# Heteroskedasticity, Part 2

## EC 421, Set 5

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17 April 2019

# Prologue

# Schedule

## Last Time

Heteroskedasticity: Issues and tests

## Today

Living with heteroskedasticity

## Upcoming

- First assignment! **Due at 11:59pm on Sunday** (4/21).
- No class on Monday (4/22).

# EC 421

## Goals

- Develop **intuition** for econometrics.
- Learn how to ***apply*** econometrics—strengths, weaknessed, *etc.*
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This course has the potential to be one of the most useful/valuable/applicable/marketable classes that you take at UO.

# Heteroskedasticity

*Review*

# Heteroskedasticity

## Review

Three review questions

**Question 1:** What is the difference between $u_i$ and $e_i$?

**Question 2:** We spend *a lot* of time discussing $u_i^2$. Why?

**Question 3:** We also spend *a lot* of time discussing $e_i^2$. Why?

# Heteroskedasticity

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$u_i$ gives the **population disturbance** for the $i^{\text{th}}$ observation. $u_i$ measures how far the $i^{\text{th}}$ observation is from the **population** line, *i.e.*,

$$u_i = y_i - \underbrace{(\beta_0 + \beta_1 x_i)}_{\text{Population line}}$$

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$e_i$ gives the **regression residual (error)** for the $i^{\text{th}}$ observation. $e_i$ measures how far the $i^{\text{th}}$ observation is from the **regression** line, *i.e.,*

$$e_i = y_i - \underbrace{\left(\hat{\beta}_0 + \hat{\beta}_1 x_i\right)}_{\text{Regression line}=\hat{y}} = y_i - \hat{y}_i$$

# Heteroskedasticity

## Review

**Question 2:** We spend *a lot* of time discussing $u_i^2$. Why?

**Answer 2:**

One of major assumptions is that our disturbances (the $u_i$'s) are homoskedastic (they have constant variance), *i.e.*, $\text{Var}(u_i|x_i) = \sigma^2$.

We also assume that the mean of these disturbances is zero, $\textbf{\textit{E}}[u_i|x_i] = \textbf{0}$.

By definition, $\text{Var}(u_i|x_i) = \textbf{\textit{E}}\left[u_i^2 - \underbrace{\textbf{\textit{E}}[u_i|x_i]^2}_{=0}\Big|x_i\right] = \textbf{\textit{E}}\left[u_i^2|x_i\right]$

Thus, if we want to learn about the variance of $u_i$, we can focus on $u_i^2$.

# Heteroskedasticity

## Review

**Question 3:** We also spend *a lot* of time discussing $e_i^2$. Why?

**Answer 3:**

We cannot observe $u_i$ (or $u_i^2$).

But $u_i^2$ tells us about the variance of $u_i$.

We use $e_i^2$ to learn about $u_i^2$ and, consequently, $\sigma_i^2$.

# Heteroskedasticity

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4. The explanatory variables are **exogenous**: $\boldsymbol{E}[u|X] = 0$ ( $\implies$ $\boldsymbol{E}[u] = 0$).

5. The disurbances have **constant variance** $\sigma^2$ and **zero covariance**, *i.e.,*

   - $\boldsymbol{E}\big[u_i^2|X_i\big] = \mathrm{Var}(u_i|X_i) = \sigma^2 \implies \mathrm{Var}(u_i) = \sigma^2$
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6. The disturbances come from a **Normal** distribution, *i.e.,* $u_i \overset{\mathrm{iid}}{\sim} \mathrm{N}\big(0, \sigma^2\big)$.

# Heteroskedasticity

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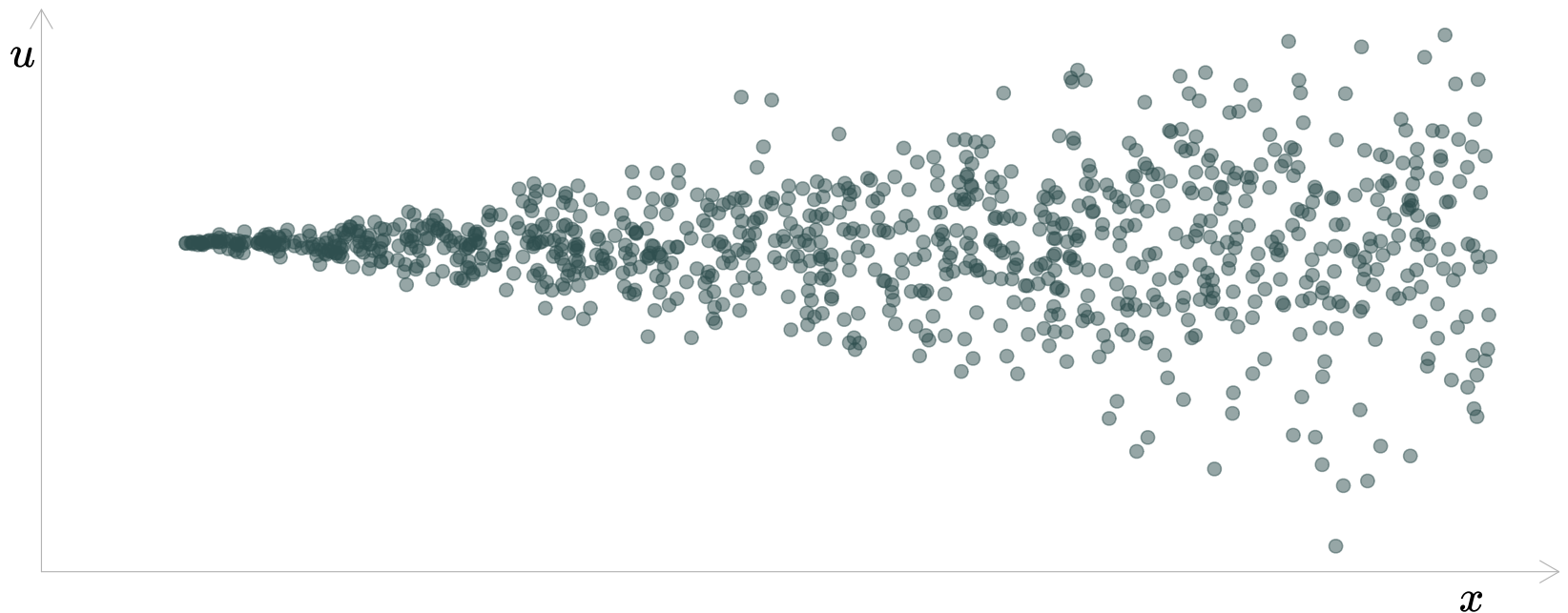**Violation of this assumption:** Our disturbances have different variances.

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Specifically, we will focus on the assumption of **constant variance** (also known as *homoskedasticity*).

**Violation of this assumption:** Our disturbances have different variances.

**Heteroskedasticity:** $\mathrm{Var}(u_i) = \sigma_i^2$ and $\sigma_i^2 \neq \sigma_j^2$ for some $i \neq j$.

# Heteroskedasticity

## Review

Classic example of heteroskedasticity: The funnel
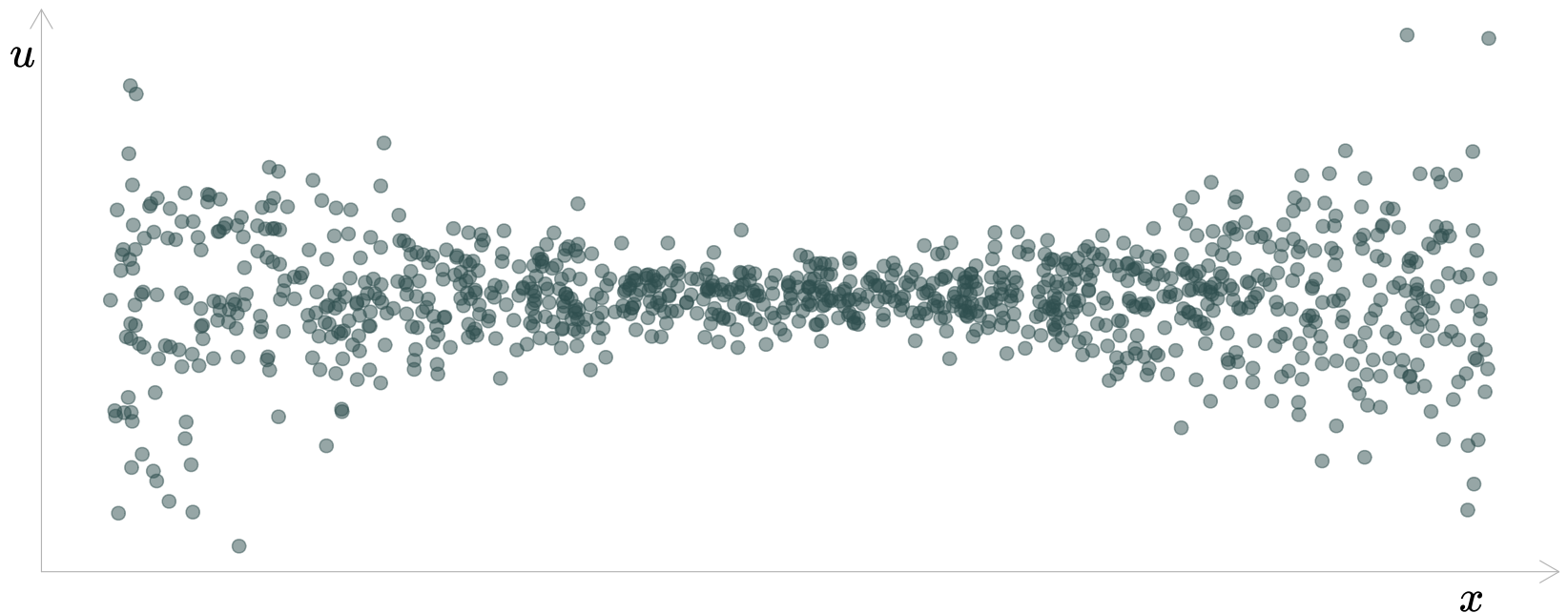
Variance of $u$ increases with $x$

# Heteroskedasticity

## Review

Another example of heteroskedasticity: (double funnel?)

