdot B^2A^2
\end{array}
$$

$$
\begin{array}{lcl}
[L_6, L_{24}] & = & \lambda^3{\cdot}(4){\cdot}(B^6 + A^6) \\
 & & +\lambda^3{\cdot}(12){\cdot}(B^4A^2 + B^2A^4) \\
 & & +\lambda^3{\cdot}(36){\cdot}(B^3A + BA^3) \\
 & & +\lambda^3{\cdot}(24){\cdot}(B^2 + A^2) \\
[L_6, L_{25}] & = & \lambda^3{\cdot}(16){\cdot}B^3A^3 \\
 & & +\lambda^3{\cdot}(36){\cdot}B^2A^2 \\
 & & +\lambda^3{\cdot}(12){\cdot}BA \\
[L_6, L_{26}] & = & \lambda^3{\cdot}(2){\cdot}(B^4 + A^4) \\
 & & +\lambda^3{\cdot}(12){\cdot}B^2A^2 \\
 & & +\lambda^3{\cdot}(12){\cdot}BA \\
[L_7, L_{11}] & = & 0 \\
[L_7, L_{12}] & = & \lambda^3{\cdot}(-2){\cdot}(B^2 - A^2) \\
[L_7, L_{13}] & = & \lambda^3{\cdot}(-4){\cdot}(B^3A - BA^3) \\
 & & +\lambda^3{\cdot}(-2){\cdot}(B^2 - A^2) \\
[L_7, L_{14}] & = & \lambda^3{\cdot}(-6){\cdot}(B^4A^2 - B^2A^4) \\
 & & +\lambda^3{\cdot}(-6){\cdot}(B^3A - BA^3) \\
[L_7, L_{15}] & = & \lambda^3{\cdot}(12){\cdot}(B^5A - BA^5) \\
 & & +\lambda^3{\cdot}(30){\cdot}(B^4 - A^4) \\
[L_7, L_{16}] & = & \lambda^3{\cdot}(-2){\cdot}(B^6 - A^6) \\
 & & +\lambda^3{\cdot}(10){\cdot}(B^4A^2 - B^2A^4) \\
 & & +\lambda^3{\cdot}(20){\cdot}(B^3A - BA^3) \\
[L_7, L_{17}] & = & \lambda^3{\cdot}(-4){\cdot}(B^5A - BA^5) \\
 & & +\lambda^3{\cdot}(-2){\cdot}(B^4 - A^4) \\
[L_7, L_{18}] & = & \lambda^3{\cdot}(8){\cdot}(B^3A - BA^3) \\
 & & +\lambda^3{\cdot}(12){\cdot}(B^2 - A^2) \\
[L_7, L_{19}] & = & \lambda^3{\cdot}(-2){\cdot}(B^4 - A^4) \\
[L_7, L_{20}] & = & 0 \\
[L_7, L_{21}] & = & \lambda^3{\cdot}(12){\cdot}(B^5A + BA^5) \\
 & & +\lambda^3{\cdot}(30){\cdot}(B^4 + A^4) \\
[L_7, L_{22}] & = & \lambda^3{\cdot}(-2){\cdot}(B^6 + A^6) \\
 & & +\lambda^3{\cdot}(10){\cdot}(B^4A^2 + B^2A^4) \\
 & & +\lambda^3{\cdot}(20){\cdot}(B^3A + BA^3) \\
[L_7, L_{23}] & = & \lambda^3{\cdot}(-4){\cdot}(B^5A + BA^5) \\
 & & +\lambda^3{\cdot}(-2){\cdot}(B^4 + A^4) \\
 & & +\lambda^3{\cdot}(16){\cdot}B^3A^3 \\
 & & +\lambda^3{\cdot}(24){\cdot}B^2A^2 \\
[L_7, L_{24}] & = & \lambda^3{\cdot}(8){\cdot}(B^3A + BA^3) \\
 & & +\lambda^3{\cdot}(12){\cdot}(B^2 + A^2) \\
[L_7, L_{25}] & = & \lambda^3{\cdot}(-2){\cdot}(B^4 + A^4) \\
 & & +\lambda^3{\cdot}(12){\cdot}B^2A^2 \\
 & & +\lambda^3{\cdot}(12){\cdot}BA \\
[L_7, L_{26}] & = & \lambda^3{\cdot}(8){\cdot}BA \\
 & & +\lambda^3{\cdot}(4) \\
[L_8, L_{11}] & = & 0 \\
[L_8, L_{12}] & = & \lambda^3{\cdot}(-4){\cdot}(B^4 + A^4)
\end{array}
$$

$$
\begin{aligned}
[L_8, L_{13}] &= \lambda^3\cdot(-8)\cdot(B^5A + BA^5) \\
&\quad +\lambda^3\cdot(-12)\cdot(B^4 + A^4) \\
[L_8, L_{14}] &= \lambda^3\cdot(-12)\cdot(B^6A^2 + B^2A^6) \\
&\quad +\lambda^3\cdot(-36)\cdot(B^5A + BA^5) \\
&\quad +\lambda^3\cdot(-24)\cdot(B^4 + A^4) \\
[L_8, L_{15}] &= \lambda^3\cdot(-24)\cdot(B^5A^3 + B^3A^5) \\
&\quad +\lambda^3\cdot(-180)\cdot(B^4A^2 + B^2A^4) \\
&\quad +\lambda^3\cdot(-480)\cdot(B^3A + BA^3) \\
&\quad +\lambda^3\cdot(-360)\cdot(B^2 + A^2) \\
[L_8, L_{16}] &= \lambda^3\cdot(-4)\cdot(B^8 + A^8) \\
&\quad +\lambda^3\cdot(-40)\cdot B^4A^4 \\
&\quad +\lambda^3\cdot(-240)\cdot B^3A^3 \\
&\quad +\lambda^3\cdot(-480)\cdot B^2A^2 \\
&\quad +\lambda^3\cdot(-240)\cdot BA \\
[L_8, L_{17}] &= \lambda^3\cdot(-8)\cdot(B^7A + BA^7) \\
&\quad +\lambda^3\cdot(-16)\cdot(B^5A^3 + B^3A^5) \\
&\quad +\lambda^3\cdot(-12)\cdot(B^6 + A^6) \\
&\quad +\lambda^3\cdot(-72)\cdot(B^4A^2 + B^2A^4) \\
&\quad +\lambda^3\cdot(-96)\cdot(B^3A + BA^3) \\
&\quad +\lambda^3\cdot(-24)\cdot(B^2 + A^2) \\
[L_8, L_{18}] &= \lambda^3\cdot(-32)\cdot B^3A^3 \\
&\quad +\lambda^3\cdot(-144)\cdot B^2A^2 \\
&\quad +\lambda^3\cdot(-192)\cdot BA \\
&\quad +\lambda^3\cdot(-48) \\
[L_8, L_{19}] &= \lambda^3\cdot(-4)\cdot(B^6 + A^6) \\
&\quad +\lambda^3\cdot(-12)\cdot(B^4A^2 + B^2A^4) \\
&\quad +\lambda^3\cdot(-36)\cdot(B^3A + BA^3) \\
&\quad +\lambda^3\cdot(-24)\cdot(B^2 + A^2) \\
[L_8, L_{20}] &= \lambda^3\cdot(-8)\cdot(B^3A + BA^3) \\
&\quad +\lambda^3\cdot(-12)\cdot(B^2 + A^2) \\
[L_8, L_{21}] &= \lambda^3\cdot(-24)\cdot(B^5A^3 - B^3A^5) \\
&\quad +\lambda^3\cdot(-180)\cdot(B^4A^2 - B^2A^4) \\
&\quad +\lambda^3\cdot(-480)\cdot(B^3A - BA^3) \\
&\quad +\lambda^3\cdot(-360)\cdot(B^2 - A^2) \\
[L_8, L_{22}] &= \lambda^3\cdot(-4)\cdot(B^8 - A^8) \\
[L_8, L_{23}] &= \lambda^3\cdot(-8)\cdot(B^7A - BA^7) \\
&\quad +\lambda^3\cdot(16)\cdot(B^5A^3 - B^3A^5) \\
&\quad +\lambda^3\cdot(-12)\cdot(B^6 - A^6) \\
&\quad +\lambda^3\cdot(72)\cdot(B^4A^2 - B^2A^4) \\
&\quad +\lambda^3\cdot(96)\cdot(B^3A - BA^3) \\
&\quad +\lambda^3\cdot(24)\cdot(B^2 - A^2) \\
[L_8, L_{24}] &= 0 \\
[L_8, L_{25}] &= \lambda^3\cdot(-4)\cdot(B^6 - A^6) \\
&\quad +\lambda^3\cdot(12)\cdot(B^4A^2 - B^2A^4) \\
&\quad +\lambda^3\cdot(36)\cdot(B^3A - BA^3) \\
&\quad +\lambda^3\cdot(24)\cdot(B^2 - A^2)
\end{aligned}
$$

