

# Asymptotics and consistency

EC 421, Set 6

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# Prologue

# Schedule

## Last Time

Living with heteroskedasticity

## Today

Asymptotics and consistency

## Next

- New problem set
- *Next topic*: Time series
- Midterm on Feb. 12

# R showcase

Need speed? **R** allows essentially infinite parallelization.

Three popular packages:

- `future` and `furrr`
- `parallel`
- `foreach`

And here's a nice [tutorial](#).

# Consistency

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This approach misses something.

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### New question:

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This *new question* forms a new way to think about the properties of estimators: **asymptotic properties** (or large-sample properties).

A "good" estimator will become indistinguishable from the parameter it estimates when  $n$  is very large (close to  $\infty$ ).

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## Probability limits

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Just as the *expected value* helped us characterize **the finite-sample distribution of an estimator** with sample size  $n$ ,

the *probability limit* helps us analyze **the asymptotic distribution of an estimator** (the distribution of the estimator as  $n$  gets "big"<sup>†</sup>).

<sup>†</sup> Here, "big"  $n$  means  $n \rightarrow \infty$ . That's *really* big data.

# Consistency

## Probability limits

Let  $B_n$  be our estimator with sample size  $n$ .

Then the **probability limit** of  $B$  is  $\alpha$  if

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The definition in (1) *essentially* says that as the **sample size** approaches infinity, the probability that  $B_n$  differs from  $\alpha$  by more than a very small number ( $\epsilon$ ) is zero.

*Practically:*  $B$ 's distribution collapses to a spike at  $\alpha$  as  $n$  approaches  $\infty$ .

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## Probability limits

Equivalent statements:

- The probability limit of  $B_n$  is  $\alpha$ .
- $\text{plim } B = \alpha$
- $B$  converges in probability to  $\alpha$ .

# Consistency

## Probability limits

Probability limits have some nice/important properties:

- $\text{plim}(X \times Y) = \text{plim}(X) \times \text{plim}(Y)$
- $\text{plim}(X + Y) = \text{plim}(X) + \text{plim}(Y)$
- $\text{plim}(c) = c$ , where  $c$  is a constant
- $\text{plim}\left(\frac{X}{Y}\right) = \frac{\text{plim}(X)}{\text{plim}(Y)}$
- $\text{plim}(f(X)) = f(\text{plim}(X))$

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*In math:* The estimator  $B$  is consistent for  $\alpha$  if  $\text{plim}B = \alpha$ .

The estimator is *inconsistent* if  $\text{plim}B \neq \alpha$ .



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## Consistent estimators

*Example:* We want to estimate the population mean  $\mu_x$  (where  $X \sim \text{Normal}$ ).

Let's compare the asymptotic distributions of two competing estimators:

1. The first observation:  $X_1$

2. The sample mean:  $\bar{X} = \frac{1}{n} \sum_{i=1}^n x_i$

3. Some other estimator:  $\tilde{X} = \frac{1}{n+1} \sum_{i=1}^n x_i$

Note that (1) and (2) are unbiased, but (3) is biased.

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$$E[\tilde{X}]$$

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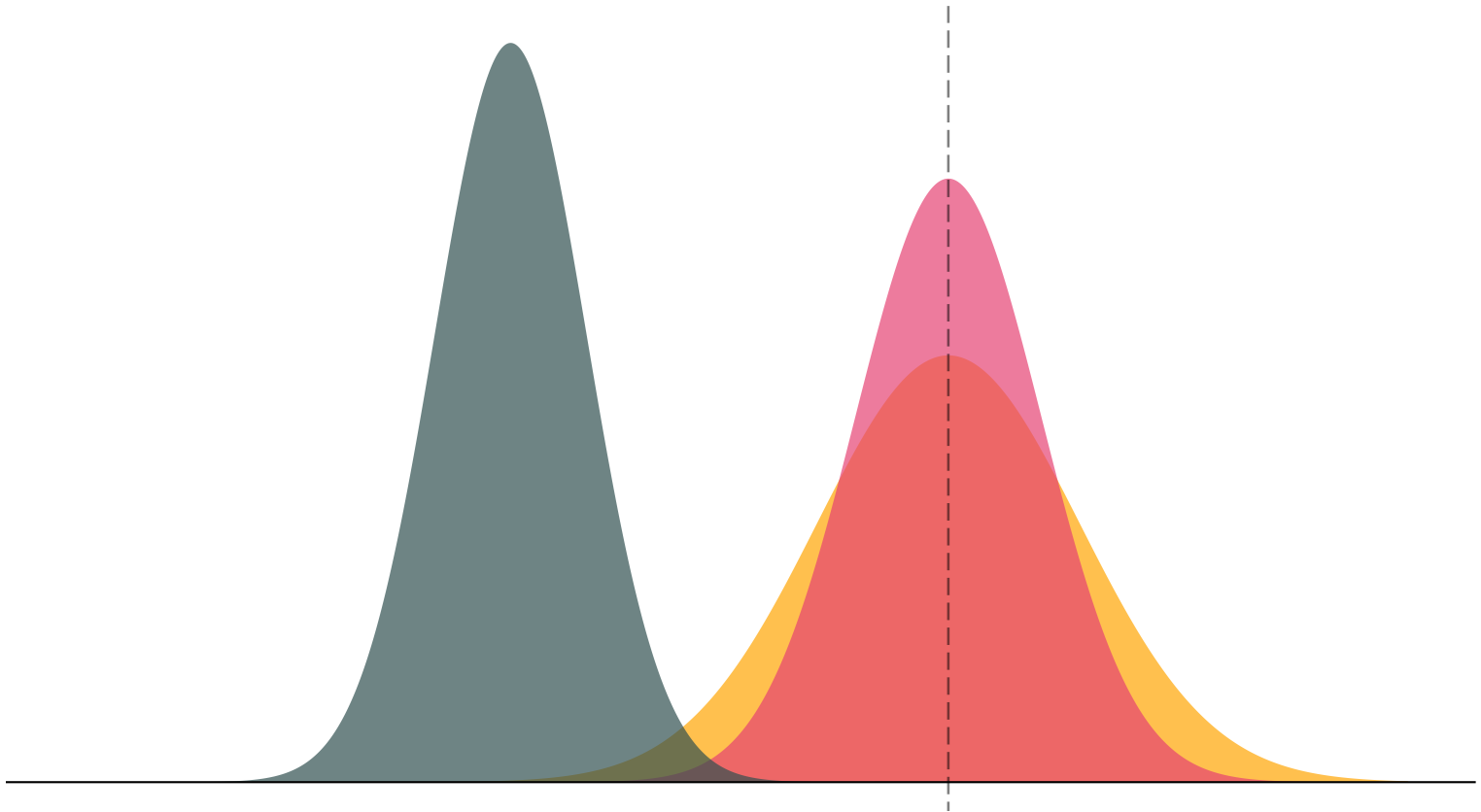
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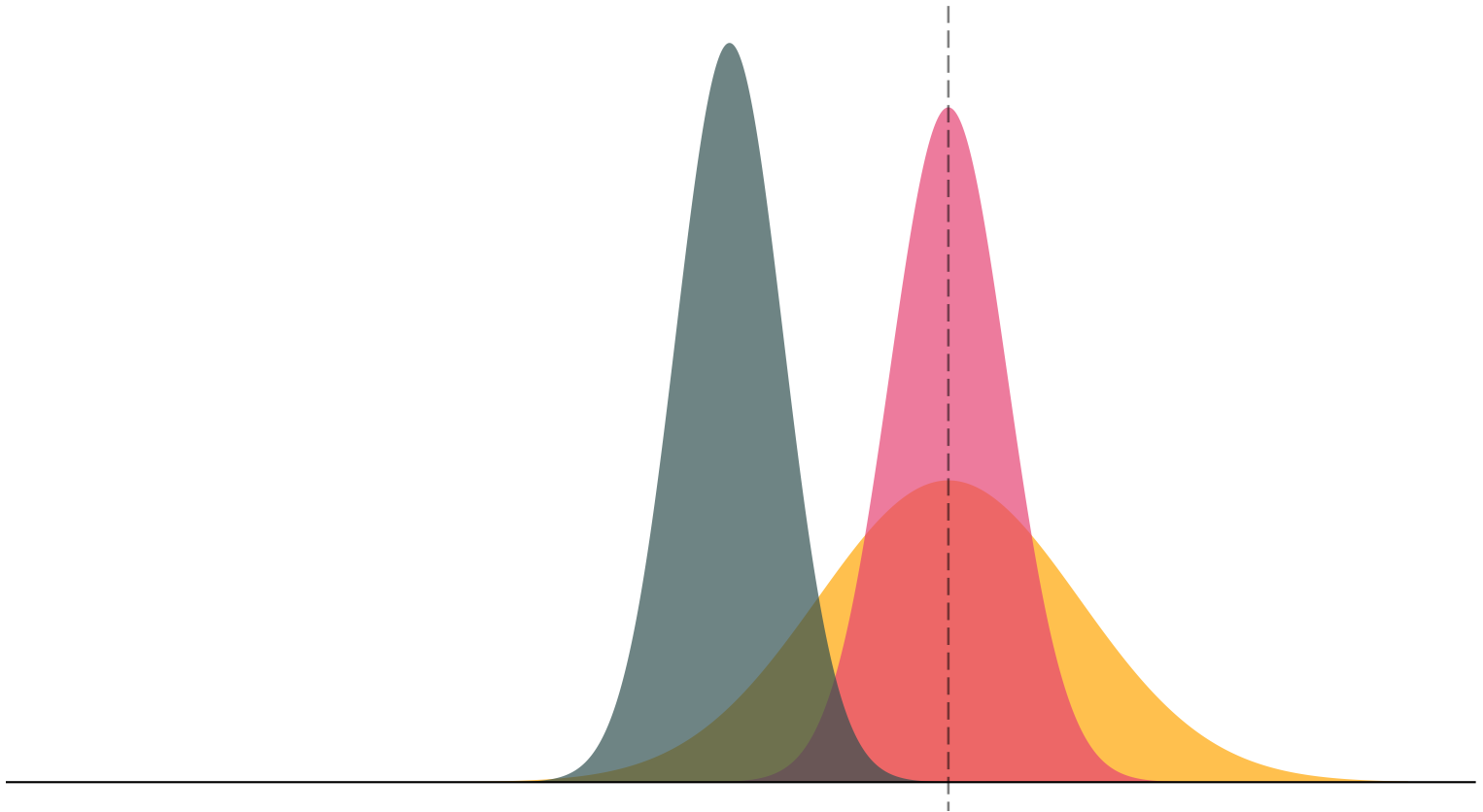
Distributions of  $X_1$ ,  $\bar{X}$ , and  $\tilde{X}$