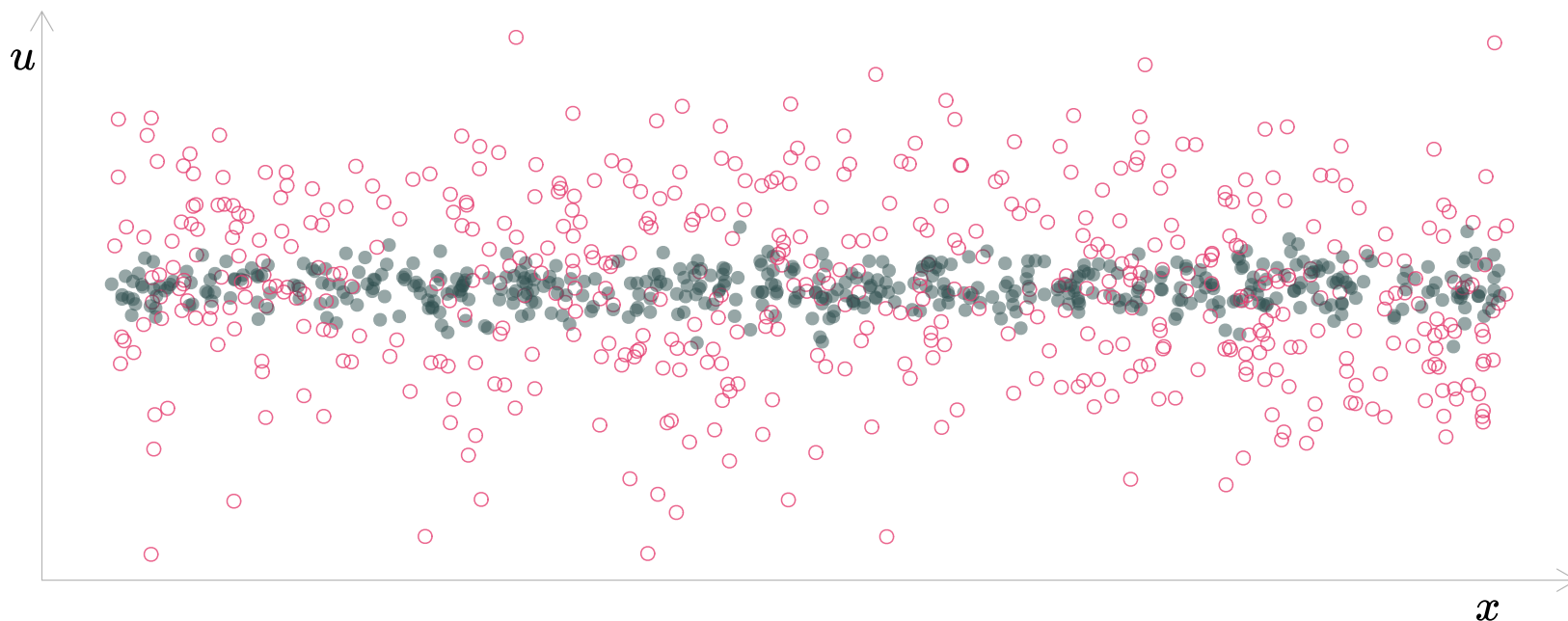Variance of $u$ increasing at the extremes of $x$

# Heteroskedasticity

## Review

Another example of heteroskedasticity:

Differing variances of $u$ by group

# Heteroskedasticity

## Review

**Heteroskedasticity** is present when the variance of $u$ changes with any combination of our explanatory variables $x_1$ through $x_k$.

# Testing for heteroskedasticity

We have some tests that may help us detect heteroskedasticity.

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- Breusch-Pagan
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What do we do if we detect it?

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- still **unbiased**
- **no longer the most efficient** unbiased linear estimator

On average, we get the right answer but with more noise (less precision).
*Also:* Our standard errors are biased.

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**Options:**

1. Check regression **specification**.
2. Find a new, more efficient **unbiased estimator** for $\beta_j$'s.
3. Live with OLS's inefficiency; find a **new variance estimator**.
    - Standard errors
    - Confidence intervals
    - Hypothesis tests

# Living with heteroskedasticity

## Misspecification

As we've discussed, the specification$^\dagger$ of your regression model matters a lot for the unbiasedness and efficiency of your estimator.

**Response #1:** Ensure your specification doesn't cause heteroskedasticity.

$\dagger$ *Specification:* Functional form and included variables.

## Misspecification

*Example:* Let the population relationship be

$$y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + u_i$$

with $\boldsymbol{E}[u_i|x_i] = 0$ and $\mathrm{Var}(u_i|x_i) = \sigma^2$.

However, we omit $x^2$ and estimate

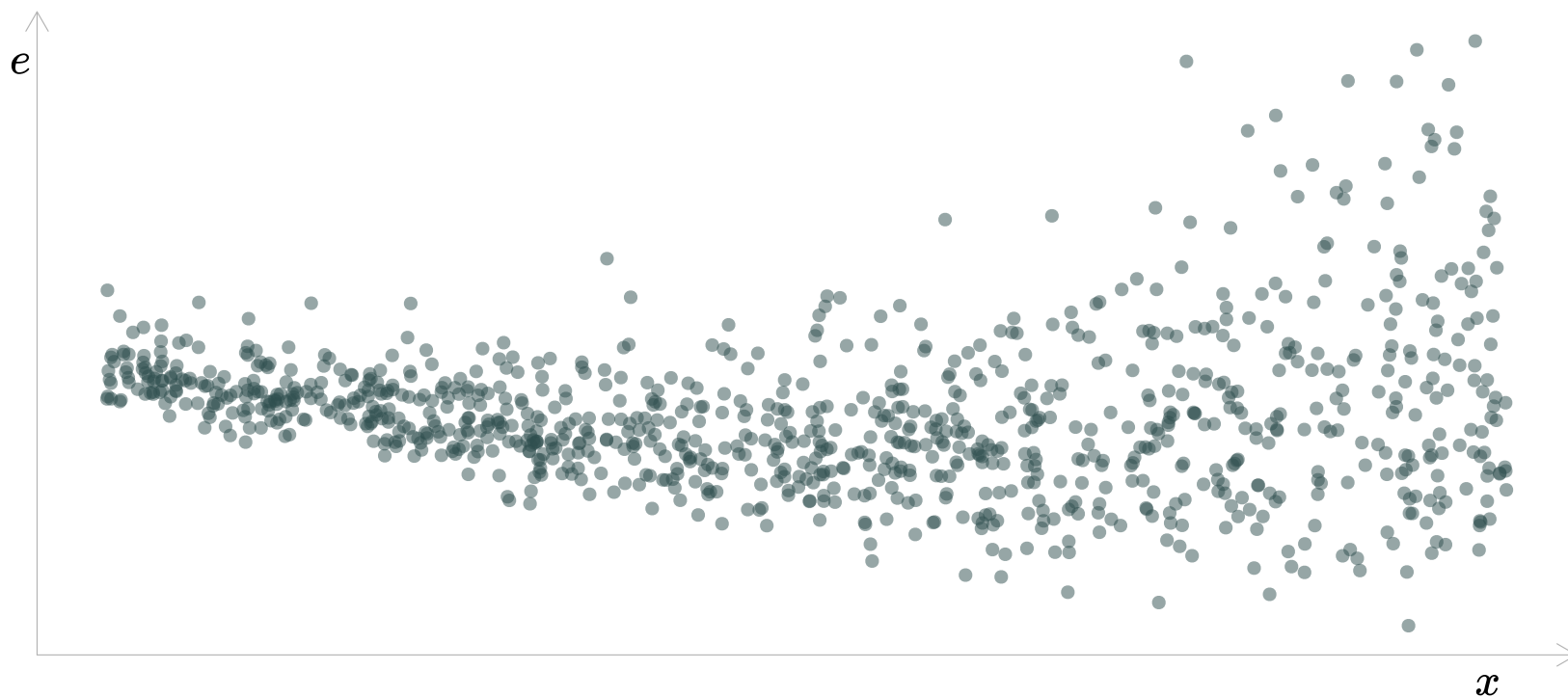$$y_i = \gamma_0 + \gamma_1 x_i + w_i$$

Then

$$w_i = u_i + \beta_2 x_i^2 \implies \mathrm{Var}(w_i) = f(x_i)$$

*I.e.*, the variance of $w_i$ changes systematically with $x_i$ (heteroskedasticity).