$$
\begin{array}{lcl}
[L_8, L_{26}] & = & \lambda^3 \cdot (8) \cdot (B^3A - BA^3) \\
 & & +\lambda^3 \cdot (12) \cdot (B^2 - A^2) \\
[L_9, L_{11}] & = & 0 \\
[L_9, L_{12}] & = & \lambda^3 \cdot (-2) \cdot (B^3A + BA^3) \\
[L_9, L_{13}] & = & \lambda^3 \cdot (-4) \cdot (B^4A^2 + B^2A^4) \\
 & & +\lambda^3 \cdot (-6) \cdot (B^3A + BA^3) \\
[L_9, L_{14}] & = & \lambda^3 \cdot (-6) \cdot (B^5A^3 + B^3A^5) \\
 & & +\lambda^3 \cdot (-18) \cdot (B^4A^2 + B^2A^4) \\
 & & +\lambda^3 \cdot (-6) \cdot (B^3A + BA^3) \\
[L_9, L_{15}] & = & \lambda^3 \cdot (6) \cdot (B^8 + A^8) \\
 & & +\lambda^3 \cdot (-18) \cdot (B^6A^2 + B^2A^6) \\
 & & +\lambda^3 \cdot (-90) \cdot (B^5A + BA^5) \\
 & & +\lambda^3 \cdot (-120) \cdot (B^4 + A^4) \\
[L_9, L_{16}] & = & \lambda^3 \cdot (2) \cdot (B^7A + BA^7) \\
 & & +\lambda^3 \cdot (-14) \cdot (B^5A^3 + B^3A^5) \\
 & & +\lambda^3 \cdot (-60) \cdot (B^4A^2 + B^2A^4) \\
 & & +\lambda^3 \cdot (-60) \cdot (B^3A + BA^3) \\
[L_9, L_{17}] & = & \lambda^3 \cdot (-2) \cdot (B^6A^2 + B^2A^6) \\
 & & +\lambda^3 \cdot (-6) \cdot (B^5A + BA^5) \\
 & & +\lambda^3 \cdot (-20) \cdot B^4A^4 \\
 & & +\lambda^3 \cdot (-72) \cdot B^3A^3 \\
 & & +\lambda^3 \cdot (-48) \cdot B^2A^2 \\
[L_9, L_{18}] & = & \lambda^3 \cdot (4) \cdot (B^6 + A^6) \\
 & & +\lambda^3 \cdot (-12) \cdot (B^4A^2 + B^2A^4) \\
 & & +\lambda^3 \cdot (-36) \cdot (B^3A + BA^3) \\
 & & +\lambda^3 \cdot (-24) \cdot (B^2 + A^2) \\
[L_9, L_{19}] & = & \lambda^3 \cdot (-16) \cdot B^3A^3 \\
 & & +\lambda^3 \cdot (-36) \cdot B^2A^2 \\
 & & +\lambda^3 \cdot (-12) \cdot BA \\
[L_9, L_{20}] & = & \lambda^3 \cdot (2) \cdot (B^4 + A^4) \\
 & & +\lambda^3 \cdot (-12) \cdot B^2A^2 \\
 & & +\lambda^3 \cdot (-12) \cdot BA \\
[L_9, L_{21}] & = & \lambda^3 \cdot (6) \cdot (B^8 - A^8) \\
 & & +\lambda^3 \cdot (-18) \cdot (B^6A^2 - B^2A^6) \\
 & & +\lambda^3 \cdot (-90) \cdot (B^5A - BA^5) \\
 & & +\lambda^3 \cdot (-120) \cdot (B^4 - A^4) \\
[L_9, L_{22}] & = & \lambda^3 \cdot (2) \cdot (B^7A - BA^7) \\
 & & +\lambda^3 \cdot (-14) \cdot (B^5A^3 - B^3A^5) \\
 & & +\lambda^3 \cdot (-60) \cdot (B^4A^2 - B^2A^4) \\
 & & +\lambda^3 \cdot (-60) \cdot (B^3A - BA^3) \\
[L_9, L_{23}] & = & \lambda^3 \cdot (-2) \cdot (B^6A^2 - B^2A^6) \\
 & & +\lambda^3 \cdot (-6) \cdot (B^5A - BA^5) \\
[L_9, L_{24}] & = & \lambda^3 \cdot (4) \cdot (B^6 - A^6) \\
 & & +\lambda^3 \cdot (-12) \cdot (B^4A^2 - B^2A^4) \\
 & & +\lambda^3 \cdot (-36) \cdot (B^3A - BA^3) \\
 & & +\lambda^3 \cdot (-24) \cdot (B^2 - A^2)
\end{array}
$$

