

Graphs

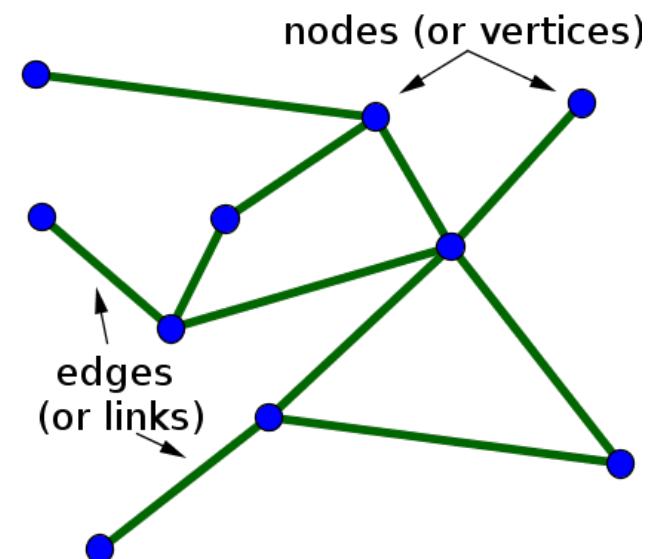
CSCI 4360/6360 Data Science II

Why graphs?

- Lots of data *is* graphs
 - Facebook, Twitter, citation data, and other social networks
 - The web, the blogosphere, the semantic web, Freebase, Wikipedia, Twitter, and other *information* networks
 - Text corpora (like RCV1), large datasets with discrete feature values, and other *bipartite* networks
 - nodes = documents or words
 - links connect document → word or word → document
 - Computer networks, biological networks (proteins, ecosystems, brains, ...), ...
 - Heterogeneous networks with multiple types of nodes
 - people, groups, documents

Properties of Graphs

- Nodes & Edges
- Set V of vertices/nodes v_1, \dots
- Set E of edges $(u, v), \dots$
 - Can be weighted/directed/labeled
- *Degree of v* is # of edges on v
 - *Indegree* and *outdegree* for weighted graphs
- *Path* is a sequence of edges $(u_1, v_1), (u_2, v_2), \dots$
- *Geodesic path between u and v* is shortest path connecting them
 - Diameter is $\max_{u, v \in V} \{\text{length of geodesic between } u, v\}$
 - Effective diameter is 90th percentile
 - Mean diameter is over connected pairs
- *(Connected) component* is subset of nodes that are all pairwise connected via paths
- *Clique* is subset of nodes that are all pairwise connected via *edges*
- *Triangle* is a clique of size three



Properties of Graphs

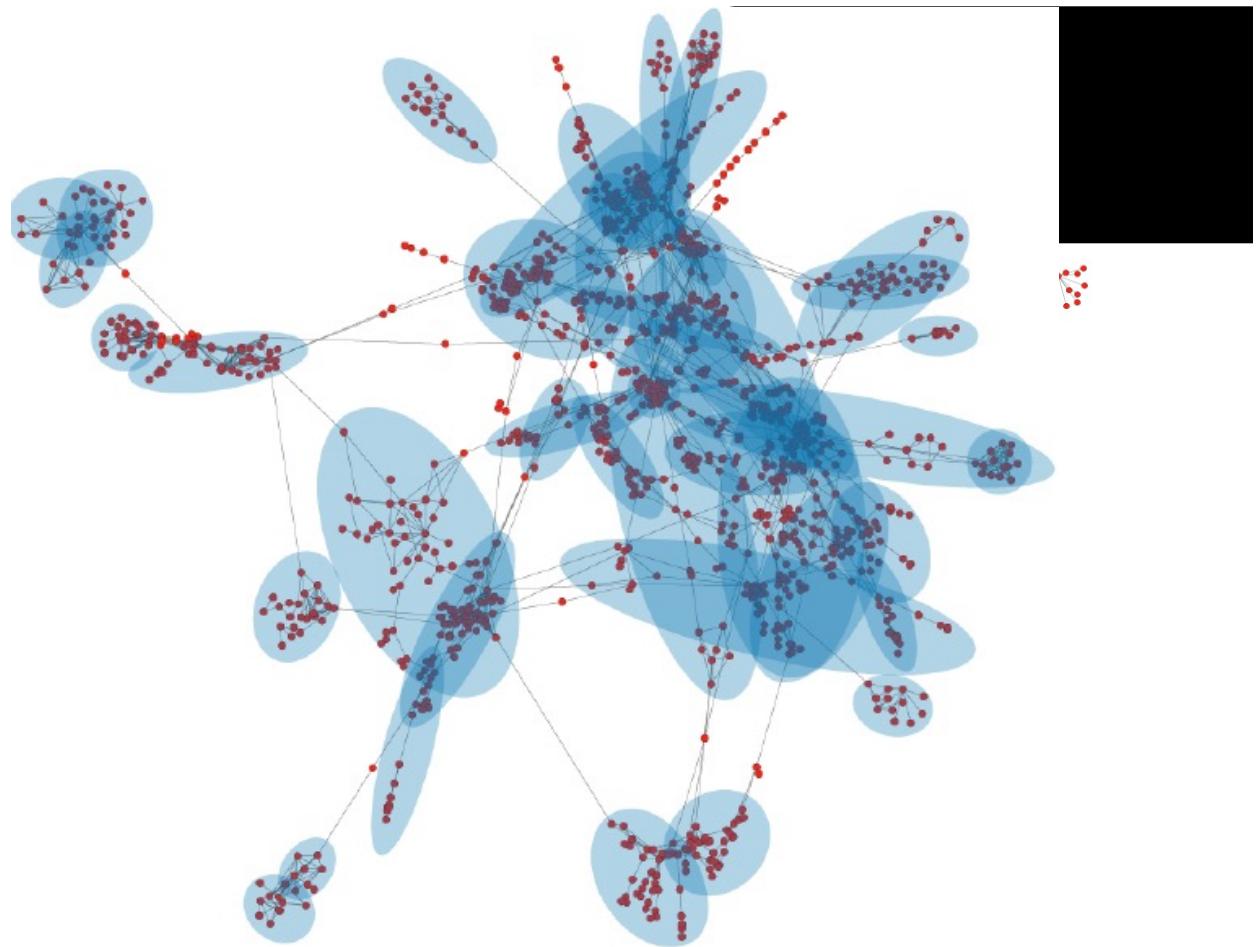
- Descriptive statistics
- Number of connected components
- Diameter
- Degree distribution
- Centrality
- ...

