

Time series

EC 421, Set 7

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Prologue

Schedule

Last Time

Asymptotics, probability limits, and consistency

Today

Time series

Soon

Problem set and midterm

Asymptotics and consistency

Review

Asymptotics and consistency

Review

1. Compare/contrast the concepts *expected value* and *probability limit*.
2. What does it mean if the estimator $\hat{\theta}$ is consistent for θ ?
3. What is required for an omitted variable to bias OLS estimates of β_j ?
4. Does omitted-variable bias affect the consistency of OLS for β_j ?
5. What can we know about the direction of omitted-variable bias?
6. How does measurement error in an explanatory variable affect the OLS estimate for that variable's effect on the outcome variable?
7. How does measurement error in an outcome variable affect OLS?

Time-series data

Time-series data

Introduction

Up to this point, we focused on **cross-sectional data**.

- Sampled *across* a population (e.g., people, counties, countries).
- Sampled at *one moment* in time (e.g., Jan. 1, 2015).
- We had n individuals, each indexed i in $\{1, \dots, n\}$.

Time-series data

Introduction

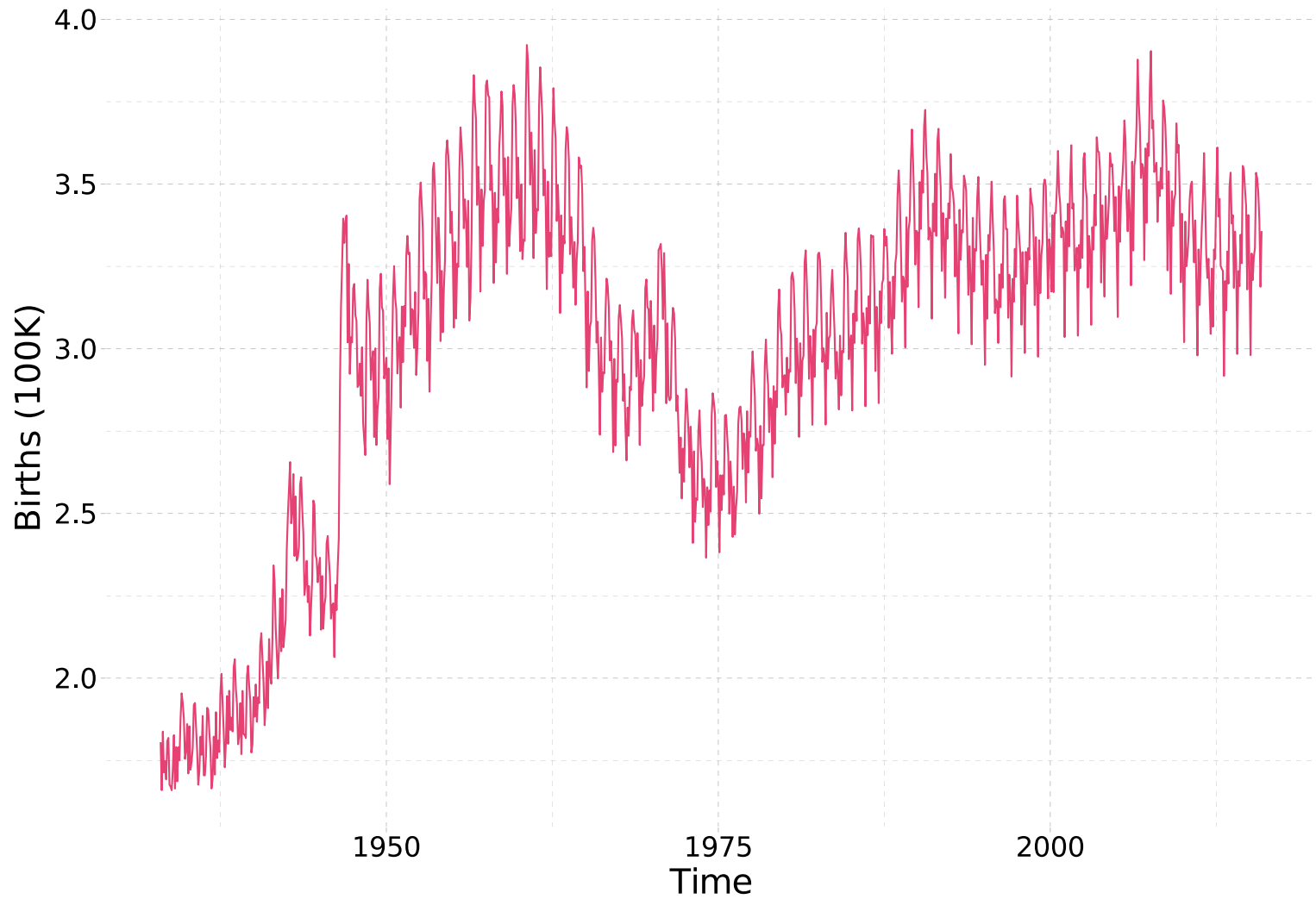
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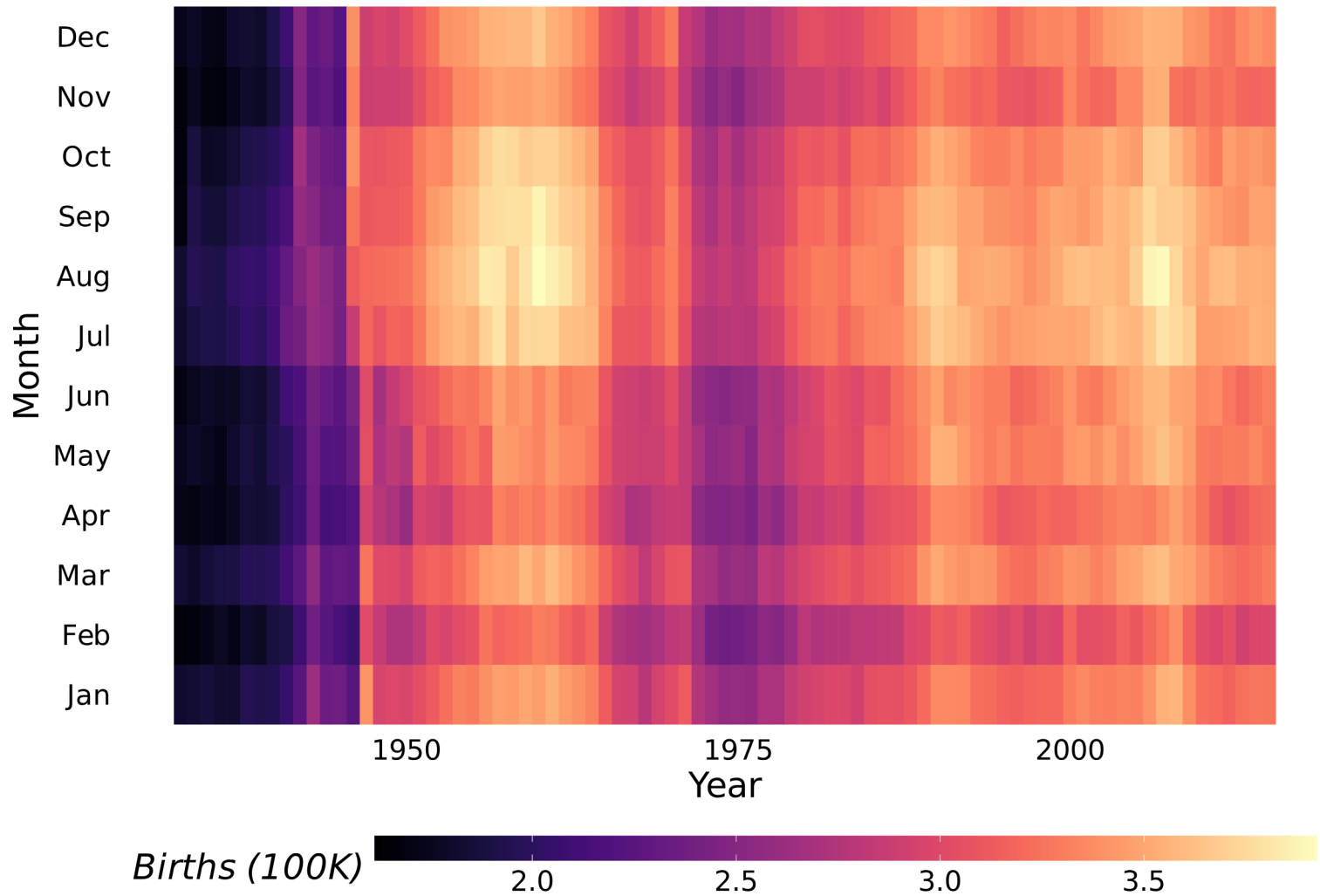
Today, we focus on a different type of data: **time-series data**.

