Heteroskedasticity

EC 421, Set 4

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17 January 2019

Prologue

R showcase

R Markdown

- Simple mark-up language for for combining/creating documents, equations, figures, R, and more
- Basics of Markdown
- *E.g.*, `**I'm bold**`, `*I'm italic*`, `I ← "code"`

Econometrics with R

- (Currently) free, online textbook
- Written and published using R (and probably R Markdown)
- *Warning:* I haven't read this book yet.

Related: Tyler Ransom has a great cheatsheet for econometrics.

Schedule

Last Time

We wrapped up our review.

Today

Heteroskedasticity

This week

First assignment! Due in a week.

Heteroskedasticity

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5. The disurbances have **constant variance** σ^2 and **zero covariance**, *i.e.*,
   - $E\left[u_i^2|X\right] = \text{Var}(u_i|X) = \sigma^2 \implies \text{Var}(u_i) = \sigma^2$
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6. The disturbances come from a **Normal** distribution, *i.e.*, $u_i \overset{\text{iid}}{\sim} \text{N}\left(0, \sigma^2\right)$.

Heteroskedasticity

Today we're focusing on assumption #5:

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Heteroskedasticity: $\text{Var}(u_i) = \sigma_i^2$ and $\sigma_i^2 \neq \sigma_j^2$ for some $i \neq j$.

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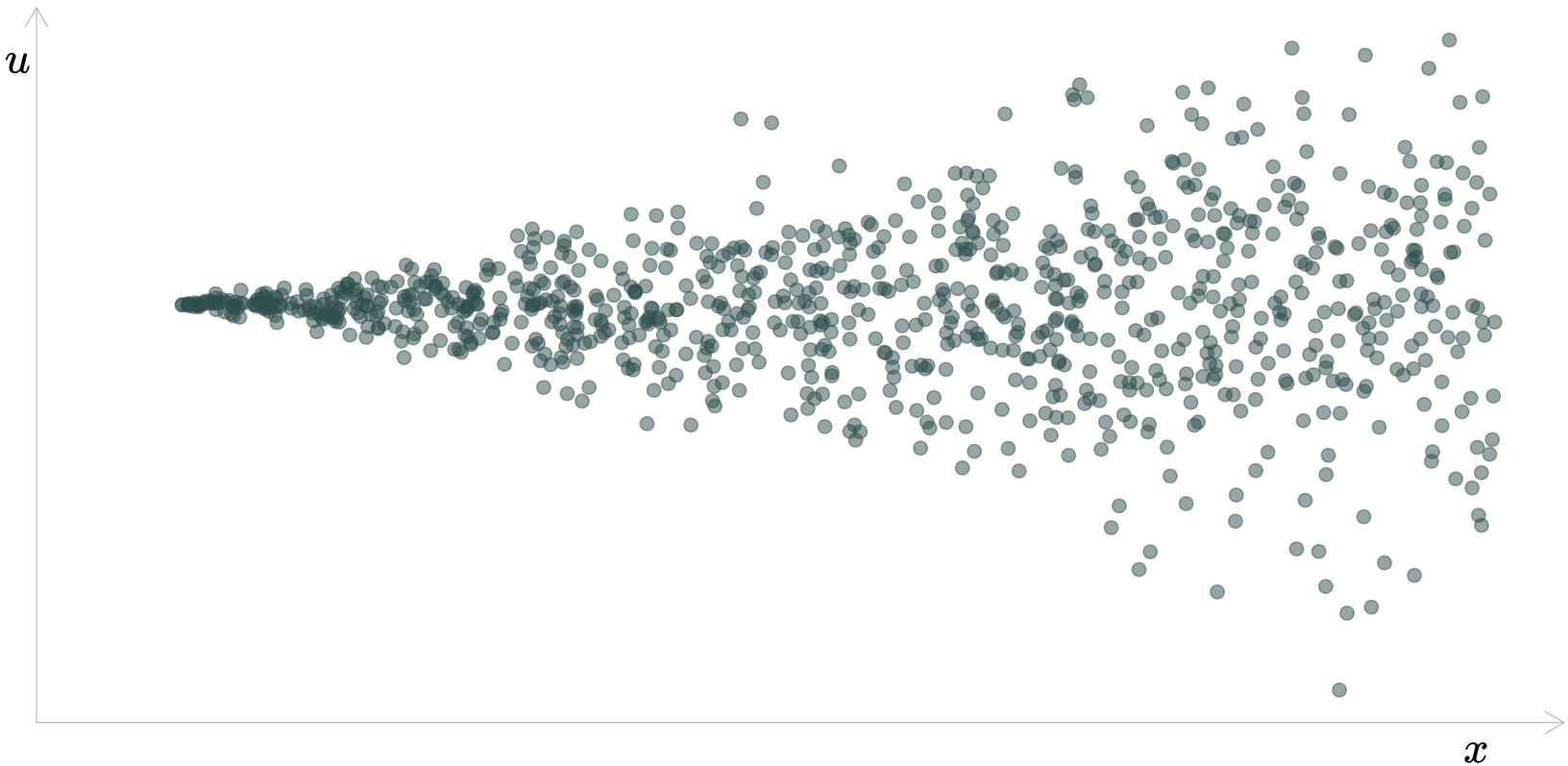
Heteroskedasticity: $\text{Var}(u_i) = \sigma_i^2$ and $\sigma_i^2 \neq \sigma_j^2$ for some $i \neq j$.

In other words: Our disturbances have different variances.