Properties of Graphs

- Models of formation and growth
- Erdos-Rayni
- Watts-Strogatz
- Preferential attachment
- Stochastic block models
- ...

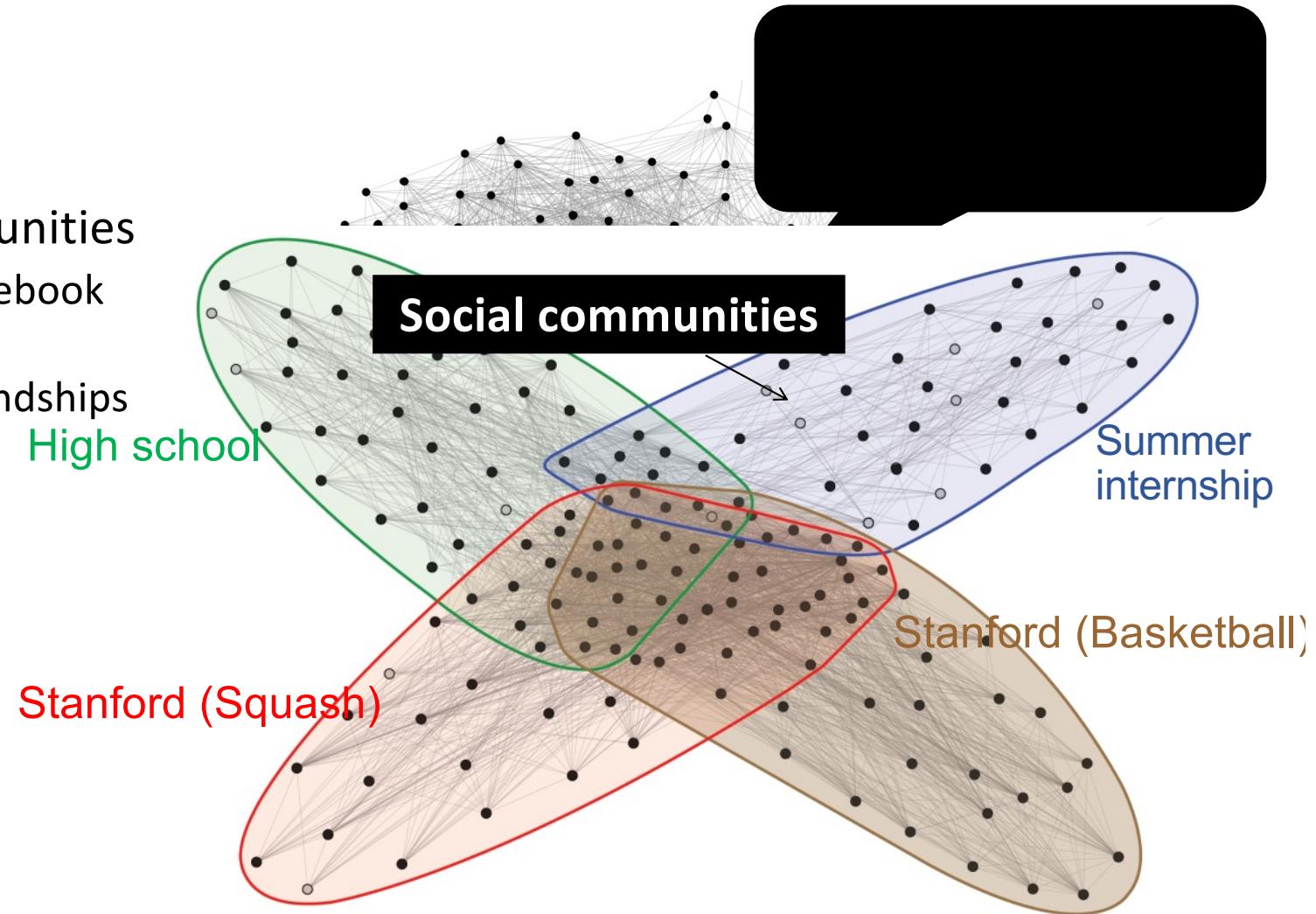
Biology

- Protein-protein interaction networks
 - Nodes: proteins
 - Edges: interactions
- Functional modules
- Cell phenotype hierarchies