$n = 2$

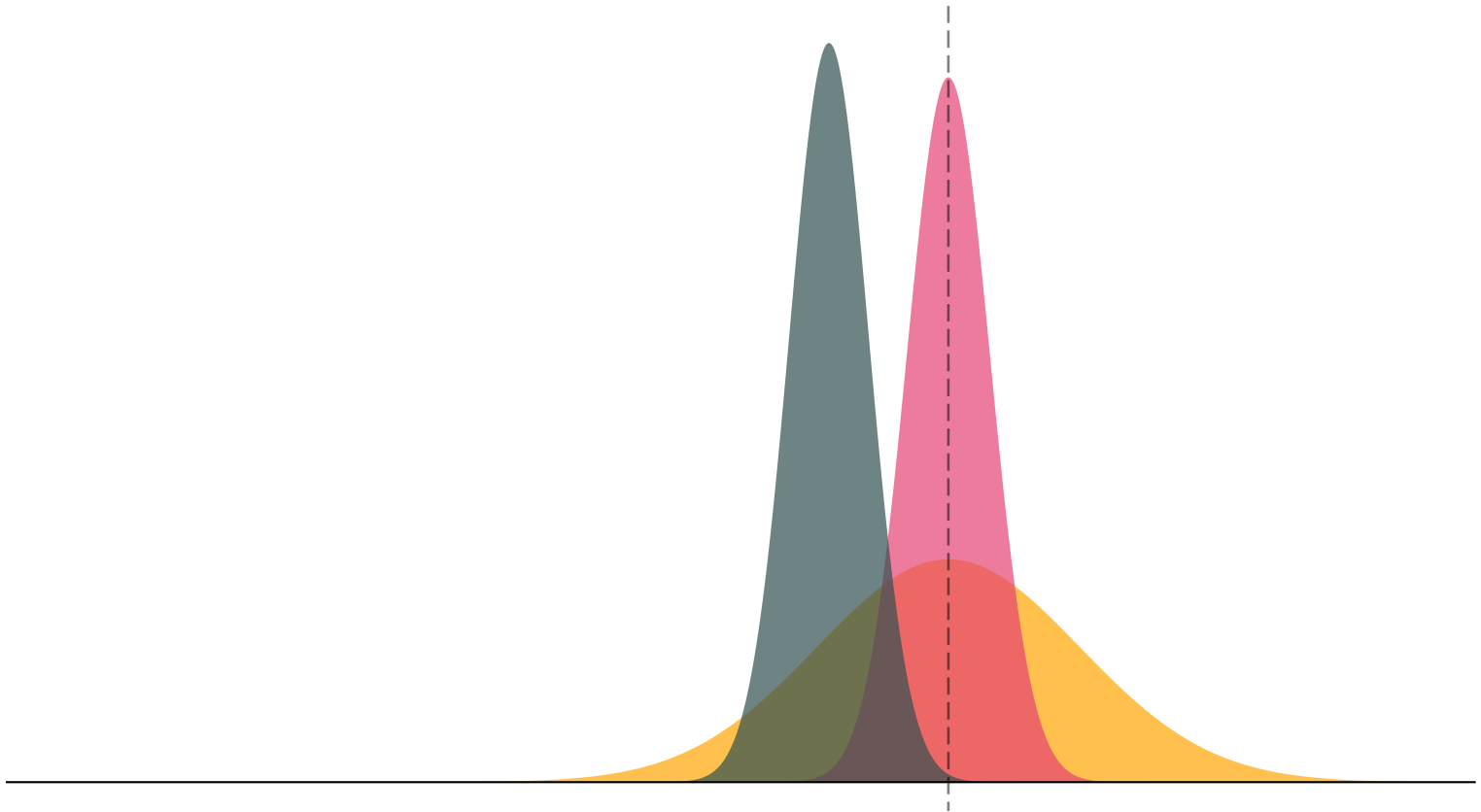


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Distributions of  $X_1$ ,  $\bar{X}$ , and  $\tilde{X}$   
 $n = 5$

