

# Time series

EC 421, Set 7

Edward Rubin

# Prologue

# Schedule

## Last Time

Asymptotics, probability limits, and consistency

## Today

Time series

# Asymptotics and consistency

## *Review*

# Asymptotics and consistency

## Review

1. Compare/contrast the concepts *expected value* and *probability limit*.
2. What does it mean if the estimator  $\hat{\theta}$  is consistent for  $\theta$ ?
3. What is required for an omitted variable to bias OLS estimates of  $\beta_j$ ?
4. Does omitted-variable bias affect the consistency of OLS for  $\beta_j$ ?
5. What can we know about the direction of omitted-variable bias?
6. How does measurement error in an explanatory variable affect the OLS estimate for that variable's effect on the outcome variable?
7. How does measurement error in an outcome variable affect OLS?

# Time-series data

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## Introduction

Up to this point, we focused on **cross-sectional data**.

- Sampled *across* a population (e.g., people, counties, countries).
- Sampled at *one moment* in time (e.g., Jan. 1, 2015).
- We had  $n$  individuals, each indexed  $i$  in  $\{1, \dots, n\}$ .

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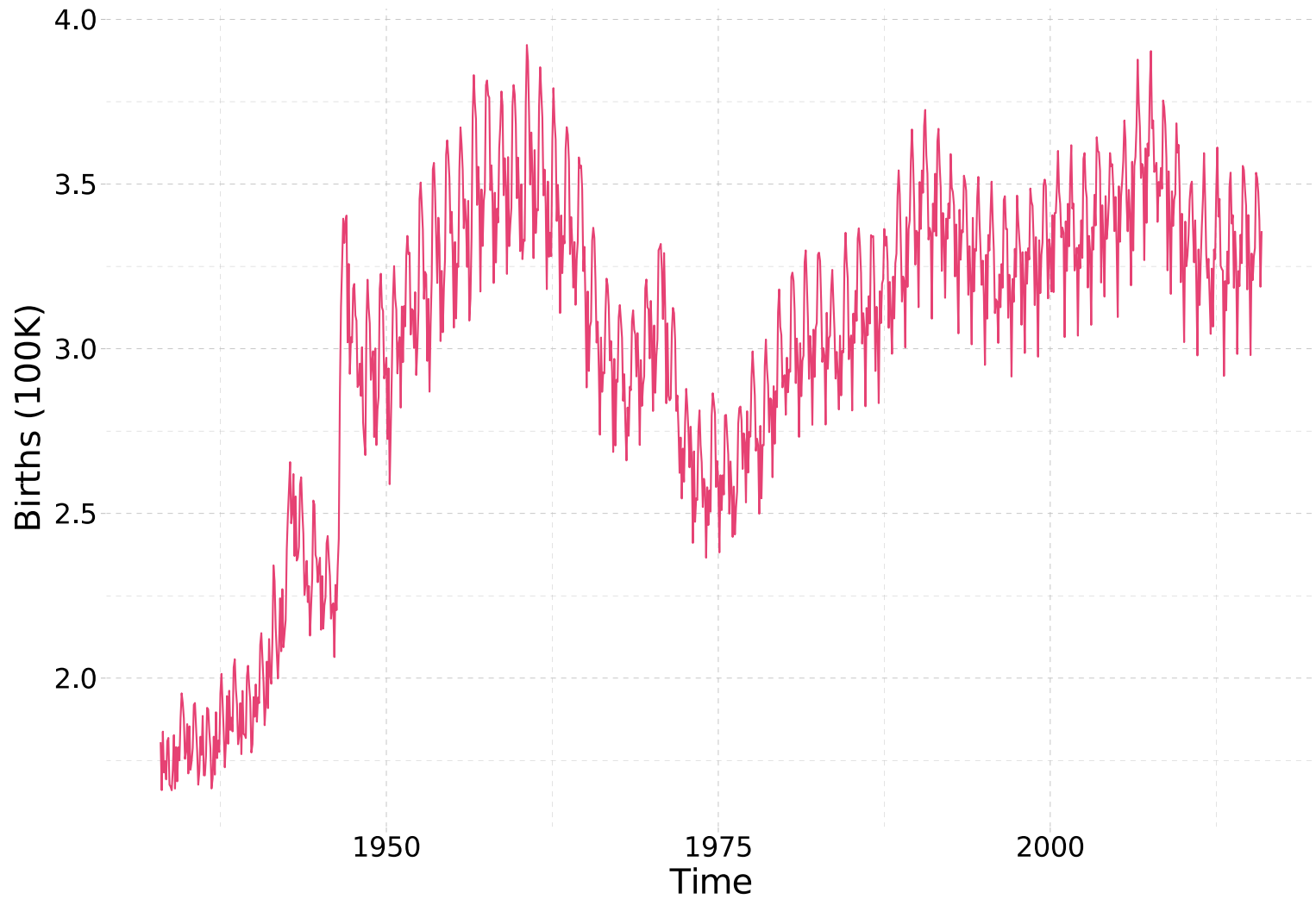
- Sampled *across* a population (e.g., people, counties, countries).
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Today, we focus on a different type of data: **time-series data**.