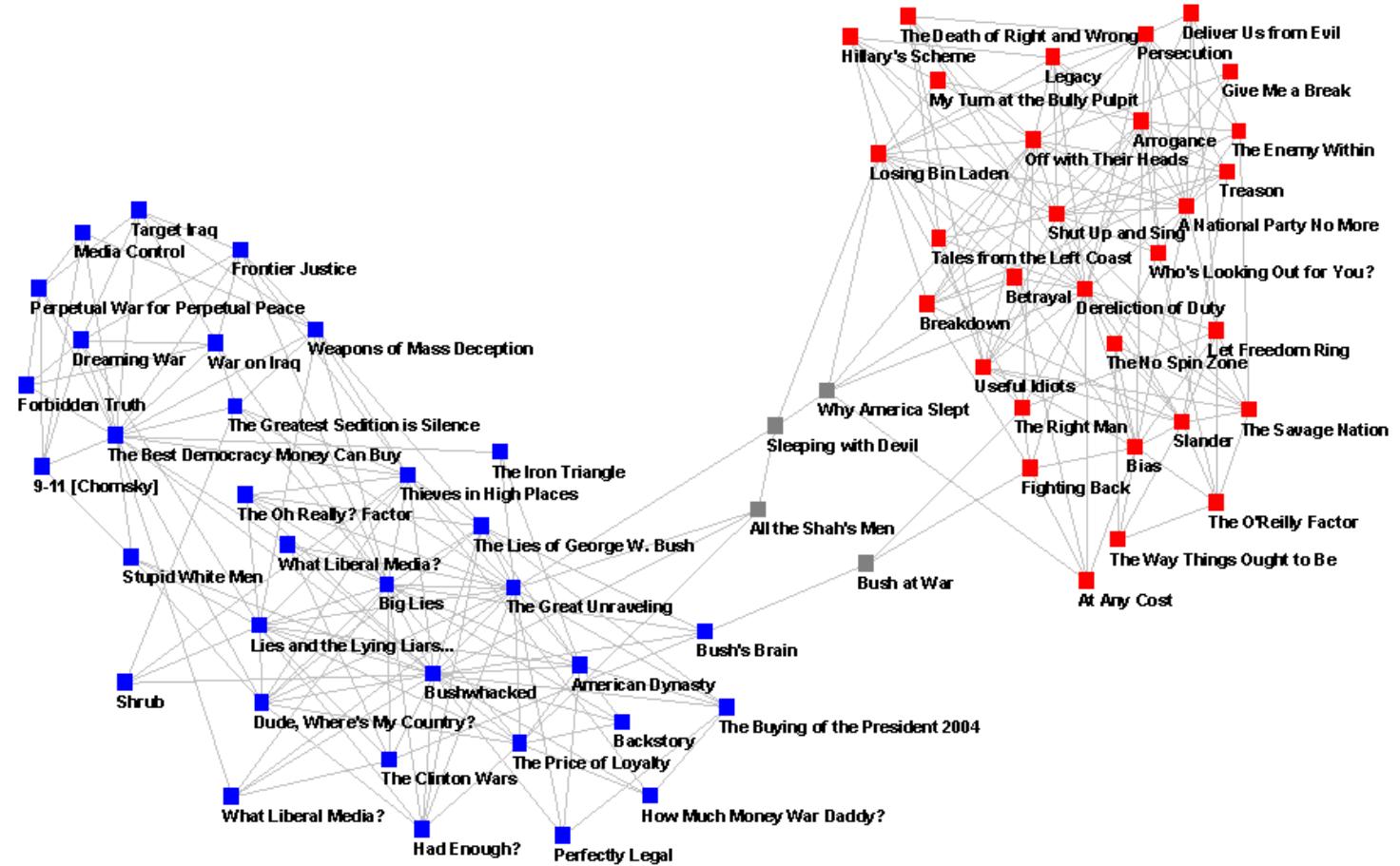


Facebook

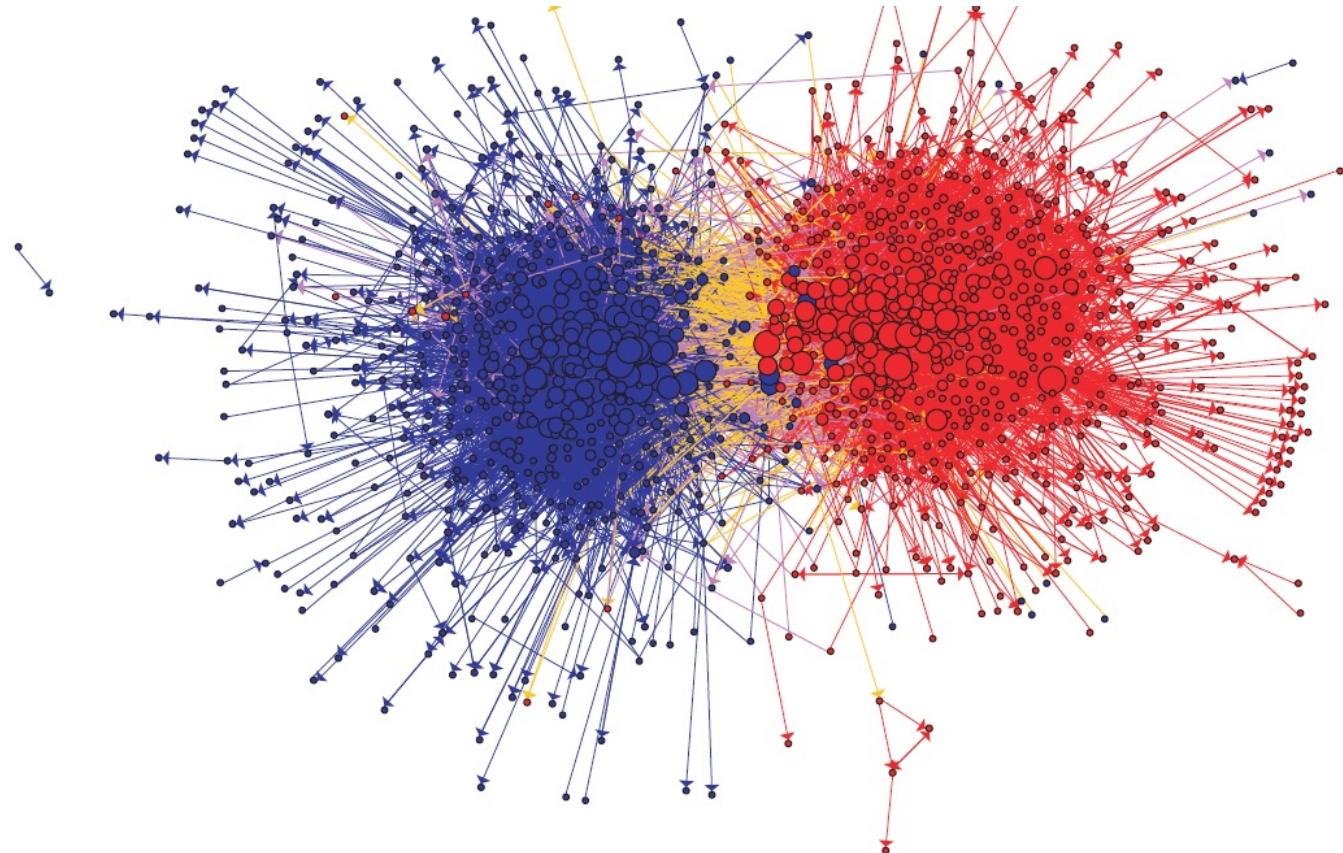
- Social communities
 - Nodes: Facebook users
 - Edges: Friendships



Blogs



Blogs



Graph Models

- Fundamental graph types

Erdos-Renyi graphs

- Take n nodes, and connect each pair with probability p
 - Mean degree is $z=p(n-1)$

$$\Pr[\text{degree}(v) = k] = p_k = \binom{n}{k} p^k (1-p)^{n-k} \approx \frac{z^k e^{-z}}{k!} \quad \text{for fixed } z, \text{ large } n$$

$$\binom{n}{k} p^k \approx \frac{n^k p^k}{k!} = \frac{(np)^k}{k!} \approx \frac{z^k}{k!}$$

$$e^{-z}$$

Erdos-Renyi graphs

- Take n nodes, and connect each pair with probability p
 - Mean degree is $z=p(n-1)$
 - Mean number of neighbors distance d from v is z^d
 - How large does d need to be so that $z^d \geq n$?
 - If $z > 1$, $d = \log(n)/\log(z)$
 - If $z < 1$, you can't do it
- So:
 - *There tend to be either many small components ($z < 1$) or one large one ($z > 1$) giant connected component*
- Another intuition:
 - If there are two large connected components, then with high probability a few random edges will link them up.