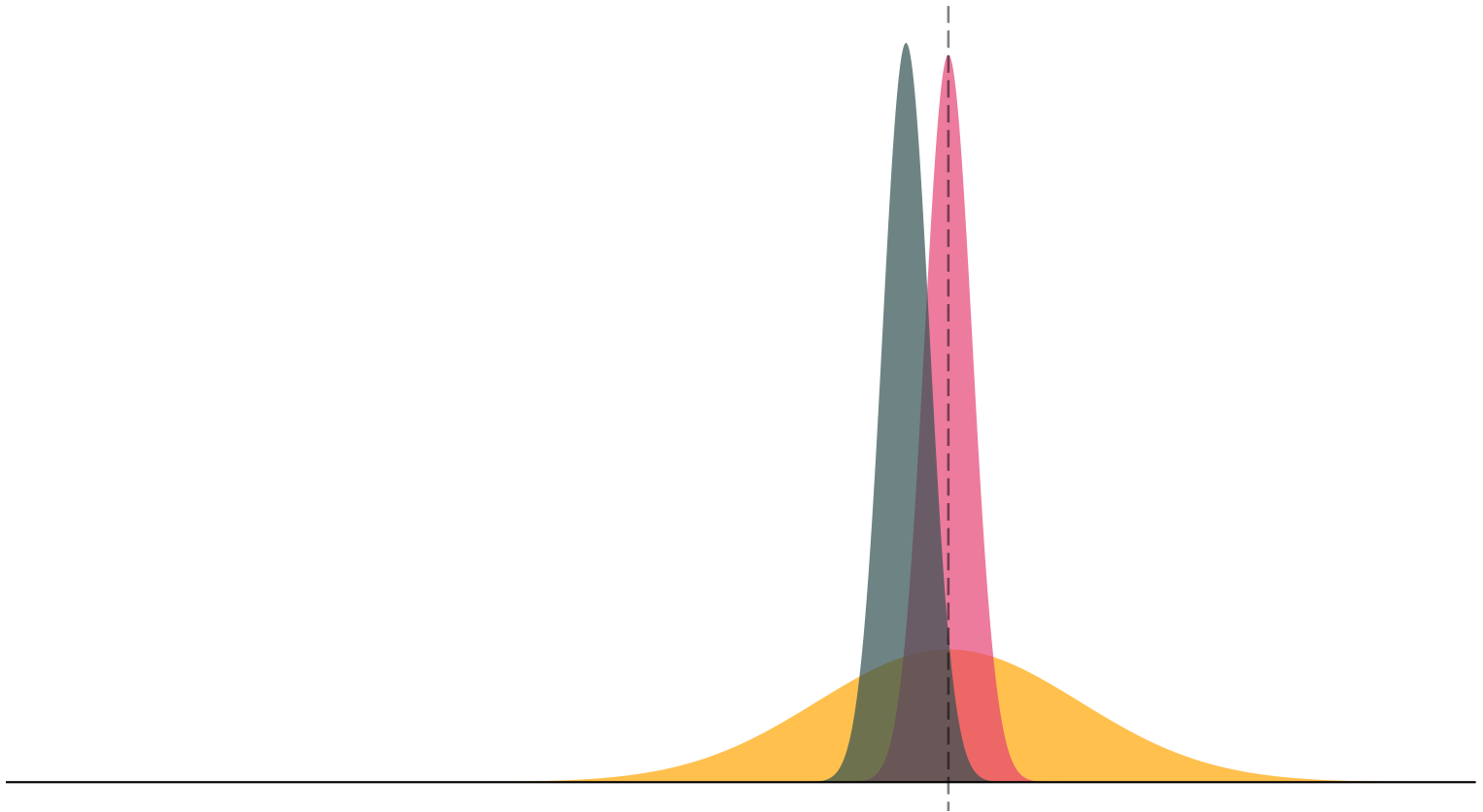


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Distributions of  $X_1$ ,  $X$ , and  $\tilde{X}$ 
$$n = 10$$


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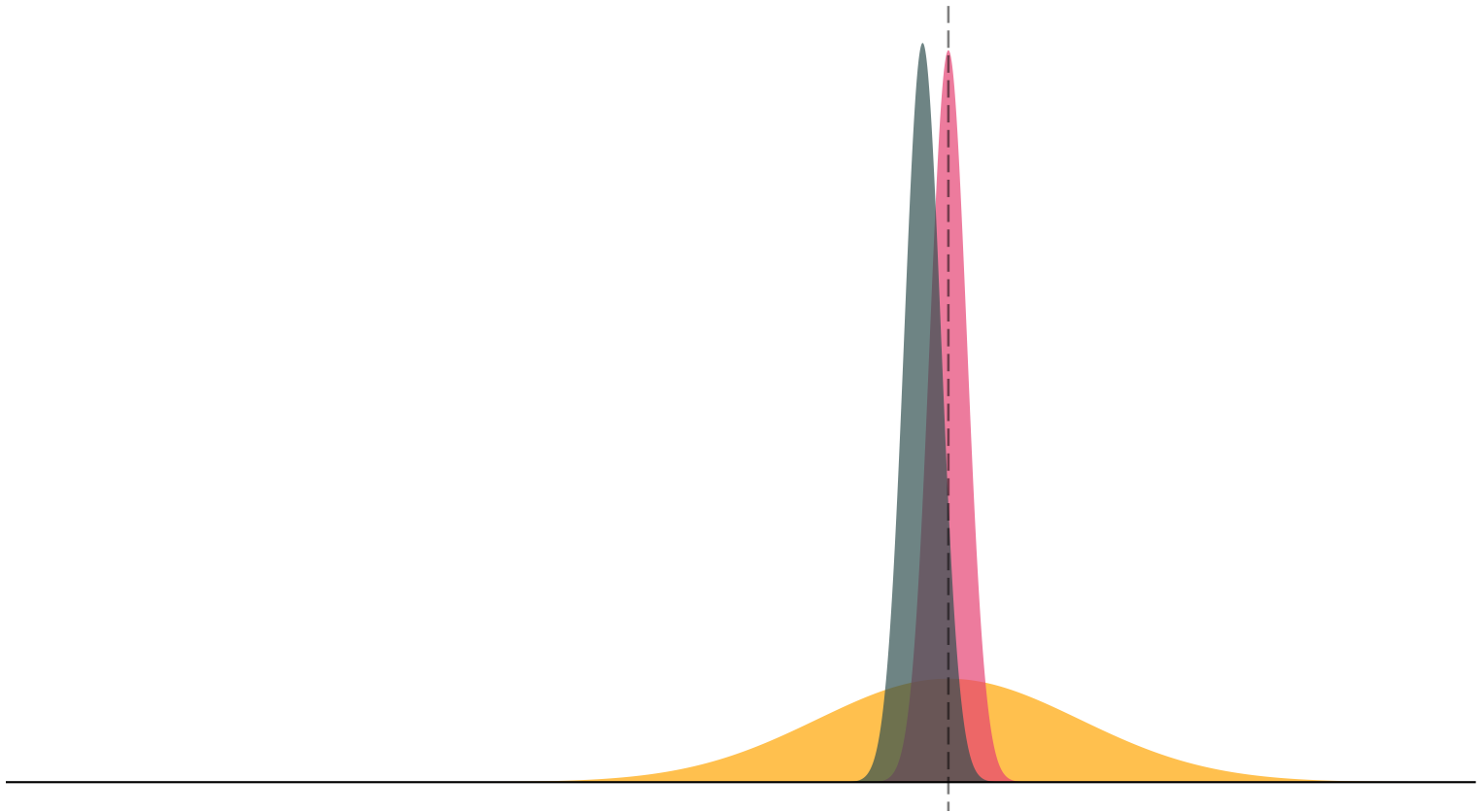
Distributions of  $X_1$ ,  $\bar{X}$ , and  $\tilde{X}$   
 $n = 30$





