Time series

EC 421, Set 7

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Prologue

Schedule

Last Time

Asymptotics, probability limits, and consistency

Today

Time series

About our class

1. EC 421 is a **hard class**.

2. EC 421 requires **more math/theory** than most other classes.

3. This **theory is important**—why/when you can trust OLS/regression.

4. With all of this theory, we get **fewer traditional examples**. Proofs and simulations *are* our examples.

Asymptotics and consistency

Review

Asymptotics and consistency

Review

1. Compare/contrast the concepts *expected value* and *probability limit*.

2. What does it mean if the estimator $\hat{\theta}$ is consistent for θ?

3. What is required for an omitted variable to bias OLS estimates of β_j?

4. Does omitted-variable bias affect the consistency of OLS for β_j?

5. What can we know about the direction of omitted-variable bias?

6. How does measurement error in an explanatory variable affect the OLS estimate for that variable's effect on the outcome variable?

7. How does measurement error in an outcome variable affect OLS?

Time-series data

Time-series data

Introduction

Up to this point, we focused on **cross-sectional data**.

- Sampled *across* a population (*e.g.*, people, counties, countries).
- Sampled at *one moment* in time (*e.g.*, Jan. 1, 2015).
- We had n *individuals*, each indexed i in $\{1, \ldots, n\}$.