## Misspecification

Truth: $\log(y_i) = \beta_0 + \beta_1 x_i + u_i$     **Misspecification:** $y_i = \beta_0 + \beta_1 x_i + v_i$

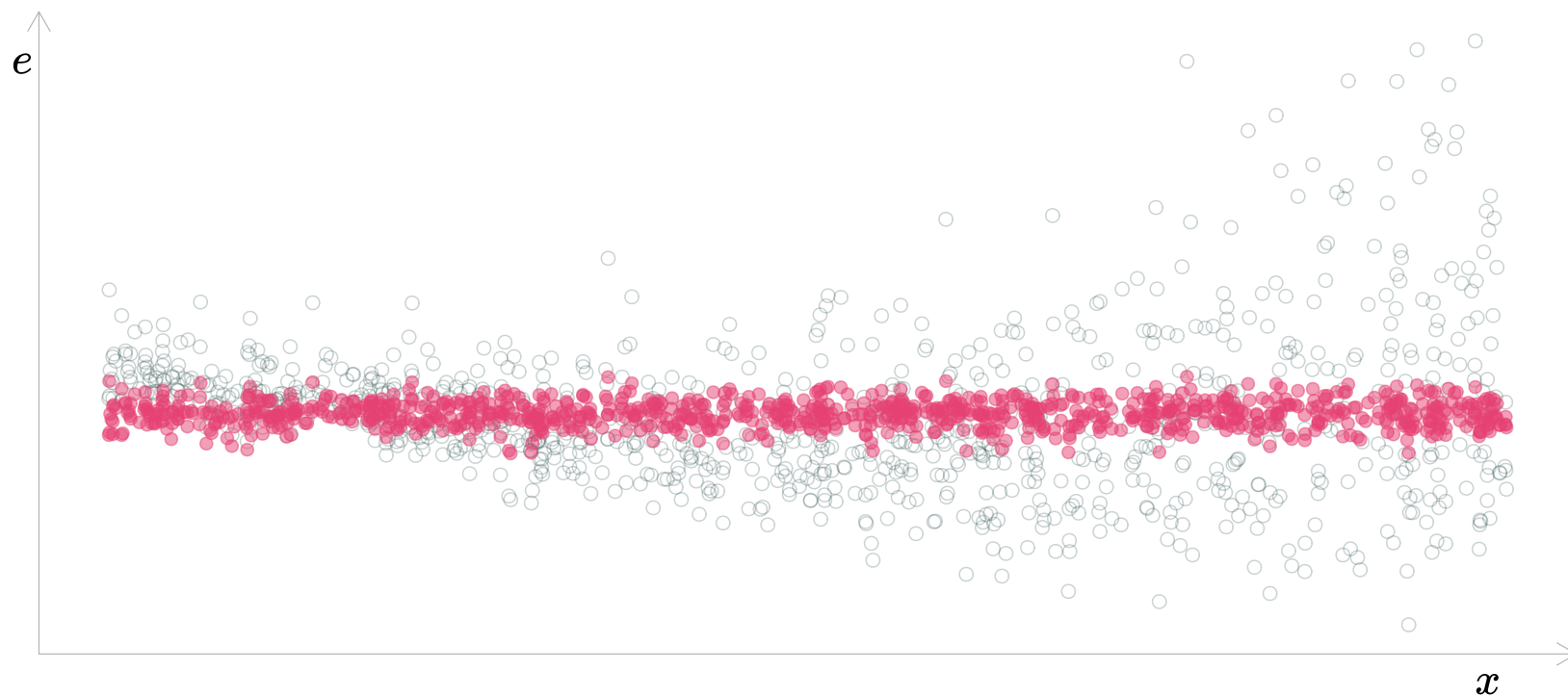# Living with heteroskedasticity

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More generally:

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**New problems:**

- We often don't know the *right* specification.
- We'd like a more formal process for addressing heteroskedasticity.

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**New problems:**

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- We'd like a more formal process for addressing heteroskedasticity.

**Conclusion:** Specification often will not "solve" heteroskedasticity. However, correctly specifying your model is still really important.

# Living with heteroskedasticity

## Weighted least squares

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Now transform (1) by dividing each observation's data by $\sigma_i$, *i.e.*,

$$\frac{y_i}{\sigma_i} = \beta_0 \frac{1}{\sigma_i} + \beta_1 \frac{x_i}{\sigma_i} + \frac{u_i}{\sigma_i} \tag{2}$$

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# Living with heteroskedasticity

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WLS is great, but we need to know $\sigma_i^2$, which is generally unlikely.

We can *slightly* relax this requirement—instead requiring

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As before, we transform our heteroskedastic model into a homoskedastic model. This time we divide each observation's data[†] by $\sqrt{h(x_i)}$.

---

[†] Divide *all* of the data by $\sqrt{h(x_i)}$, including the intercept.

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**Homoskedasticity!**

# Living with heteroskedasticity

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$$y_i = \beta_0 + \beta_1 x_{1i} + u_i \quad \text{vs.} \quad \frac{y_i}{\sigma_i} = \beta_0 \frac{1}{\sigma_i} + \beta_1 \frac{x_{1i}}{\sigma_i} + \frac{u_i}{\sigma_i}$$

*Notes:*

1. WLS **transforms** a heteroskedastic model into a homoskedastic model.
2. **Weighting:** WLS downweights observations with higher variance $u_i$'s.
3. **Big requirement:** WLS requires that we *know $\sigma_i^2$* for each observation.
4. WLS is generally **infeasible**. *Feasible* GLS (FGLS) offers a solution.
5. Under its assumptions: WLS is the **best linear unbiased estimator**.

# Living with heteroskedasticity

## Heteroskedasticity-robust standard errors

**Response #3:**

- Ignore OLS's inefficiency (in the presence of heteroskedasticity).
- Focus on **unbiased estimates for our standard errors**.
- In the process: Correct inference.

# Living with heteroskedasticity

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**A:** The **standard deviation of an estimator's distribution**.

# Living with heteroskedasticity

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- Focus on **unbiased estimates for our standard errors**.
- In the process: Correct inference.

**Q:** What is a standard error?

**A:** The **standard deviation of an estimator's distribution**.

Estimators (like $\hat{\beta}_1$) are random variables, so they have distributions.

Standard errors give us a sense of how much variability is in our estimator.

## Heteroskedasticity-robust standard errors

*Recall*: We can write the OLS estimator for $\beta_1$ as

$$\hat{\beta}_1 = \beta_1 + \frac{\sum_i \left(x_i - \overline{x}\right) u_i}{\sum_i \left(x_i - \overline{x}\right)^2} = \beta_1 + \frac{\sum_i \left(x_i - \overline{x}\right) u_i}{\text{SST}_x} \tag{3}$$

## Heteroskedasticity-robust standard errors

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Let $\text{Var}(u_i|x_i) = \sigma_i^2$.

# Living with heteroskedasticity

## Heteroskedasticity-robust standard errors

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Let $\mathrm{Var}(u_i | x_i) = \sigma_i^2$.

We can use (3) to write the variance of $\hat{\beta}_1$, *i.e.*,

$$\mathrm{Var}\left(\hat{\beta}_1 \big| x_i\right) = \frac{\sum_i \left(x_i - \overline{x}\right)^2 \sigma_i^2}{\mathrm{SST}_x^2} \tag{4}$$

# Living with heteroskedasticity

## Heteroskedasticity-robust standard errors

If we want unbiased estimates for our standard errors, we need an unbiased estimate for

$$\frac{\sum_i \left(x_i - \overline{x}\right)^2 \sigma_i^2}{\text{SST}_x^2}$$

Our old friend Hal White provided such an estimator:[†]

$$\widehat{\text{Var}}\left(\hat{\beta}_1\right) = \frac{\sum_i \left(x_i - \overline{x}\right)^2 e_i^2}{\text{SST}_x^2}$$

where the $e_i$ comes from the OLS regression of interest.

[†] This specific equation is for simple linear regression.

# Living with heteroskedasticity

## Heteroskedasticity-robust standard errors

Our heteroskedasticity-robust estimators for the standard error of $\beta_j$.

**Case 1** Simple linear regression, $y_i = \beta_0 + \beta_1 x_i + u_i$

$$\widehat{\text{Var}}\left(\hat{\beta}_1\right) = \frac{\sum_i \left(x_i - \overline{x}\right)^2 e_i^2}{\text{SST}_x^2}$$

**Case 2** Multiple (linear) regression, $y_i = \beta_0 + \beta_1 x_{1i} + \cdots + \beta_k x_{ki} + u_i$

$$\widehat{\text{Var}}\left(\hat{\beta}_j\right) = \frac{\sum_i \hat{r}_{ij}^2 e_i^2}{\text{SST}_{x_j^2}}$$

where $\hat{r}_{ij}$ denotes the ith residual from regressing $x_j$ on all other explanatory variables.

# Living with heteroskedasticity

## Heteroskedasticity-robust standard errors

With these standard errors, we can return to correct statistical inferencel

*E.g.*, we can update our previous $t$ statistic formula with our new heteroskedasticity-robust standard erros.

$$t = \frac{\text{Estimate} - \text{Hypothesized value}}{\text{Standard error}}$$

# Living with heteroskedasticity

## Heteroskedasticity-robust standard errors

**Notes**

- We are still using **OLS estimates for** $\beta_j$
- Our het.-robust standard errors use a **different estimator**.
- Homoskedasticity
  - Plain OLS variance estimator is more efficient.
  - Het.-robust is still unbiased.
- Heteroskedasticity
  - Plain OLS variance estimator is biased.
  - Het.-robust variance estimator is unbiased.