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 - If $z < 1$, you can't do it
- So:
 - *If $z > 1$, diameters tend to be small (relative to n)*

64 of 296 chains
succeed, avg chain
length is 6.2

Sociometry, Vol. 32, No. 4. (Dec., 1969), pp. 425-443.

**An Experimental Study of the
Small World Problem***

JEFFREY TRAVERS

Harvard University

AND

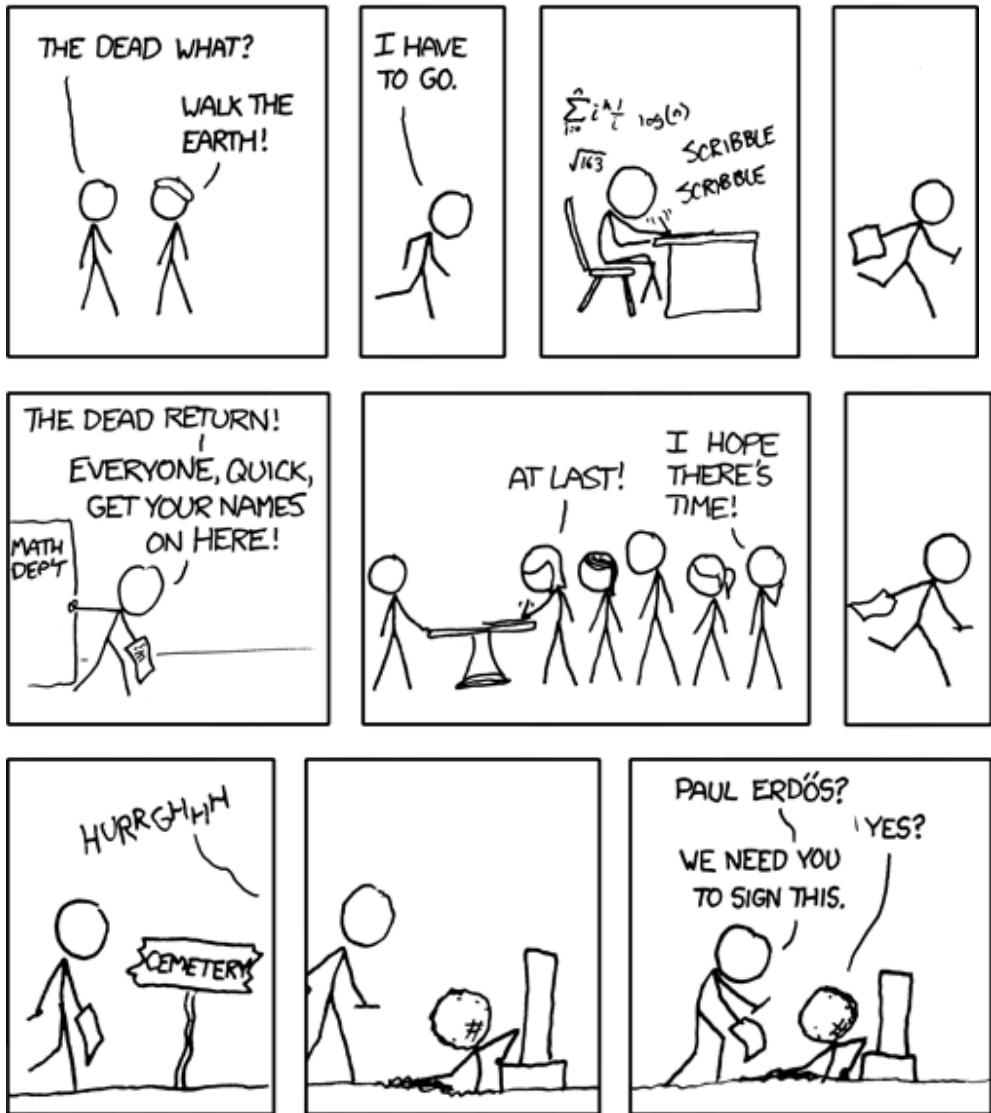
STANLEY MILGRAM

The City University of New York

Arbitrarily selected individuals ($N=296$) in Nebraska and Boston are asked to generate acquaintance chains to a target person in Massachusetts, employing “the small world method” (Milgram, 1967). Sixty-four chains reach the target person. Within this group the mean number of intermediaries between starters and targets is 5.2. Boston starting chains reach the target

Illustrations of the Small World

- Milgram's experiment
- Erdős numbers
 - <http://www.ams.org/mathscinet/searchauthors.html>
- Bacon numbers
 - <http://oracleofbacon.org/>
- LinkedIn
 - <http://www.linkedin.com/>
 - Privacy issues: the whole network is *not* visible to all