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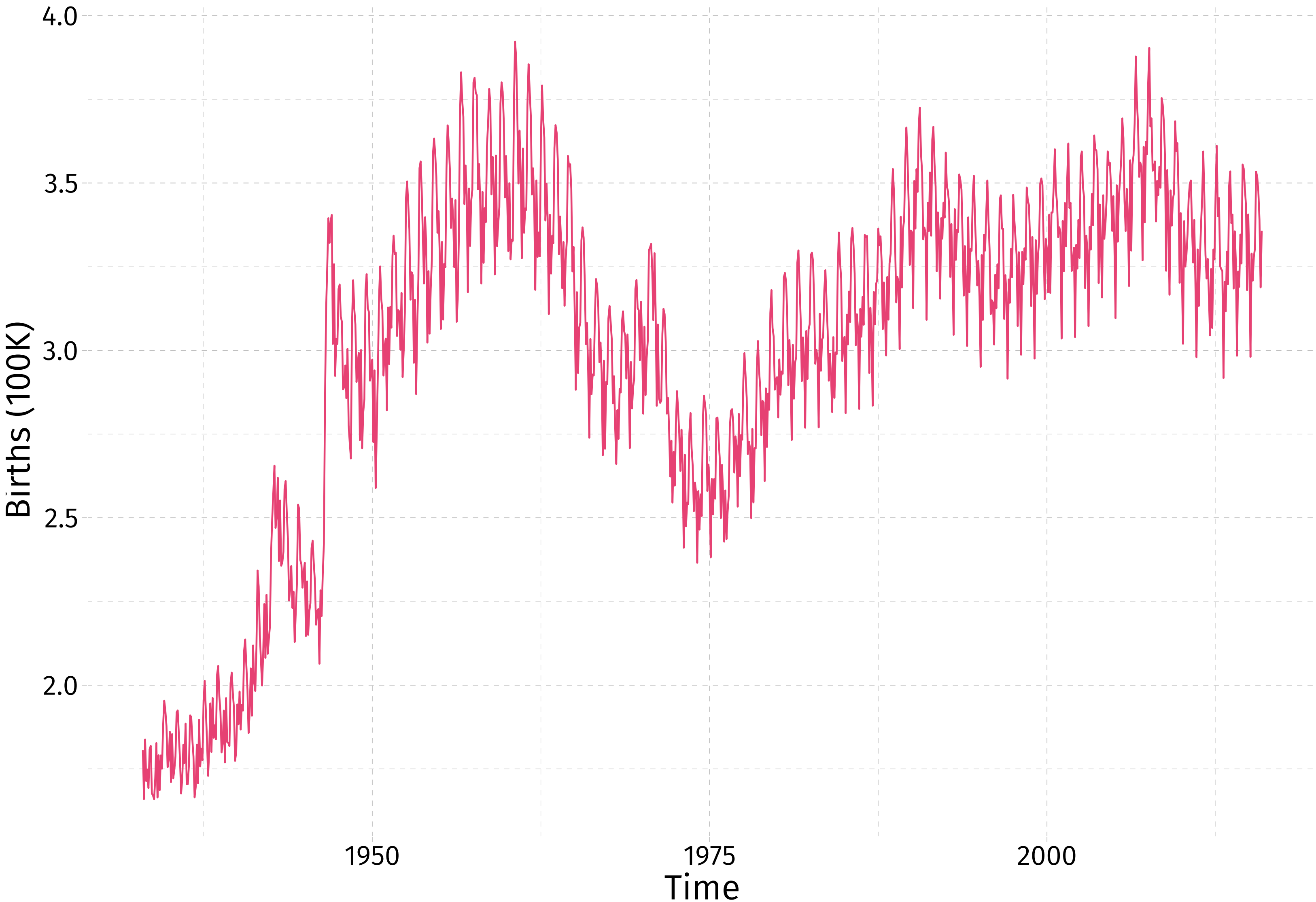
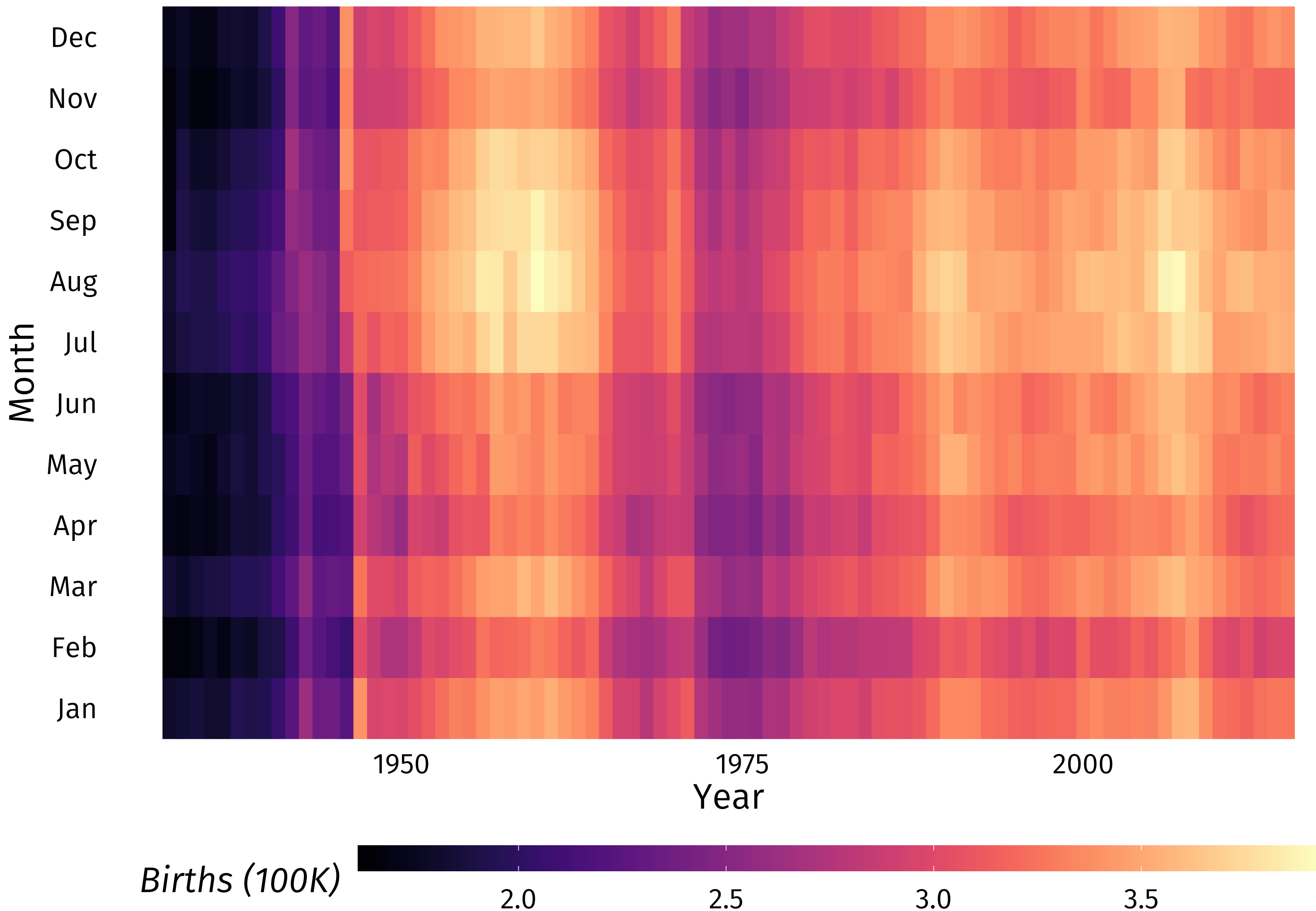
Today, we focus on a different type of data: **time-series data**.