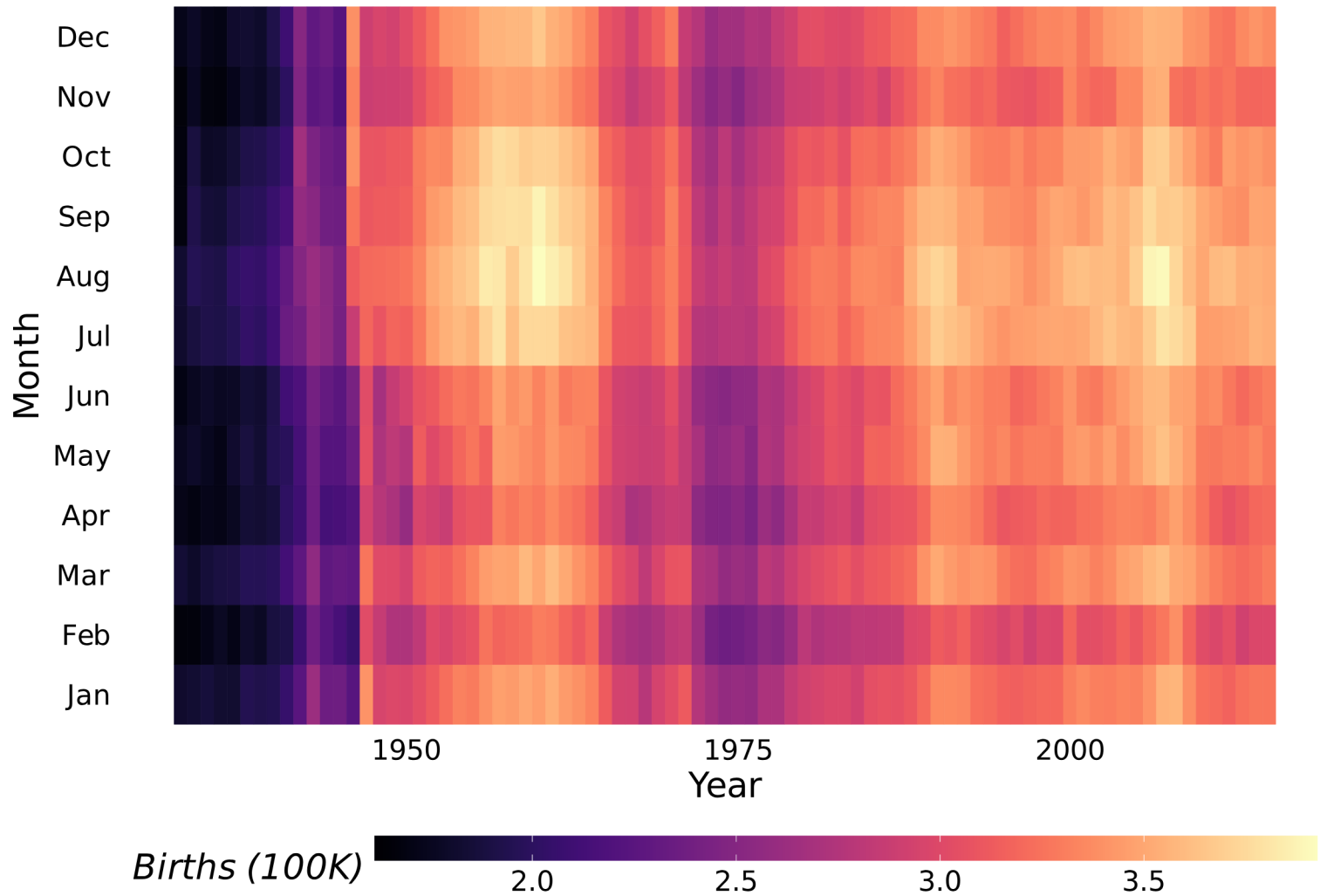
- Sampled within **one unit/individual** (e.g., Oregon).
- Observe **multiple times** for the same unit (e.g., Oregon: 1990–2020).
- We have  **$T$  time periods**, each indexed  $t$  in  $\{1, \dots, T\}$ .