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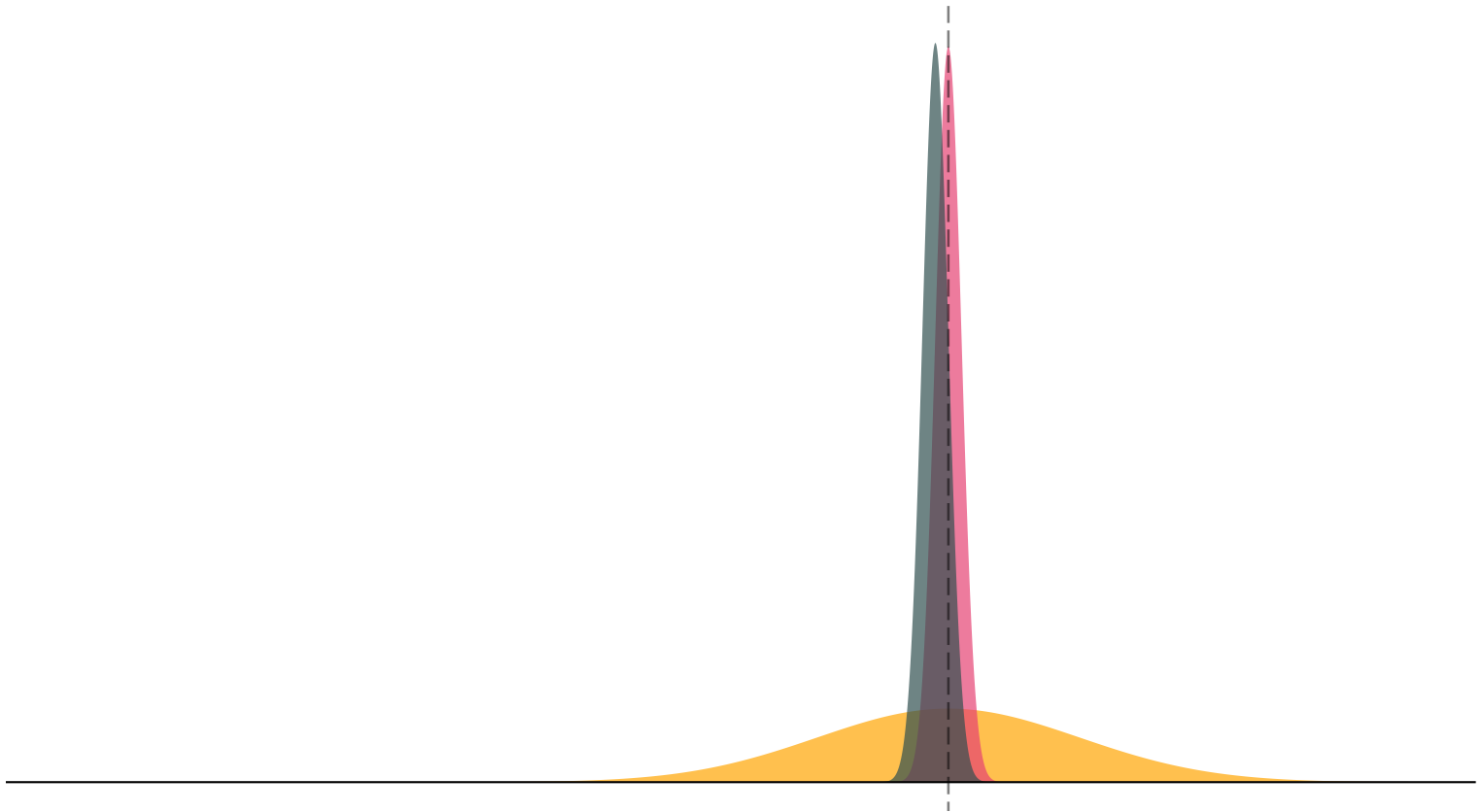
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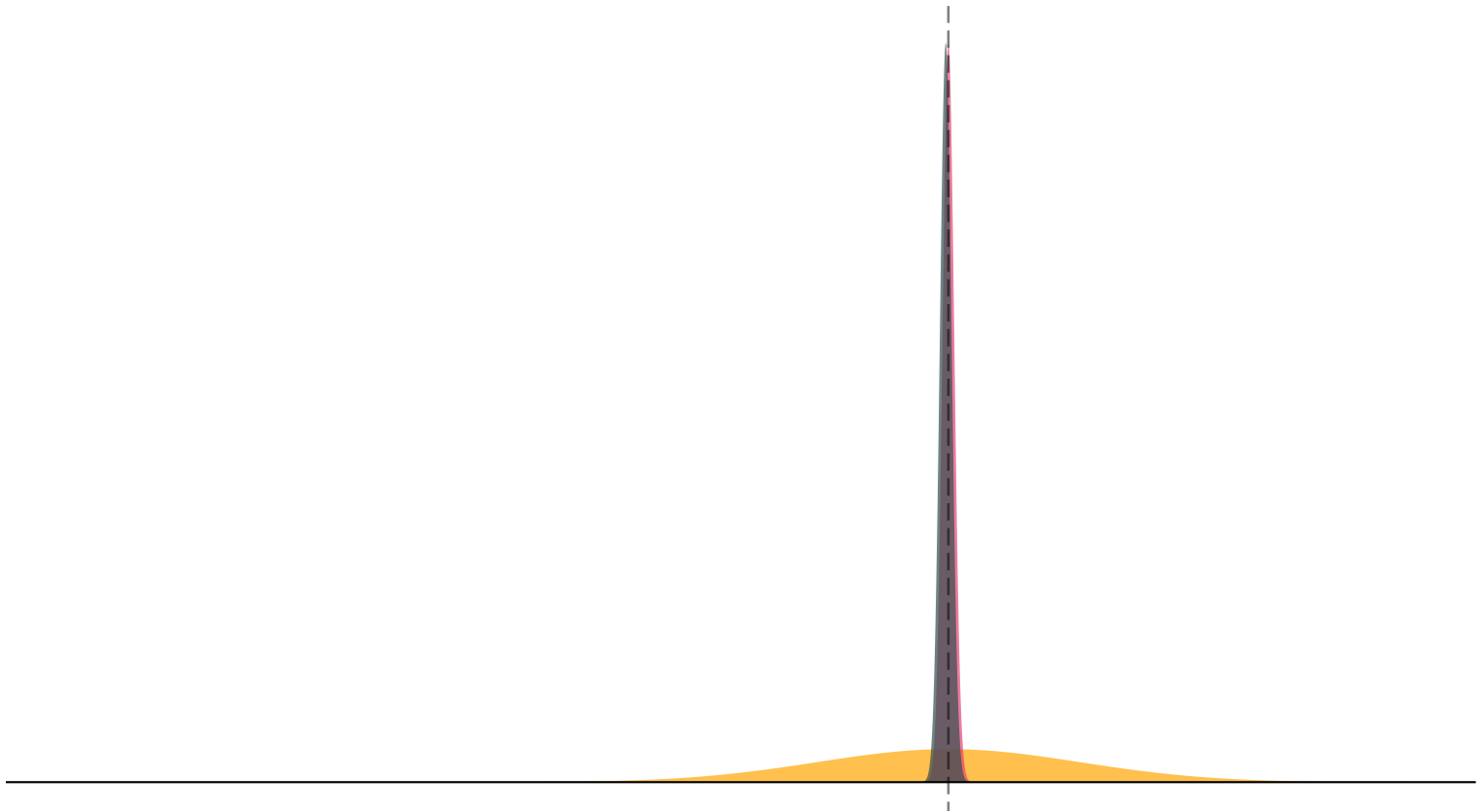