Heteroskedasticity

Classic example of heteroskedasticity: The funnel

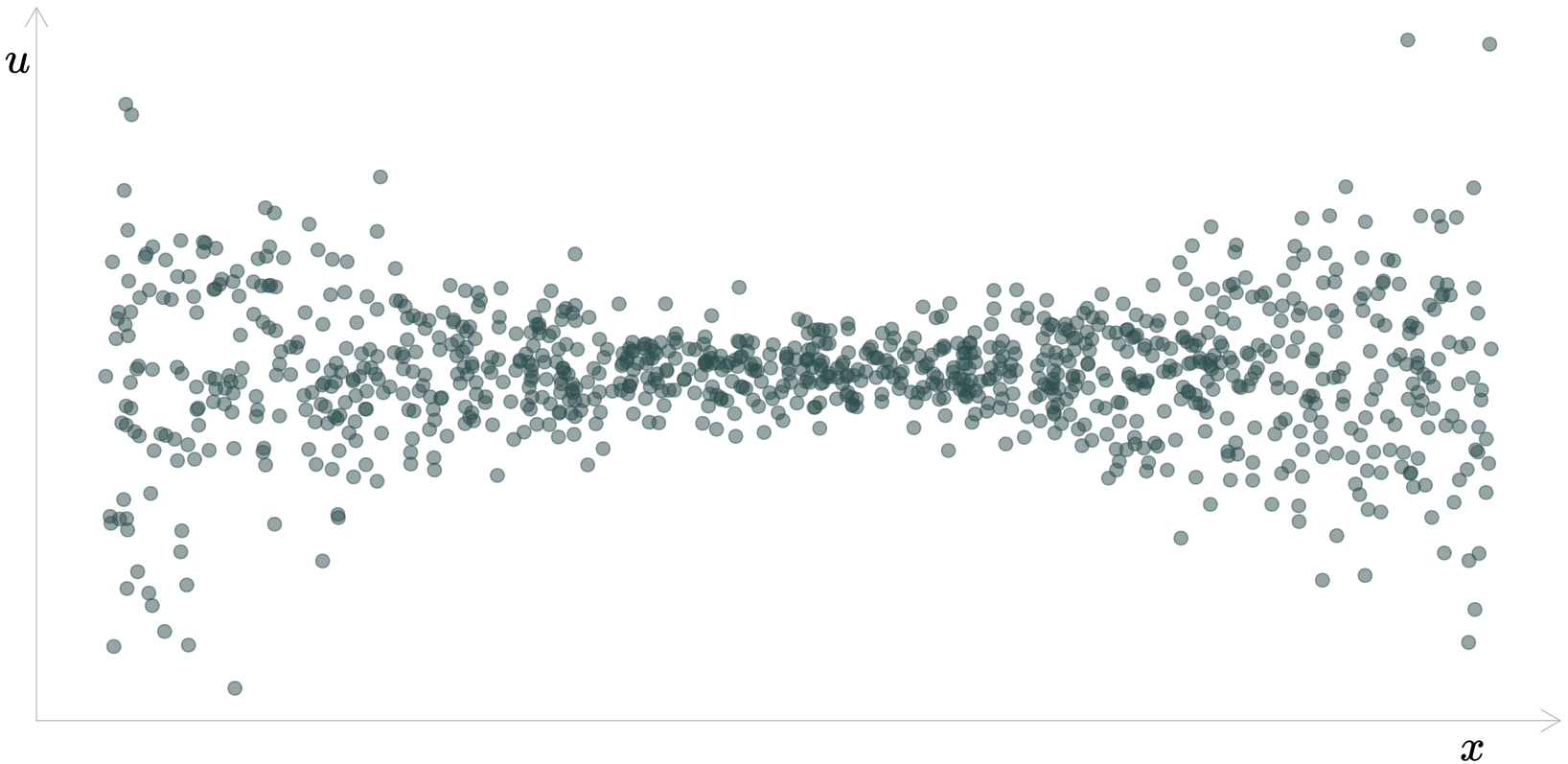
Variance of u increases with x



Heteroskedasticity

Another example of heteroskedasticity: (double funnel?)