# Living with heteroskedasticity

## Heteroskedasticity-robust standard errors

These standard errors go by many names

- Heteroskedasticity-robust standard errors
- Het.-robust standard errors
- White standard errors
- Eicker-White standard errors
- Huber standard errors
- Eicker-Huber-White standards errors
- (some other combination of Eicker, Huber, and White)

**Do not say:** "Robust standard errors". The problem: "robust" to what?

# Living with heteroskedasticity

*Examples*

# Living with heteroskedasticity

## Examples

Back to our test-scores dataset...

```r
# Load packages
library(pacman)
p_load(tidyverse, Ecdat)
# Select and rename desired variables; assign to new dataset; format as tibble
test_df ← Caschool %>% select(
  test_score = testscr, ratio = str, income = avginc, enrollment = enrltot
) %>% as_tibble()
# View first 2 rows of the dataset
head(test_df, 2)
```

```
#> # A tibble: 2 x 4
#>   test_score ratio income enrollment
#>        <dbl> <dbl>  <dbl>      <int>
#> 1       691.  17.9  22.7         195
#> 2       661.  21.5   9.82        240
```

# Living with heteroskedasticity

## Example: Model specification
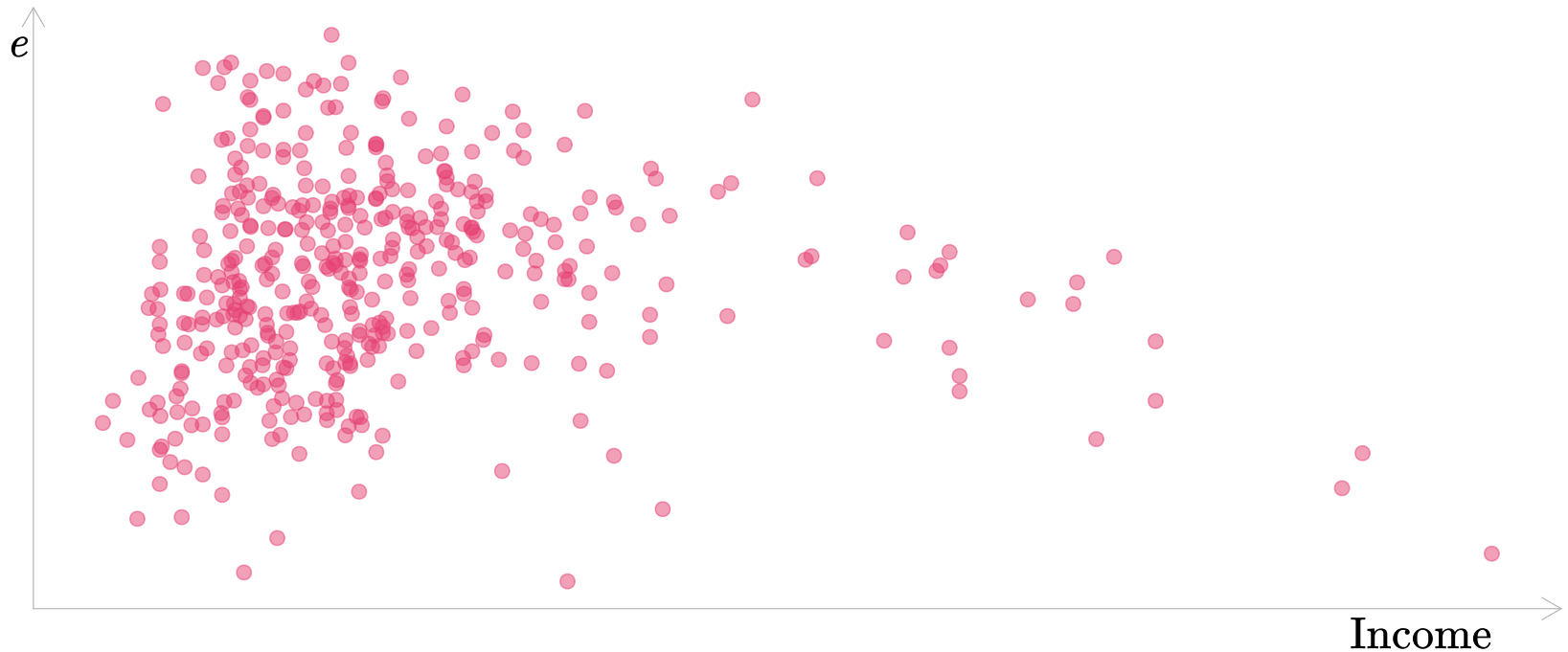
We found significant evidence of heteroskedasticity.

Let's check if it was due to misspecifying our model.

# Living with heteroskedasticity

## Example: Model specification

Model₁: $\text{Score}_i = \beta_0 + \beta_1 \text{Ratio}_i + \beta_2 \text{Income}_i + u_i$
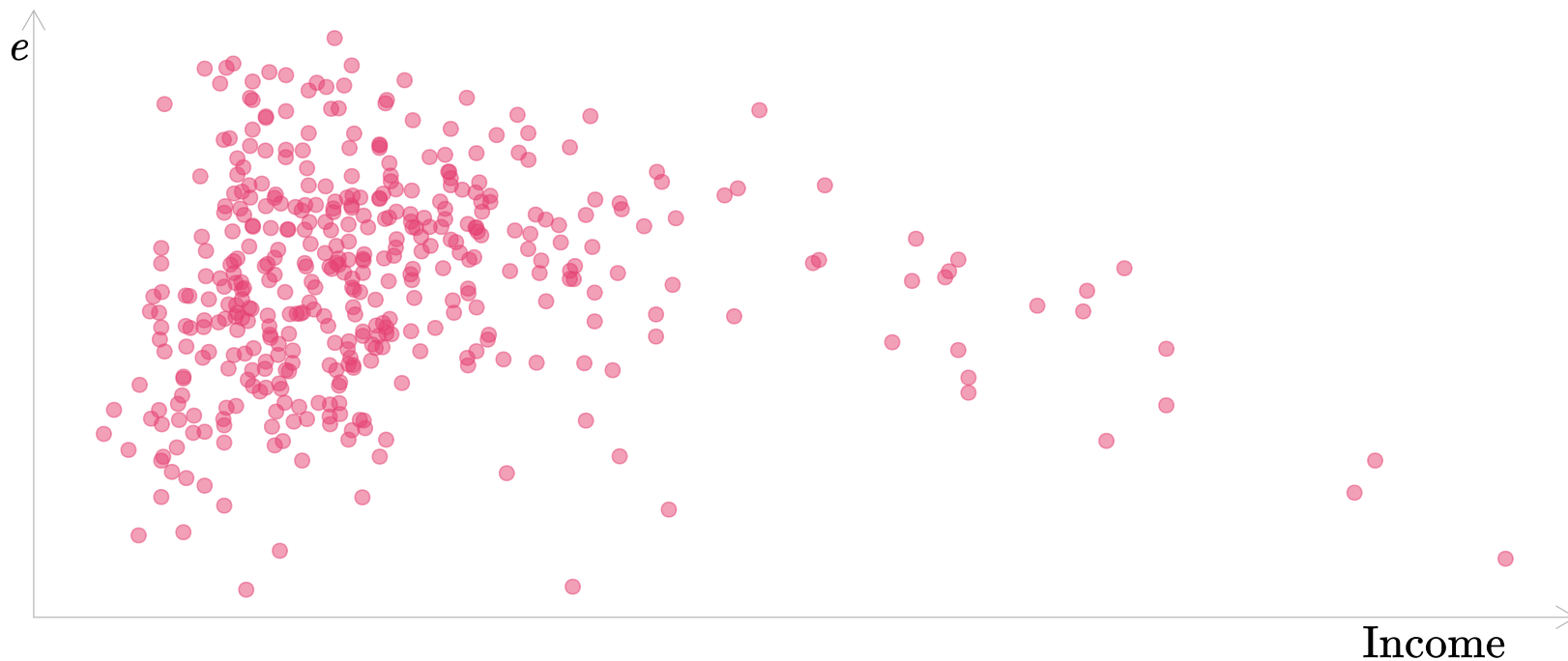
```
lm(test_score ~ ratio + income, data = test_df)
```

## Example: Model specification

Model$_2$: $\log(\text{Score}_i) = \beta_0 + \beta_1 \text{Ratio}_i + \beta_2 \text{Income}_i + u_i$

```
lm(log(test_score) ~ ratio + income, data = test_df)
```
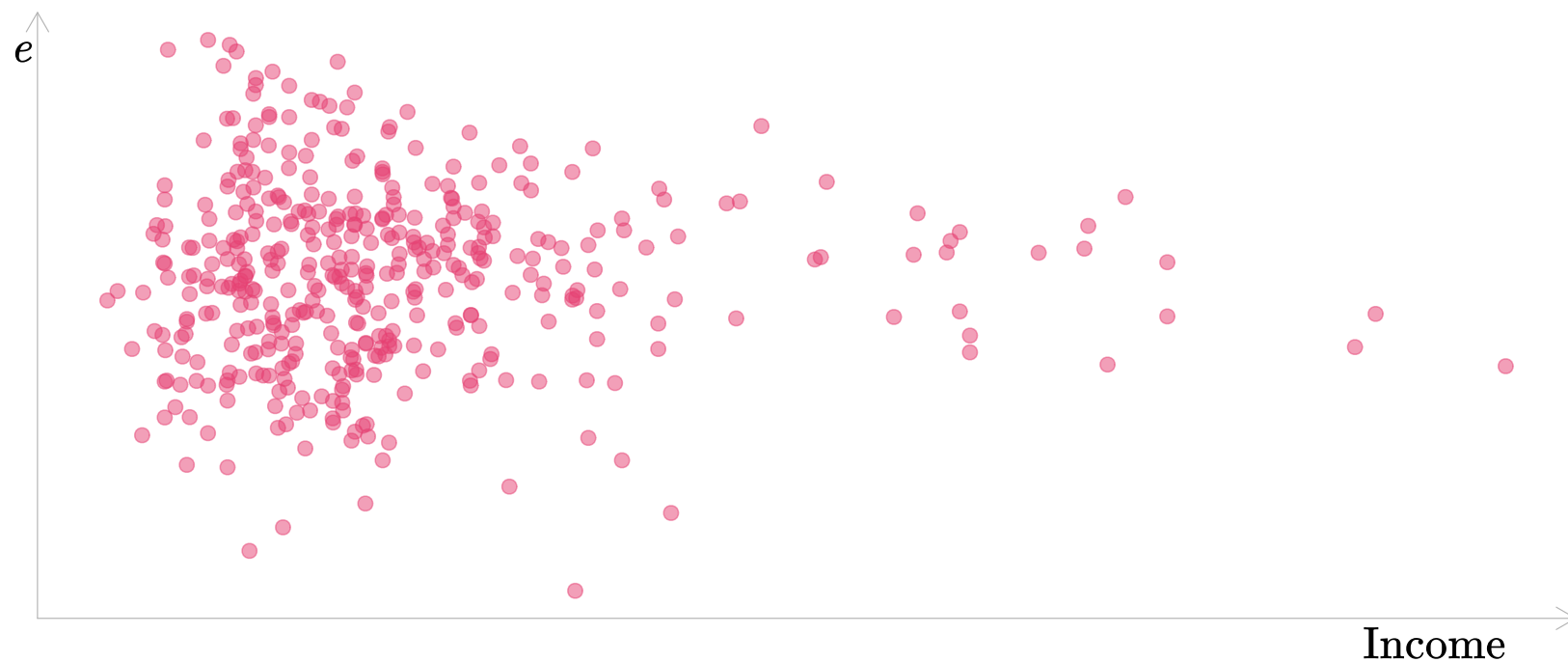
## Example: Model specification

