Stretch Bar

Xuchen Han

February 2021

1 Problem Setup

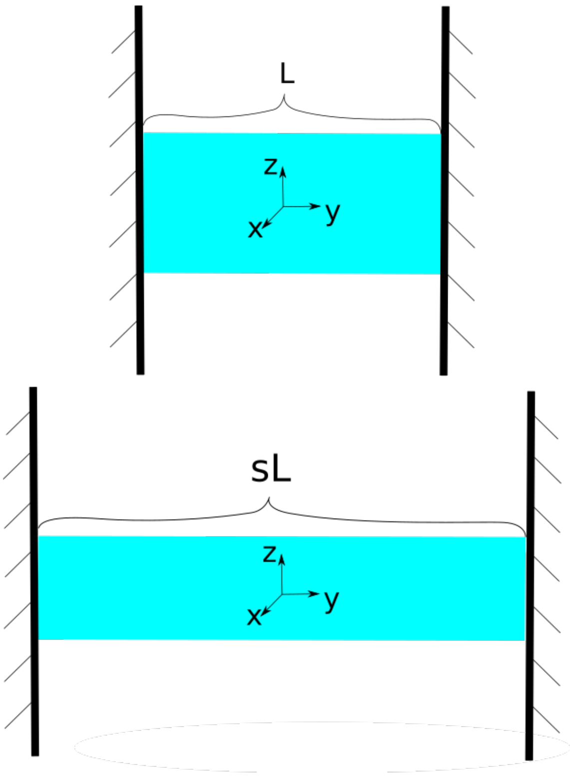

Figure 1: Problem description.

We consider an elastic block stretched on the side by Dirichlet boundary conditions. The bar occupies the domain $[-\frac{Lx}{2}, \frac{Lx}{2}] \times [-\frac{Ly}{2}, \frac{Ly}{2}] \times [-\frac{Lz}{2}, \frac{Lz}{2}]$. We assume linear elasticity with Young's modulus E and Poisson ratio ν. We apply Dirichlet boundary condition at $y = \frac{L}{2}$ and $y = -\frac{L}{2}$ so that the bar is stretched by a factor of $s = 1 + \epsilon$ in the y-direction. In other words, the boundary condition applied on the displacement $\mathbf{u} = (u_1(x, y, z), u_2(x, y, z), u_3(x, y, z))^T$ takes the form

$$u_2(x, -\frac{L}{2}, z) = -\frac{\epsilon L}{2}$$
$$u_2(x, \frac{L}{2}, z) = -\frac{\epsilon L}{2}.$$

We use the symmetry of the problem and restrict our attention to a quarter of the original domain $[0, \frac{Lx}{2}] \times [-\frac{Ly}{2}, \frac{Ly}{2}] \times [0, \frac{Lz}{2}]$ and add boundary conditions

$$u_1(0, y, z) = 0,$$
$$u_3(x, y, 0) = 0,$$

All other boundaries assume zero Neumann (no stress) boundary conditions. We assume zero gravity for simplicity.

2 Solution

We use $\mathbf{u} = (u_1(x,y,z), u_2(x,y,z), u_3(x,y,z))^T$ to denote displacement, $\mathbf{P}(x,y,z)$ to denote first Piola-Kirchhoff stress density, $\boldsymbol{\epsilon}$ to denote the infinitesimal strain $\left(\epsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i})\right)$. There is no external force as we assume zero gravity.

Under the stretching boundary condition, the bar should experience constant uni-axial stress. Therefore, we guess the stress has the form

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In other words, the stress has a constant $2-2$ component and zeros everywhere else. With this guess, the static equilibrium equation

$$\nabla \cdot \mathbf{P} = 0$$

is obviously satisfied. We then plug this guess into the constitutive model

$$\mathbf{P} = 2\mu\boldsymbol{\epsilon} + \lambda \mathrm{tr}(\boldsymbol{\epsilon})\mathbf{I},$$

where μ and λ are the Lame parameters related to E and ν via

$$\mu = \frac{E}{2(1+\nu)}, \tag{1}$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \tag{2}$$

and get the following equations on the diagonal:

$$P_{11} = 2\mu\epsilon_{11} + \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) = 0, \tag{3}$$

$$P_{22} = 2\mu\epsilon_{22} + \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) = \sigma, \tag{4}$$

$$P_{33} = 2\mu\epsilon_{33} + \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) = 0, \tag{5}$$

and the following equations on the off-diagonal:

$$\frac{P_{12}}{2\mu} = \epsilon_{12} = \epsilon_{21} = 0, \tag{6}$$

$$\frac{P_{13}}{2\mu} = \epsilon_{13} = \epsilon_{31} = 0, \tag{7}$$

$$\frac{P_{23}}{2\mu} = \epsilon_{23} = \epsilon_{32} = 0. \tag{8}$$

Solving equations (3) through (5), and utilizing the identity in (1) and (2), we get

$$\epsilon_{22} = \frac{\sigma}{E}, \tag{9}$$

$$\epsilon_{11} = \epsilon_{33} = \frac{-\nu\sigma}{E}. \tag{10}$$

Integrating equations (9) and (10), we get

$$u_1(x,y,z) = \frac{-\nu\sigma}{E}x + h_1(y,z), \tag{11}$$

$$u_2(x,y,z) = \frac{\sigma}{E}y + h_2(x,z), \tag{12}$$

$$u_3(x,y,z) = \frac{-\nu\sigma}{E}z + h_3(x,y), \tag{13}$$

where h_1, h_2, h_3 are arbitrary functions resulting from integration. Using boundary conditions $u_2(x, \frac{L}{2}, z) = \frac{\epsilon L}{2}$ and $u_2(x, -\frac{L}{2}, z) = -\frac{\epsilon L}{2}$, we get

$$h_2(x,z) = 0,$$
$$\sigma = \epsilon E.$$

Using boundary conditions $u_1(0, y, z) = 0$ and $u_3(x, y, 0) = 0$, we get

$$
\begin{aligned}
h_1(y, z) &= 0, \\
h_3(x, y) &= 0.
\end{aligned}
$$

Therefore, the final displacement is given by:

$$
\begin{aligned}
u_1(x, y, z) &= -\nu\epsilon x, \\
u_2(x, y, z) &= \epsilon y, \\
u_3(x, y, z) &= -\nu\epsilon z.
\end{aligned}
$$

We easily verify the displacement above satisfies equations (6) through (8).