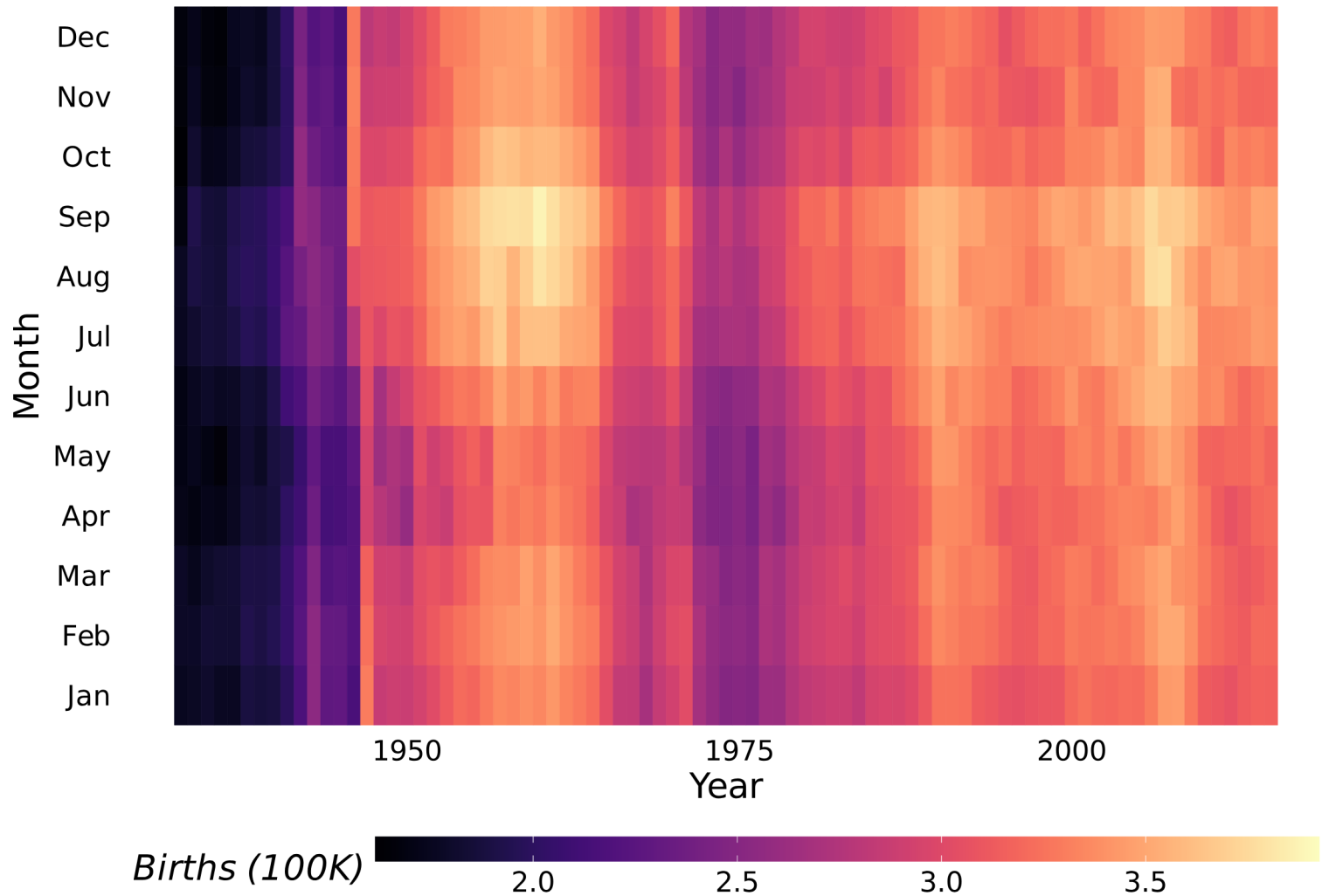
## US monthly births, 1933–2015: Classic time-series graph