$$
\begin{array}{lcl}
[L_9, L_{25}] & = & 0 \\
[L_9, L_{26}] & = & \lambda^3 \cdot (2) \cdot (B^4 - A^4) \\
[L_{10}, L_{11}] & = & 0 \\
[L_{10}, L_{12}] & = & \lambda^3 \cdot (-2) \cdot (B^2 + A^2) \\
[L_{10}, L_{13}] & = & \lambda^3 \cdot (-4) \cdot (B^3 A + B A^3) \\
 & & +\lambda^3 \cdot (-2) \cdot (B^2 + A^2) \\
[L_{10}, L_{14}] & = & \lambda^3 \cdot (-6) \cdot (B^4 A^2 + B^2 A^4) \\
 & & +\lambda^3 \cdot (-6) \cdot (B^3 A + B A^3) \\
[L_{10}, L_{15}] & = & \lambda^3 \cdot (-12) \cdot (B^5 A + B A^5) \\
 & & +\lambda^3 \cdot (-30) \cdot (B^4 + A^4) \\
[L_{10}, L_{16}] & = & \lambda^3 \cdot (-2) \cdot (B^6 + A^6) \\
 & & +\lambda^3 \cdot (-10) \cdot (B^4 A^2 + B^2 A^4) \\
 & & +\lambda^3 \cdot (-20) \cdot (B^3 A + B A^3) \\
[L_{10}, L_{17}] & = & \lambda^3 \cdot (-4) \cdot (B^5 A + B A^5) \\
 & & +\lambda^3 \cdot (-2) \cdot (B^4 + A^4) \\
 & & +\lambda^3 \cdot (-16) \cdot B^3 A^3 \\
 & & +\lambda^3 \cdot (-24) \cdot B^2 A^2 \\
[L_{10}, L_{18}] & = & \lambda^3 \cdot (-8) \cdot (B^3 A + B A^3) \\
 & & +\lambda^3 \cdot (-12) \cdot (B^2 + A^2) \\
[L_{10}, L_{19}] & = & \lambda^3 \cdot (-2) \cdot (B^4 + A^4) \\
 & & +\lambda^3 \cdot (-12) \cdot B^2 A^2 \\
 & & +\lambda^3 \cdot (-12) \cdot B A \\
[L_{10}, L_{20}] & = & \lambda^3 \cdot (-8) \cdot B A \\
 & & +\lambda^3 \cdot (-4) \\
[L_{10}, L_{21}] & = & \lambda^3 \cdot (-12) \cdot (B^5 A - B A^5) \\
 & & +\lambda^3 \cdot (-30) \cdot (B^4 - A^4) \\
[L_{10}, L_{22}] & = & \lambda^3 \cdot (-2) \cdot (B^6 - A^6) \\
 & & +\lambda^3 \cdot (-10) \cdot (B^4 A^2 - B^2 A^4) \\
 & & +\lambda^3 \cdot (-20) \cdot (B^3 A - B A^3) \\
[L_{10}, L_{23}] & = & \lambda^3 \cdot (-4) \cdot (B^5 A - B A^5) \\
 & & +\lambda^3 \cdot (-2) \cdot (B^4 - A^4) \\
[L_{10}, L_{24}] & = & \lambda^3 \cdot (-8) \cdot (B^3 A - B A^3) \\
 & & +\lambda^3 \cdot (-12) \cdot (B^2 - A^2) \\
[L_{10}, L_{25}] & = & \lambda^3 \cdot (-2) \cdot (B^4 - A^4) \\
[L_{10}, L_{26}] & = & 0
\end{array}
$$

Table 5: A partial list of commutators from elements of the second order Lie Algebra.

By inspection, we can see that our second order Lie Algebra must be extended. The complete list is shown in Table 6.

		Nonperturbative Terms (λ^k where $k=0$)
$L_0^{(3)}$	$=$	$I = 1$
$L_1^{(3)}$	$=$	BA
		First Order Terms (λ^k where $k=1$)
$L_2^{(3)}$	$=$	$\lambda\cdot I = \lambda$
$L_3^{(3)}$	$=$	$\lambda\cdot BA$
$L_4^{(3)}$	$=$	$\lambda\cdot B^2A^2$
$L_5^{(3)}$	$=$	$\lambda\cdot(B^4 + A^4)$
$L_6^{(3)}$	$=$	$\lambda\cdot(B^3A + BA^3)$
$L_7^{(3)}$	$=$	$\lambda\cdot(B^2 + A^2)$
$L_8^{(3)}$	$=$	$\lambda\cdot(B^4 - A^4)$
$L_9^{(3)}$	$=$	$\lambda\cdot(B^3A - BA^3)$
$L_{10}^{(3)}$	$=$	$\lambda\cdot(B^2 - A^2)$
		Second Order Terms (λ^k where $k=2$)
$L_{11}^{(3)}$	$=$	$\lambda^2\cdot I = \lambda^2$
$L_{12}^{(3)}$	$=$	$\lambda^2\cdot BA$
$L_{13}^{(3)}$	$=$	$\lambda^2\cdot B^2A^2$
$L_{14}^{(3)}$	$=$	$\lambda^2\cdot B^3A^3$
$L_{15}^{(3)}$	$=$	$\lambda^2\cdot(B^6 + A^6)$
$L_{16}^{(3)}$	$=$	$\lambda^2\cdot(B^5A + BA^5)$
$L_{17}^{(3)}$	$=$	$\lambda^2\cdot(B^4A^2 + B^2A^4)$
$L_{18}^{(3)}$	$=$	$\lambda^2\cdot(B^4 + A^4)$
$L_{19}^{(3)}$	$=$	$\lambda^2\cdot(B^3A + BA^3)$
$L_{20}^{(3)}$	$=$	$\lambda^2\cdot(B^2 + A^2)$
$L_{21}^{(3)}$	$=$	$\lambda^2\cdot(B^6 - A^6)$
$L_{22}^{(3)}$	$=$	$\lambda^2\cdot(B^5A - BA^5)$
$L_{23}^{(3)}$	$=$	$\lambda^2\cdot(B^4A^2 - B^2A^4)$
$L_{24}^{(3)}$	$=$	$\lambda^2\cdot(B^4 - A^4)$
$L_{25}^{(3)}$	$=$	$\lambda^2\cdot(B^3A - BA^3)$
$L_{26}^{(3)}$	$=$	$\lambda^2\cdot(B^2 - A^2)$

Third Order Terms (λ^k where $k = 3$)		
$L_{27}^{(3)}$	$=$	$\lambda^3 \cdot I = \lambda^3$
$L_{28}^{(3)}$	$=$	$\lambda^3 \cdot BA$
$L_{29}^{(3)}$	$=$	$\lambda^3 \cdot B^2 A^2$
$L_{30}^{(3)}$	$=$	$\lambda^3 \cdot B^3 A^3$
$L_{31}^{(3)}$	$=$	$\lambda^3 \cdot B^4 A^4$
$L_{32}^{(3)}$	$=$	$\lambda^3 \cdot (B^8 + A^8)$
$L_{33}^{(3)}$	$=$	$\lambda^3 \cdot (B^7 A + BA^7)$
$L_{34}^{(3)}$	$=$	$\lambda^3 \cdot (B^6 A^2 + B^2 A^6)$
$L_{35}^{(3)}$	$=$	$\lambda^3 \cdot (B^5 A^3 + B^3 A^5)$
$L_{36}^{(3)}$	$=$	$\lambda^3 \cdot (B^6 + A^6)$
$L_{37}^{(3)}$	$=$	$\lambda^3 \cdot (B^5 A + BA^5)$
$L_{38}^{(3)}$	$=$	$\lambda^3 \cdot (B^4 A^2 + B^2 A^4)$
$L_{39}^{(3)}$	$=$	$\lambda^3 \cdot (B^4 + A^4)$
$L_{40}^{(3)}$	$=$	$\lambda^3 \cdot (B^3 A + BA^3)$
$L_{41}^{(3)}$	$=$	$\lambda^3 \cdot (B^2 + A^2)$
$L_{42}^{(3)}$	$=$	$\lambda^3 \cdot (B^8 - A^8)$
$L_{43}^{(3)}$	$=$	$\lambda^3 \cdot (B^7 A - BA^7)$
$L_{44}^{(3)}$	$=$	$\lambda^3 \cdot (B^6 A^2 - B^2 A^6)$
$L_{45}^{(3)}$	$=$	$\lambda^3 \cdot (B^5 A^3 - B^3 A^5)$
$L_{46}^{(3)}$	$=$	$\lambda^3 \cdot (B^6 - A^6)$
$L_{47}^{(3)}$	$=$	$\lambda^3 \cdot (B^5 A - BA^5)$
$L_{48}^{(3)}$	$=$	$\lambda^3 \cdot (B^4 A^2 - B^2 A^4)$
$L_{49}^{(3)}$	$=$	$\lambda^3 \cdot (B^4 - A^4)$
$L_{50}^{(3)}$	$=$	$\lambda^3 \cdot (B^3 A - BA^3)$
$L_{51}^{(3)}$	$=$	$\lambda^3 \cdot (B^2 - A^2)$

Table 6: The list of all terms in the third order Lie algebra, derived (by inspection) from Table 5.