Model$_3$: $\log(\text{Score}_i) = \beta_0 + \beta_1 \text{Ratio}_i + \beta_2 \log(\text{Income}_i) + u_i$

```
lm(log(test_score) ~ ratio + log(income), data = test_df)
```

# Living with heteroskedasticity

## Example: Model specification

Let's test this new specification with the White test for heteroskedasticity.

$$\text{Model}_3: \log(\text{Score}_i) = \beta_0 + \beta_1 \text{Ratio}_i + \beta_2 \log(\text{Income}_i) + u_i$$

# Living with heteroskedasticity

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Let's test this new specification with the White test for heteroskedasticity.

$$\text{Model}_3 \colon \log(\text{Score}_i) = \beta_0 + \beta_1 \text{Ratio}_i + \beta_2 \log(\text{Income}_i) + u_i$$

The regression for the White test

# Living with heteroskedasticity

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$$\text{Model}_3: \log(\text{Score}_i) = \beta_0 + \beta_1 \text{Ratio}_i + \beta_2 \log(\text{Income}_i) + u_i$$

The regression for the White test

$$e_i^2 = \alpha_0 + \alpha_1 \text{Ratio}_i + \alpha_2 \log(\text{Income}_i) + \alpha_3 \text{Ratio}_i^2 + \alpha_4 (\log(\text{Income}_i))^2$$
$$+ \alpha_5 \left(\text{Ratio}_i \times \log(\text{Income}_i)\right) + v_i$$

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The regression for the White test

$$\begin{aligned}
e_i^2 =& \alpha_0 + \alpha_1 \text{Ratio}_i + \alpha_2 \log(\text{Income}_i) + \alpha_3 \text{Ratio}_i^2 + \alpha_4 (\log(\text{Income}_i))^2 \\
& + \alpha_5 \left( \text{Ratio}_i \times \log(\text{Income}_i) \right) + v_i
\end{aligned}$$

yields $R_e^2 \approx 0.029$

## Example: Model specification

Let's test this new specification with the White test for heteroskedasticity.

$$\text{Model}_3\colon \log(\mathrm{Score}_i) = \beta_0 + \beta_1 \mathrm{Ratio}_i + \beta_2 \log(\mathrm{Income}_i) + u_i$$

The regression for the White test

$$
\begin{aligned}
e_i^2 =& \alpha_0 + \alpha_1 \mathrm{Ratio}_i + \alpha_2 \log(\mathrm{Income}_i) + \alpha_3 \mathrm{Ratio}_i^2 + \alpha_4 (\log(\mathrm{Income}_i))^2 \\
& + \alpha_5 \left( \mathrm{Ratio}_i \times \log(\mathrm{Income}_i) \right) + v_i
\end{aligned}
$$

yields $R_e^2 \approx 0.029$ and test statistic of

# Living with heteroskedasticity

## Example: Model specification

Let's test this new specification with the White test for heteroskedasticity.

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e_i^2 =& \alpha_0 + \alpha_1 \text{Ratio}_i + \alpha_2 \log(\text{Income}_i) + \alpha_3 \text{Ratio}_i^2 + \alpha_4 (\log(\text{Income}_i))^2 \\
& + \alpha_5 \left( \text{Ratio}_i \times \log(\text{Income}_i) \right) + v_i
\end{aligned}
$$

yields $R_e^2 \approx 0.029$ and test statistic of $\widehat{\text{LM}} = n \times R_e^2 \approx 12.2$.

## Example: Model specification

Let's test this new specification with the White test for heteroskedasticity.

$$\text{Model}_3\text{: } \log(\text{Score}_i) = \beta_0 + \beta_1\text{Ratio}_i + \beta_2\log(\text{Income}_i) + u_i$$

The regression for the White test

$$\begin{aligned}
e_i^2 =& \alpha_0 + \alpha_1\text{Ratio}_i + \alpha_2\log(\text{Income}_i) + \alpha_3\text{Ratio}_i^2 + \alpha_4(\log(\text{Income}_i))^2 \\
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\end{aligned}$$

yields $R_e^2 \approx 0.029$ and test statistic of $\widehat{\text{LM}} = n \times R_e^2 \approx 12.2$.

Under H$_0$, $\text{LM}$ is distributed as

# Living with heteroskedasticity

## Example: Model specification

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$$\text{Model}_3\text{: } \log(\text{Score}_i) = \beta_0 + \beta_1 \text{Ratio}_i + \beta_2 \log(\text{Income}_i) + u_i$$

The regression for the White test

$$\begin{aligned} e_i^2 =& \alpha_0 + \alpha_1 \text{Ratio}_i + \alpha_2 \log(\text{Income}_i) + \alpha_3 \text{Ratio}_i^2 + \alpha_4 (\log(\text{Income}_i))^2 \\ & + \alpha_5 \left(\text{Ratio}_i \times \log(\text{Income}_i)\right) + v_i \end{aligned}$$

yields $R_e^2 \approx 0.029$ and test statistic of $\widehat{\text{LM}} = n \times R_e^2 \approx 12.2$.

Under $H_0$, $\text{LM}$ is distributed as $\chi_5^2$

## Example: Model specification

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yields $R_e^2 \approx 0.029$ and test statistic of $\widehat{\text{LM}} = n \times R_e^2 \approx 12.2$.

Under $H_0$, $\text{LM}$ is distributed as $\chi_5^2 \implies$ $p$-value $\approx 0.033$.

# Living with heteroskedasticity

## Example: Model specification

Let's test this new specification with the White test for heteroskedasticity.

$$\text{Model}_3\text{: } \log(\text{Score}_i) = \beta_0 + \beta_1 \text{Ratio}_i + \beta_2 \log(\text{Income}_i) + u_i$$

The regression for the White test

$$
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e_i^2 =& \alpha_0 + \alpha_1 \text{Ratio}_i + \alpha_2 \log(\text{Income}_i) + \alpha_3 \text{Ratio}_i^2 + \alpha_4 (\log(\text{Income}_i))^2 \\
& + \alpha_5 \left( \text{Ratio}_i \times \log(\text{Income}_i) \right) + v_i
\end{aligned}
$$

yields $R_e^2 \approx 0.029$ and test statistic of $\widehat{\text{LM}} = n \times R_e^2 \approx 12.2$.

Under H$_0$, $\text{LM}$ is distributed as $\chi_5^2 \implies$ $p$-value $\approx 0.033$.

$\therefore$

## Example: Model specification

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& + \alpha_5 \left( \text{Ratio}_i \times \log(\text{Income}_i) \right) + v_i
\end{aligned}
$$

yields $R_e^2 \approx 0.029$ and test statistic of $\widehat{\text{LM}} = n \times R_e^2 \approx 12.2$.

Under H$_0$, $\text{LM}$ is distributed as $\chi_5^2 \implies$ $p$-value $\approx 0.033$.

$\therefore$ **Reject H$_0$.**

## Example: Model specification

Let's test this new specification with the White test for heteroskedasticity.

$$\text{Model}_3: \log(\text{Score}_i) = \beta_0 + \beta_1 \text{Ratio}_i + \beta_2 \log(\text{Income}_i) + u_i$$

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yields $R_e^2 \approx 0.029$ and test statistic of $\widehat{\text{LM}} = n \times R_e^2 \approx 12.2$.

Under $H_0$, $\text{LM}$ is distributed as $\chi_5^2 \implies$ $p$-value $\approx 0.033$.

$\therefore$ **Reject $H_0$. Conclusion:**

# Living with heteroskedasticity

## Example: Model specification

Let's test this new specification with the White test for heteroskedasticity.

$$\text{Model}_3\text{: } \log(\text{Score}_i) = \beta_0 + \beta_1 \text{Ratio}_i + \beta_2 \log(\text{Income}_i) + u_i$$

The regression for the White test

$$\begin{aligned} e_i^2 =&\alpha_0 + \alpha_1 \text{Ratio}_i + \alpha_2 \log(\text{Income}_i) + \alpha_3 \text{Ratio}_i^2 + \alpha_4 (\log(\text{Income}_i))^2 \\ &+ \alpha_5 \left(\text{Ratio}_i \times \log(\text{Income}_i)\right) + v_i \end{aligned}$$

yields $R_e^2 \approx 0.029$ and test statistic of $\widehat{\text{LM}} = n \times R_e^2 \approx 12.2$.