## US monthly births, 1933–2015: Newfangled time-series graph



## US monthly births per 30 days, 1933–2015: Newfangled time-series graph



# Time-series models

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$$\text{Births}_t = \beta_0 + \beta_1 \text{Income}_t + u_t$$

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where  $t - 1$  denotes the time period prior to  $t$  (*lagged* income or births).



# Time-series models

## Assumptions

1. **New:** Weakly persistent outcomes—essentially,  $x_{t+k}$  in the distant period  $t + k$  is weakly correlated with period  $x_t$  (when  $k$  is "big").
2.  $y_t$  is a **linear function** of its parameters and disturbance.
3. There is **no perfect collinearity** in our data.
4. The  $u_t$  have conditional mean of zero (**exogeneity**),  $\mathbf{E}[u_t|X] = 0$ .
5. The  $u_t$  are **homoskedastic** with **zero correlation** between  $u_t$  and  $u_s$ , *i.e.*,  $\text{Var}(u_t|X) = \text{Var}(u_t) = \sigma^2$  and  $\text{Cor}(u_t, u_s|X) = 0$ .
6. **Normality of disturbances**, *i.e.*,  $u_t \stackrel{\text{iid}}{\sim} N(0, \sigma^2)$ .

# Time-series models

## Model options

Time-series modeling boils down to two classes of models.

1. **Static models:** Do not allow for persistent effect.
2. **Dynamic models:** Allow for persistent effects.

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2. **Dynamic models:** Allow for persistent effects.
  - Models with **lagged explanatory** variables
  - **Autoregressive, distributed-lag** (ADL) models

# Time-series models

## Model options

Option 1: **Static models**

**Static models** assume the outcome depends upon **only the current period**.

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Can be a very restrictive way to consider time-series data.

# Time-series models

## Model options

Option 2: **Dynamic models**

**Dynamic models** allow the outcome to depend upon other periods.



# Time-series models

## Model options

Option 2a: **Dynamic models** with lagged explanatory variables

These models allow the outcome to depend upon the explanatory variable(s) in other periods.

$$\text{Births}_t = \beta_0 + \beta_1 \text{Income}_t + \beta_2 \text{Income}_{t-1} + \beta_3 \text{Income}_{t-2} + \beta_4 \text{Income}_{t-3} + u_t$$

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*Note:* We still assume current births don't affect future births.

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### Option 2b: **Autoregressive distributed-lag (ADL) models**

These models allow the outcome to depend upon the explanatory variable(s) and/or the outcome variable in prior periods.

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Here, current income affects **current** births and **future** births.

In addition, **current births affect future births**—we're allowing lags of the outcome variable.



# Autoregressive distributed-lag models

## Numbers of lags

ADL models are often specified as  $\text{ADL}(p, q)$ , where

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Write out the model for period  $t - 1$ :

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which we can substitute in for  $\text{Births}_{t-1}$  in the first equation, *i.e.*,

$$\text{Births}_t = \beta_0 + \beta_1 \text{Income}_t + \underbrace{\beta_2(\beta_0 + \beta_1 \text{Income}_{t-1} + \beta_2 \text{Births}_{t-2} + u_{t-1})}_{\text{Births}_{t-1}} + u_t$$

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We could then substitute in the equation for  $\text{Births}_{t-2}$ ,  $\text{Births}_{t-3}$ , ...

