Number Systems

Number systems in mathematics are often derived out of necessity when solving equations. For example, the equation $2x - 3 = 1$ has the solution $x = 2$ which is a natural number, the set of which is denoted $\mathbb{N} = \{1, 2, 3, \dots\}$.

However, this set of numbers does not contain the solution to the equation $2x + 1 = 1$, which is $x = 0$. In this case, a new term needs to be added to give a new set $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$.

This idea can be extended to generate other number systems that can help solve other types of equations. The most common number systems are:

- **Natural Numbers**
- **Whole Numbers**
- **Integers**
- **Rational Numbers**

There is also the set of **Irrational Numbers** which consists of infinite non-repeating decimals such as $\mathrm{e}, \pi, \ln(2), \sqrt{2}$ and φ and can be used to solve equations such as $x^2 - 2 = 0$. (In general, there is no simple way of representing the set of irrational numbers in a closed form.) The combination of the rational and irrational numbers gives the set of **Real Numbers** denoted $\mathbb{R}$.

Even with all these number systems, there is still one class of equations that has not been addressed. For instance, consider the equation $x^2 + 4 = 0$, this equation has no solution in the real numbers since there is no real number x whose square is -4. For this purpose, an additional term would have to be added to the list of numbers, a term known as the **Imaginary Unit** denoted i (note that sometimes, this would be denoted j in engineering contexts but i will be used here). This value can be regarded as a number whose square is -1, thus generating two new number systems:

- **Imaginary Numbers**: $\mathbb{I} = \left\{x\mathrm{i} : x \in \mathbb{R},\ \mathrm{i}^2 = -1\right\}$ to solve $x^2 + 4 = 0$
- **Complex Numbers**: $\mathbb{C} = \left\{x + \mathrm{i}y \mid x, y \in \mathbb{R},\ \mathrm{i}^2 = -1\right\}$ Note that the imaginary unit i is a number that satisfies $\mathrm{i}^2 = -1$ and *not* $\mathrm{i} = \sqrt{-1}$.

The reason $\mathrm{i} = \sqrt{-1}$ is technically incorrect is because it may lead to contradictions:

$$-1 = \mathrm{i}^2 = \mathrm{i} \times \mathrm{i} = \sqrt{-1} \times \sqrt{-1} = \sqrt{-1 \times -1} = \sqrt{1} = 1.$$

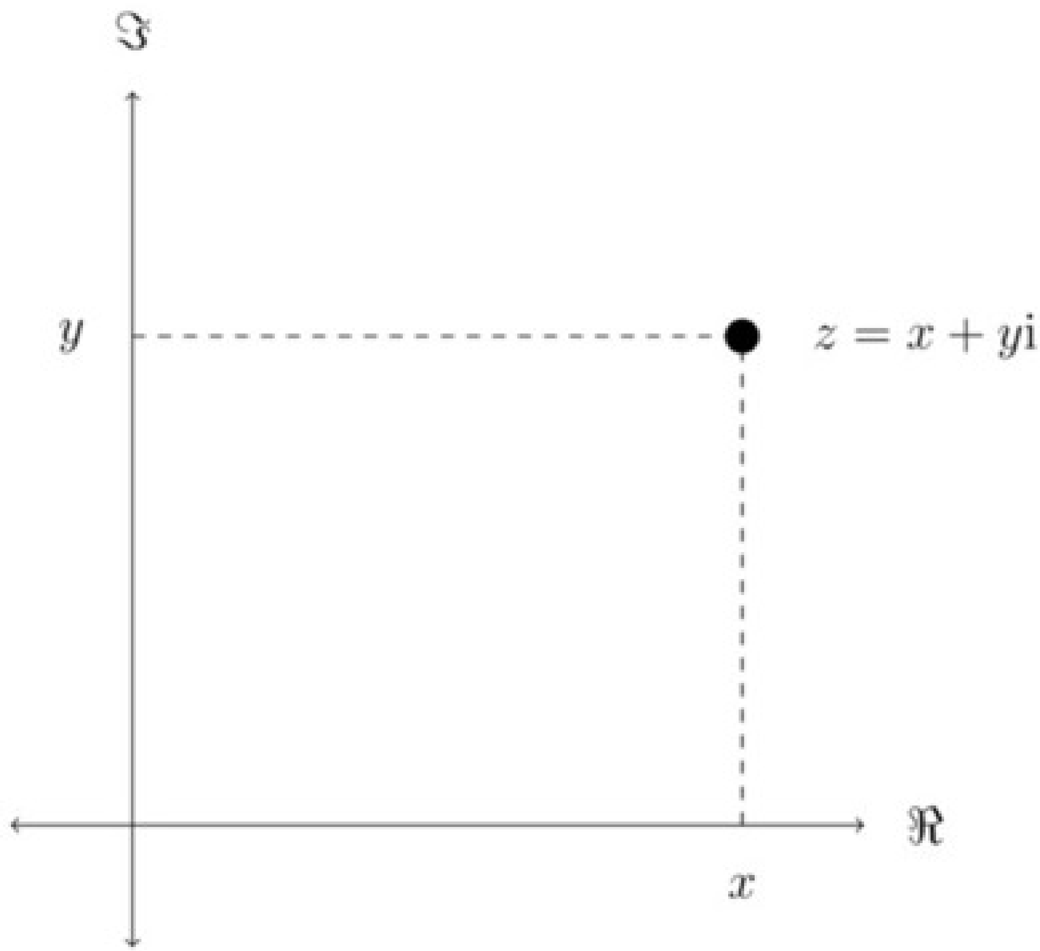
The Argand Diagram

A complex number $z \in \mathbb{C}$ is usually written in **Standard Form** (also called **Cartesian Form**) as

$$z = x + y\mathrm{i} \quad \text{where} \quad x, y \in \mathbb{R}.$$

The term x is known as the **Real Part** of z and y is known as the **Imaginary Part** of z. The real and imaginary parts of a complex number are represented by $\mathfrak{R}$ and $\mathfrak{I}$ respectively, so

$$\text{if} \quad z = x + y\mathrm{i} \quad \text{then} \quad x = \mathfrak{R}(z) \quad \text{and} \quad y = \mathfrak{I}(z).$$



If $y = 0$, then z is **Purely Real** and if $x = 0$, then z is called **Purely Imaginary**.

The complex number z can be drawn on the **Complex Plane** as a point on a 2-dimensional space where the horizontal axis is the **Real Axis** and the vertical axis is the **Imaginary Axis**. This particular type of diagram is called an **Argand Diagram**.