Under H$_0$, $\text{LM}$ is distributed as $\chi_5^2 \implies$ $p$-value $\approx 0.033$.

$\therefore$ **Reject H$_0$. Conclusion:** There is statistically significant evidence of heteroskedasticity at the five-percent level.

# Living with heteroskedasticity

## Example: Model specification

Okay, we tried adjusting our specification, but there is still evidence of heteroskedasticity.

**Next:** In general, you will turn to heteroskedasticity-robust standard errors.

- OLS is still unbiased for the **coefficients** (the $\beta_j$'s)

- Heteroskedasticity-robust standard errors are unbiased for the **standard errors** of the $\hat{\beta}_j$'s, *i.e.*, $\sqrt{\mathrm{Var}\left(\hat{\beta}_j\right)}$.

# Living with heteroskedasticity

## Example: Het.-robust standard errors

Let's return to our model

$$\text{Score}_i = \beta_0 + \beta_1 \text{Ratio}_i + \beta_2 \text{Income}_i + u_i$$

We can use the `lfe` package in R to calculate standard errors.

# Living with heteroskedasticity

## Example: Het.-robust standard errors

$$\text{Score}_i = \beta_0 + \beta_1 \text{Ratio}_i + \beta_2 \text{Income}_i + u_i$$

1. Run the regression with `felm()` (instead of `lm()`)

```
# Load 'lfe' package
p_load(lfe)
# Regress log score on ratio and log income
test_reg ← felm(test_score ~ ratio + income, data = test_df)
```

# Living with heteroskedasticity

## Example: Het.-robust standard errors

$$\mathrm{Score}_i = \beta_0 + \beta_1 \mathrm{Ratio}_i + \beta_2 \mathrm{Income}_i + u_i$$

1. Run the regression with `felm()` (instead of `lm()`)

```
# Load 'lfe' package
p_load(lfe)
# Regress log score on ratio and log income
test_reg ← felm(test_score ~ ratio + income, data = test_df)
```

*Notice* that `felm()` uses the same syntax as `lm()` for this regression.

## Example: Het.-robust standard errors

$$\text{Score}_i = \beta_0 + \beta_1 \text{Ratio}_i + \beta_2 \text{Income}_i + u_i$$

2. Estimate het.-robust standard errors with `robust = T` option in `summary()`

```
# Het-robust standard errors with 'robust = T'
summary(test_reg, robust = T)
```

```
#>                Estimate Robust s.e t value Pr(>|t|)
#> (Intercept) 638.7292      7.3012  87.482   <2e-16 ***
#> ratio        -0.6487      0.3533  -1.836   0.0671 .
#> income        1.8391      0.1147  16.029   <2e-16 ***
```

# Living with heteroskedasticity

## Example: Het.-robust standard errors

Ceofficients and **heteroskedasticity-robust standard errors**:

```
summary(test_reg, robust = T)
```

```
#>               Estimate Robust s.e t value Pr(>|t|)
#> (Intercept) 638.7292       7.3012  87.482    <2e-16 ***
#> ratio        -0.6487       0.3533  -1.836    0.0671 .
#> income        1.8391       0.1147  16.029    <2e-16 ***
```

Ceofficients and **plain OLS standard errors** (assumes homoskedasticity):

```
summary(test_reg, robust = F)
```

```
#>               Estimate Std. Error t value Pr(>|t|)
#> (Intercept) 638.72915      7.44908  85.746    <2e-16 ***
#> ratio        -0.64874      0.35440  -1.831    0.0679 .
#> income        1.83911      0.09279  19.821    <2e-16 ***
```

# Living with heteroskedasticity

## Example: WLS

We mentioned that WLS is often not possible—we need to know the functional form of the heteroskedasticity—either

**A**. $\sigma_i^2$

or

**B**. $h(x_i)$, where $\sigma_i^2 = \sigma^2 h(x_i)$

# Living with heteroskedasticity

## Example: WLS

We mentioned that WLS is often not possible—we need to know the functional form of the heteroskedasticity—either

**A**. $\sigma_i^2$

or

**B**. $h(x_i)$, where $\sigma_i^2 = \sigma^2 h(x_i)$

There *are* occasions in which we can know $h(x_i)$.

## Example: WLS

Imagine individuals in a population have homoskedastic disturbances.

However, instead of observing individuals' data, we observe (in data) groups' averages (*e.g.*, cities, counties, school districts).

If these groups have different sizes, then our dataset will be heteroskedastic—in a predictable fashion.

**Recall:** The variance of the sample mean depends upon the sample size,

$$\mathrm{Var}\left(\overline{x}\right) = \frac{\sigma_x^2}{n}$$

# Living with heteroskedasticity

## Example: WLS

Imagine individuals in a population have homoskedastic disturbances.

However, instead of observing individuals' data, we observe (in data) groups' averages (*e.g.*, cities, counties, school districts).

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**Recall:** The variance of the sample mean depends upon the sample size,

$$\text{Var}\left(\overline{x}\right) = \frac{\sigma_x^2}{n}$$

**Example:** Our school testing data is averaged at the school level.

# Living with heteroskedasticity

## Example: WLS

*Example:* Our school testing data is averaged at the school level.

Even if individual students have homoskedastic disturbances, the schools would have heteroskedastic disturbances, *i.e.,*

**Individual-level model:**    $\text{Score}_i = \beta_0 + \beta_1 \text{Ratio}_i + \beta_2 \text{Income}_i + u_i$

**School-level model:**    $\overline{\text{Score}}_s = \beta_0 + \beta_1 \overline{\text{Ratio}}_s + \beta_2 \overline{\text{Income}}_s + \overline{u}_s$

where the $s$ subscript denotes an individual school (just as $i$ indexes an individual person).

$$\text{Var}\left(\overline{u}_s\right) = \frac{\sigma^2}{n_s}$$

# Living with heteroskedasticity

## Example: WLS

For WLS, we're looking for a function $h(x_s)$ such that $\mathrm{Var}\left(\overline{u}_s | x_s\right) = \sigma^2 h(x_s)$.

# Living with heteroskedasticity

## Example: WLS

For WLS, we're looking for a function $h(x_s)$ such that $\text{Var}\big(\overline{u}_s|x_s\big) = \sigma^2 h(x_s)$.

We just showed$^\dagger$ that $\text{Var}\big(\overline{u}_s|x_s\big) = \dfrac{\sigma^2}{n_s}$.

$\dagger$ Assuming the individuals' disturbances are homoskedastic.

# Living with heteroskedasticity

## Example: WLS

For WLS, we're looking for a function $h(x_s)$ such that $\text{Var}(\overline{u}_s|x_s) = \sigma^2 h(x_s)$.

We just showed[†] that $\text{Var}(\overline{u}_s|x_s) = \dfrac{\sigma^2}{n_s}$.

Thus, $h(x_s) = 1/n_s$, where $n_s$ is the number of students in school $s$.

[†] Assuming the individuals' disturbances are homoskedastic.

# Living with heteroskedasticity

## Example: WLS

For WLS, we're looking for a function $h(x_s)$ such that $\text{Var}(\overline{u}_s | x_s) = \sigma^2 h(x_s)$.

We just showed[†] that $\text{Var}(\overline{u}_s | x_s) = \dfrac{\sigma^2}{n_s}$.

Thus, $h(x_s) = 1/n_s$, where $n_s$ is the number of students in school $s$.

To implement WLS, we divide each observation's data by $1/\sqrt{h(x_s)}$, meaning we need to multiply each school's data by $\sqrt{n_s}$.

† Assuming the individuals' disturbances are homoskedastic.

# Living with heteroskedasticity

## Example: WLS

For WLS, we're looking for a function $h(x_s)$ such that $\text{Var}\left(\overline{u}_s | x_s\right) = \sigma^2 h(x_s)$.

We just showed[†] that $\text{Var}\left(\overline{u}_s | x_s\right) = \dfrac{\sigma^2}{n_s}$.

Thus, $h(x_s) = 1/n_s$, where $n_s$ is the number of students in school $s$.

To implement WLS, we divide each observation's data by $1/\sqrt{h(x_s)}$, meaning we need to multiply each school's data by $\sqrt{n_s}$.

The variable `enrollment` in the `test_df` dataset is our $n_s$.

[†] Assuming the individuals' disturbances are homoskedastic.

# Living with heteroskedasticity

## Example: WLS

Using WLS to estimate $\mathbf{Score}_i = \beta_0 + \beta_1 \mathbf{Ratio}_i + \beta_2 \mathbf{Income}_i + u_i$

**Step 1:** Multiply each variable by $1/\sqrt{h(x_i)} = \sqrt{\mathbf{Enrollment}_i}$

```
# Create WLS transformed variables, multiplying by sqrt of 'pop'
test_df ← mutate(test_df,
  test_score_wls = test_score * sqrt(enrollment),
  ratio_wls      = ratio * sqrt(enrollment),
  income_wls     = income * sqrt(enrollment),
  intercept_wls  = 1 * sqrt(enrollment)
)
```

Notice that we are creating a transformed intercept.