# Autoregressive distributed-lag models

## Complexity

Eventually we arrive at

$$\begin{aligned}\text{Births}_t = & \beta_0 (1 + \beta_2 + \beta_2^2 + \beta_2^3 + \dots) + \\ & \beta_1 (\text{Income}_t + \beta_2 \text{Income}_{t-1} + \beta_2^2 \text{Income}_{t-2} + \dots) + \\ & u_t + \beta_2 u_{t-1} + \beta_2^2 u_{t-2} + \dots\end{aligned}$$

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The point?

By including just **one lag of the dependent variable**—as in a ADL(1, 0)—we implicitly include for *many lags* of the explanatory variables and disturbances.<sup>†</sup>

<sup>†</sup> These lags enter into the equation in a very specific way—not the most flexible specification.

# Autoregressive distributed-lag models

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*Partial-adjustment models* help us model this situation.

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## The partial-adjustment model

### *Example*

We want to know how the **desired number of cigarettes**,  $\widetilde{\text{Cig}}_t$ , changes with the current period's cigarette tax, e.g.,

$$\widetilde{\text{Cig}}_t = \beta_0 + \beta_1 \text{Tax}_t + u_t \quad (\text{A})$$

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Imagine **actual cigarette consumption**,  $\text{Cig}_t$ , doesn't change immediately (e.g., habit persistence). Instead, consumption depends upon **current desired level** and **previous consumption level**

$$\text{Cig}_t = \lambda \widetilde{\text{Cig}}_t + (1 - \lambda) \text{Cig}_{t-1} \quad (\text{B})$$

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Substituting  $\widetilde{\text{Cig}}_t$  from (A) into (B) yields

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The equation in (C) is ADL(1, 0).

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We can also estimate/recover the speed-of-adjustment coefficient  $\lambda$ .

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We need both of these parts to be true for OLS to be unbiased.

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## Unbiased coefficients

We need both parts of our exogeneity assumption for OLS to be unbiased:

$$\mathbf{E}[\hat{\beta}_1 | X] = \beta_1 + \mathbf{E}\left[\frac{\sum_t (x_t - \bar{x}) u_t}{\sum_t (x_t - \bar{x})^2} \middle| X\right]$$

*i.e.*, to guarantee the numerator equals zero, we need  $\mathbf{E}[u_t | X] = 0$ —for both  $\mathbf{E}[u_t | X_t] = 0$  and  $\mathbf{E}[u_t | X_s] = 0$  ( $s \neq t$ ).

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# OLS in time series

## Unbiased coefficients

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Thus, **OLS is biased for dynamic models with lagged outcome variables.**



# OLS in time series

## Unbiased coefficients

To see why dynamic models with lagged outcome variables violate our exogeneity assumption, consider two periods of our simple ADL(1, 0) model.

$$\text{Births}_t = \beta_0 + \beta_1 \text{Income}_t + \beta_2 \text{Births}_{t-1} + u_t \quad (1)$$

$$\text{Births}_{t+1} = \beta_0 + \beta_1 \text{Income}_{t+1} + \beta_2 \text{Births}_t + u_{t+1} \quad (2)$$

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This correlation violates the second part of our exogeneity requirement.

# OLS in time series

## Consistent coefficients

All is not lost.

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**Contemporaneous exogeneity:** each disturbance is uncorrelated with the explanatory variables **in the same period**, *i.e.*,

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With contemporaneous exogeneity, OLS estimates for the coefficients in a time series model are consistent.