- Sampled within one unit/individual (*e.g.*, Oregon).
- Observe multiple times for the same unit (*e.g.*, Oregon: 1990–2020).
- We have T time periods, each indexed t in $\{1, \ldots, T\}$.

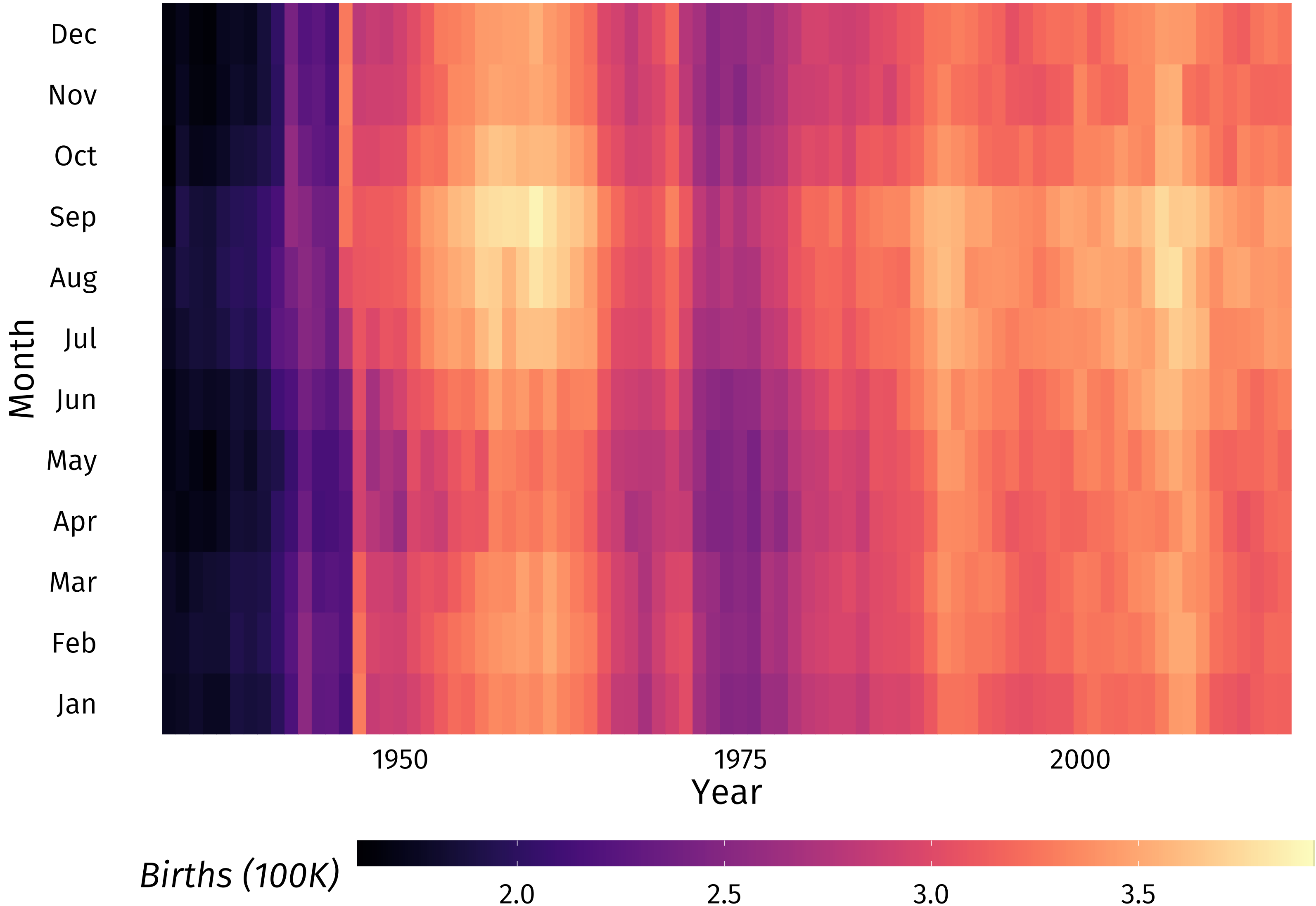
US monthly births, 1933–2015: Classic time-series graph