The terms in Table 6 form a complete representation, as it satisfies the two conditions from 2.1, where $\lambda^3 \neq 0$ and $\lambda^4 = 0$.

4.2 Step 2: Construct a General Lie Group Element

Once again, we may discard terms that commute with H_0, (the number operator terms, L_0 through L_4, L_{11} through L_{14}, and L_{27} through L_{31}) when constructing the general Lie group element. Further, to keep things managable, we can apply the knowledge that many terms must be zero in order to simplify to the lower order approximations.

$$\begin{aligned}
U \quad = \exp(\quad & \gamma_8 L_8 + \gamma_9 L_9 + \gamma_{10} L_{10} \\
& + \gamma_{21} L_{21} + \gamma_{23} L_{23} + \gamma_{25} L_{25} + \gamma_{26} L_{26} \\
& + \gamma_{32} L_{32} + \gamma_{33} L_{33} + \gamma_{34} L_{34} + \gamma_{35} L_{35} \\
& + \gamma_{36} L_{36} + \gamma_{37} L_{37} + \gamma_{38} L_{38} + \gamma_{39} L_{39} \\
& + \gamma_{40} L_{40} + \gamma_{41} L_{41} + \gamma_{42} L_{42} + \gamma_{43} L_{43} \\
& + \gamma_{44} L_{44} + \gamma_{45} L_{45} + \gamma_{46} L_{46} + \gamma_{47} L_{47} \\
& + \gamma_{48} L_{48} + \gamma_{49} L_{49} + \gamma_{50} L_{50} + \gamma_{51} L_{51})
\end{aligned} \tag{17}$$

4.3 Step 3: Use the Hammard Lemma to Compute our Lie Group Element

There are 27 coefficients to compute in this group element. We proceed as before.

4.3.1 Step 3.1: Expand $U^\dagger H_0 U$ by the Hammard Lemma

$$U^\dagger H_0 U = H_0 + [-X, H_0] + \frac{1}{2!}([-X, [-X, H_0]]) + \frac{1}{3!}([-X, [-X, [-X, H_0]]]) \tag{18}$$

where $X = \gamma_8 L_8 + \gamma_9 L_9 + \gamma_{10} L_{10} + \gamma_{21} L_{21} + \gamma_{23} L_{23} + \gamma_{25} L_{25} + \gamma_{26} L_{26} + \gamma_{32} L_{32} + \gamma_{33} L_{33} + \gamma_{34} L_{34} + \gamma_{35} L_{35} + \gamma_{36} L_{36} + \gamma_{37} L_{37} + \gamma_{38} L_{38} + \gamma_{39} L_{39} + \gamma_{40} L_{40} + \gamma_{41} L_{41} + \gamma_{42} L_{42} + \gamma_{43} L_{43} + \gamma_{44} L_{44} + \gamma_{45} L_{45} + \gamma_{46} L_{46} + \gamma_{47} L_{47} + \gamma_{48} L_{48} + \gamma_{49} L_{49} + \gamma_{50} L_{50} + \gamma_{51} L_{51}$.

Taking the first term:

$$
\begin{aligned}
[-X, H_0] \quad = \quad & \lambda^3 \cdot (8 \cdot \gamma_{42}) \cdot (B^8 + A^8) + \lambda^3 \cdot (8 \cdot \gamma_{32}) \cdot (B^8 - A^8) \\
& + \lambda^3 \cdot (6 \cdot \gamma_{43}) \cdot (B^7 A + B A^7) + \lambda^3 \cdot (6 \cdot \gamma_{33}) \cdot (B^7 A - B A^7) \\
& + \lambda^3 \cdot (4 \cdot \gamma_{44}) \cdot (B^6 A^2 + B^2 A^6) + \lambda^3 \cdot (4 \cdot \gamma_{34}) \cdot (B^6 A^2 - B^2 A^6) \\
& + \lambda^3 \cdot (2 \cdot \gamma_{45}) \cdot (B^5 A^3 + B^3 A^5) + \lambda^3 \cdot (2 \cdot \gamma_{35}) \cdot (B^5 A^3 - B^3 A^5) \\
& + \lambda^2 \cdot (6 \cdot \gamma_{21}) \cdot (B^6 + A^6) \\
& + \lambda^3 \cdot (6 \cdot \gamma_{46}) \cdot (B^6 + A^6) + \lambda^3 \cdot (6 \cdot \gamma_{36}) \cdot (B^6 - A^6) \\
& + \lambda^3 \cdot (4 \cdot \gamma_{47}) \cdot (B^5 A + B A^5) + \lambda^3 \cdot (4 \cdot \gamma_{37}) \cdot (B^5 A - B A^5) \\
& + \lambda^2 \cdot (2 \cdot \gamma_{23}) \cdot (B^4 A^2 + B^2 A^4) \\
& + \lambda^3 \cdot (2 \cdot \gamma_{48}) \cdot (B^4 A^2 + B^2 A^4) + \lambda^3 \cdot (2 \cdot \gamma_{38}) \cdot (B^4 A^2 - B^2 A^4) \\
& + \lambda \cdot (4 \cdot \gamma_8) \cdot (B^4 + A^4) \\
& + \lambda^3 \cdot (4 \cdot \gamma_{49}) \cdot (B^4 + A^4) + \lambda^3 \cdot (4 \cdot \gamma_{39}) \cdot (B^4 - A^4) \\
& + \lambda \cdot (2 \cdot \gamma_9) \cdot (B^3 A + B A^3) \\
& + \lambda^2 \cdot (2 \cdot \gamma_{25}) \cdot (B^3 A + B A^3) \\
& + \lambda^3 \cdot (2 \cdot \gamma_{50}) \cdot (B^3 A + B A^3) + \lambda^3 \cdot (2 \cdot \gamma_{40}) \cdot (B^3 A - B A^3) \\
& + \lambda \cdot (2 \cdot \gamma_{10}) \cdot (B^2 + A^2) \\
& + \lambda^2 \cdot (2 \cdot \gamma_{26}) \cdot (B^2 + A^2) \\
& + \lambda^3 \cdot (2 \cdot \gamma_{51}) \cdot (B^2 + A^2) + \lambda^3 \cdot (2 \cdot \gamma_{41}) \cdot (B^2 - A^2)
\end{aligned}
$$