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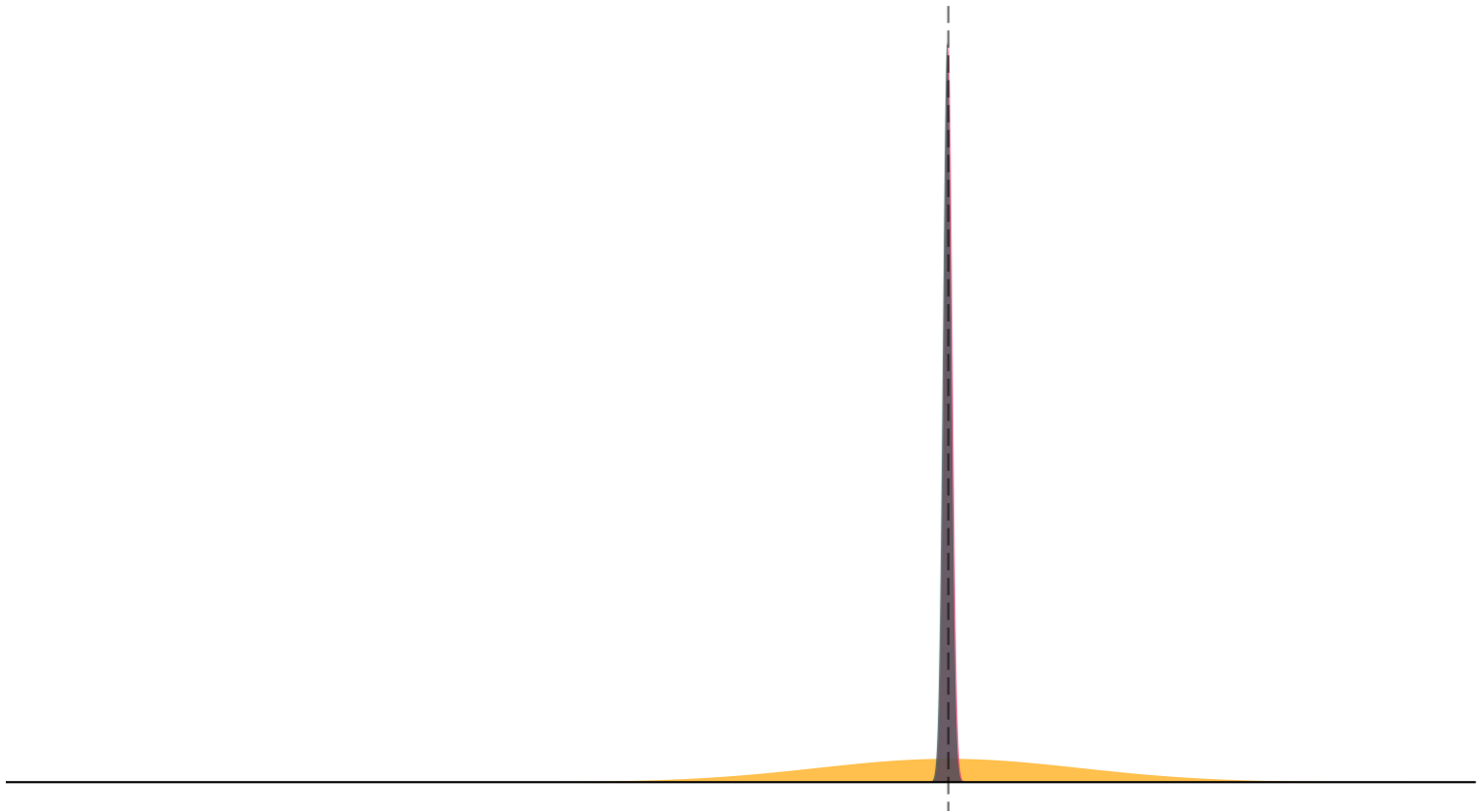
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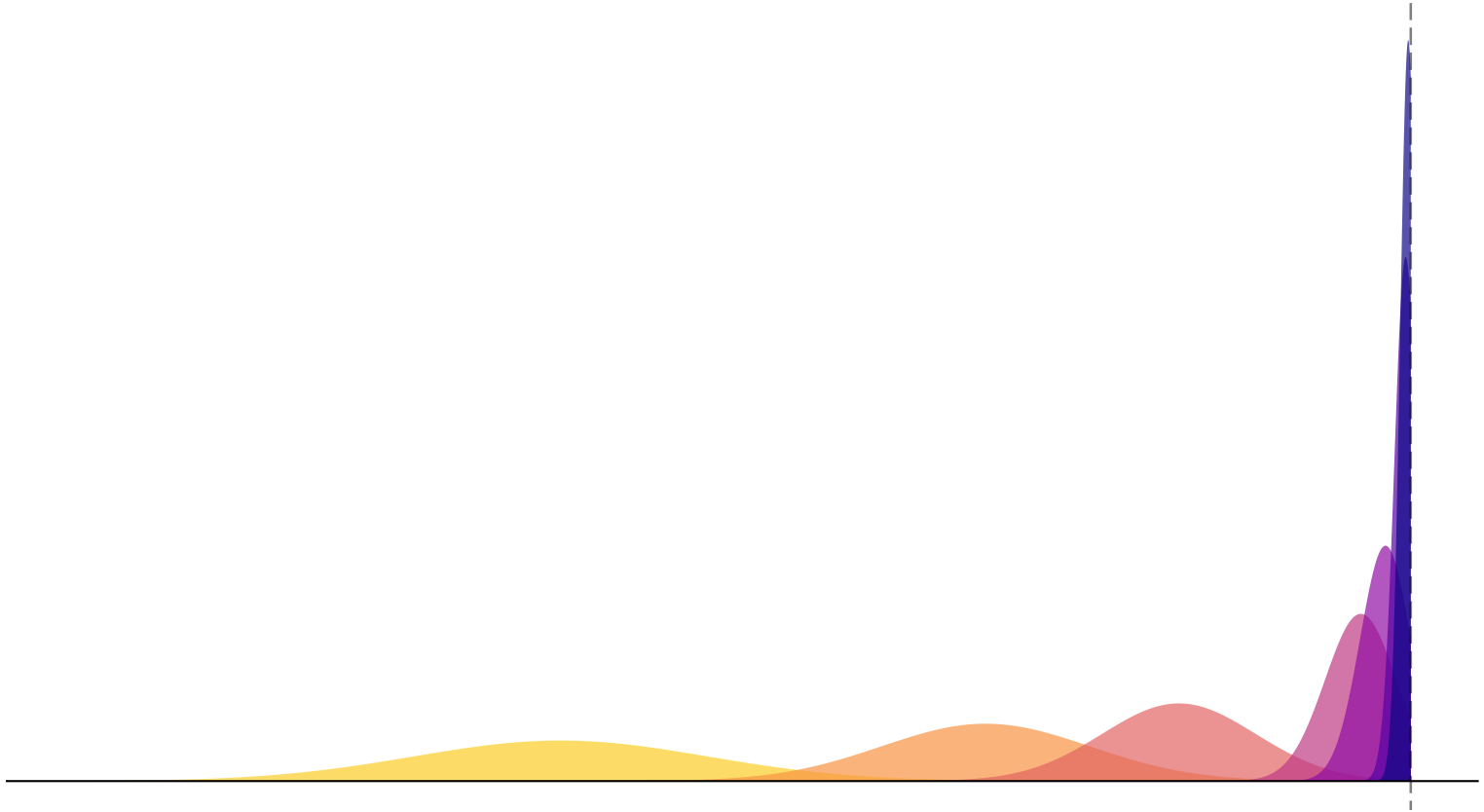
$n = 1000$