US monthly births, 1933–2015: Newfangled time-series graph



US monthly births per 30 days, 1933–2015: Newfangled time-series graph



Time-series models

Introduction

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where $t-1$ denotes the time period prior to t (*lagged* income or births).

Time-series models

Assumptions

1. **New: Weakly persistent outcomes**—essentially, x_{t+k} in the distant period $t+k$ is weakly correlated with period x_t (when k is "big").

2. y_t is a **linear function** of its parameters and disturbance.

3. There is **no perfect collinearity** in our data.

4. The u_t have conditional mean of zero (**exogeneity**), $\boldsymbol{E}[u_t|X] = 0$.

5. The u_t are **homoskedastic** with **zero correlation** between u_t and u_s, *i.e.*, $\mathrm{Var}(u_t|X) = \mathrm{Var}(u_t) = \sigma^2$ and $\mathrm{Cor}(u_t,\, u_s|X) = 0$.

6. **Normality of disturbances**, *i.e.*, $u_t \overset{\text{iid}}{\sim} N\left(0,\, \sigma^2\right)$.

Time-series models

Model options

Time-series modeling boils down to two classes of models.

1. **Static models:** Do not allow for persistent effect.

2. **Dynamic models:** Allow for persistent effects.

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    - Models with **lagged explanatory** variables

    - **Autoregressive, distributed-lag** (ADL) models

Model options

Option 1: Static models

Static models assume the outcome depends upon only the current period.

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Can be a very restrictive way to consider time-series data.

Time-series models

Model options

Option 2: Dynamic models

Dynamic models allow the outcome to depend upon other periods.

Model options

Option 2a: Dynamic models with lagged explanatory variables

These models allow the outcome to depend upon the explanatory variable(s) in other periods.

$$\begin{aligned}\text{Births}_t =& \beta_0 + \beta_1 \text{Income}_t + \beta_2 \text{Income}_{t-1} + \\ & \beta_3 \text{Income}_{t-2} + \beta_4 \text{Income}_{t-3} + u_t\end{aligned}$$

Time-series models

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Estimate *total* effects by summing lags' coefficients, *e.g.*, $\beta_1 + \beta_2 + \beta_3 + \beta_4$.

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Note: We still assume current births don't affect future births.

Time-series models

Model options

Option 2b: Autoregressive distributed-lag (ADL) models

These models allow the outcome to depend upon the explanatory variable(s) and/or the outcome variable in prior periods.

$$\text{Births}_t = \beta_0 + \beta_1 \text{Income}_t + \beta_2 \text{Income}_{t-1} + \beta_3 \text{Births}_{t-1} + u_t$$

Time-series models

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Here, current income affects affects **current** births and **future** births.

In addition, current births affect future births—we're allowing lags of the outcome variable.

Numbers of lags

ADL models are often specified as $\mathrm{ADL}(p, q)$, where

- p is the (maximum) number of **lags** for the outcome variable.
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Autoregressive distributed-lag models

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$$\begin{aligned}\text{Births}_t =& \beta_0 + \beta_1 \text{Income}_t + \beta_2 \text{Income}_{t-1} + \beta_3 \text{Income}_{t-2} \\ &+ \beta_4 \text{Births}_{t-1} + \beta_5 \text{Births}_{t-2} + u_t\end{aligned}$$

Autoregressive distributed-lag models

Complexity

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Consider ADL(1, 0): $\text{Births}_t = \beta_0 + \beta_1 \text{Income}_t + \beta_2 \text{Births}_{t-1} + u_t$