And now, the second term:

$$
\begin{aligned}
\frac{1}{2!}[-X,[-X,H_0]] \quad = \quad & \lambda^3\cdot(-12\cdot\gamma_9\cdot\gamma_{21})\cdot(B^8+A^8) \\
& +\lambda^3\cdot(-8\cdot\gamma_8\cdot\gamma_{23})\cdot(B^7A+BA^7) \\
& +\lambda^3\cdot(72\cdot\gamma_9\cdot\gamma_{21})\cdot(B^6A^2+B^2A^6) \\
& +\lambda^3\cdot(48\cdot\gamma_8\cdot\gamma_{23}+120\cdot\gamma_8\cdot\gamma_{21})\cdot(B^5A^3+B^3A^5) \\
& +\lambda^3\cdot(-12\cdot\gamma_8\cdot\gamma_{23}-4\cdot\gamma_8\cdot\gamma_{25})\cdot(B^6+A^6) \\
& +\lambda^2\cdot(-4\cdot\gamma_8\cdot\gamma_9)\cdot(B^6+A^6) \\
& +\lambda^3\cdot(360\cdot\gamma_9\cdot\gamma_{21}+48\cdot\gamma_{10}\cdot\gamma_{21})\cdot(B^5A+BA^5) \\
& +\lambda^3\cdot(216\cdot\gamma_8\cdot\gamma_{23}+900\cdot\gamma_8\cdot\gamma_{21}+36\cdot\gamma_8\cdot\gamma_{25})\cdot(B^4A^2+B^2A^4) \\
& +\lambda^2\cdot(36\cdot\gamma_8\cdot\gamma_9)\cdot(B^4A^2+B^2A^4) \\
& +\lambda^3\cdot(480\cdot\gamma_9\cdot\gamma_{21}+120\cdot\gamma_{10}\cdot\gamma_{21})\cdot(B^4+A^4) \\
& +\lambda^3\cdot(288\cdot\gamma_8\cdot\gamma_{23}+2400\cdot\gamma_8\cdot\gamma_{21}+108\cdot\gamma_8\cdot\gamma_{25}+24\cdot\gamma_8\cdot\gamma_{26})\cdot(B^3A+BA^3) \\
& +\lambda^2\cdot(108\cdot\gamma_8\cdot\gamma_9+24\cdot\gamma_8\cdot\gamma_{10})\cdot(B^3A+BA^3) \\
& +\lambda^3\cdot(72\cdot\gamma_8\cdot\gamma_{23}+1800\cdot\gamma_8\cdot\gamma_{21}+72\cdot\gamma_8\cdot\gamma_{25}+36\cdot\gamma_8\cdot\gamma_{26})\cdot(B^2+A^2) \\
& +\lambda^2\cdot(72\cdot\gamma_8\cdot\gamma_9+36\cdot\gamma_8\cdot\gamma_{10})\cdot(B^2+A^2) \\
& +\lambda^3\cdot(40\cdot\gamma_9\cdot\gamma_{23})\cdot B^4A^4 \\
& +\lambda^3\cdot(144\cdot\gamma_9\cdot\gamma_{23}+32\cdot\gamma_{10}\cdot\gamma_{23}+32\cdot\gamma_9\cdot\gamma_{25})\cdot B^3A^3 \\
& +\lambda^2\cdot(64\cdot\gamma_8^2+16\cdot\gamma_9^2)\cdot B^3A^3 \\
& +\lambda^3\cdot(96\cdot\gamma_9\cdot\gamma_{23}+48\cdot\gamma_{10}\cdot\gamma_{23}+72\cdot\gamma_9\cdot\gamma_{25}+24\cdot\gamma_9\cdot\gamma_{26}+24\cdot\gamma_{10}\cdot\gamma_{25})\cdot B^2A^2 \\
& +\lambda^2\cdot(288\cdot\gamma_8^2+36\cdot\gamma_9^2+24\cdot\gamma_9\cdot\gamma_{10})\cdot B^2A^2 \\
& +\lambda^2\cdot(384\cdot\gamma_8^2+12\cdot\gamma_9^2+24\cdot\gamma_9\cdot\gamma_{10}+8\cdot\gamma_{10}^2)\cdot BA \\
& +\lambda^3\cdot(24\cdot\gamma_9\cdot\gamma_{25}+24\cdot\gamma_9\cdot\gamma_{26}+24\cdot\gamma_{10}\cdot\gamma_{25}+16\cdot\gamma_{10}\cdot\gamma_{26})\cdot BA \\
& +\lambda^2\cdot(96\cdot\gamma_8^2+4\cdot\gamma_{10}^2) \\
& +\lambda^3\cdot(8\cdot\gamma_{10}\cdot\gamma_{26})
\end{aligned}
$$

Next, the third term, multiplied by 3!, to simplify:

$$
\begin{aligned}
[-X,[-X,[-X,H_0]]] \quad = \quad & \lambda^3\cdot(48\cdot\gamma_8\cdot\gamma_9^2)\cdot(B^8+A^8) \\
& +\lambda^3\cdot(576\cdot\gamma_8^2\cdot\gamma_9)\cdot(B^7A+BA^7) \\
& +\lambda^3\cdot(1536\cdot\gamma_8^3+384\cdot\gamma_8\cdot\gamma_9^2)\cdot(B^6A^2+B^2A^6) \\
& +\lambda^3\cdot(1728\cdot\gamma_8^2\cdot\gamma_9+192\cdot\gamma_9^3)\cdot(B^5A^3+B^3A^5) \\
& +\lambda^3\cdot(1728\cdot\gamma_8^2\cdot\gamma_9+192\cdot\gamma_8^2\cdot\gamma_{10})\cdot(B^6+A^6) \\
& +\lambda^3\cdot(9216\cdot\gamma_8^3+1440\cdot\gamma_8\cdot\gamma_9^2+576\cdot\gamma_8\cdot\gamma_9\cdot\gamma_{10})\cdot(B^5A+BA^5) \\
& +\lambda^3\cdot(10944\cdot\gamma_8^2\cdot\gamma_9+864\cdot\gamma_9^3+1344\cdot\gamma_8^2\cdot\gamma_{10}+384\cdot\gamma_9^2\cdot\gamma_{10})\cdot(B^4A^2+B^2A^4) \\
& +\lambda^3\cdot(13056\cdot\gamma_8^3+480\cdot\gamma_8\cdot\gamma_9^2+960\cdot\gamma_8\cdot\gamma_9\cdot\gamma_{10}+160\cdot\gamma_8\cdot\gamma_{10}^2)\cdot(B^4+A^4) \\
& +\lambda^3\cdot(17760\cdot\gamma_8^2\cdot\gamma_9+672\cdot\gamma_9^3+5376\cdot\gamma_8^2\cdot\gamma_{10}+864\cdot\gamma_9^2\cdot\gamma_{10}+224\cdot\gamma_9\cdot\gamma_{10}^2)\cdot(B^3A+ \\
& +\lambda^3\cdot(4704\cdot\gamma_8^2\cdot\gamma_{10}+192\cdot\gamma_9^2\cdot\gamma_{10}+192\cdot\gamma_9\cdot\gamma_{10}^2+32\cdot\gamma_{10}^3+5760\cdot\gamma_8^2\cdot\gamma_9)\cdot(B^2+A^2) \\
& +\lambda^3\cdot(1440\cdot\gamma_8\cdot\gamma_9^2)\cdot B^4A^4 \\
& +\lambda^3\cdot(8640\cdot\gamma_8\cdot\gamma_9^2+1920\cdot\gamma_8\cdot\gamma_9\cdot\gamma_{10})\cdot B^3A^3 \\
& +\lambda^3\cdot(12960\cdot\gamma_8\cdot\gamma_9^2+6912\cdot\gamma_8\cdot\gamma_9\cdot\gamma_{10}+576\cdot\gamma_8\cdot\gamma_{10}^2)\cdot B^2A^2 \\
& +\lambda^3\cdot(4320\cdot\gamma_8\cdot\gamma_9^2+5184\cdot\gamma_8\cdot\gamma_9\cdot\gamma_{10}+1152\cdot\gamma_8\cdot\gamma_{10}^2)\cdot BA \\
& +\lambda^3\cdot(576\cdot\gamma_8\cdot\gamma_9\cdot\gamma_{10}+288\cdot\gamma_8\cdot\gamma_{10}^2)
\end{aligned}
$$