# Living with heteroskedasticity

## Example: WLS

Using WLS to estimate $\mathbf{Score}_i = \beta_0 + \beta_1 \mathbf{Ratio}_i + \beta_2 \mathbf{Income}_i + u_i$

**Step 2:** Run our WLS (transformed) regression

```
# WLS regression
wls_reg ← lm(
  test_score_wls ~ -1 + intercept_wls + ratio_wls + income_wls,
  data = test_df
)
```

# Living with heteroskedasticity

## Example: WLS

Using WLS to estimate $\mathbf{Score}_i = \beta_0 + \beta_1 \mathbf{Ratio}_i + \beta_2 \mathbf{Income}_i + u_i$

**Step 2:** Run our WLS (transformed) regression

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# WLS regression
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  test_score_wls ~ -1 + intercept_wls + ratio_wls + income_wls,
  data = test_df
)
```

*Note:* The `-1` in our regression tells R not to add an intercept, since we are adding a transformed intercept (`intercept_wls`).

# Living with heteroskedasticity

## Example: WLS

The **WLS estimates and standard errors:**

```
#>                     Estimate Std. Error t value Pr(>|t|)
#>  intercept_wls 618.78331     8.26929  74.829   <2e-16 ***
#>  ratio_wls      -0.21314     0.37676  -0.566    0.572
#>  income_wls      2.26493     0.09065  24.985   <2e-16 ***
```

# Living with heteroskedasticity

## Example: WLS

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```

The **OLS estimates** and **het.-robust standard errors:**

```
#>               Estimate Robust s.e t value Pr(>|t|)
#>  (Intercept) 638.7292     7.3012  87.482   <2e-16 ***
#>  ratio        -0.6487     0.3533  -1.836   0.0671 .
#>  income        1.8391     0.1147  16.029   <2e-16 ***
```

# Living with heteroskedasticity

## Example: WLS

Alternative to doing your own weighting: feed `lm()` some `weights`.

```
lm(test_score ~ ratio + income, data = test_df, weights = enrollment)
```

# Living with heteroskedasticity

In this example

- **Heteroskedasticity-robust standard errors** did not change our standard errors very much (relative to plain OLS standard errors).

- **WLS** changed our answers a bit—coefficients and standard errors.

# Living with heteroskedasticity

In this example

- **Heteroskedasticity-robust standard errors** did not change our standard errors very much (relative to plain OLS standard errors).

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These examples highlighted a few things:

1. Using the correct estimator for your standard errors really matters.[†]

2. Econometrics doesn't always offer an obviously *correct* route.

[†] Sit in on an economics seminar, and you will see what I mean.

# Living with heteroskedasticity

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- **Heteroskedasticity-robust standard errors** did not change our standard errors very much (relative to plain OLS standard errors).

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These examples highlighted a few things:

1. Using the correct estimator for your standard errors really matters.[†]

2. Econometrics doesn't always offer an obviously *correct* route.

To see #1, let's run a simulation.

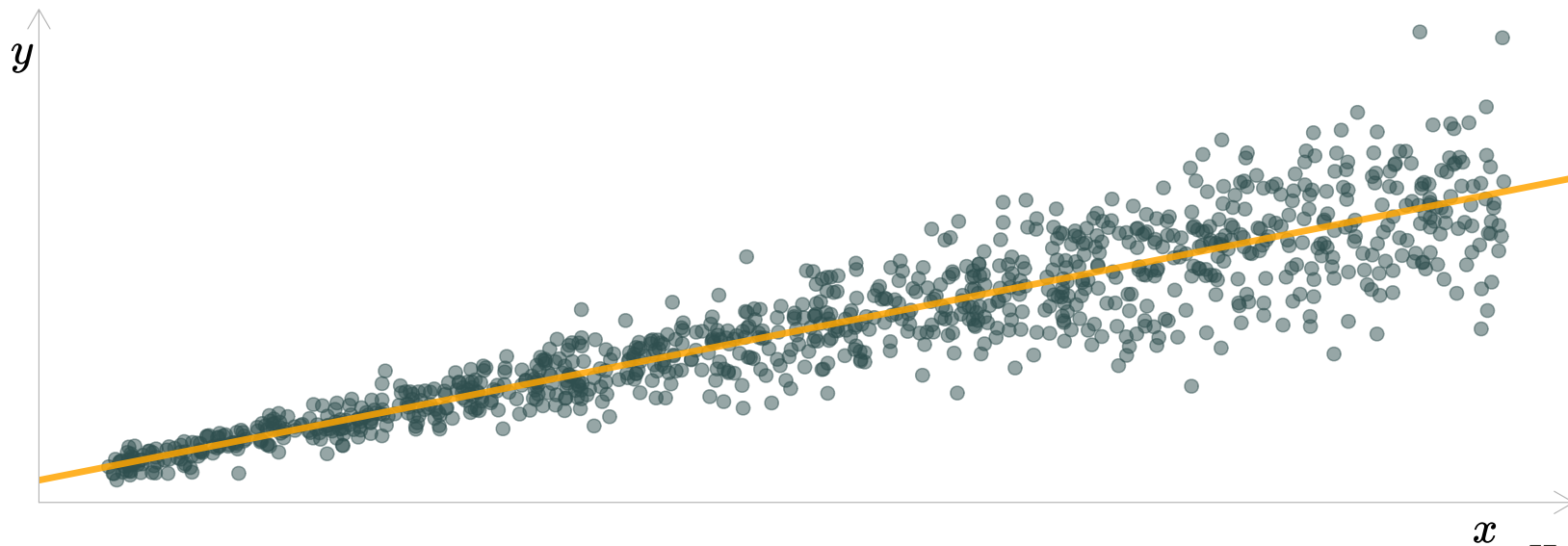[†] Sit in on an economics seminar, and you will see what I mean.

# Living with heteroskedasticity

## Simulation

Let's examine a simple linear regression model with heteroskedasticity.

$$y_i = \underbrace{\beta_0}_{=1} + \underbrace{\beta_1}_{=10} x_i + u_i$$

where $\mathrm{Var}(u_i|x_i) = \sigma_i^2 = \sigma^2 x_i^2$.
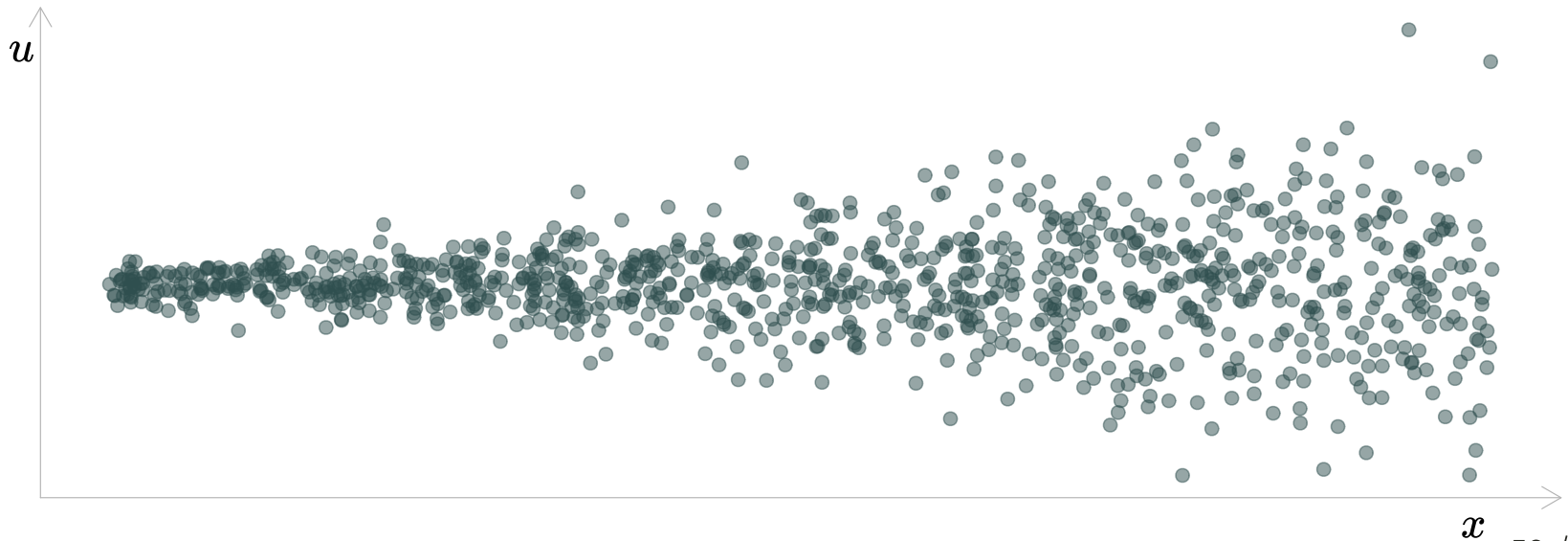
# Living with heteroskedasticity

## Simulation

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# Living with heteroskedasticity

## Simulation

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# Living with heteroskedasticity

## Simulation

*Note regarding WLS:*

Since $\mathbf{Var}(u_i|x_i) = \sigma^2 x_i^2$,

$$\mathbf{Var}(u_i|x_i) = \sigma^2 h(x_i) \implies h(x_i) = x_i^2$$

WLS multiplies each variable by $1/\sqrt{h(x_i)} = 1/x_i$.

## Simulation

In this simulation, we want to compare

1. The **efficiency** of

   - OLS
   - WLS with correct weights: $h(x_i) = x_i$
   - WLS with incorrect weights: $h(x_i) = \sqrt{x_i}$

2. How well our **standard errors** perform (via confidence intervals) with

   - Plain OLS standard errors
   - Heteroskedasticity-robust standard errors
   - WLS standard errors

# Living with heteroskedasticity

## Simulation

The simulation plan:

```
Do 10,000 times:

1. Generate a sample of size 30 from the population

2. Calculate/save OLS and WLS (×2) estimates for β₁

3. Calculate/save standard errors for β₁ using

      ○ Plain OLS standard errors
      ○ Heteroskedasticity-robust standard errors
      ○ WLS (correct)
      ○ WLS (incorrect)
```

# Living with heteroskedasticity

## Simulation

**For one iteration of the simulation:**

Code to generate the data...