Write out the model for period $t-1$:

$$\text{Births}_{t-1} = \beta_0 + \beta_1 \text{Income}_{t-1} + \beta_2 \text{Births}_{t-2} + u_{t-1}$$

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which we can substitute in for Births_{t-1} in the first equation, *i.e.*,

$$\begin{aligned} \text{Births}_t =& \beta_0 + \beta_1 \text{Income}_t + \\ & \beta_2 \underbrace{\left(\beta_0 + \beta_1 \text{Income}_{t-1} + \beta_2 \text{Births}_{t-2} + u_{t-1} \right)}_{\text{Births}_{t-1}} + u_t \end{aligned}$$

Autoregressive distributed-lag models

Complexity

Continuing...

$$
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& \beta_2 \underbrace{\left(\beta_0 + \beta_1 \text{Income}_{t-1} + \beta_2 \text{Births}_{t-2} + u_{t-1}\right)}_{\text{Births}_{t-1}} + u_t \\
=& \beta_0 \left(1 + \beta_2\right) + \beta_1 \text{Income}_t + \beta_1 \beta_2 \text{Income}_{t-1} + \\
& \beta_2^2 \text{Births}_{t-2} + u_t + \beta_2 u_{t-1}
\end{aligned}
$$

Autoregressive distributed-lag models

Complexity

Continuing...

$$\begin{aligned}\text{Births}_t =& \beta_0 + \beta_1 \text{Income}_t + \\ & \beta_2 \underbrace{(\beta_0 + \beta_1 \text{Income}_{t-1} + \beta_2 \text{Births}_{t-2} + u_{t-1})}_{\text{Births}_{t-1}} + u_t \\ =& \beta_0 \left(1 + \beta_2\right) + \beta_1 \text{Income}_t + \beta_1 \beta_2 \text{Income}_{t-1} + \\ & \beta_2^2 \text{Births}_{t-2} + u_t + \beta_2 u_{t-1}\end{aligned}$$

We could then substitute in the equation for Births_{t-2}, Births_{t-3}, ...

Autoregressive distributed-lag models

Complexity

Eventually we arrive at

$$\begin{aligned}
\text{Births}_t = &\beta_0 \left(1 + \beta_2 + \beta_2^2 + \beta_2^3 + \cdots\right) + \\
&\beta_1 \left(\text{Income}_t + \beta_2 \text{Income}_{t-1} + \beta_2^2 \text{Income}_{t-2} + \cdots\right) + \\
&u_t + \beta_2 u_{t-1} + \beta_2^2 u_{t-2} + \cdots
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The point?

Autoregressive distributed-lag models

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& u_t + \beta_2 u_{t-1} + \beta_2^2 u_{t-2} + \cdots
\end{aligned}
$$

The point?

By including just **one lag of the dependent variable**—as in a ADL(1, 0)—we implicitly include for *many lags* of the explanatory variables and disturbances.[†]

† These lags enter into the equation in a very specific way—not the most flexible specification.

Autoregressive distributed-lag models

The partial-adjustment model

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Autoregressive distributed-lag models

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The partial-adjustment model

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Partial-adjustment models help us model this situation.

Autoregressive distributed-lag models

The partial-adjustment model

Example

We want to know how the **desired number of cigarettes**, $\widetilde{\text{Cig}}_t$, changes with the current period's cigarette tax, *e.g.*,

$$\widetilde{\text{Cig}}_t = \beta_0 + \beta_1 \text{Tax}_t + u_t \qquad \textbf{(A)}$$

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Imagine actual cigarette consumption, Cig_t, doesn't change immediately (*e.g.*, habit persistence). Instead, consumption depends upon current desired level and previous consumption level

$$\text{Cig}_t = \lambda \widetilde{\text{Cig}}_t + (1 - \lambda)\, \text{Cig}_{t-1} \tag{B}$$

Autoregressive distributed-lag models

The partial-adjustment model

Example, continued

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Substituting $\widetilde{\text{Cig}}_t$ from (A) into (B) yields

$$\begin{aligned} \text{Cig}_t &= \lambda\left(\beta_0 + \beta_1 \text{Tax}_t + u_t\right) + (1 - \lambda)\, \text{Cig}_{t-1} \\ &= \lambda\beta_0 + \lambda\beta_1 \text{Tax}_t + (1 - \lambda)\, \text{Cig}_{t-1} + \lambda u_t \qquad \text{(C)} \end{aligned}$$

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The equation in $(\mathbf{C})$ is ADL(1, 0).