- Sampled within **one unit/individual** (e.g., Oregon).
- Observe **multiple times** for the same unit (e.g., Oregon: 1990–2020).
- We have **T time periods**, each indexed t in $\{1, \dots, T\}$.

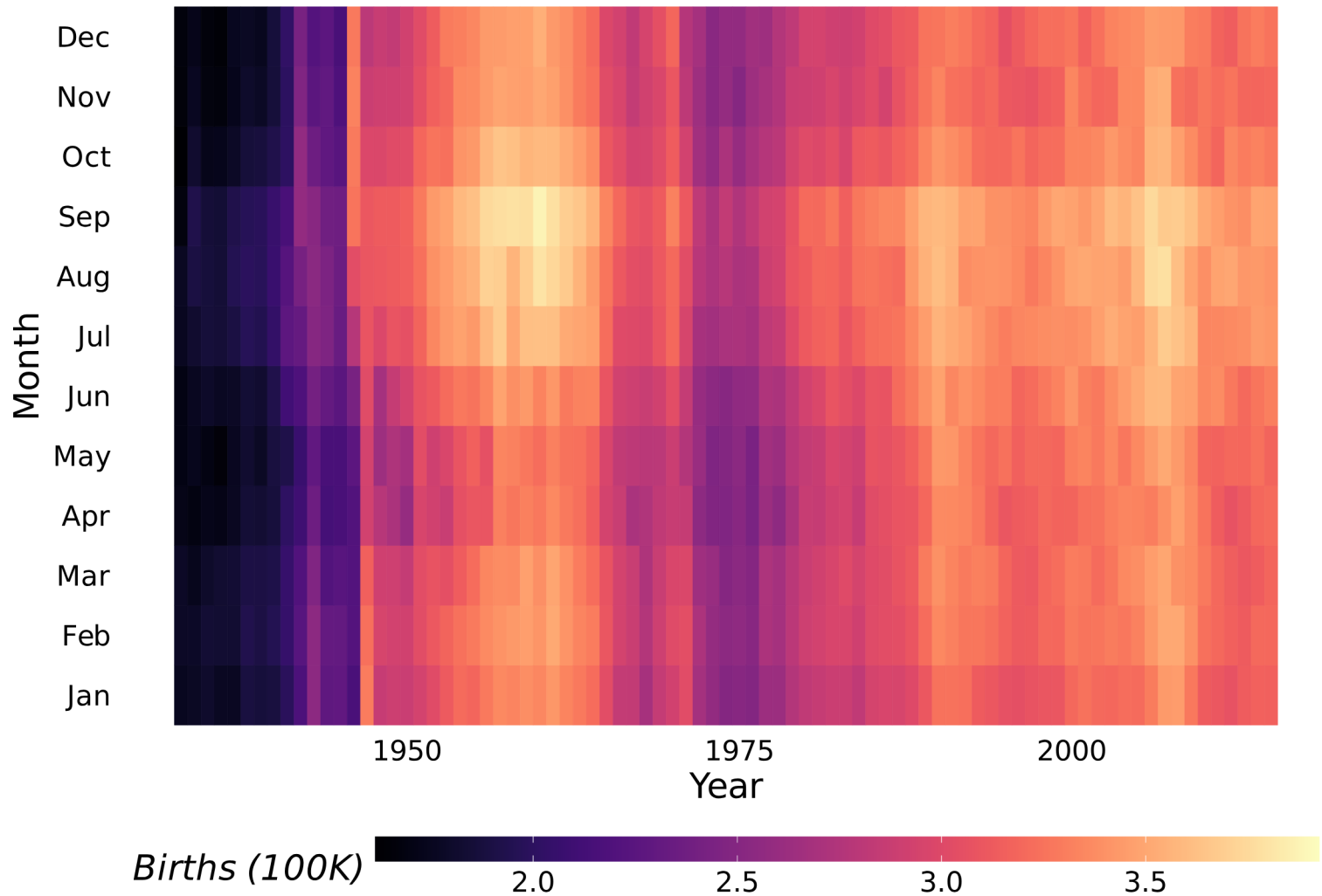
US monthly births, 1933–2015: Classic time-series graph



US monthly births, 1933–2015: Newfangled time-series graph



US monthly births per 30 days, 1933–2015: Newfangled time-series graph



Time-series models

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where $t - 1$ denotes the time period prior to t (*lagged* income or births).