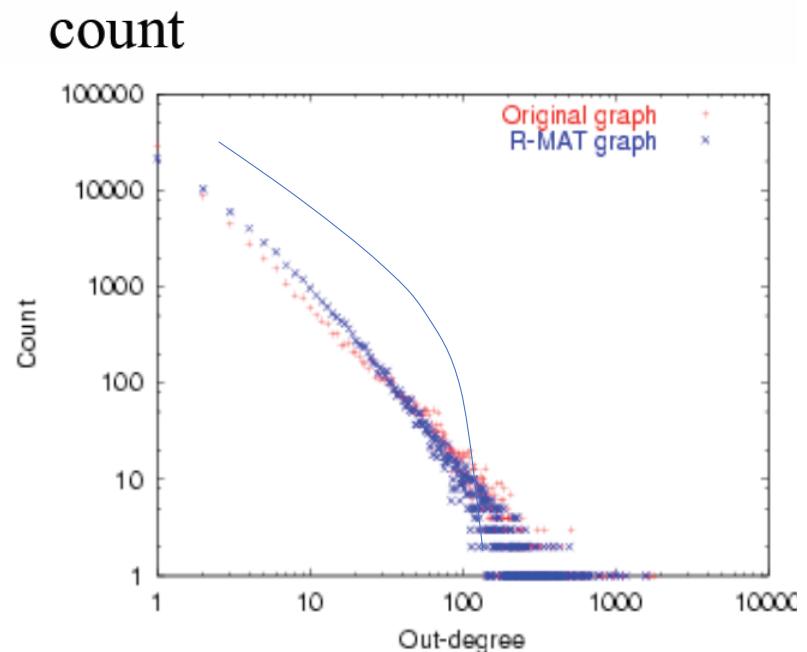
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social	film actors	undirected	449 913	25 516 482	113.43	3.8	2.3	0.20	0.78
	company directors	undirected	7 673	55 392	14.44	4.40	—	0.59	0.88
	math coauthorship	undirected	253 339	496 489	3.92	7.7	—	0.15	0.34
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	citation network	directed	783 339	6 716 193	8.57		3.0/—		
	Roget's Thesaurus	directed	1 022	5 103	4.99	4.47	—	0.13	0.15
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Erdos-Renyi graphs

- A good model of degree distribution in "natural" networks?

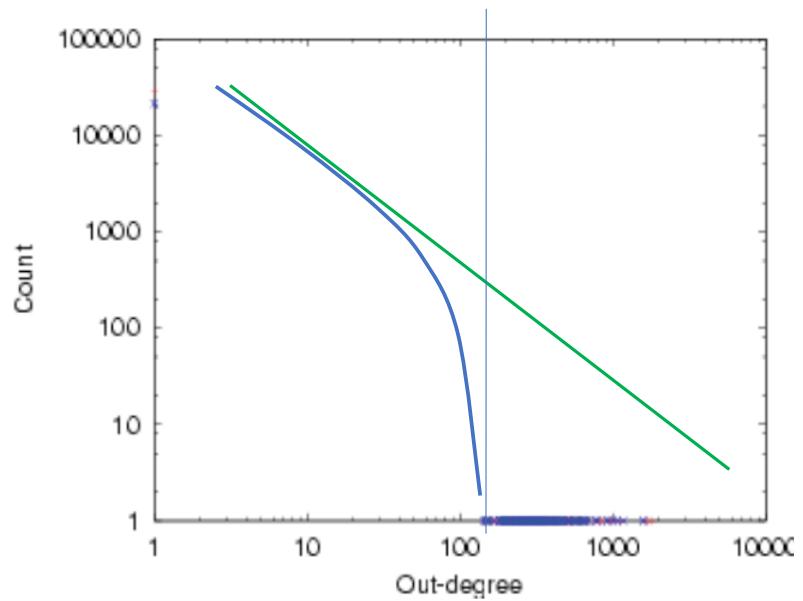
Degree distribution

- Plot cumulative degree
 - X axis is degree
 - Y axis is #nodes that have degree at least k
- Typically use a log-log scale
 - Straight lines are a power law; normal curve dives to zero at some point



Degree distribution

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 - X axis is degree
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- Typically use a log-log scale
 - Straight lines are a power law; normal curve dives to zero at some point
 - This defines a “scale” for the network



