Sheet 1

Due Tuesday 16th January

Hand in solutions to 1b, 2, 3, 5a, 6b

Note: Show means prove!

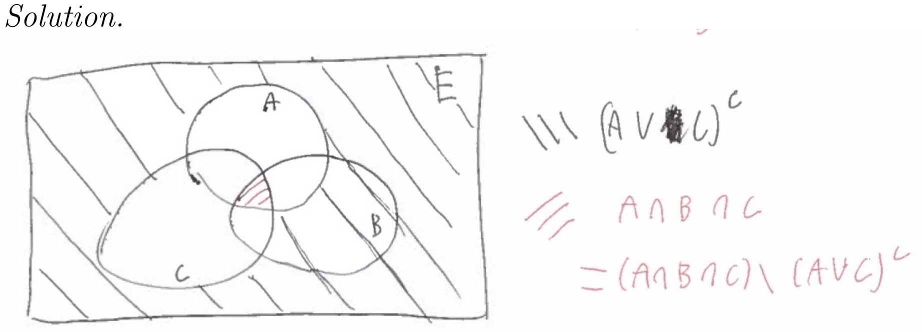
1. Shade the following sets on Venn diagrams:

   (a) $(A \cap B) \cup (A \,\Delta\, B)$

   Solution.

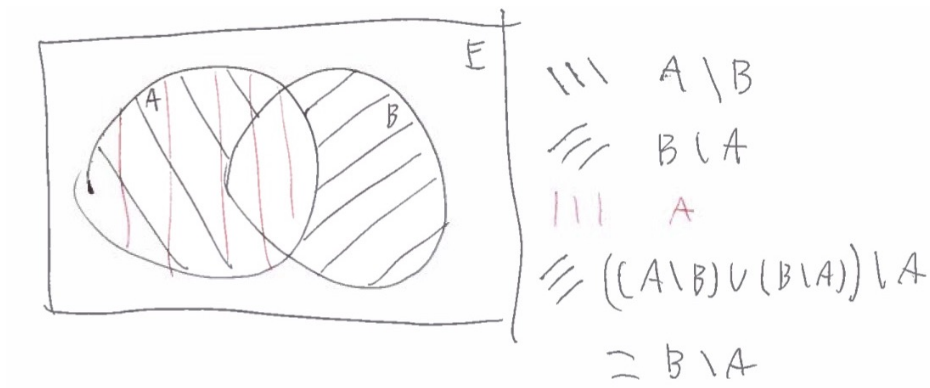
   **(b) $(A \cap B \cap C) \setminus (A \cup C)^c$ (2 marks)

   Solution.



   (c) $\Big((A \setminus B) \cup (B \setminus A)\Big) \setminus A$

   Solution.



**2. Show that

$$A \,\Delta\, B = (A \setminus B) \cup (B \setminus A). \qquad \text{(2 marks)}$$

Solution. We first show $(A \setminus B) \cup (B \setminus A) \subseteq A \Delta B$. If $x \in (A \setminus B) \cup (B \setminus A)$ then either $x \in A$ and $x \notin B$ or $x \in B$ and $x \notin A$. In both cases $x \in A \cup B$ and $x \notin A \cap B$. Thus $x \in (A \cup B) \setminus (A \cap B) = A \,\Delta\, B$.

Now we show that $A \,\Delta\, B \subseteq (A \setminus B) \cup (B \setminus A)$. If $x \in A \,\Delta\, B$ then $x \in A$ or $x \in B$, and also $x \notin A \cap B$. The last property means that either $x \notin A$ or $x \notin B$. Note that $x \in A$ or $x \in B$ but $x \notin A$ means that $x \in B \setminus A$. Similarly, $x \in A$ or $x \in B$ but $x \notin B$ means that $x \in A \setminus B$. Thus $x \in B \setminus A$ or $x \in A \setminus B$, i.e. $x \in (A \setminus B) \cup (B \setminus A)$.

**3. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$, where A, B and C are any three sets. (2 marks)

Solution.

$$\begin{gathered}
(x, y) \in A \times (B \cap C) \iff \\
x \in A \text{ and } y \in B \cap C \iff \\
x \in A \text{ and } (y \in B \text{ and } y \in C) \iff \\
(x, y) \in A \times C \text{ and } (x, y) \in A \times B \iff \\
(x, y) \in (A \times B) \cup (A \times C).
\end{gathered}$$

4. Prove that

$$\left(\bigcap_{i \in I} A_i \right)^c = \bigcup_{i \in I} A_i^c.$$

Solution.

$$x \in (\bigcap_{i \in I} A_i)^c \iff$$
$$x \notin \bigcap_{i \in I} A_i \iff$$
$$x \notin A_i \text{ for some } i \in I \iff$$
$$x \in \bigcup_{i \in I} A_i^c.$$

5. Determine the following sets:

**(a) $A = (\mathbb{Z} \setminus \{-1, 10, 3\}) \setminus (\mathbb{Z} \setminus \{2, 3, 6\})$; (2 marks)
Solution. If $x \in A$ then $x \in \mathbb{Z}$ and $x \notin \mathbb{Z} \setminus \{2, 3, 6\}$, so we have $x \in \{2, 3, 6\}$. If $x = 2$ or 6 then $x \in \mathbb{Z} \setminus \{-1, 10, 3\}$ and $x \notin \mathbb{Z} \setminus \{2, 3, 6\}$, thus $x \in A$. If $x = 3$ then $x \notin \mathbb{Z} \setminus \{-1, 10, 3\}$ so $x \notin A$. In conclusion $A = \{2, 6\}$.

(b) $B = \bigcup_{i \in \mathbb{N}} \{i, i+1, i+2\}$;
Solution. $\bigcup_{i \in \mathbb{N}} \{i, i+1, i+2\} = \mathbb{N}$ (clear)

(c) $C = \bigcap_{i \in \mathbb{N}} \{i, i+1, i+2\}$.
Solution. $\bigcap_{i \in \mathbb{N}} \{i, i+1, i+2\} = \emptyset$ (clear)

6. (a) Let $B_n = (-\frac{1}{n}, 1]$ for each $n \in \mathbb{N}$. Show that

$$\bigcap_{n \in \mathbb{N}} B_n = [0, 1].$$

Solution. If $x \in [0, 1]$ then $x \in B_n = (-\frac{1}{n}, 0]$, because $-\frac{1}{n} < x \leq 1$. Thus $x \in \bigcup_{n \in \mathbb{N}} B_n$.
If $x \in \bigcup_{n \in \mathbb{N}} B_n$, then $x \in B_n$ for all $n \in \mathbb{N}$. Then $-\frac{1}{n} < x \leq 1$ for all $n \in \mathbb{N}$. But this implies that $0 \leq x \leq 1$, so $x \in [0, 1]$.

**(b) Let $T_n = [\frac{1}{n}, 1]$for each $n \in \mathbb{N}$. Show that

$$\bigcup_{n \in \mathbb{N}} T_n = (0, 1].$$ (2 marks)

Solution. Let $x \in (0,1]$, that is $0 < x \leq 1$. Then $x \in [\frac{1}{n}, 1]$ if $n \in \mathbb{N}$ satisfies $n \geq \frac{1}{x}$. Thus $x \in T_n$ if $n \in \mathbb{N}$ satisfies $n \geq \frac{1}{x}$. Thus $x \in \bigcup_{n\in\mathbb{N}} T_n$.
If $x \in \bigcup_{n\in\mathbb{N}}$ then $x \in T_n$ for some $n \in \mathbb{N}$. Then $x \in [\frac{1}{n}, 1] \subset (0,1]$. Thus $x \in (0,1]$.