Time-series models

Assumptions

1. **New:** Weakly persistent outcomes—essentially, x_{t+k} in the distant period $t + k$ is weakly correlated with period x_t (when k is "big").
2. y_t is a **linear function** of its parameters and disturbance.
3. There is **no perfect collinearity** in our data.
4. The u_t have conditional mean of zero (**exogeneity**), $\mathbf{E}[u_t|X] = 0$.
5. The u_t are **homoskedastic** with **zero correlation** between u_t and u_s , *i.e.*, $\text{Var}(u_t|X) = \text{Var}(u_t) = \sigma^2$ and $\text{Cor}(u_t, u_s|X) = 0$.
6. **Normality of disturbances**, *i.e.*, $u_t \stackrel{\text{iid}}{\sim} N(0, \sigma^2)$.

Time-series models

Model options

Time-series modeling boils down to two classes of models.

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2. **Dynamic models:** Allow for persistent effects.

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 - Models with **lagged explanatory** variables
 - **Autoregressive, distributed-lag** (ADL) models

Time-series models

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Option 1: **Static models**

Static models assume the outcome depends upon **only the current period**.

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Can be a very restrictive way to consider time-series data.

Time-series models

Model options

Option 2: **Dynamic models**

Dynamic models allow the outcome to depend upon other periods.

Time-series models

Model options

Option 2a: **Dynamic models** with lagged explanatory variables

These models allow the outcome to depend upon the explanatory variable(s) in other periods.

$$\text{Births}_t = \beta_0 + \beta_1 \text{Income}_t + \beta_2 \text{Income}_{t-1} + \beta_3 \text{Income}_{t-2} + \beta_4 \text{Income}_{t-3} + u_t$$

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Estimate *total* effects by summing lags' coefficients, e.g., $\beta_1 + \beta_2 + \beta_3 + \beta_4$.

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Note: We still assume current births don't affect future births.

Time-series models

Model options

Option 2b: **Autoregressive distributed-lag (ADL) models**

These models allow the outcome to depend upon the explanatory variable(s) and/or the outcome variable in prior periods.

$$\text{Births}_{t_{\text{red}}} = \beta_0 + \beta_1 \text{Income}_{t_{\text{red}}} + \beta_2 \text{Income}_{t_{\text{blue}}-1} + \beta_3 \text{Births}_{t_{\text{blue}}-1} + u_{t_{\text{red}}}$$

Time-series models

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Here, current income affects **current** births and **future** births.

In addition, **current births affect future births**—we're allowing lags of the outcome variable.

Autoregressive distributed-lag models

Numbers of lags

ADL models are often specified as $\text{ADL}(p, q)$, where

- p is the (maximum) number of **lags** for the outcome variable.
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$$\begin{aligned} \text{Births}_t = & \beta_0 + \beta_1 \text{Income}_t + \beta_2 \text{Income}_{t-1} + \beta_3 \text{Income}_{t-2} \\ & + \beta_4 \text{Births}_{t-1} + \beta_5 \text{Births}_{t-2} + u_t \end{aligned}$$

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Write out the model for period $t - 1$:

$$\text{Births}_{t-1} = \beta_0 + \beta_1 \text{Income}_{t-1} + \beta_2 \text{Births}_{t-2} + u_{t-1}$$

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which we can substitute in for Births_{t-1} in the first equation, *i.e.*,

$$\text{Births}_t = \beta_0 + \beta_1 \text{Income}_t + \underbrace{\beta_2(\beta_0 + \beta_1 \text{Income}_{t-1} + \beta_2 \text{Births}_{t-2} + u_{t-1})}_{\text{Births}_{t-1}} + u_t$$

Autoregressive distributed-lag models

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Continuing...

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We could then substitute in the equation for Births_{t-2} , Births_{t-3} , ...