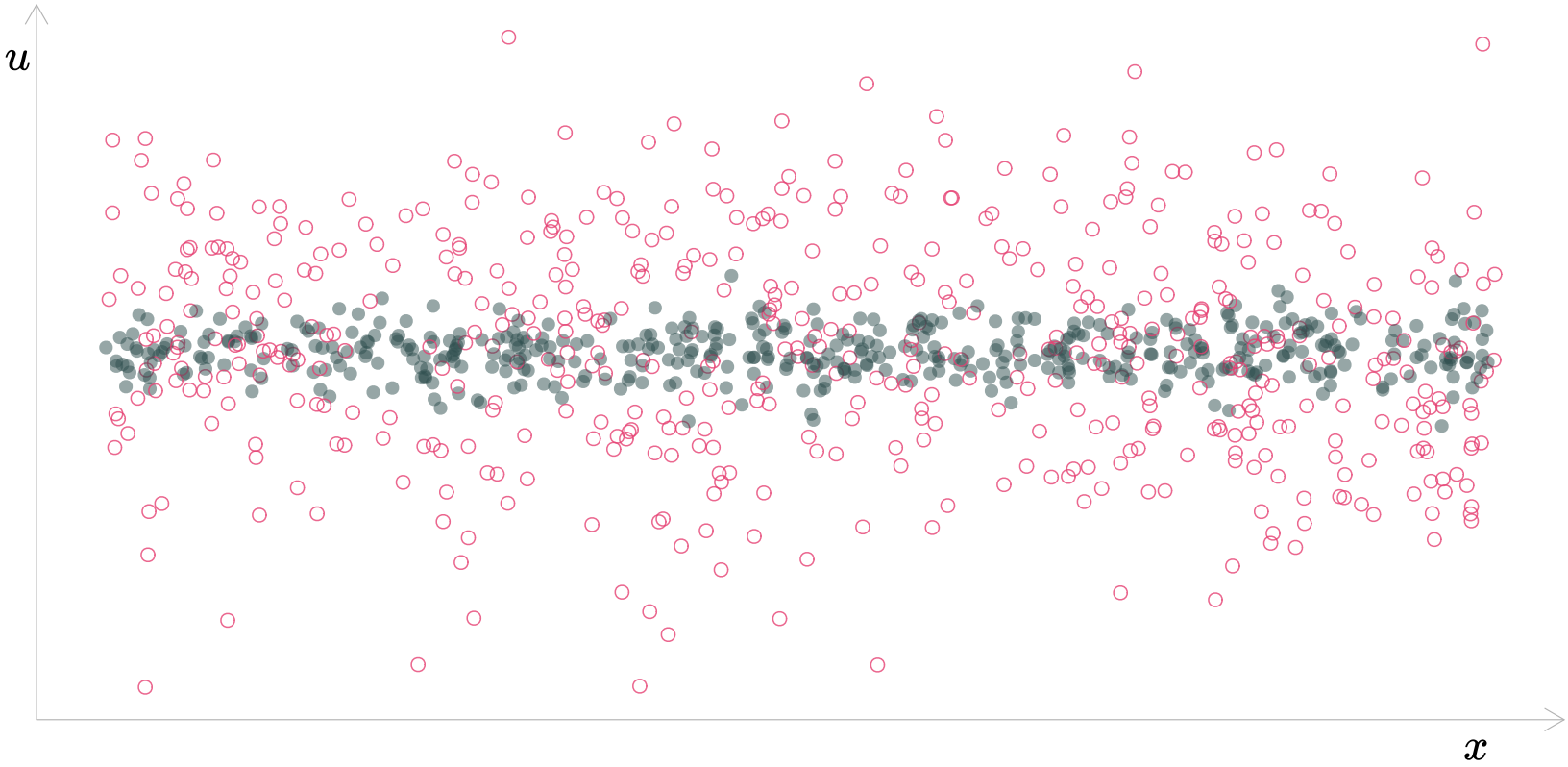
Variance of u increasing at the extremes of x



Heteroskedasticity

Another example of heteroskedasticity:

Differing variances of u by group



Heteroskedasticity

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(Very common in practice)

Heteroskedasticity

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So what are the consquences of heteroskedasticity? Bias? Inefficiency?

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Recall$_1$: OLS being unbiased means $E\left[\hat{\beta}_k \middle| X\right] = \beta_k$ for all k.

Recall$_2$: We previously showed $\hat{\beta}_1 = \dfrac{\sum_i \left(y_i - \overline{y}\right)\left(x_i - \overline{x}\right)}{\sum_i \left(x_i - \overline{x}\right)^2}$

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It will actually help us to rewrite this estimator as

$$\hat{\beta}_1 = \beta_1 + \frac{\sum_i \left(x_i - \overline{x}\right) u_i}{\sum_i \left(x_i - \overline{x}\right)^2}$$

Heteroskedasticity

Proof: Assuming $y_i = \beta_0 + \beta_1 x_i + u_i$

$$
\begin{aligned}
\hat{\beta}_1 &= \frac{\sum_i \left(y_i - \overline{y}\right)\left(x_i - \overline{x}\right)}{\sum_i \left(x_i - \overline{x}\right)^2} \\
&= \frac{\sum_i \left(\left[\beta_0 + \beta_1 x_i + u_i\right] - \left[\beta_0 + \beta_1 \overline{x} + \overline{u}\right]\right)\left(x_i - \overline{x}\right)}{\sum_i \left(x_i - \overline{x}\right)^2} \\
&= \frac{\sum_i \left(\beta_1\left[x_i - \overline{x}\right] + \left[u_i - \overline{u}\right]\right)\left(x_i - \overline{x}\right)}{\sum_i \left(x_i - \overline{x}\right)^2} \\
&= \frac{\sum_i \left(\beta_1\left[x_i - \overline{x}\right]^2 + \left[x_i - \overline{x}\right]\left[u_i - \overline{u}\right]\right)}{\sum_i \left(x_i - \overline{x}\right)^2} \\
&= \beta_1 + \frac{\sum_i \left(x_i - \overline{x}\right)\left(u_i - \overline{u}\right)}{\sum_i \left(x_i - \overline{x}\right)^2}
\end{aligned}
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Heteroskedasticity

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\hat{\beta}_1 = \cdots = \beta_1 + \frac{\sum_i \left(x_i - \overline{x}\right)\left(u_i - \overline{u}\right)}{\sum_i \left(x_i - \overline{x}\right)^2} \\
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We now want to see if heteroskedasticity biases the OLS estimator for β_1.

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E\left[\hat{\beta}_1 \middle| X\right] &= E\left[\beta_1 + \frac{\sum_i \left(x_i - \overline{x}\right) u_i}{\sum_i \left(x_i - \overline{x}\right)^2} \middle| X\right] \\
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Phew. **OLS is still unbiased** for the β_k.

Consequences: Efficiency

OLS's efficiency and inference do not survive heteroskedasticity.