$$p_k \propto k^{-\alpha}$$

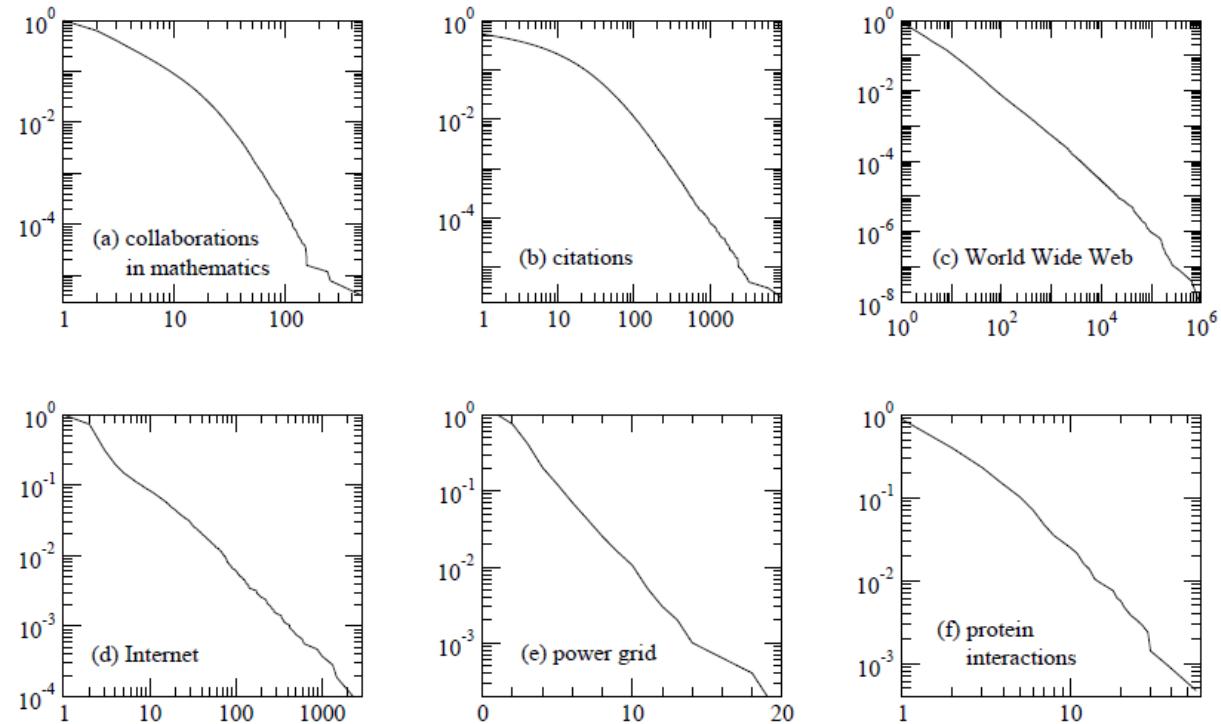


FIG. 6 Cumulative degree distributions for six different networks. The horizontal axis for each panel is vertex degree k (or in-degree for the citation and Web networks, which are directed) and the vertical axis is the cumulative probability distribution of degrees, i.e., the fraction of vertices that have degree greater than or equal to k . The networks shown are: (a) the collaboration network of mathematicians [182]; (b) citations between 1981 and 1997 to all papers cataloged by the Institute for Scientific Information [351]; (c) a 300 million vertex subset of the World Wide Web, *circa* 1999 [74]; (d) the Internet at the level of autonomous systems, April 1999 [86]; (e) the power grid of the western United States [416]; (f) the interaction network of proteins in the metabolism of the yeast *S. Cerevisiae* [212]. Of these networks, three of them, (c), (d) and (f), appear to have power-law degree distributions, as indicated by their approximately straight-line forms on the doubly logarithmic scales, and one (b) has a power-law tail but deviates markedly from power-law behavior for small degree. Network (e) has an exponential degree distribution (note the log-linear scales used in this panel) and network (a) appears to have a truncated power-law degree distribution of some type, or possibly two separate power-law regimes with different exponents.

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Graphs

- Some common properties of graphs:
 - **Distribution of node degrees**
 - Distribution of cliques (e.g., triangles)
 - **Distribution of paths**
 - **Diameter** (max shortest-path)
 - Effective diameter (90^{th} percentile)
 - **Connected components**
 - ...
- Some types of graphs to consider:
 - Real graphs (social & otherwise)
 - Generated graphs:
 - Erdos-Renyi “Bernoulli” or “Poisson”
 - Watts-Strogatz “small world” graphs
 - Barabasi-Albert “preferential attachment”
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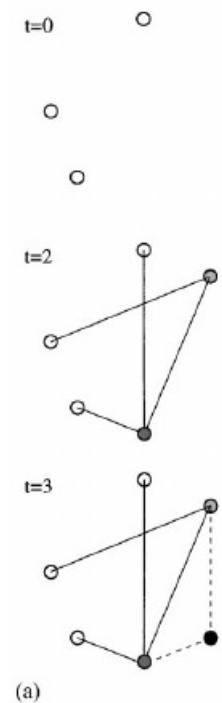
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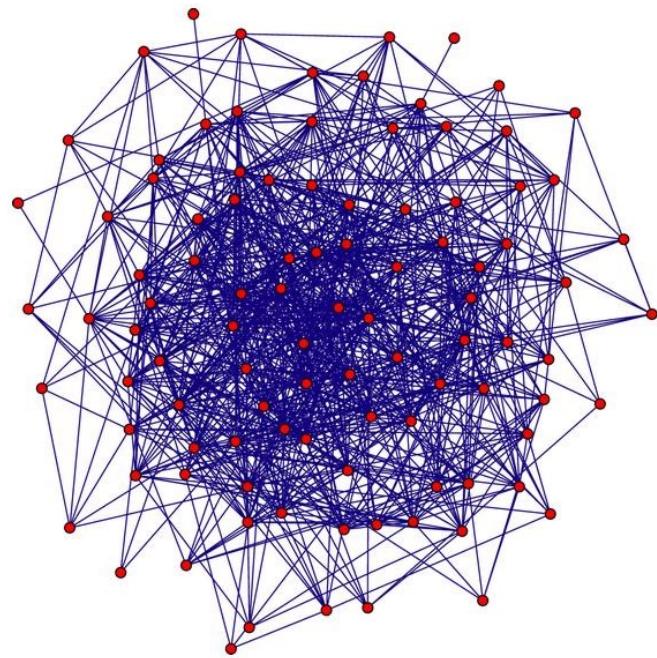
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Barabasi-Albert Networks

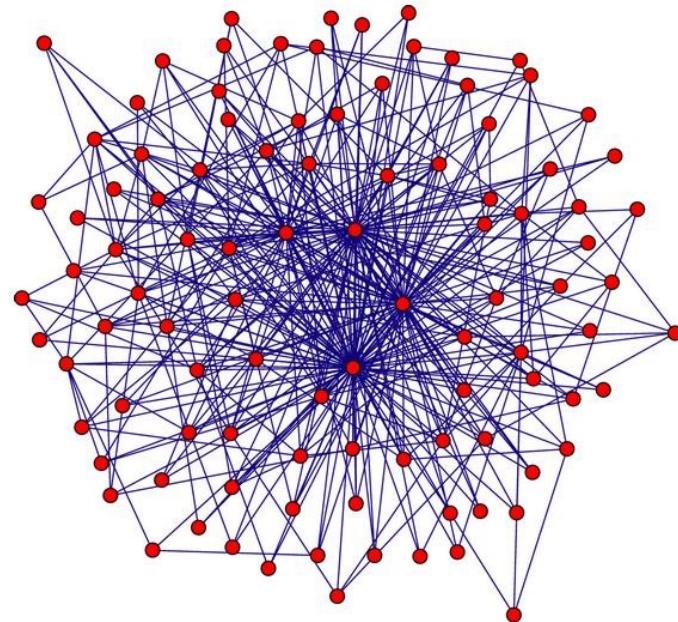
- *Science* **286** (1999)
 - Start from a small number of node, add a new node with m links
 - **Preferential Attachment**
 - Probability of these links to connect to existing nodes is proportional to the node's degree
- $$\prod_i (k_i) = \frac{k_i}{\sum_j k_j}$$
- ‘Rich gets richer’
 - This creates ‘hubs’: few nodes with very large degrees





Random graph
(Erdos Renyi)

Preferential attachment
(Barabasi-Albert)



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Homophily

- One definition: excess edges between similar nodes
- Another definition: excess edges between common neighbors of v

$$CC(v) = \frac{\text{\# triangles connected to } v}{\text{\# pairs connected to } v}$$

$$CC(V, E) = \frac{1}{|V|} \sum_v CC(v)$$

$$CC'(V, E) = \frac{\text{\# triangles in graph}}{\text{\# length 3 paths in graph}}$$