```
# Parameters
b0 ← 1
b1 ← 10
s2 ← 1
# Sample size
n ← 30
# Generate data
sample_df ← tibble(
  x = runif(n, 0.5, 1.5),
  y = b0 + b1 * x + rnorm(n, 0, sd = s2 * x^2)
)
```

## Simulation

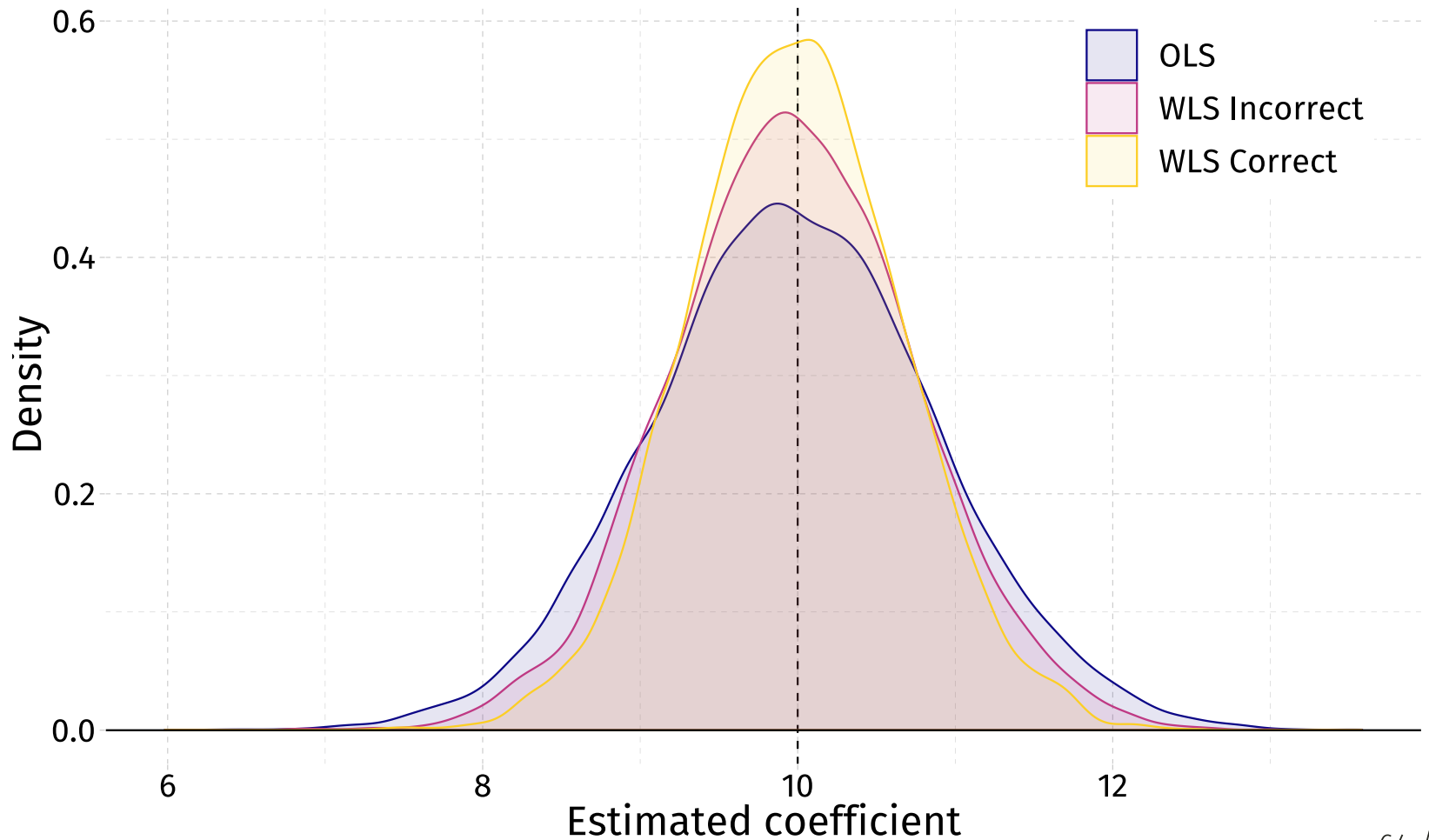**For one iteration of the simulation:**

Code to estimate our coefficients and standard errors...

```r
# OLS
ols ← felm(y ~ x, data = sample_df)
# WLS: Correct weights
wls_t ← lm(y ~ x, data = sample_df, weights = 1/x^2)
# WLS: Correct weights
wls_f ← lm(y ~ x, data = sample_df, weights = 1/x)
# Coefficients and standard errors
summary(ols, robust = F)
summary(ols, robust = T)
summary(wls_t)
summary(wls_f)
```
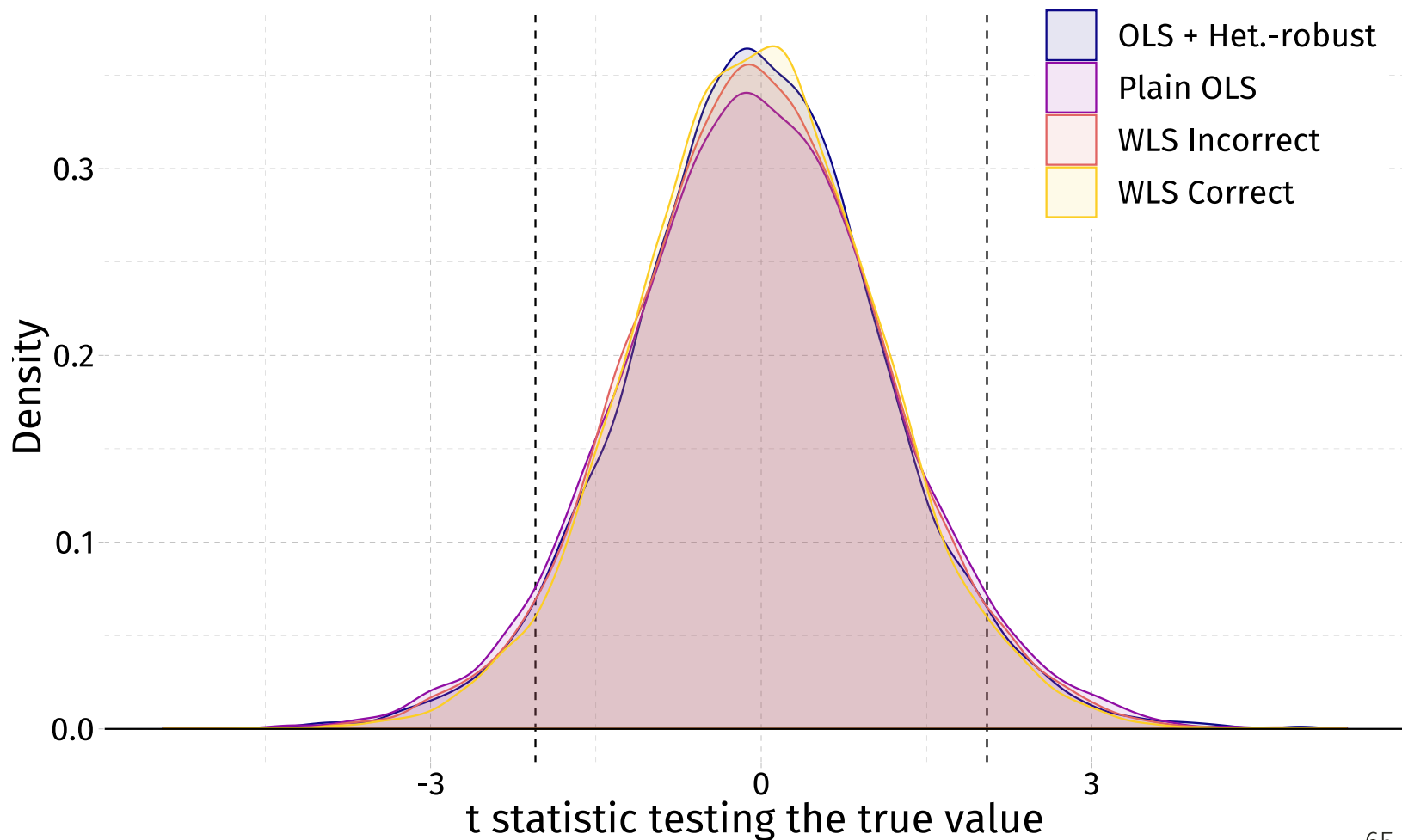
Then save the results.

## Simulation: Coefficients

## Simulation: Inference

# Living with heteroskedasticity

## Simulation: Results

Summarizing our simulation results (10,000 iterations)

**Estimation**: Summary of $\hat{\beta}_1$'s

| Estimator | Mean | S.D. |
|-----------|------|------|
| OLS | 9.975 | 0.904 |
| WLS Correct | 9.982 | 0.676 |
| WLS Incorrect | 9.978 | 0.772 |

# Living with heteroskedasticity

## Simulation: Results

Summarizing our simulation results (10,000 iterations)

**Estimation**: Summary of $\hat{\beta}_1$'s

| Estimator | Mean | S.D. |
|---|---|---|
| OLS | 9.975 | 0.904 |
| WLS Correct | 9.982 | 0.676 |
| WLS Incorrect | 9.978 | 0.772 |

**Inference:** % of times we reject $\beta_1$

| Estimators | % Reject |
|---|---|
| OLS + Het.-robust | 7.8 |
| OLS + Homosk. | 8.9 |
| WLS Correct | 6.6 |
| WLS Incorrect | 7.9 |