- In the presence of heteroskedasticity, OLS is **no longer the most efficient** (best) linear unbiased estimator.

Heteroskedasticity

Consequences: Efficiency

OLS's efficiency and inference do not survive heteroskedasticity.

- In the presence of heteroskedasticity, OLS is **no longer the most efficient** (best) linear unbiased estimator.
- It would be more informative (efficient) to **weight observations** inversely to their u_i's variance.
  - Downweight high-variance u_i's (too noisy to learn much).
  - Upweight observations with low-variance u_i's (more 'trustworthy').
  - Now you have the idea of weighted least squares (WLS)

Consequences: Inference

OLS **standard errors are biased** in the presence of heteroskedasticity.

- Wrong confidence intervals
- Problems for hypothesis testing (both t and F tests)

Heteroskedasticity

Consequences: Inference

OLS **standard errors are biased** in the presence of heteroskedasticity.

- Wrong confidence intervals
- Problems for hypothesis testing (both t and F tests)
- It's hard to learn much without sound inference.

Heteroskedasticity

Solutions

1. **Tests** to determine whether heteroskedasticity is present.

2. **Remedies** for (1) efficiency and (2) inference

Testing for heteroskedasticity

Testing for heteroskedasticity

While we *might* have solutions for heteroskedasticity, the efficiency of our estimators depends upon whether or not heteroskedasticity is present.

1. The **Goldfeld-Quandt test**
2. The **Breush-Pagan test**
3. The **White test**

Testing for heteroskedasticity

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1. The **Goldfeld-Quandt test**
2. The **Breush-Pagan test**
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Each of these tests centers on the fact that we can **use the OLS residual e_i to estimate the population disturbance u_i**.

Testing for heteroskedasticity

The Goldfeld-Quandt test

Focuses on a specific type of heteroskedasticity: whether the variance of u_i differs **between two groups**.[†]

Remember how we used our residuals to estimate the σ^2?

$$s^2 = \frac{\text{SSE}}{n-1} = \frac{\sum_i e_i^2}{n-1}$$

We will use this same idea to determine whether there is evidence that our two groups differ in the variances of their disturbances, effectively comparing s_1^2 and s_2^2 from our two groups.

[†]: The G-Q test was one of the early tests of heteroskedasticity (1965).

Testing for heteroskedasticity

The Goldfeld-Quandt test

Operationally,

1. Order your the observations by x
2. Split the data into two groups of size n*
    - G_1: The first third
    - G_2: The last third
3. Run separate regressions of y on x for G_1 and G_2
4. Record SSE_1 and SSE_2
5. Calculate the G-Q test statistic

Testing for heteroskedasticity

The Goldfeld-Quandt test

The G-Q test statistic

$$F_{(n^\star - k,\, n^\star - k)} = \frac{\text{SSE}_2/(n^\star - k)}{\text{SSE}_1/(n^\star - k)} = \frac{\text{SSE}_2}{\text{SSE}_1}$$

follows an F distribution (under the null hypothesis) with $n^\star - k$ and $n^\star - k$ degrees of freedom.[†]

Testing for heteroskedasticity

The Goldfeld-Quandt test

The G-Q test statistic

$$F_{(n^\star - k,\, n^\star - k)} = \frac{\text{SSE}_2 / (n^\star - k)}{\text{SSE}_1 / (n^\star - k)} = \frac{\text{SSE}_2}{\text{SSE}_1}$$

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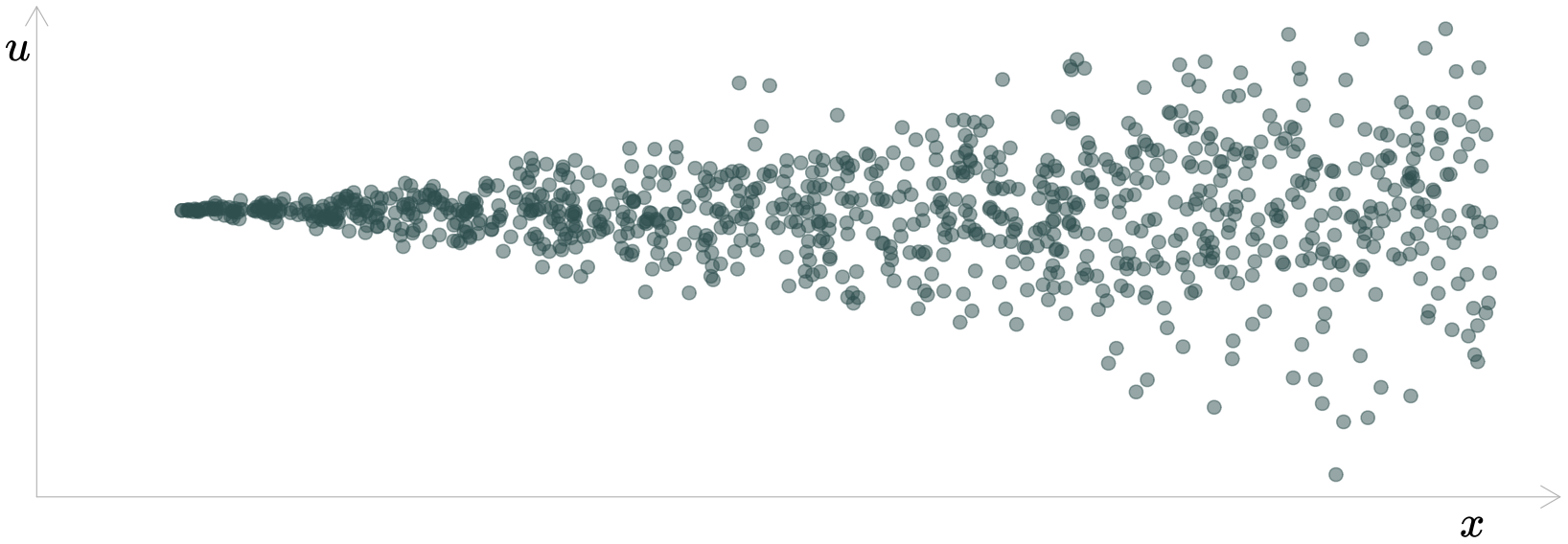
Notes

- The G-Q test requires the disturbances follow normal distributions.
- The G-Q assumes a very specific type/form of heteroskedasticity.
- Performs very well if we know the form of potentially heteroskedasticity.