# Consistency

The distributions of  $\tilde{X}$

For  $n$  in  $\{2, 5, 10, 50, 100, 500, 1000\}$



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- An estimator can be biased and inconsistent (e.g.,  $X - 50$ ).

**Best-case scenario:** The estimator is unbiased and consistent.

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## Why consistency (asymptotics)?

1. We cannot always find an unbiased estimator. In these situations, we generally (at least) want consistency.
2. Expected values can be hard/undefined. Probability limits are less constrained, e.g.,

$$E[g(X)h(Y)] \text{ vs. } \text{plim}(g(X)h(Y))$$

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**Caution:** As we saw, consistent estimators can be biased in small samples.

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1. Consistency
2. Asymptotic Normality

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Let's prove #1 for OLS with simple, linear regression, *i.e.*,

$$y_i = \beta_0 + \beta_1 x_i + u_i$$

# OLS in asymptopia

## Proof of consistency

First, recall our previous derivation of  $\hat{\beta}_1$ ,

$$\hat{\beta}_1 = \beta_1 + \frac{\sum_i (x_i - \bar{x}) u_i}{\sum_i (x_i - \bar{x})^2}$$

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$$\text{plim} \hat{\beta}_1 = \beta_1 + \frac{0}{\text{Var}(x)} = \beta_1 \quad \text{🧐}$$

so long as  $\text{Var}(x) \neq 0$  (which we've assumed).

# OLS in asymptopia

## Asymptotic normality

Up to this point, we made a very specific assumption about the distribution of  $u_i$ —the  $u_i$  came from a normal distribution.

We can relax this assumption—allowing the  $u_i$  to come from any distribution (still assume exogeneity, independence, and homoskedasticity).

We will focus on the **asymptotic distribution** of our estimators (how they are distributed as  $n$  gets large), rather than their finite-sample distribution.

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As  $n$  approaches  $\infty$ , the distribution of the OLS estimator converges to a normal distribution.

# OLS in asymptopia

## Recap

With a more limited set of assumptions, OLS is **consistent** and is **asymptotically normally distributed**.

## Current assumptions

1. Our data were **randomly sampled** from the population.
2.  $y_i$  is a **linear function** of its parameters and disturbance.
3. There is **no perfect collinearity** in our data.
4. The  $u_i$  have conditional mean of zero (**exogeneity**),  $E[u_i | X_i] = 0$ .
5. The  $u_i$  are **homoskedastic** with **zero correlation** between  $u_i$  and  $u_j$ .

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1.  $x_2$  affects  $y$ , i.e.,  $\beta_2 \neq 0$ .

# Omitted-variable bias, redux

## Inconsistency?

Imagine we have a population whose true model is

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*Recall*<sub>1</sub>: **Omitted-variable bias** occurs when we omit a variable in our linear regression model (e.g., leaving out  $x_2$ ) such that

1.  $x_2$  affects  $y$ , i.e.,  $\beta_2 \neq 0$ .
2. Correlates with an included explanatory variable, i.e.,  $\text{Cov}(x_1, x_2) \neq 0$ .

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## Inconsistency?

Imagine we have a population whose true model is

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*Recall*<sub>2</sub>: We defined the **bias** of an estimator  $W$  for parameter  $\theta$

# Omitted-variable bias, redux

## Inconsistency?

Imagine we have a population whose true model is

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*Recall*<sub>2</sub>: We defined the **bias** of an estimator  $W$  for parameter  $\theta$

$$\text{Bias}(W)_{\theta} = \mathbf{E}[W] - \theta$$

# Omitted-variable bias, redux

## Inconsistency?

Imagine we have a population whose true model is

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + u_i$$

We know that omitted-variable bias causes **biased estimates**.

*Question:* Do *omitted variables* also cause **inconsistent estimates**?

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Imagine we have a population whose true model is

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We know that omitted-variable bias causes **biased estimates**.

*Question:* Do *omitted variables* also cause **inconsistent estimates**?

*Answer:* Find  $\text{plim} \hat{\beta}_1$  in a regression that omits  $x_2$ .

# Omitted-variable bias, redux

## Inconsistency?

Imagine we have a population whose true model is

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + u_i$$

but we instead specify the model as

$$y_i = \beta_0 + \beta_1 x_{1i} + w_i$$

where  $w_i = \beta_2 x_{2i} + u_i$ .

# Omitted-variable bias, redux

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$$y_i = \hat{\beta}_0 + \hat{\beta}_1 x_{1i} + \hat{w}_i$$



# Omitted-variable bias, redux

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*Our question:* Is  $\hat{\beta}_1$  consistent for  $\beta_1$  when we omit  $x_2$ ?

# Omitted-variable bias, redux

## Inconsistency?

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*Our question:* Is  $\hat{\beta}_1$  consistent for  $\beta_1$  when we omit  $x_2$ ?

$$\text{plim}(\hat{\beta}_1) \stackrel{?}{=} \beta_1$$

# Omitted-variable bias, redux

## Inconsistency?

**Truth:**  $y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + u_i$

**Specified:**  $y_i = \beta_0 + \beta_1 x_{1i} + w_i$

We already showed  $\text{plim} \hat{\beta}_1 = \beta_1 + \frac{\text{Cov}(x_1, w)}{\text{Var}(x_1)}$

where  $w$  is the disturbance.