Now, combining these terms (after multiplying them all by 3! to keep the coefficients simple):
$6 \cdot \big([-X, H_0] + \frac{1}{2!}[-X,[-X,H_0]] + \frac{1}{3!}[-X,[-X,[-X,H_0]]]\big)$

$$
\begin{aligned}
= \quad & \lambda^3{\cdot}(48{\cdot}\gamma_{42} - 72{\cdot}\gamma_9{\cdot}\gamma_{21} + 48{\cdot}\gamma_8{\cdot}\gamma_9^2){\cdot}(B^8 + A^8) + \lambda^3{\cdot}(48{\cdot}\gamma_{32}){\cdot}(B^8 - A^8) \\
& +\lambda^3{\cdot}(36{\cdot}\gamma_{43} - 48{\cdot}\gamma_8{\cdot}\gamma_{23} + 576{\cdot}\gamma_8^2{\cdot}\gamma_9){\cdot}(B^7A + BA^7) + \lambda^3{\cdot}(36{\cdot}\gamma_{33}){\cdot}(B^7A - BA^7) \\
& +\lambda^3{\cdot}(24{\cdot}\gamma_{44} + 432{\cdot}\gamma_9{\cdot}\gamma_{21} + 1536{\cdot}\gamma_8^3 + 384{\cdot}\gamma_8{\cdot}\gamma_9^2){\cdot}(B^6A^2 + B^2A^6) + \lambda^3{\cdot}(24{\cdot}\gamma_{34}){\cdot}(B^6A^2 - B^2A^6) \\
& +\lambda^3{\cdot}(12{\cdot}\gamma_{45} + 288{\cdot}\gamma_8{\cdot}\gamma_{23} + 720{\cdot}\gamma_8{\cdot}\gamma_{21} + 1728{\cdot}\gamma_8^2{\cdot}\gamma_9 + 192{\cdot}\gamma_9^3){\cdot}(B^5A^3 + B^3A^5) + \lambda^3{\cdot}(12{\cdot}\gamma_{35}){\cdot}(B^5A^3 \\
& +\lambda^2{\cdot}(36{\cdot}\gamma_{21} - 24{\cdot}\gamma_8{\cdot}\gamma_9){\cdot}(B^6 + A^6) \\
& +\lambda^3{\cdot}(36{\cdot}\gamma_{46} - 72{\cdot}\gamma_8{\cdot}\gamma_{23} - 24{\cdot}\gamma_8{\cdot}\gamma_{25} + 1728{\cdot}\gamma_8^2{\cdot}\gamma_9 + 192{\cdot}\gamma_8^2{\cdot}\gamma_{10}){\cdot}(B^6 + A^6) + \lambda^3{\cdot}(36{\cdot}\gamma_{36}){\cdot}(B^6 - A^6) \\
& +\lambda^3{\cdot}(24{\cdot}\gamma_{47} + 2160{\cdot}\gamma_9{\cdot}\gamma_{21} + 288{\cdot}\gamma_{10}{\cdot}\gamma_{21} + 9216{\cdot}\gamma_8^3 + 1440{\cdot}\gamma_8{\cdot}\gamma_9^2 + 576{\cdot}\gamma_8{\cdot}\gamma_9{\cdot}\gamma_{10}){\cdot}(B^5A + BA^5) + \\
& +\lambda^2{\cdot}(12{\cdot}\gamma_{23} + 216{\cdot}\gamma_8{\cdot}\gamma_9){\cdot}(B^4A^2 + B^2A^4) \\
& +\lambda^3{\cdot}(12{\cdot}\gamma_{48} + 1296{\cdot}\gamma_8{\cdot}\gamma_{23} + 5400{\cdot}\gamma_8{\cdot}\gamma_{21} + 216{\cdot}\gamma_8{\cdot}\gamma_{25} + 10944{\cdot}\gamma_8^2{\cdot}\gamma_9 + 864{\cdot}\gamma_9^3 + 1344{\cdot}\gamma_8^2{\cdot}\gamma_{10} + 384 \\
& +\lambda{\cdot}(24{\cdot}\gamma_8){\cdot}(B^4 + A^4) \\
& +\lambda^3{\cdot}(24{\cdot}\gamma_{49} + 2880{\cdot}\gamma_9{\cdot}\gamma_{21} + 720{\cdot}\gamma_{10}{\cdot}\gamma_{21} + 13056{\cdot}\gamma_8^3 + 480{\cdot}\gamma_8{\cdot}\gamma_9^2 + 960{\cdot}\gamma_8{\cdot}\gamma_9{\cdot}\gamma_{10} + 160{\cdot}\gamma_8{\cdot}\gamma_{10}^2){\cdot}(B^4 \\
& +\lambda{\cdot}(12{\cdot}\gamma_9){\cdot}(B^3A + BA^3) \\
& +\lambda^2{\cdot}(12{\cdot}\gamma_{25} + 648{\cdot}\gamma_8{\cdot}\gamma_9 + 144{\cdot}\gamma_8{\cdot}\gamma_{10}){\cdot}(B^3A + BA^3) \\
& +\lambda^3{\cdot}(12{\cdot}\gamma_{50} + 1728{\cdot}\gamma_8{\cdot}\gamma_{23} + 14400{\cdot}\gamma_8{\cdot}\gamma_{21} + 648{\cdot}\gamma_8{\cdot}\gamma_{25} + 144{\cdot}\gamma_8{\cdot}\gamma_{26} + 17760{\cdot}\gamma_8^2{\cdot}\gamma_9 + 672{\cdot}\gamma_9^3 + 5376 \\
& +\lambda{\cdot}(12{\cdot}\gamma_{10}){\cdot}(B^2 + A^2) \\
& +\lambda^2{\cdot}(12{\cdot}\gamma_{26} + 432{\cdot}\gamma_8{\cdot}\gamma_9 + 216{\cdot}\gamma_8{\cdot}\gamma_{10}){\cdot}(B^2 + A^2) \\
& +\lambda^3{\cdot}(12{\cdot}\gamma_{51} + 432{\cdot}\gamma_8{\cdot}\gamma_{23} + 10800{\cdot}\gamma_8{\cdot}\gamma_{21} + 432{\cdot}\gamma_8{\cdot}\gamma_{25} + 216{\cdot}\gamma_8{\cdot}\gamma_{26} + 4704{\cdot}\gamma_8^2{\cdot}\gamma_{10} + 192{\cdot}\gamma_9^2{\cdot}\gamma_{10} + 19 \\
& +\lambda^3{\cdot}(240{\cdot}\gamma_9{\cdot}\gamma_{23} + 1440{\cdot}\gamma_8{\cdot}\gamma_9^2){\cdot}B^4A^4 \\
& +\lambda^3{\cdot}(864{\cdot}\gamma_9{\cdot}\gamma_{23} + 192{\cdot}\gamma_{10}{\cdot}\gamma_{23} + 192{\cdot}\gamma_9{\cdot}\gamma_{25} + 8640{\cdot}\gamma_8{\cdot}\gamma_9^2 + 1920{\cdot}\gamma_8{\cdot}\gamma_9{\cdot}\gamma_{10}){\cdot}B^3A^3 \\
& +\lambda^2{\cdot}(384{\cdot}\gamma_8^2 + 96{\cdot}\gamma_9^2){\cdot}B^3A^3 \\
& +\lambda^3{\cdot}(576{\cdot}\gamma_9{\cdot}\gamma_{23} + 288{\cdot}\gamma_{10}{\cdot}\gamma_{23} + 432{\cdot}\gamma_9{\cdot}\gamma_{25} + 144{\cdot}\gamma_9{\cdot}\gamma_{26} + 144{\cdot}\gamma_{10}{\cdot}\gamma_{25} + 12960{\cdot}\gamma_8{\cdot}\gamma_9^2 + 6912{\cdot}\gamma_8{\cdot}\gamma_9 \\
& +\lambda^2{\cdot}(1728{\cdot}\gamma_8^2 + 216{\cdot}\gamma_9^2 + 144{\cdot}\gamma_9{\cdot}\gamma_{10}){\cdot}B^2A^2 \\
& +\lambda^2{\cdot}(2304{\cdot}\gamma_8^2 + 72{\cdot}\gamma_9^2 + 144{\cdot}\gamma_9{\cdot}\gamma_{10} + 48{\cdot}\gamma_{10}^2){\cdot}BA \\
& +\lambda^3{\cdot}(144{\cdot}\gamma_9{\cdot}\gamma_{25} + 144{\cdot}\gamma_9{\cdot}\gamma_{26} + 144{\cdot}\gamma_{10}{\cdot}\gamma_{25} + 96{\cdot}\gamma_{10}{\cdot}\gamma_{26} + 4320{\cdot}\gamma_8{\cdot}\gamma_9^2 + 5184{\cdot}\gamma_8{\cdot}\gamma_9{\cdot}\gamma_{10} + 1152{\cdot}\gamma_8 \\
& +\lambda^2{\cdot}(576{\cdot}\gamma_8^2 + 24{\cdot}\gamma_{10}^2) \\
& +\lambda^3{\cdot}(48{\cdot}\gamma_{10}{\cdot}\gamma_{26} + 576{\cdot}\gamma_8{\cdot}\gamma_9{\cdot}\gamma_{10} + 288{\cdot}\gamma_8{\cdot}\gamma_{10}^2)
\end{aligned}
$$