[†]: Goldfelt and Quadt suggested $n^\star$ of $(3/8)n$.

Testing for heteroskedasticity

The Goldfeld-Quandt test



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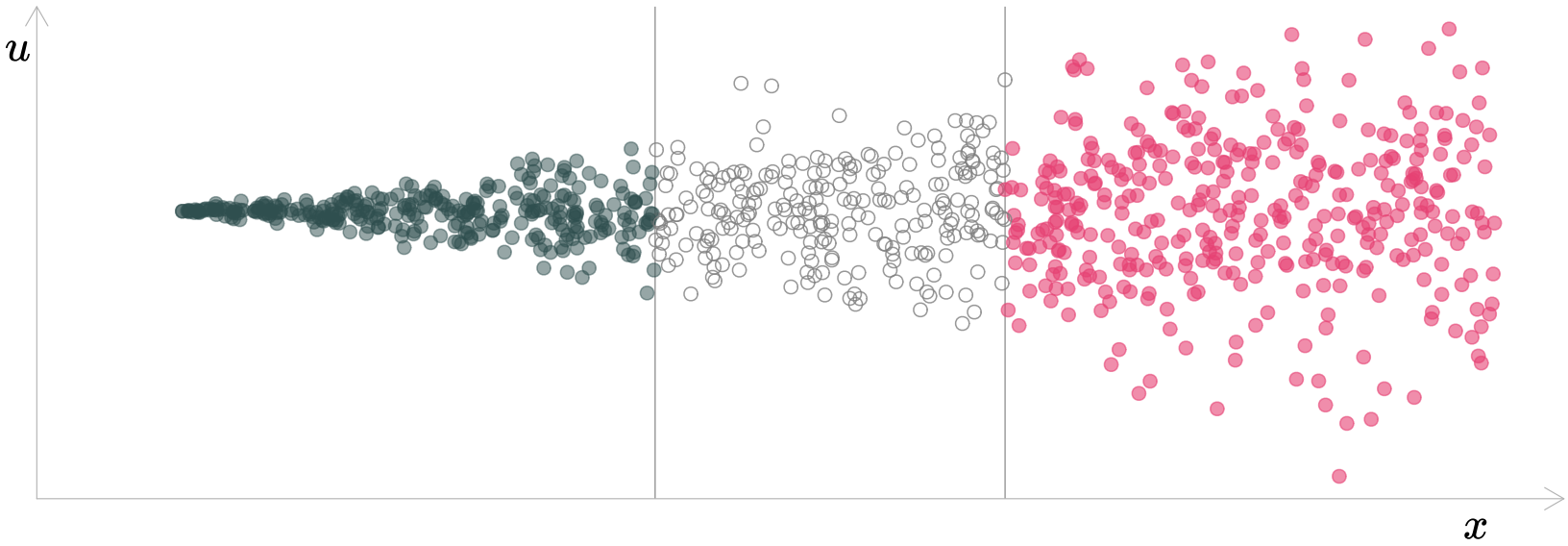


$$F_{375,\,375} = \frac{\text{SSE}_2 = 18,203.4}{\text{SSE}_1 = 1,039.5} \approx 17.5 \implies p\text{-value} < 0.001$$

$\therefore$ We reject H_0: $\sigma_1^2 = \sigma_2^2$ and conclude there is statistically significant evidence of heteroskedasticity.