# Omitted-variable bias, redux

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$$\text{plim} \hat{\beta}_1 = \beta_1 + \frac{\text{Cov}(x_1, w)}{\text{Var}(x_1)}$$

where  $w$  is the disturbance. Here, we know  $w = \beta_2 x_2 + u$ . Thus,

# Omitted-variable bias, redux

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$$\text{plim} \hat{\beta}_1 = \beta_1 + \frac{\text{Cov}(x_1, \beta_2 x_2 + u)}{\text{Var}(x_1)}$$

# Omitted-variable bias, redux

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**Truth:**  $y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + u_i$

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Now, we make use of  $\text{Cov}(X, Y + Z) = \text{Cov}(X, Y) + \text{Cov}(X, Z)$

# Omitted-variable bias, redux

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$$\text{Cov}(x_1, \beta_2 x_2) + \text{Cov}(x_1, u)$$

# Omitted-variable bias, redux

Inconsistency?

$$\text{plim}\hat{\beta}_1 = \beta_1 + \frac{\text{Cov}(x_1, \beta_2 x_2) + \text{Cov}(x_1, u)}{\text{Var}(x_1)}$$

Now we use the fact that  $\text{Cov}(X, cY) = c\text{Cov}(X, Y)$  for a constant  $c$ .



# Omitted-variable bias, redux

## Inconsistency?

$$\text{plim}\hat{\beta}_1 = \beta_1 + \frac{\text{Cov}(x_1, \beta_2 x_2) + \text{Cov}(x_1, u)}{\text{Var}(x_1)}$$

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$$\text{plim}\hat{\beta}_1 = \beta_1 + \frac{\beta_2 \text{Cov}(x_1, x_2) + \text{Cov}(x_1, u)}{\text{Var}(x_1)}$$

# Omitted-variable bias, redux

## Inconsistency?

$$\text{plim}\hat{\beta}_1 = \beta_1 + \frac{\text{Cov}(x_1, \beta_2 x_2) + \text{Cov}(x_1, u)}{\text{Var}(x_1)}$$

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As before, our exogeneity (conditional mean zero) assumption implies  $\text{Cov}(x_1, u) = 0$ , which gives us

# Omitted-variable bias, redux

## Inconsistency?

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$$\text{plim}\hat{\beta}_1 = \beta_1 + \frac{\beta_2 \text{Cov}(x_1, x_2) + \text{Cov}(x_1, u)}{\text{Var}(x_1)}$$

As before, our exogeneity (conditional mean zero) assumption implies  $\text{Cov}(x_1, u) = 0$ , which gives us

$$\beta_2 \text{Cov}(x_1, x_2)$$

# Omitted-variable bias, redux

## Inconsistency?

Thus, we find that

$$\text{plim}\hat{\beta}_1 = \beta_1 + \beta_2 \frac{\text{Cov}(x_1, x_2)}{\text{Var}(x_1)}$$

In other words, an **omitted variable will cause OLS to be inconsistent if both** of the following statements are true:

1. The omitted variable **affects our outcome**, *i.e.*,  $\beta_2 \neq 0$ .
2. The omitted variable correlates with included explanatory variables, *i.e.*,  $\text{Cov}(x_1, x_2) \neq 0$ .

If both of these statements are true, then the OLS estimate  $\hat{\beta}_1$  will not

# Omitted-variable bias, redux

## Signing the bias

Sometimes we're stuck with omitted variable bias.<sup>†</sup>

$$\text{plim}\hat{\beta}_1 = \beta_1 + \beta_2 \frac{\text{Cov}(x_1, x_2)}{\text{Var}(x_1)}$$

When this happens, we can often at least know the direction of the inconsistency.

<sup>†</sup> You will often hear the term "omitted-variable bias" when we're actually talking about inconsistency (rather than bias).

# Omitted-variable bias, redux

## Signing the bias

Begin with

$$\text{plim}\hat{\beta}_1 = \beta_1 + \beta_2 \frac{\text{Cov}(x_1, x_2)}{\text{Var}(x_1)}$$

We know  $\text{Var}(x_1) > 0$ . Suppose  $\beta_2 > 0$  and  $\text{Cov}(x_1, x_2) > 0$ . Then

# Omitted-variable bias, redux

## Signing the bias

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$$\text{plim}\hat{\beta}_1 = \beta_1 + (+) \frac{(+)}{(+)} \implies \text{plim}\hat{\beta}_1 > \beta_1$$

$\therefore$  In this case, OLS is **biased upward** (estimates are too large).

# Omitted-variable bias, redux

## Signing the bias

Begin with

$$\text{plim}\hat{\beta}_1 = \beta_1 + \beta_2 \frac{\text{Cov}(x_1, x_2)}{\text{Var}(x_1)}$$

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$$\begin{array}{cc} \text{Cov}(x_1, x_2) > 0 & \text{Cov}(x_1, x_2) < 0 \\ \beta_2 > 0 & \text{Upward} \end{array}$$



# Omitted-variable bias, redux

## Signing the bias

Begin with

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# Omitted-variable bias, redux

## Signing the bias

Begin with

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$$\text{plim}\hat{\beta}_1 = \beta_1 + (-) \frac{(+)}{(+)} \implies \text{plim}\hat{\beta}_1 < \beta_1$$

$\therefore$  In this case, OLS is **biased downward** (estimates are too small).

$$\begin{array}{cc} \text{Cov}(x_1, x_2) > 0 & \text{Cov}(x_1, x_2) < 0 \\ \beta_2 > 0 & \text{Upward} \end{array}$$

# Omitted-variable bias, redux

## Signing the bias

Begin with

$$\text{plim}\hat{\beta}_1 = \beta_1 + \beta_2 \frac{\text{Cov}(x_1, x_2)}{\text{Var}(x_1)}$$

We know  $\text{Var}(x_1) > 0$ . Suppose  $\beta_2 > 0$  and  $\text{Cov}(x_1, x_2) < 0$ . Then

$$\text{plim}\hat{\beta}_1 = \beta_1 + (+) \frac{(-)}{(+)} \implies \text{plim}\hat{\beta}_1 < \beta_1$$

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	$\text{Cov}(x_1, x_2) > 0$	$\text{Cov}(x_1, x_2) < 0$
$\beta_2 > 0$	Upward	Downward

# Omitted-variable bias, redux

## Signing the bias

Begin with

$$\text{plim}\hat{\beta}_1 = \beta_1 + \beta_2 \frac{\text{Cov}(x_1, x_2)}{\text{Var}(x_1)}$$

We know  $\text{Var}(x_1) > 0$ . Suppose  $\beta_2 < 0$  and  $\text{Cov}(x_1, x_2) < 0$ . Then

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	$\text{Cov}(x_1, x_2) > 0$	$\text{Cov}(x_1, x_2) < 0$
$\beta_2 > 0$	Upward	Downward

# Omitted-variable bias, redux

## Signing the bias

Thus, in cases where we have a sense of

1. the sign of  $\text{Cov}(x_1, x_2)$
2. the sign of  $\beta_2$

we know in which direction inconsistency pushes our estimates.