# OLS in time series

## Consistent coefficients

To see why OLS is consistent with contemporaneous exogeneity, consider the OLS estimate for  $\beta_1$  in

$$\text{Births}_t = \beta_0 + \beta_1 \text{Births}_{t-1} + u_t$$

which we've shown (a few times) can be written

$$\hat{\beta}_1 = \beta_1 + \frac{\sum_t \left( \text{Births}_{t-1} - \overline{\text{Births}} \right) u_t}{\sum_t \left( \text{Births}_{t-1} - \overline{\text{Births}} \right)^2}$$

# OLS in time series

## Consistent coefficients

$$\begin{aligned}\text{plim } \hat{\beta}_1 &= \text{plim} \left( \beta_1 + \frac{\sum_t \left( \text{Births}_{t-1} - \overline{\text{Births}} \right) u_t}{\sum_t \left( \text{Births}_{t-1} - \overline{\text{Births}} \right)^2} \right) \\ &= \beta_1 + \frac{\text{plim} \left[ \sum_t \left( \text{Births}_{t-1} - \overline{\text{Births}} \right) u_t / T \right]}{\text{plim} \left[ \sum_t \left( \text{Births}_{t-1} - \overline{\text{Births}} \right)^2 / T \right]} \\ &= \beta_1 + \frac{\text{Cov}(\text{Births}_{t-1}, u_t)}{\text{Var}(\text{Births}_t)}\end{aligned}$$

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**Contemporaneous exogeneity** gives us  $\text{Cov}(\text{Births}_{t-1}, u_t) = 0$ .

# OLS in time series

## Consistent coefficients

Thus, if we assume **contemporaneous exogeneity**, OLS is consistent for the coefficients, *even for models with lagged dependent variables*.

The end.

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  - Partial-adjustment models
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7. *Extra*: ADL in equilibrium

# Autoregressive distributed-lag models

## Equilibrium effects

ADL models also offer interesting insights for long-run/equilibrium effects.

$$\text{Births}_t = \beta_0 + \beta_1 \text{Income}_t + \beta_2 \text{Births}_{t-1} + u_t$$

In this ADL(1, 0) model,  $\beta_1$  gives the **short-run effect** of income on the number of births.



# Autoregressive distributed-lag models

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In this ADL(1, 0) model,  $\beta_1$  gives the **short-run effect** of income on the number of births. *i.e.*, how income in time  $t$  affects births in time  $t$ .

# Autoregressive distributed-lag models

## Equilibrium effects

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Now rearrange...

$$\text{Births}^* - \beta_2 \text{Births}^* = \beta_0 + \beta_1 \text{Income}^*$$

$$(1 - \beta_2) \text{Births}^* = \beta_0 + \beta_1 \text{Income}^*$$

$$\text{Births}^* = \frac{\beta_0}{(1 - \beta_2)} + \frac{\beta_1}{(1 - \beta_2)} \text{Income}^*$$

# Autoregressive distributed-lag models

## Equilibrium effects

**Short-run** effect of income on births:

$$\text{Births}_t = \beta_0 + \beta_1 \text{Income}_t + \beta_2 \text{Births}_{t-1} + u_t$$

**Long-run** effect of income on births:

$$\text{Births}^* = \frac{\beta_0}{(1 - \beta_2)} + \frac{\beta_1}{(1 - \beta_2)} \text{Income}^*$$

# Autoregressive distributed-lag models

## Equilibrium effects

Another way to see this result:

We already showed

$$\text{Births}_t = \beta_0 + \beta_1 \text{Income}_t + \beta_2 \text{Births}_{t-1}$$

gives us

$$\begin{aligned} \text{Births}_t = & \beta_0 (1 + \beta_2 + \beta_2^2 + \beta_2^3 + \cdots) + \\ & \beta_1 (\text{Income}_t + \beta_2 \text{Income}_{t-1} + \beta_2^2 \text{Income}_{t-2} + \cdots) + \\ & u_t + \beta_2 u_{t-1} + \beta_2^2 u_{t-2} + \cdots \end{aligned}$$

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In equilibrium:  $\text{Income}_t = \text{Income}_{t-k} = \text{Income}^*$  for all  $k$ .



# Autoregressive distributed-lag models

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Substituting  $\text{Income}_t = \text{Income}^*$  for all  $k$   
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So long as  $-1 < \beta_2 < 1$ .<sup>†</sup>

<sup>†</sup> This simplification comes from  $\sum_{k=0}^{\infty} p^k = \frac{1}{1-p}$  for  $-1 < p < 1$ .