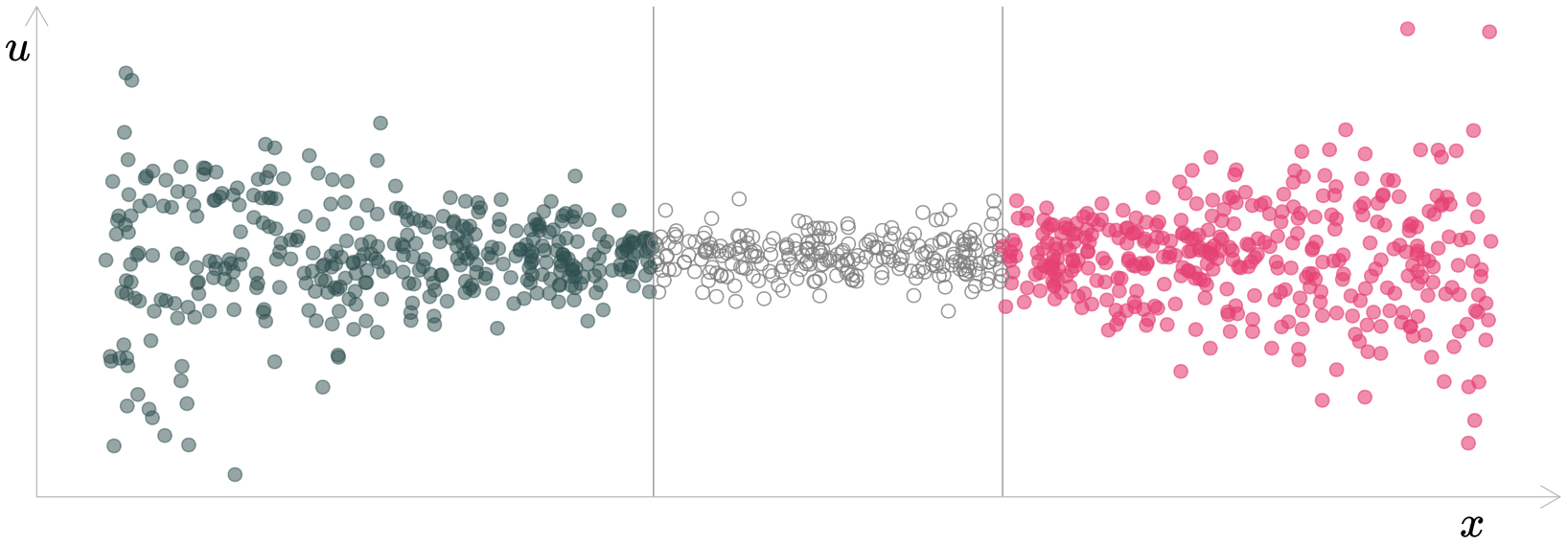
Testing for heteroskedasticity

The Goldfeld-Quandt test

The problem...

Testing for heteroskedasticity

The Goldfeld-Quandt test



$$F_{375,\,375} = \frac{\text{SSE}_2 = 14,516.8}{\text{SSE}_1 = 14,937.1} \approx 1 \implies p\text{-value} \approx 0.609$$

$\therefore$ We fail to reject H_0: $\sigma_1^2 = \sigma_2^2$ while heteroskedasticity is present.

Testing for heteroskedasticity

The Breush-Pagan test

Breusch and Pagan (1981) attempted to solve this issue of being too specific with the functional form of the heteroskedasticity.

- Allows the data to show if/how the variance of u_i correlates with X.
- If σ_i^2 correlates with X, then we have heteroskedasticity.
- Regresses e_i^2 on $X = [1, x_1, x_2, \ldots, x_k]$ and tests for joint significance.

Testing for heteroskedasticity

The Breush-Pagan test

How to implement:

1. Regress y on an intercept, x_1, x_2, …, x_k.
2. Record residuals e.
3. Regress e^2 on an intercept, x_1, x_2, …, x_k.

$$e_i^2 = \alpha_0 + \alpha_1 x_{1i} + \alpha_2 x_{2i} + \cdots + \alpha_k x_{ki} + v_i$$

4. Record R^2.
5. Test hypothesis H_0: $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$

Testing for heteroskedasticity

The Breush-Pagan test

The B-P test statistic[†] is

$$\mathrm{LM} = n \times R_e^2$$

where R_e^2 is the R^2 from the regression

$$e_i^2 = \alpha_0 + \alpha_1 x_{1i} + \alpha_2 x_{2i} + \cdots + \alpha_k x_{ki} + v_i$$

Under the null, LM is asymptotically distributed as χ_k^2.

Testing for heteroskedasticity

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Under the null, LM is asymptotically distributed as χ_k^2.

This test statistic tests H_0: $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$.

Rejecting the null hypothesis implies evidence of heteroskedasticity.

[†]: This specific form of the test statistic actually comes form Koenker (1981).

Testing for heteroskedasticity

The Breush-Pagan test

Problem: We're still assuming a fairly restrictive **functional form** between our explanatory variables X and the variances of our disturbances σ_i^2.

Testing for heteroskedasticity

The Breush-Pagan test

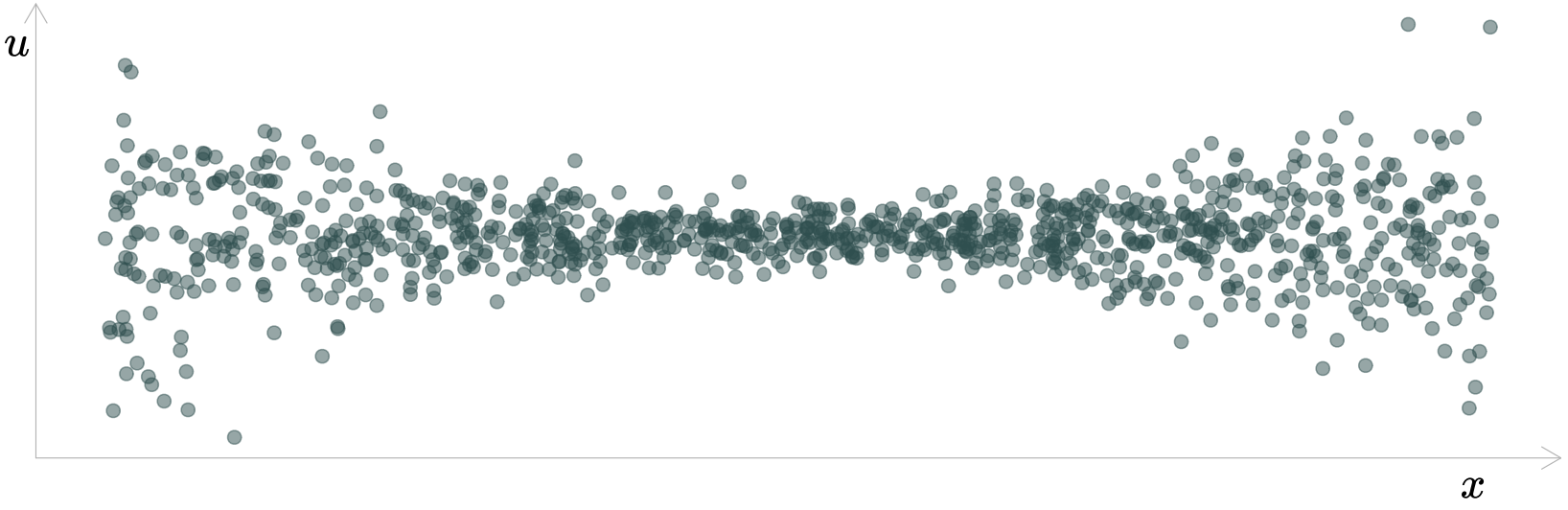
Problem: We're still assuming a fairly restrictive **functional form** between our explanatory variables X and the variances of our disturbances σ_i^2.