Last, computing $3! \cdot \Lambda_4$:

$$
\begin{aligned}
6 \cdot \left(H_4 - U^\dagger H_0 U\right) &= 6 \cdot (\Lambda_4) \\
&= 6 \cdot \left(\tfrac{\lambda}{4}(A+B)^4 - \left([-X, H_0] + \tfrac{1}{2!}[-X,[-X,H_0]] + \tfrac{1}{3!}[-X,[-X,[-X,H_0]]]\right)\right) \\
&= \lambda^3{\cdot}(-48{\cdot}\gamma_{42} + 72{\cdot}\gamma_9{\cdot}\gamma_{21} - 48{\cdot}\gamma_8{\cdot}\gamma_9^2){\cdot}(B^8 + A^8) + \lambda^3{\cdot}(-48{\cdot}\gamma_{32}){\cdot}(B^8 - A^8) \\
&\quad +\lambda^3{\cdot}(-36{\cdot}\gamma_{43} + 48{\cdot}\gamma_8{\cdot}\gamma_{23} - 576{\cdot}\gamma_8^2{\cdot}\gamma_9){\cdot}(B^7A + BA^7) + \lambda^3{\cdot}(-36{\cdot}\gamma_{33}){\cdot}(B^7A - B \\
&\quad +\lambda^3{\cdot}(-24{\cdot}\gamma_{44} - 432{\cdot}\gamma_9{\cdot}\gamma_{21} - 1536{\cdot}\gamma_8^3 - 384{\cdot}\gamma_8{\cdot}\gamma_9^2){\cdot}(B^6A^2 + B^2A^6) + \lambda^3{\cdot}(-24{\cdot}\gamma_3 \\
&\quad +\lambda^3{\cdot}(-12{\cdot}\gamma_{45} - 288{\cdot}\gamma_8{\cdot}\gamma_{23} - 720{\cdot}\gamma_8{\cdot}\gamma_{21} - 1728{\cdot}\gamma_8^2{\cdot}\gamma_9 - 192{\cdot}\gamma_9^3){\cdot}(B^5A^3 + B^3A^5) \\
&\quad +\lambda^2{\cdot}(-36{\cdot}\gamma_{21} + 24{\cdot}\gamma_8{\cdot}\gamma_9){\cdot}(B^6 + A^6) \\
&\quad +\lambda^3{\cdot}(-36{\cdot}\gamma_{46} + 72{\cdot}\gamma_8{\cdot}\gamma_{23} + 24{\cdot}\gamma_8{\cdot}\gamma_{25} - 1728{\cdot}\gamma_8^2{\cdot}\gamma_9 - 192{\cdot}\gamma_8^2{\cdot}\gamma_{10}){\cdot}(B^6 + A^6) + \lambda^3 \\
&\quad +\lambda^3{\cdot}(-24{\cdot}\gamma_{47} - 2160{\cdot}\gamma_9{\cdot}\gamma_{21} - 288{\cdot}\gamma_{10}{\cdot}\gamma_{21} - 9216{\cdot}\gamma_8^3 - 1440{\cdot}\gamma_8{\cdot}\gamma_9^2 - 576{\cdot}\gamma_8{\cdot}\gamma_9{\cdot}\gamma_1 \\
&\quad +\lambda^2{\cdot}(-12{\cdot}\gamma_{23} - 216{\cdot}\gamma_8{\cdot}\gamma_9){\cdot}(B^4A^2 + B^2A^4) \\
&\quad +\lambda^3{\cdot}(-12{\cdot}\gamma_{48} - 1296{\cdot}\gamma_8{\cdot}\gamma_{23} - 5400{\cdot}\gamma_8{\cdot}\gamma_{21} - 216{\cdot}\gamma_8{\cdot}\gamma_{25} - 10944{\cdot}\gamma_8^2{\cdot}\gamma_9 - 864{\cdot}\gamma_9^3 - \\
&\quad +\lambda{\cdot}(1.5 - 24{\cdot}\gamma_8){\cdot}(B^4 + A^4) \\
&\quad +\lambda^3{\cdot}(-24{\cdot}\gamma_{49} - 2880{\cdot}\gamma_9{\cdot}\gamma_{21} - 720{\cdot}\gamma_{10}{\cdot}\gamma_{21} - 13056{\cdot}\gamma_8^3 - 480{\cdot}\gamma_8{\cdot}\gamma_9^2 - 960{\cdot}\gamma_8{\cdot}\gamma_9{\cdot}\gamma_1 \\
&\quad +\lambda{\cdot}(6 - 12{\cdot}\gamma_9){\cdot}(B^3A + BA^3) \\
&\quad +\lambda^2{\cdot}(-12{\cdot}\gamma_{25} - 648{\cdot}\gamma_8{\cdot}\gamma_9 - 144{\cdot}\gamma_8{\cdot}\gamma_{10}){\cdot}(B^3A + BA^3) \\
&\quad +\lambda^3{\cdot}(-12{\cdot}\gamma_{50} - 1728{\cdot}\gamma_8{\cdot}\gamma_{23} - 14400{\cdot}\gamma_8{\cdot}\gamma_{21} - 648{\cdot}\gamma_8{\cdot}\gamma_{25} - 144{\cdot}\gamma_8{\cdot}\gamma_{26} - 17760{\cdot}\gamma_8^2 \\
&\quad +\lambda{\cdot}(9 - 12{\cdot}\gamma_{10}){\cdot}(B^2 + A^2) \\
&\quad +\lambda^2{\cdot}(-12{\cdot}\gamma_{26} - 432{\cdot}\gamma_8{\cdot}\gamma_9 - 216{\cdot}\gamma_8{\cdot}\gamma_{10}){\cdot}(B^2 + A^2) \\
