Sheet 2

Due 17.30 Tuesday 23rd January

Hand in solutions to questions 2, 4b, 5.

Please write your student ID number on your work and staple it together. Writing your name is optional. Hand in work in the MATH6301 mailbox in the 5th floor common room in the maths department.

1. List all the bijections from $\{1,2,3\}$ to $\{1,2,3\}$.

   Solution. In table notation:

$$\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix} = \mathrm{id}_{\{1,2,3\}}, \qquad \begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}, \qquad \begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix},$$
$$\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}, \qquad \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}, \qquad \begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix}.$$

   For the element in the bottom left corner we have three choices. Then, for the element in the middle of the bottom row, we have two choices.

**2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 + 5 & \text{if } x > 0 \\ 2x & \text{if } x \leq 0. \end{cases}$$

   Is f injective? Is it surjective? Find an appropriate inverse of f. (2 marks)

   Solution. The function is injective (perhaps the best way to see this is to draw the graph), so it has a left inverse. The range is $(-\infty, 0] \cup (5, \infty)$, so the function is not surjective and does not have a right inverse. To find the left inverse we must solve the equations $y = x^2 + 5$ for $x > 0$ and $y = 2x$ for $x \leq 0$. The unique solution of the first equation is $x = \sqrt{y-5}$, and the unique solution of the second one is $x = y/2$. We claim that the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(y) = \begin{cases} y/2 & y \leq 0, \\ 0 & 0 < y \leq 5, \\ \sqrt{y-5} & y > 5 \end{cases}$$

   is a left inverse for f. Indeed, for any $x \in \mathbb{R}$ we have $(g \circ f)(x) = x$.

3. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 - 2 & \text{if } x > 0 \\ x - 1 & \text{if } x \leq 0. \end{cases}$$

   Is f injective? Is it surjective? Find an appropriate inverse of f.

Solution. The function is surjective (perhaps the best way to see this is to draw the graph), so that it has a right inverse. We have $f(1) = f(0) = -1$ so the function is not injective and does not have a left inverse. To find the right inverse we must solve the equations $y = x^2-2$ for $x > 0$ and $y = x-1$ for $x \leq 0$. The unique solution of the first equation is $x = \sqrt{y+5}$, and the unique solution of the second one is $x = y + 1$. We claim that the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(y) = \begin{cases} y+1 & y \leq -1, \\ \sqrt{y+2} & y > -1 \end{cases}$$

is a right inverse for f. Indeed, for any $x \in \mathbb{R}$ we have

$$(f \circ g)(y) = \begin{cases} (y-1) - 1 = y & y \leq -1 \\ (\sqrt{y+2})^2 - 2 = y & y > -1. \end{cases}$$

4. Let $f : Y \to Z$ and $g : X \to Y$ be any two functions.

   (a) Show that if f and g are both injective then $f \circ g$ is injective.
   Solution. Assume f and g are injective. Suppose $f(g(x)) = f(g(y))$. Since f is injective we know that $g(x) = g(y)$. Then since g is injective we know that $x = y$. Hence $f \circ g$ is injective.

   **(b) Show that if f, g are both surjective then $f \circ g$ is surjective. (2 marks)
   Solution. Assume f and g are surjective. Choose any $z \in Z$. Since f is surjective there is an element $y \in Y$ such that $f(y) = z$. Since g is surjective there is $x \in X$ such that $g(x) = y$. Hence $f(g(x)) = f(y) = z$. This shows that $f \circ g$ is surjective.

   (c) Show that if f and g are both bijective then $f \circ g$ is bijective.
   Solution. Asssume f and g are bijective. By 4a we know that $f \circ g$ is bijective, and by 4b we know that $f \circ g$ is surjective. Therefore $f \circ g$ is bijective.

**5. Consider the permutations

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 7 & 2 & 8 & 3 & 6 & 5 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 3 & 5 & 6 & 4 & 2 & 1 \end{pmatrix}.$$

   **(a) Calculate $\sigma\tau$, $\tau\sigma$, σ^2, σ^{-1} in table notation. (1 mark)
   Solution.

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 5 & 2 & 3 & 6 & 8 & 7 & 4 \end{pmatrix}, \quad \tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 2 & 7 & 1 & 3 & 4 & 6 & 8 \end{pmatrix},$$
$$\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 5 & 7 & 1 & 2 & 6 & 3 & 4 \end{pmatrix}, \quad \sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 3 & 5 & 1 & 7 & 6 & 2 & 4 \end{pmatrix}.$$

   **(b) Write σ in disjoint cycle notation. (1 mark)

Solution. $\sigma = (4\ 8\ 1)(2\ 7\ 5\ 3)$.

**(c) Write τ in disjoint cycle notation. (1 mark)

Solution. $\tau = (4\ 5\ 6)(1\ 8)(2\ 7)$.

**(d) Calculate τ^{222} and τ^{1343}, writing your answers in table notation. (1 mark)

Solution.

$$\begin{aligned}
\tau^{222} &= (4\ 5\ 6)^{3\times 74}(1\ 8)^{2\times 111}(2\ 7)^{2\times 111} \\
&= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}, \\
\tau^{1343} &= (4\ 5\ 6)^{3\times 447+2}(1\ 8)^{2\times 671+1}(2\ 7)^{2\times 671+1} \\
&= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 3 & 6 & 4 & 5 & 2 & 1 \end{pmatrix}.
\end{aligned}$$

**(e) Find the order of σ. (1 mark)

Solution. 12

**(f) Find the order of τ. (1 mark)

Solution. 6

6. What is the largest number which is the order of an element of S_8? Write down an element of that order in disjoint cycle notation.

   Solution. The order of an element σ of S_n is defined to be the smallest natural number k such that $\sigma^k = \text{id}$. For S_8 the largest order is 15. An element of order 15 is $(1\ 2\ 3)(4\ 5\ 6\ 7\ 8)$.