Result: B-P *may* still miss fairly simple forms of heteroskedasticity.

Testing for heteroskedasticity

The Breush-Pagan test

Breusch-Pagan tests are still **sensitive to functional form**.



$e_i^2 = \hat{\alpha}_0 + \hat{\alpha}_1 x_{1i}$	$\widehat{\text{LM}} = 1.26$	$\text{p-value} \approx 0.261$
$e_i^2 = \hat{\alpha}_0 + \hat{\alpha}_1 x_{1i} + \hat{\alpha}_2 x_{1i}^2$	$\widehat{\text{LM}} = 185.8$	$\text{p-value} < 0.001$

Testing for heteroskedasticity

The White test

So far we've been testing for specific relationships between our explanatory variables and the variances of the disturbances, *e.g.*,

- H_0: $\sigma_1^2 = \sigma_2^2$ for two groups based upon x_j (**G-Q**)
- H_0: $\alpha_1 = \cdots = \alpha_k = 0$ from $e_i^2 = \alpha_0 + \alpha_1 x_{1i} + \cdots + \alpha_k x_{ki} + v_i$ (**B-P**)

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Q: Can't we just test this hypothesis?

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However, we actually want to know if

$$\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_n^2$$

Q: Can't we just test this hypothesis? **A:** Sort of.

Testing for heteroskedasticity

The White test

Toward this goal, Hal White took advantage of the fact that we can **replace the homoskedasticity requirement with a weaker assumption**:

- **Old:** $\text{Var}(u_i|X) = \sigma^2$
- **New:** u^2 is *uncorrelated* with the explanatory variables (*i.e.*, x_j for all j), their squares (*i.e.*, x_j^2), and the first-degree interactions (*i.e.*, $x_j x_h$).

Testing for heteroskedasticity

The White test

Toward this goal, Hal White took advantage of the fact that we can **replace the homoskedasticity requirement with a weaker assumption**:

- **Old:** $\text{Var}(u_i|X) = \sigma^2$
- **New:** u^2 is *uncorrelated* with the explanatory variables (*i.e.*, x_j for all j), their squares (*i.e.*, x_j^2), and the first-degree interactions (*i.e.*, $x_j x_h$).

This new assumption is easier to explicitly test (*hint:* regression).

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An outline of White's test for heteroskedasticity:

1. Regress y on x_1, x_2, …, x_k. Save residuals e.

2. Regress squared residuals on all explanatory variables, their squares, and interactions.

$$e^2 = \alpha_0 + \sum_{h=1}^{k} \alpha_h x_h + \sum_{j=1}^{k} \alpha_j x_j^2 + \sum_{\ell=1}^{k-1} \sum_{m=\ell+1}^{k} x_\ell x_m + v_i$$

3. Record R_e^2.

4. Calculate test statistic to test H_0: $\alpha_p = 0$ for all $p \neq 0$.

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Just as with the Bruesch-Pagan test, White's test statistic is

$$\mathrm{LM} = n \times R_e^2 \qquad \text{Under } \mathrm{H}_0,\ \mathrm{LM} \overset{\mathrm{d}}{\sim} \chi_k^2$$

but now the R_e^2 comes from the regression of e^2 on the explanatory variables, their squares, and their interactions.

$$e^2 = \alpha_0 + \underbrace{\sum_{h=1}^{k} \alpha_h x_h}_{\text{Expl. variables}} + \underbrace{\sum_{j=1}^{k} \alpha_j x_j^2}_{\text{Squared terms}} + \underbrace{\sum_{\ell=1}^{k-1} \sum_{m=\ell+1}^{k} x_\ell x_m}_{\text{Interactions}} + v_i$$

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Note: If a variable is equal to its square (*e.g.*, binary variables), then you don't (can't) include it. THe same rule applies for interactions.

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Example: Consider the model[†] $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + u$

Step 1: Estimate the model; obtain residuals (e).

Step 2: Regress e^2 on explanatory variables, squares, and interactions.

$$\begin{aligned} e^2 =& \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_1^2 + \alpha_5 x_2^2 + \alpha_6 x_3^2 \\ &+ \alpha_7 x_1 x_2 + \alpha_8 x_1 x_3 + \alpha_9 x_2 x_3 + v \end{aligned}$$

Record the R^2 from this equation (call it R_e^2).

Step 3: Test H_0: $\alpha_1 = \alpha_2 = \cdots = \alpha_9 = 0$ using $\text{LM} = nR_e^2 \overset{\text{d}}{\sim} \chi_3^2$.