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Example, continued

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The equation in $(\mathbf{C})$ is ADL(1, 0).

We can also estimate/recover the speed-of-adjustment coefficient λ.

OLS in time series

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Unbiased coefficients

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1. The disturbance u_t is independent of the explanatory variables in the **same period** (*i.e.*, X_t).

2. The disturbance u_t is independent of the explanatory variables in the **other periods** (*i.e.*, X_s for $s \neq t$).

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We need both of these parts to be true for OLS to be unbiased.

OLS in time series

Unbiased coefficients

We need both parts of our exogeneity assumption for OLS to be unbiased:

$$\boldsymbol{E}\left[\hat{\beta}_1 \middle| X\right] = \beta_1 + \boldsymbol{E}\left[\frac{\sum_t \left(x_t - \overline{x}\right) u_t}{\sum_t \left(x_t - \overline{x}\right)^2} \middle| X\right]$$

I.e., to guarantee the numerator equals zero, we need $\boldsymbol{E}[u_t|X] = \mathbf{0}$—for both $\boldsymbol{E}[u_t|X_t] = 0$ *and* $\boldsymbol{E}[u_t|X_s] = 0$ $(s \neq t)$.

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The second part of our exogeneity assumption—requiring that u_t is independent of all regressors in other periods—fails with dynamic models with lagged outcome variables.

OLS in time series

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We need both parts of our exogeneity assumption for OLS to be unbiased:

$$\boldsymbol{E}\left[\hat{\beta}_1 \middle| X\right] = \beta_1 + \boldsymbol{E}\left[\frac{\sum_t \left(x_t - \overline{x}\right) u_t}{\sum_t \left(x_t - \overline{x}\right)^2} \middle| X\right]$$

I.e., to guarantee the numerator equals zero, we need $\boldsymbol{E}[u_t|X] = 0$—for both $\boldsymbol{E}[u_t|X_t] = 0$ *and* $\boldsymbol{E}[u_t|X_s] = 0$ $(s \neq t)$.

The second part of our exogeneity assumption—requiring that u_t is independent of all regressors in other periods—fails with dynamic models with lagged outcome variables.

Thus, **OLS is biased for dynamic models with lagged outcome variables**.

Unbiased coefficients

To see why dynamic models with lagged outcome variables violate our exogeneity assumption, consider two periods of our simple ADL(1, 0) model.

$$\text{Births}_t = \beta_0 + \beta_1 \text{Income}_t + \beta_2 \text{Births}_{t-1} + u_t \tag{1}$$

$$\text{Births}_{t+1} = \beta_0 + \beta_1 \text{Income}_{t+1} + \beta_2 \text{Births}_t + u_{t+1} \tag{2}$$

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$\therefore$ The disturbance in t (u_t) correlates with a regressor in $t+1$ (Births_t).

OLS in time series

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This correlation violates the second part of our exogeneity requirement.

OLS in time series

Consistent coefficients

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For OLS to be **consistent**, we only need **contemporaneous exogeneity**.

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With contemporaneous exogeneity, OLS estimates for the coefficients in a time series model are consistent.

Consistent coefficients

To see why OLS is consistent with contemporaneous exogeneity, consider the OLS estimate for β_1 in

$$\text{Births}_t = \beta_0 + \beta_1 \text{Births}_{t-1} + u_t$$

which we've shown (a few times) can be written

$$\hat{\beta}_1 = \beta_1 + \frac{\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}} \right) u_t}{\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}} \right)^2}$$

OLS in time series

Consistent coefficients

$$\operatorname{plim} \hat{\beta}_1 = \operatorname{plim} \left(\beta_1 + \frac{\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}} \right) u_t}{\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}} \right)^2} \right)$$

$$= \beta_1 + \frac{\operatorname{plim} \left[\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}} \right) u_t / T \right]}{\operatorname{plim} \left[\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}} \right)^2 / T \right]}$$

$$= \beta_1 + \frac{\text{Cov}(\text{Births}_{t-1},\, u_t)}{\text{Var}(\text{Births}_t)}$$