&\quad +\lambda^3{\cdot}(-12{\cdot}\gamma_{51} - 432{\cdot}\gamma_8{\cdot}\gamma_{23} - 10800{\cdot}\gamma_8{\cdot}\gamma_{21} - 432{\cdot}\gamma_8{\cdot}\gamma_{25} - 216{\cdot}\gamma_8{\cdot}\gamma_{26} - 4704{\cdot}\gamma_8^2{\cdot}\gamma \\
&\quad +\lambda^3{\cdot}(-240{\cdot}\gamma_9{\cdot}\gamma_{23} - 1440{\cdot}\gamma_8{\cdot}\gamma_9^2){\cdot}B^4A^4 \\
&\quad +\lambda^3{\cdot}(-864{\cdot}\gamma_9{\cdot}\gamma_{23} - 192{\cdot}\gamma_{10}{\cdot}\gamma_{23} - 192{\cdot}\gamma_9{\cdot}\gamma_{25} - 8640{\cdot}\gamma_8{\cdot}\gamma_9^2 - 1920{\cdot}\gamma_8{\cdot}\gamma_9{\cdot}\gamma_{10}){\cdot}B^3 \\
&\quad +\lambda^2{\cdot}(-384{\cdot}\gamma_8^2 - 96{\cdot}\gamma_9^2){\cdot}B^3A^3 \\
&\quad +\lambda{\cdot}(9){\cdot}B^2A^2 \\
&\quad +\lambda^3{\cdot}(-576{\cdot}\gamma_9{\cdot}\gamma_{23} - 288{\cdot}\gamma_{10}{\cdot}\gamma_{23} - 432{\cdot}\gamma_9{\cdot}\gamma_{25} - 144{\cdot}\gamma_9{\cdot}\gamma_{26} - 144{\cdot}\gamma_{10}{\cdot}\gamma_{25} - 12960 \\
&\quad +\lambda^2{\cdot}(-1728{\cdot}\gamma_8^2 - 216{\cdot}\gamma_9^2 - 144{\cdot}\gamma_9{\cdot}\gamma_{10}){\cdot}B^2A^2 \\
&\quad +\lambda{\cdot}(18){\cdot}BA \\
&\quad +\lambda^2{\cdot}(-2304{\cdot}\gamma_8^2 - 72{\cdot}\gamma_9^2 - 144{\cdot}\gamma_9{\cdot}\gamma_{10} - 48{\cdot}\gamma_{10}^2){\cdot}BA \\
&\quad +\lambda^3{\cdot}(-144{\cdot}\gamma_9{\cdot}\gamma_{25} - 144{\cdot}\gamma_9{\cdot}\gamma_{26} - 144{\cdot}\gamma_{10}{\cdot}\gamma_{25} - 96{\cdot}\gamma_{10}{\cdot}\gamma_{26} - 4320{\cdot}\gamma_8{\cdot}\gamma_9^2 - 5184{\cdot}\gamma \\
&\quad +\lambda{\cdot}(4.5) \\
&\quad +\lambda^2{\cdot}(-576{\cdot}\gamma_8^2 - 24{\cdot}\gamma_{10}^2) \\
&\quad +\lambda^3{\cdot}(-48{\cdot}\gamma_{10}{\cdot}\gamma_{26} - 576{\cdot}\gamma_8{\cdot}\gamma_9{\cdot}\gamma_{10} - 288{\cdot}\gamma_8{\cdot}\gamma_{10}^2)
\end{aligned}
$$

4.3.2 Step 3.2: Tune γ_k so Λ_4 is a Number Operator

Tuning the γ values is easily done by setting each term that is not a number operator in Λ_4 equal to zero, and solving for γ_k. The results are shown in Table 7.

γ_8	$=$	$\frac{1}{16}$	γ_{32}	$=$	0	γ_{42}	$=$	0
γ_9	$=$	$\frac{1}{2}$	γ_{33}	$=$	0	γ_{43}	$=$	$-\frac{5}{64}$
γ_{10}	$=$	$\frac{3}{4}$	γ_{34}	$=$	0	γ_{44}	$=$	$-\frac{29}{64}$
γ_{21}	$=$	$\frac{1}{48}$	γ_{35}	$=$	0	γ_{45}	$=$	$-\frac{97}{64}$
γ_{23}	$=$	$-\frac{9}{16}$	γ_{36}	$=$	0	γ_{46}	$=$	$-\frac{35}{128}$
γ_{25}	$=$	$-\frac{9}{4}$	γ_{37}	$=$	0	γ_{47}	$=$	$-\frac{87}{32}$
γ_{26}	$=$	$-\frac{63}{32}$	γ_{38}	$=$	0	γ_{48}	$=$	$-\frac{1455}{128}$
			γ_{39}	$=$	0	γ_{49}	$=$	$-\frac{427}{128}$
			γ_{40}	$=$	0	γ_{50}	$=$	$-\frac{2225}{128}$
			γ_{41}	$=$	0	γ_{51}	$=$	$-\frac{855}{256}$

Table 7: Values for γ_k.

The resulting form for Λ_4 is

$$\begin{aligned}\Lambda_4 &= \lambda(\frac{3}{2}B^2A^2 + 3BA + \frac{3}{4}) \\ &+\lambda^2(\frac{-17}{4}B^3A^3 + \frac{-153}{8}B^2A^2 + (-18)BA + \frac{-21}{8}) \\ &+\lambda^3(\frac{15}{2}B^4A^4 + 60B^3A^3 + \frac{513}{4}B^2A^2 + \frac{153}{2}BA + \frac{63}{8})\end{aligned} \tag{19}$$

Not surprisingly, setting λ^3 equal to zero will return the second order perturbation result. Setting λ^2 equal to zero will return the first order perturbation result.