### Direction of bias

	$\text{Cov}(x_1, x_2) > 0$	$\text{Cov}(x_1, x_2) < 0$
$\beta_2 > 0$	Upward	Downward
$\beta_2 < 0$	Downward	Upward

# Measurement error

**Measurement error** in our explanatory variables presents another case in which OLS is inconsistent.

Consider the population model:  $y_i = \beta_0 + \beta_1 z_i + u_i$

- We want to observe  $z_i$  but cannot.
- Instead, we *measure* the variable  $x_i$ , which is  $z_i$  plus some error (noise):

$$x_i = z_i + \omega_i$$

- Assume  $\mathbf{E}[\omega_i] = 0$ ,  $\text{Var}(\omega_i) = \sigma_\omega^2$ , and  $\omega$  is independent of  $z$  and  $u$ .

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- Assume  $\mathbf{E}[\omega_i] = 0$ ,  $\text{Var}(\omega_i) = \sigma_\omega^2$ , and  $\omega$  is independent of  $z$  and  $u$ .

OLS regression of  $y$  and  $x$  will produce inconsistent estimates for  $\beta_1$ .

# Measurement error

## Proof

$$y_i = \beta_0 + \beta_1 z_i + u_i$$



# Measurement error

## Proof

$$\begin{aligned} y_i &= \beta_0 + \beta_1 z_i + u_i \\ &= \beta_0 + \beta_1 (x_i - \omega_i) + u_i \end{aligned}$$

# Measurement error

## Proof

$$\begin{aligned}y_i &= \beta_0 + \beta_1 z_i + u_i \\&= \beta_0 + \beta_1 (x_i - \omega_i) + u_i \\&= \beta_0 + \beta_1 x_i + (u_i - \beta_1 \omega_i)\end{aligned}$$

# Measurement error

## Proof

$$\begin{aligned}y_i &= \beta_0 + \beta_1 z_i + u_i \\&= \beta_0 + \beta_1 (x_i - \omega_i) + u_i \\&= \beta_0 + \beta_1 x_i + (u_i - \beta_1 \omega_i) \\&= \beta_0 + \beta_1 x_i + \varepsilon_i\end{aligned}$$

where  $\varepsilon_i = u_i - \beta_1 \omega_i$

# Measurement error

## Proof

$$\begin{aligned}y_i &= \beta_0 + \beta_1 z_i + u_i \\&= \beta_0 + \beta_1 (x_i - \omega_i) + u_i \\&= \beta_0 + \beta_1 x_i + (u_i - \beta_1 \omega_i) \\&= \beta_0 + \beta_1 x_i + \varepsilon_i\end{aligned}$$

where  $\varepsilon_i = u_i - \beta_1 \omega_i$

What happens when we estimate  $y_i = \hat{\beta}_0 + \hat{\beta}_1 x_i + e_i$ ?

$$\text{plim} \hat{\beta}_1 = \beta_1 + \frac{\text{Cov}(x, \varepsilon)}{\text{Var}(x)}$$

We will derive the numerator and denominator separately...

# Measurement error

## Proof

The covariance of our noisy variable  $x$  and the disturbance  $\varepsilon$ .

$$\text{Cov}(x, \varepsilon)$$

# Measurement error

## Proof

The covariance of our noisy variable  $x$  and the disturbance  $\varepsilon$ .

$$\text{Cov}(x, \varepsilon) = \text{Cov}\left([z + \omega], [u - \beta_1 \omega]\right)$$

# Measurement error

## Proof

The covariance of our noisy variable  $x$  and the disturbance  $\varepsilon$ .

$$\begin{aligned}\text{Cov}(x, \varepsilon) &= \text{Cov}\left([z + \omega], [u - \beta_1 \omega]\right) \\ &= \text{Cov}(z, u) - \beta_1 \text{Cov}(z, \omega) + \text{Cov}(\omega, u) - \beta_1 \text{Var}(\omega)\end{aligned}$$

# Measurement error

## Proof

The covariance of our noisy variable  $x$  and the disturbance  $\varepsilon$ .

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## Proof

The covariance of our noisy variable  $x$  and the disturbance  $\varepsilon$ .

$$\begin{aligned}\text{Cov}(x, \varepsilon) &= \text{Cov}\left([z + \omega], [u - \beta_1 \omega]\right) \\ &= \text{Cov}(z, u) - \beta_1 \text{Cov}(z, \omega) + \text{Cov}(\omega, u) - \beta_1 \text{Var}(\omega) \\ &= 0 + 0 + 0 - \beta_1 \sigma_\omega^2 \\ &= -\beta_1 \sigma_\omega^2\end{aligned}$$

# Measurement error

## Proof

Now for the denominator,  $\text{Var}(x)$ .

$\text{Var}(x)$

# Measurement error

## Proof

Now for the denominator,  $\text{Var}(x)$ .

$$\text{Var}(x) = \text{Var}(z + \omega)$$

# Measurement error

## Proof

Now for the denominator,  $\text{Var}(x)$ .

$$\begin{aligned}\text{Var}(x) &= \text{Var}(z + \omega) \\ &= \text{Var}(z) + \text{Var}(\omega) + 2\text{Cov}(z, \omega)\end{aligned}$$

# Measurement error

## Proof

Now for the denominator,  $\text{Var}(x)$ .

$$\begin{aligned}\text{Var}(x) &= \text{Var}(z + \omega) \\ &= \text{Var}(z) + \text{Var}(\omega) + 2\text{Cov}(z, \omega) \\ &= \sigma_z^2 + \sigma_\omega^2\end{aligned}$$

# Measurement error

## Proof

Putting the numerator and denominator back together,

$$\begin{aligned}\text{plim}\hat{\beta}_1 &= \beta_1 + \frac{\text{Cov}(x, \varepsilon)}{\text{Var}(x)} \\&= \beta_1 + \frac{-\beta_1\sigma_\omega^2}{\sigma_z^2 + \sigma_\omega^2} \\&= \beta_1 - \beta_1 \frac{\sigma_\omega^2}{\sigma_z^2 + \sigma_\omega^2} \\&= \beta_1 \frac{\sigma_z^2 + \sigma_\omega^2}{\sigma_z^2 + \sigma_\omega^2} - \beta_1 \frac{\sigma_\omega^2}{\sigma_z^2 + \sigma_\omega^2} \\&= \beta_1 \frac{\sigma_z^2}{\sigma_z^2 + \sigma_\omega^2}\end{aligned}$$

# Measurement error

## Summary

$$\therefore \text{plim} \hat{\beta}_1 = \beta_1 \frac{\sigma_z^2}{\sigma_z^2 + \sigma_\omega^2}.$$

What does this equation tell us?

# Measurement error

## Summary

$$\therefore \text{plim} \hat{\beta}_1 = \beta_1 \frac{\sigma_z^2}{\sigma_z^2 + \sigma_\omega^2}.$$

What does this equation tell us?

**Measurement error in our explanatory variables** biases the coefficient estimates toward zero.

- This type of bias/inconsistency is often called **attenuation bias**.
- If **the measurement error correlates with the explanatory variables**, we have bigger problems with inconsistency/bias.



# Measurement error

## Summary

What about **measurement in the outcome variable**?

It doesn't really matter—it just increases our standard errors.

# Measurement error

## It's everywhere

### General cases

1. We cannot perfectly observe a variable.
2. We use one variable as a *proxy* for another.

### Specific examples

- GDP
- Population
- Crime/police statistics
- Air quality
- Health data
- Proxy *ability* with test scores