[†]: To simplify notation here, I'm dropping the i subscripts.

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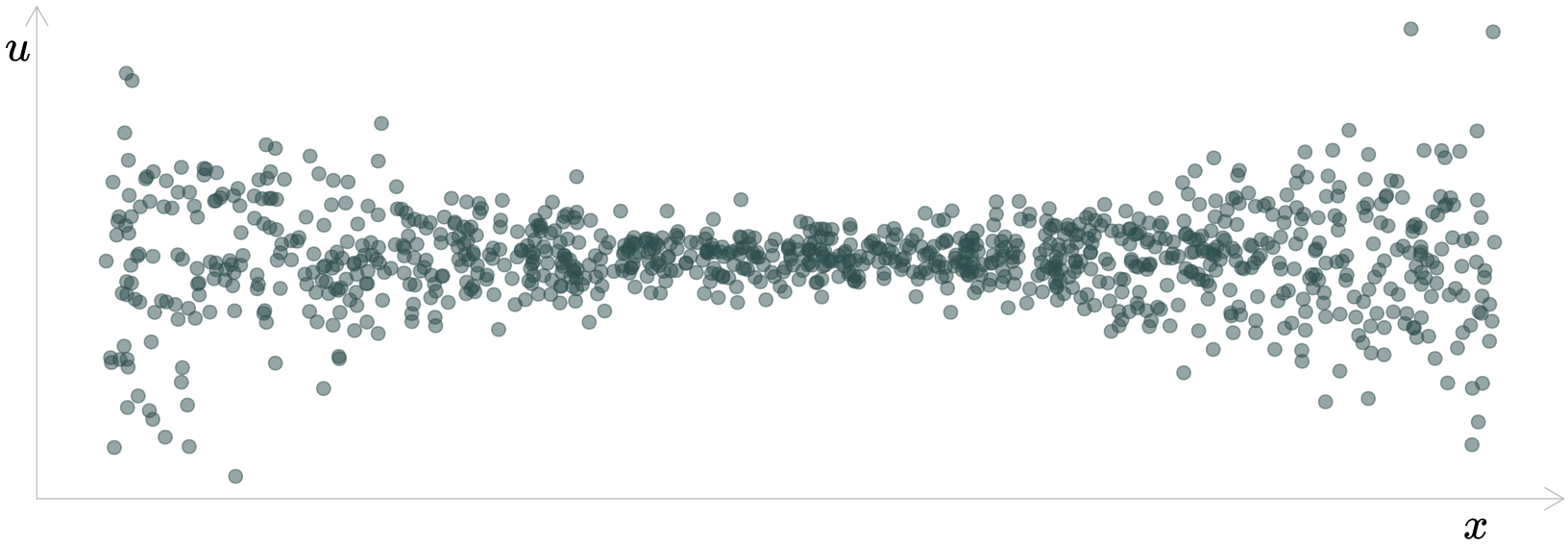
The White test



We've already done the White test for this simple linear regression.

$$e_i^2 = \hat{\alpha}_0 + \hat{\alpha}_1 x_{1i} + \hat{\alpha}_2 x_{1i}^2 \qquad \widehat{\text{LM}} = 185.8 \qquad \text{p-value} < 0.001$$

Testing for heteroskedasticity

Okay. We have some tests that may help us detect heteroskedasticity.

What do we do if we detect it?

Living with heteroskedasticity

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- still **unbiased**
- **no longer the most efficient** unbiased linear estimator

On average, we get the right answer but with more noise (less precision).

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Options:

1. Check regression **specification**.
2. Find a new, more efficient unbiased **estimator**.
3. Live with OLS's inefficiency and make **corrections for inference**.
    - Standard errors
    - Confidence intervals
    - Hypothesis tests

Living with heteroskedasticity

Misspecification

As we've discussed, the specification[†] of your regression model matters a lot for the unbiasedness and efficiency of your estimator.

Response #1: Ensure your function form doesn't cause heteroskedasticity.

[†]: *Specification:* Functional form and included variables.

Living with heteroskedasticity

Misspecification

Example: If the population relationship is

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + u$$

But you omit x^2 and estimate

$$y = \gamma_0 + \gamma_1 x + w$$

Then

$$w = u + \beta_2 x^2 \qquad \implies \qquad \mathbf{Var}(w) = f(x)$$

The variance of w changes systematically with x (heteroskedasticity).

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More generally:

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New problems:

- We often don't know the *right* specification.
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New problems:

- We often don't know the *right* specification.
- We'd like a more formal process for addressing heteroskedasticity.

Conclusion: Adjusting the specification often doesn't solve the problem.

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Weighted least squares

Weighted least squares (WLS) presents another approach.

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Let the true population relationship be

$$y_i = \beta_0 + \beta_1 x_{1i} + u_i \tag{1}$$

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Now transform (1) by dividing each observation's data by σ_i, *i.e.*,

$$\frac{y_i}{\sigma_i} = \beta_0 \frac{1}{\sigma_i} + \beta_1 \frac{x_{1i}}{\sigma_i} + \frac{u_i}{\sigma_i} \tag{2}$$

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$$\text{Var}\left(\frac{u_i}{\sigma_i}\right) = \frac{1}{\sigma_i^2}\text{Var}(u_i) = \frac{1}{\sigma_i^2}\sigma_i^2 = 1 \quad \forall i$$

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Notes:

1. **Inverse-variance weighting:** WLS downweights observations with higher variance in their errors.

2. **Big requirement:** WLS requires that we *know* σ_i^2 for each observation.