Homophily

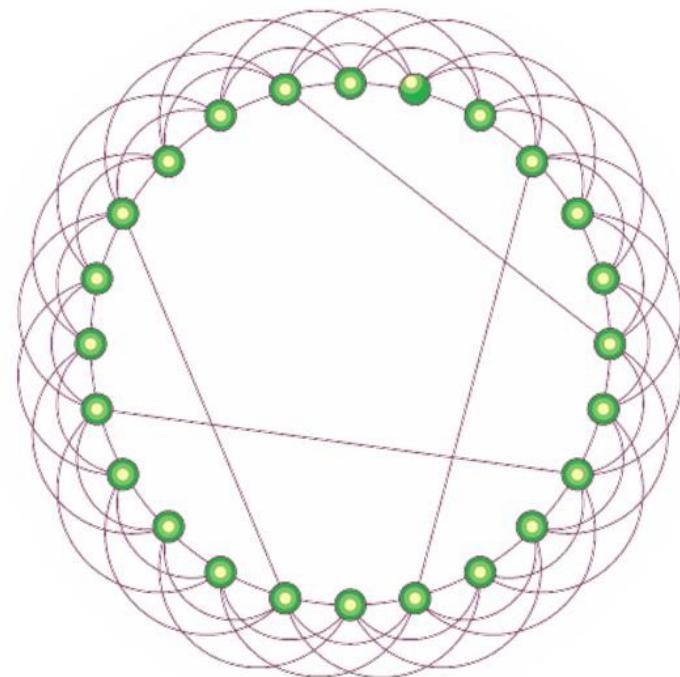
- In a random Erdos-Renyi graph:

$$CC'(V, E) = \frac{\# \text{triangles in graph}}{\# \text{length 3 paths in graph}} \approx \frac{1}{n} \text{ for large } n$$

- Probably not realistic!
- In a natural graph, two of your mutual friends might also be friends
 - Both in the same class or organization
 - You introduced them
 - They introduced you

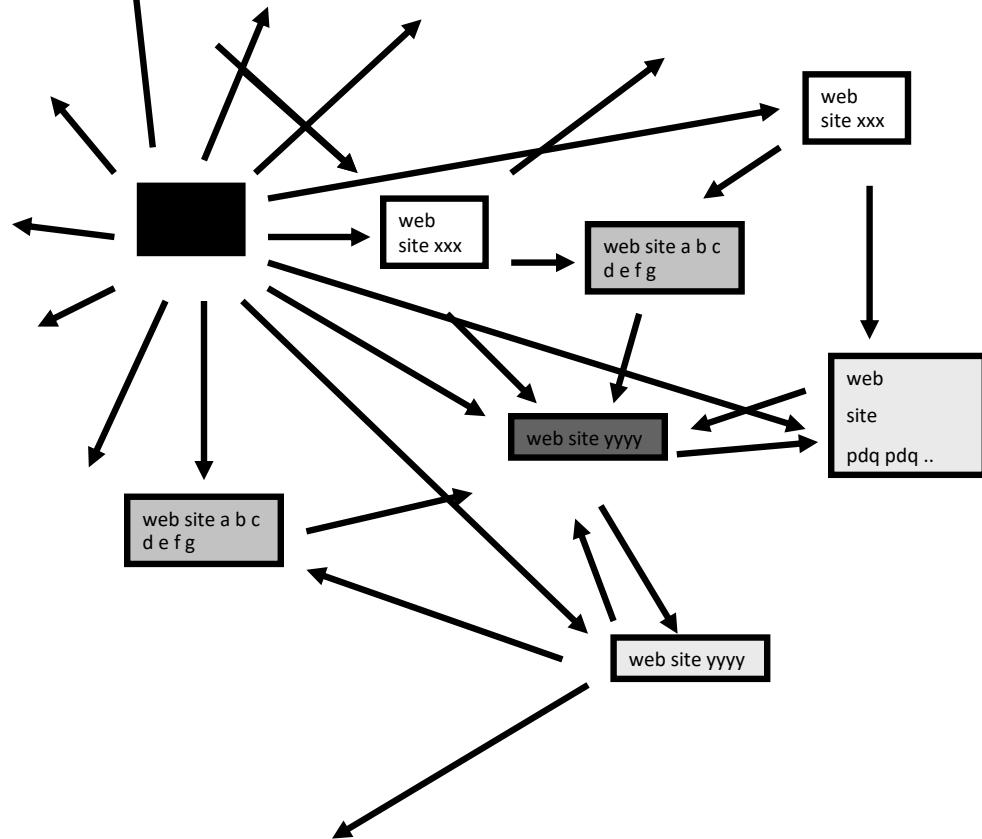
Watts-Strogatz model

- Start with a ring
- Connect each node to k nearest neighbors
 - \rightarrow homophily
- Add some random shortcuts from one point to another
 - \rightarrow small diameter
- Degree distribution *not* scale-free
- Generalizes to d dimensions



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Google's PageRank



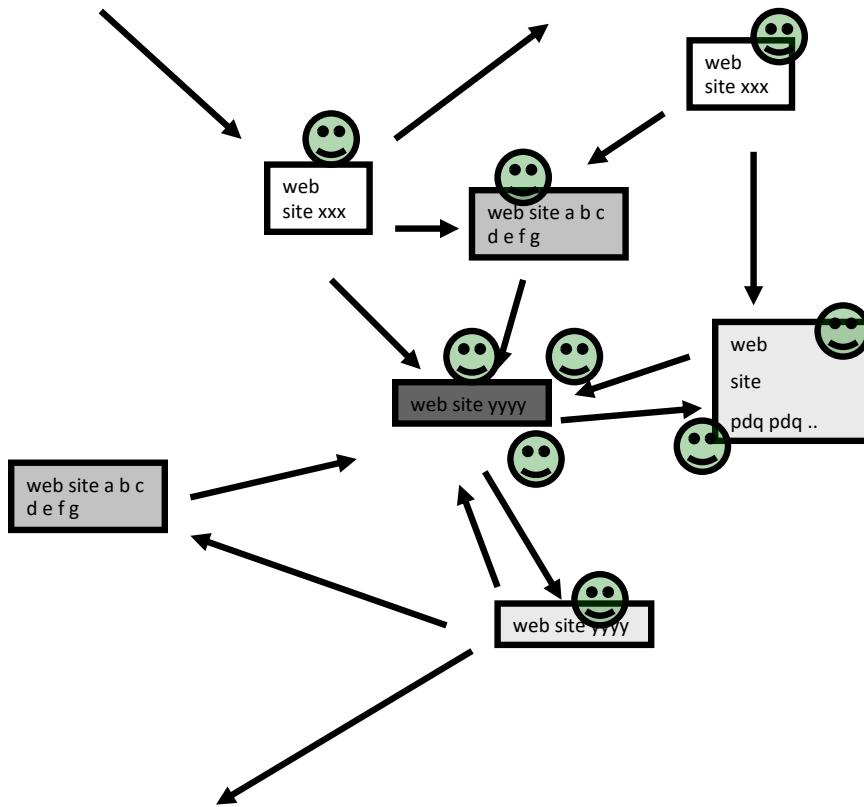
Inlinks are “good”
(recommendations)

Inlinks from a “good” site
are better than inlinks from
a “bad” site

but inlinks from sites with
many outlinks are not as
“good”...