Autoregressive distributed-lag models

Complexity

Eventually we arrive at

$$\begin{aligned}\text{Births}_t = & \beta_0 (1 + \beta_2 + \beta_2^2 + \beta_2^3 + \dots) + \\ & \beta_1 (\text{Income}_t + \beta_2 \text{Income}_{t-1} + \beta_2^2 \text{Income}_{t-2} + \dots) + \\ & u_t + \beta_2 u_{t-1} + \beta_2^2 u_{t-2} + \dots\end{aligned}$$

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The point?

Autoregressive distributed-lag models

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The point?

By including just **one lag of the dependent variable**—as in a ADL(1, 0)—we implicitly include for *many lags* of the explanatory variables and disturbances.[†]

[†] These lags enter into the equation in a very specific way—not the most flexible specification.

Autoregressive distributed-lag models

The partial-adjustment model

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Autoregressive distributed-lag models

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Partial-adjustment models help us model this situation.

Autoregressive distributed-lag models

The partial-adjustment model

Example

We want to know how the **desired number of cigarettes**, $\widetilde{\text{Cig}}_t$, changes with the current period's cigarette tax, e.g.,

$$\widetilde{\text{Cig}}_t = \beta_0 + \beta_1 \text{Tax}_t + u_t \quad (\text{A})$$

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Imagine **actual cigarette consumption**, Cig_t , doesn't change immediately (e.g., habit persistence). Instead, consumption depends upon **current desired level** and **previous consumption level**

$$\text{Cig}_t = \lambda \widetilde{\text{Cig}}_t + (1 - \lambda) \text{Cig}_{t-1} \quad (\text{B})$$

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Substituting $\widetilde{\text{Cig}}_t$ from (A) into (B) yields

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The equation in (C) is ADL(1, 0).

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We can also estimate/recover the speed-of-adjustment coefficient λ .

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We need both of these parts to be true for OLS to be unbiased.

OLS in time series

Unbiased coefficients

We need both parts of our exogeneity assumption for OLS to be unbiased:

$$\mathbf{E}[\hat{\beta}_1 | X] = \beta_1 + \mathbf{E}\left[\frac{\sum_t (x_t - \bar{x}) u_t}{\sum_t (x_t - \bar{x})^2} \middle| X\right]$$

i.e., to guarantee the numerator equals zero, we need $\mathbf{E}[u_t | X] = 0$ —for both $\mathbf{E}[u_t | X_t] = 0$ and $\mathbf{E}[u_t | X_s] = 0$ ($s \neq t$).

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Thus, **OLS is biased for dynamic models with lagged outcome variables.**

OLS in time series

Unbiased coefficients

To see why dynamic models with lagged outcome variables violate our exogeneity assumption, consider two periods of our simple ADL(1, 0) model.

$$\text{Births}_t = \beta_0 + \beta_1 \text{Income}_t + \beta_2 \text{Births}_{t-1} + u_t \quad (1)$$

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OLS in time series

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This correlation violates the second part of our exogeneity requirement.

OLS in time series

Consistent coefficients

All is not lost.

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For OLS to be **consistent**, we only need **contemporaneous exogeneity**.

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With contemporaneous exogeneity, OLS estimates for the coefficients in a time series model are consistent.

OLS in time series

Consistent coefficients

To see why OLS is consistent with contemporaneous exogeneity, consider the OLS estimate for β_1 in

$$\text{Births}_t = \beta_0 + \beta_1 \text{Births}_{t-1} + u_t$$

which we've shown (a few times) can be written

$$\hat{\beta}_1 = \beta_1 + \frac{\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}} \right) u_t}{\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}} \right)^2}$$

OLS in time series

Consistent coefficients

$$\begin{aligned}\text{plim } \hat{\beta}_1 &= \text{plim} \left(\beta_1 + \frac{\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}} \right) u_t}{\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}} \right)^2} \right) \\&= \beta_1 + \frac{\text{plim} \left[\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}} \right) u_t / T \right]}{\text{plim} \left[\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}} \right)^2 / T \right]} \\&= \beta_1 + \frac{\text{Cov}(\text{Births}_{t-1}, u_t)}{\text{Var}(\text{Births}_t)}\end{aligned}$$