OLS in time series

Consistent coefficients

$$
\begin{aligned}
\text{plim}\,\hat{\beta}_1 &= \text{plim}\left(\beta_1 + \frac{\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}}\right) u_t}{\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}}\right)^2}\right) \\
&= \beta_1 + \frac{\text{plim}\left[\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}}\right) u_t / T\right]}{\text{plim}\left[\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}}\right)^2 / T\right]} \\
&= \beta_1 + \frac{\text{Cov}(\text{Births}_{t-1},\, u_t)}{\text{Var}(\text{Births}_t)} \\
&= \beta_1 \quad \textbf{if } \text{Cov}(\text{Births}_{t-1},\, u_t) = 0
\end{aligned}
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OLS in time series

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$$\operatorname{plim} \hat{\beta}_1 = \operatorname{plim} \left(\beta_1 + \frac{\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}} \right) u_t}{\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}} \right)^2} \right)$$

$$= \beta_1 + \frac{\operatorname{plim} \left[\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}} \right) u_t / T \right]}{\operatorname{plim} \left[\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}} \right)^2 / T \right]}$$

$$= \beta_1 + \frac{\text{Cov}(\text{Births}_{t-1},\, u_t)}{\text{Var}(\text{Births}_t)}$$

$$= \beta_1 \quad \textbf{if } \text{Cov}(\text{Births}_{t-1},\, u_t) = 0$$

Contemporaneous exogeneity gives us $\text{Cov}(\text{Births}_{t-1},\, u_t) = 0$.

Consistent coefficients

Thus, if we assume **contemporaneous exogeneity**, **OLS is consistent** for the coefficients, *even for models with lagged dependent variables*.

The end.

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Equilibrium effects

ADL models also offer interesting insights for long-run/equilibrium effects.

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In this ADL(1, 0) model, β_1 gives the **short-run effect** of income on the number of births. *I.e.*, how income in time t affects births in time t.

Autoregressive distributed-lag models

Equilibrium effects

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Now rearrange...

$$\begin{aligned}
\text{Births}^\star - \beta_2 \text{Births}^\star &= \beta_0 + \beta_1 \text{Income}^\star \\
\left(1 - \beta_2\right) \text{Births}^\star &= \beta_0 + \beta_1 \text{Income}^\star \\
\text{Births}^\star &= \frac{\beta_0}{\left(1 - \beta_2\right)} + \frac{\beta_1}{\left(1 - \beta_2\right)} \text{Income}^\star
\end{aligned}$$

Autoregressive distributed-lag models

Equilibrium effects

Short-run effect of income on births:

$$\text{Births}_t = \beta_0 + \beta_1 \text{Income}_t + \beta_2 \text{Births}_{t-1} + u_t$$

Long-run effect of income on births:

$$\text{Births}^\star = \frac{\beta_0}{(1-\beta_2)} + \frac{\beta_1}{(1-\beta_2)} \text{Income}^\star$$

Autoregressive distributed-lag models

Equilibrium effects

Another way to see this result:

We already showed

$$\text{Births}_t = \beta_0 + \beta_1 \text{Income}_t + \beta_2 \text{Births}_{t-1}$$

gives us

$$\begin{aligned}\text{Births}_t =& \beta_0 \left(1 + \beta_2 + \beta_2^2 + \beta_2^3 + \cdots\right) + \\ & \beta_1 \left(\text{Income}_t + \beta_2 \text{Income}_{t-1} + \beta_2^2 \text{Income}_{t-2} + \cdots\right) + \\ & u_t + \beta_2 u_{t-1} + \beta_2^2 u_{t-2} + \cdots\end{aligned}$$

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In equilibrium: $\text{Income}_t = \text{Income}_{t-k} = \text{Income}^\star$ for all k.

Autoregressive distributed-lag models

Equilibrium effects

Substituting $\mathbf{Income}_t = \mathbf{Income}^{\star}$ for all $\boldsymbol{k}$
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So long as $-1 < \beta_2 < 1$.[†]

† This simplification comes from $\sum_{k=0}^{\infty} p^k = \frac{1}{p}$ for $-1 < k < 1$.