“Good” and “bad” are
relative.

Google's PageRank

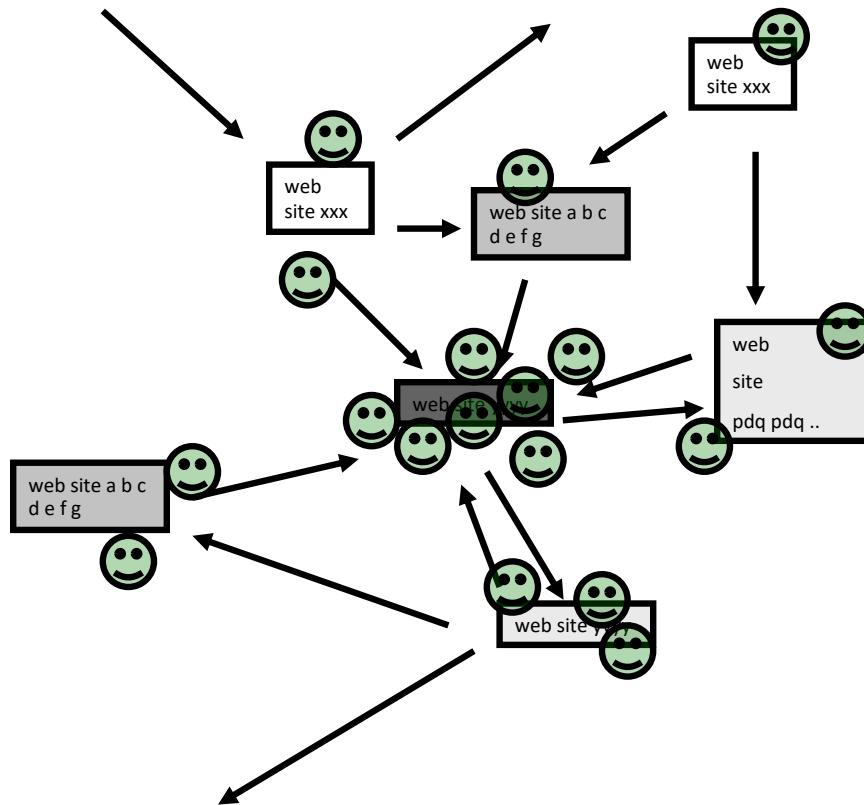


Imagine a “pagehopper” that always either

- follows a random link, or
- jumps to random page

Google's PageRank

(Brin & Page, <http://www-db.stanford.edu/~backrub/google.html>)



Imagine a “pagehopper” that always either

- follows a random link, or
- jumps to random page

PageRank ranks pages by the amount of time the pagehopper spends on a page:

- or, if there were many pagehoppers, PageRank is the expected “crowd size”

Random Walks

G : a graph

P : transition probability matrix

$$P(u,v) = \begin{cases} \frac{1}{d_u} & \text{if } u : v, \\ 0 & \text{otherwise.} \end{cases} \quad d_u := \text{the degree of } u.$$

A lazy walk:

$$W = \frac{I + P}{2}$$

Random Walks: PageRank

A (bored) surfer

- either surf a random webpage
with probability α
- or surf a linked webpage
with probability $1 - \alpha$



α : the jumping constant

$$p = \alpha \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n} \right) + (1 - \alpha) pW$$

Random Walks: PageRank

Two equivalent ways to define PageRank $p=pr(\alpha, s)$

$$(1) \quad p = \alpha s + (1 - \alpha)pW$$

$$(2) \quad p = \alpha \sum_{t=0}^{\infty} (1 - \alpha)^t (sW^t)$$

$s = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$  the (original) PageRank

$s = \text{some "seed", e.g., } (1, 0, \dots, 0)$

 personalized PageRank

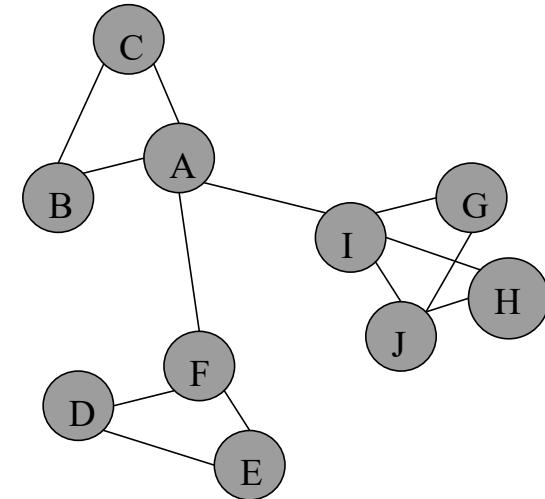
Graph = Matrix

Vector = Node → Weight

	A	B	C	D	E	F	G	H	I	J
A	-	1	1				1			
B	1	-	1							
C	1	1	-							
D				-	1	1				
E				1	-	1				
F	1			1	1	-				
G						-		1	1	
H							-	1	1	
I						1	1	-	1	
J						1	1	1		-

M

V



PageRank

- Let $\mathbf{u} = (1/N, \dots, 1/N)$
 - dimension = #nodes N
- Let A = adjacency matrix: $[a_{ij}=1 \Leftrightarrow i \text{ links to } j]$
- Let $\mathbf{W} = [w_{ij} = a_{ij}/\text{outdegree}(i)]$
 - w_{ij} is probability of jump from i to j
- Let $\mathbf{v}^0 = (1, 1, \dots, 1)$
 - or anything else you want
- Repeat until converged:
 - Let $\mathbf{v}^{t+1} = c\mathbf{u} + (1-c)\mathbf{W}\mathbf{v}^t$
 - c is probability of jumping “anywhere randomly”

Next: spectral clustering!