OLS in time series

Consistent coefficients

$$\begin{aligned}\text{plim } \hat{\beta}_1 &= \text{plim} \left(\beta_1 + \frac{\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}} \right) u_t}{\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}} \right)^2} \right) \\&= \beta_1 + \frac{\text{plim} \left[\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}} \right) u_t / T \right]}{\text{plim} \left[\sum_t \left(\text{Births}_{t-1} - \overline{\text{Births}} \right)^2 / T \right]} \\&= \beta_1 + \frac{\text{Cov}(\text{Births}_{t-1}, u_t)}{\text{Var}(\text{Births}_t)} \\&= \beta_1 \quad \text{if } \text{Cov}(\text{Births}_{t-1}, u_t) = 0\end{aligned}$$

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Contemporaneous exogeneity gives us $\text{Cov}(\text{Births}_{t-1}, u_t) = 0$.

OLS in time series

Consistent coefficients

Thus, if we assume **contemporaneous exogeneity**, OLS is consistent for the coefficients, *even for models with lagged dependent variables*.

The end.

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Autoregressive distributed-lag models

Equilibrium effects

ADL models also offer interesting insights for long-run/equilibrium effects.

$$\text{Births}_t = \beta_0 + \beta_1 \text{Income}_t + \beta_2 \text{Births}_{t-1} + u_t$$

In this ADL(1, 0) model, β_1 gives the **short-run effect** of income on the number of births.

Autoregressive distributed-lag models

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In this ADL(1, 0) model, β_1 gives the **short-run effect** of income on the number of births. *i.e.*, how income in time t affects births in time t .

Autoregressive distributed-lag models

Equilibrium effects

Starting with

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Now rearrange...

$$\text{Births}^* - \beta_2 \text{Births}^* = \beta_0 + \beta_1 \text{Income}^*$$

$$(1 - \beta_2) \text{Births}^* = \beta_0 + \beta_1 \text{Income}^*$$

$$\text{Births}^* = \frac{\beta_0}{(1 - \beta_2)} + \frac{\beta_1}{(1 - \beta_2)} \text{Income}^*$$

Autoregressive distributed-lag models

Equilibrium effects

Short-run effect of income on births:

$$\text{Births}_t = \beta_0 + \beta_1 \text{Income}_t + \beta_2 \text{Births}_{t-1} + u_t$$

Long-run effect of income on births:

$$\text{Births}^* = \frac{\beta_0}{(1 - \beta_2)} + \frac{\beta_1}{(1 - \beta_2)} \text{Income}^*$$

Autoregressive distributed-lag models

Equilibrium effects

Another way to see this result:

We already showed

$$\text{Births}_t = \beta_0 + \beta_1 \text{Income}_t + \beta_2 \text{Births}_{t-1}$$

gives us

$$\begin{aligned} \text{Births}_t = & \beta_0 (1 + \beta_2 + \beta_2^2 + \beta_2^3 + \cdots) + \\ & \beta_1 (\text{Income}_t + \beta_2 \text{Income}_{t-1} + \beta_2^2 \text{Income}_{t-2} + \cdots) + \\ & u_t + \beta_2 u_{t-1} + \beta_2^2 u_{t-2} + \cdots \end{aligned}$$

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In equilibrium: $\text{Income}_t = \text{Income}_{t-k} = \text{Income}^*$ for all k .

Autoregressive distributed-lag models

Equilibrium effects

Substituting $\text{Income}_t = \text{Income}^*$ for all k
(and assuming no disturbances in equilibrium):

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Substituting $\text{Income}_t = \text{Income}^*$ for all k
(and assuming no disturbances in equilibrium):

$$\begin{aligned}\text{Births}_t &= \beta_0 (1 + \beta_2 + \beta_2^2 + \beta_2^3 + \dots) + \\ &\quad \beta_1 (\text{Income}^* + \beta_2 \text{Income}^* + \beta_2^2 \text{Income}^* + \dots) + \\ &= \beta_0 \left(\frac{1}{1 - \beta_2} \right) + \\ &\quad \beta_1 \left(\frac{1}{1 - \beta_2} \right) \text{Income}^*\end{aligned}$$

So long as $-1 < \beta_2 < 1$.[†]

[†] This simplification comes from $\sum_{k=0}^{\infty} p^k = \frac{1}{1-p}$ for $-1 < p < 1$.