Sheet 4

Due 17.30 Tuesday 6th February

Hand in solutions to questions 1, 3, 4b, 6b.

Please write your student ID number on your work and staple it together.

**1. Let X be $\mathbb{Z} \times \mathbb{Z}$, i.e. X is the set of all ordered pairs of the form (x, y) with $x, y \in \mathbb{Z}$. Define the relation R on X as follows:

$$(x_1, x_2)R(y_1, y_2) \quad \text{iff} \quad x_1^3 + x_2^3 = y_1^3 + y_2^3.$$

Is R an equivalence relation? Justify your answer. (3 marks)

Solution. Reflexive: for any $(x, y) \in X$ we have $x^3 + y^3 = x^3 + y^3$ hence $(x, y)R(x, y)$.

Symmetric: Suppose $(x_1, x_2)R(y_1, y_2)$, so that $x_1^3 + x_2^3 = y_1^3 + y_2^3$. Then we have $y_1^3 + y_2^3 = x_1^3 + x_2^3$ and so $(y_1, y_2)R(x_1, x_2)$.

Transitive: Suppose $(x_1, x_2)R(y_1, y_2)$ and $(y_1, y_2)R(z_1, z_2)$. Then we have $x_1^3 + x_2^3 = y_1^3 + y_2^3 = z_1^3 + z_2^3$ and so $(x_1, x_2)R(z_1, z_2)$.

So R is an equivalence relation.

2. Let $X = \mathbb{R} \times \mathbb{R}$. Define the relation R on X as follows:

$$(x_1, y_1)R(x_2, y_2) \quad \text{iff} \quad y_1 - y_2 = 2(x_1 - x_2).$$

(a) Is R an equivalence relation?

Solution. Reflexive: for any (x, y) in X we have $y - y = 2(x - x) = 0$ and hence $(x, y)R(x, y)$.

Symmetric: Suppose $(x_1, y_1)R(x_2, y_2)$, i.e. $y_1 - y_2 = 2(x_1 - x_2)$. Multiplying both sides by -1 we get $y_2 - y_1 = 2(x_2 - x_1)$, so that $(x_2, y_2)R(x_1, y_2)$.

Transitive: Suppose $(x_1, y_1)R(x_2, y_2)$ and $(x_2, y_2)R(x_3, y_3)$, so that $y_1 - y_2 = 2(x_1 - x_2)$ and $y_2 - y_3 = 2(x_2 - x_3)$. Adding up these two equations we get $(y_1 - y_2) + (y_2 - y_3) = 2(x_1 - x_2) - 2(x_2 - x_3)$, that is $y_1 - y_3 = 2(x_1 - x_3)$ and hence $(x_1, y_1)R(x_3, y_3)$.

So this is an equivalence relation.

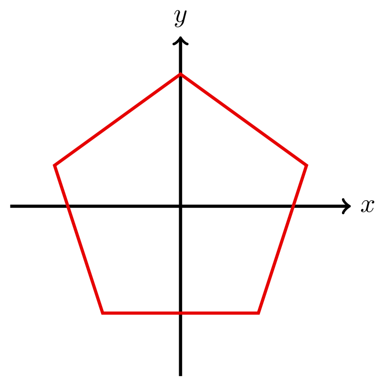
(b) If it is, what is the equivalence class of the point $(3, 1)$?

Solution. The equivalence class of $(3, 1)$ consists of all points which satisfy the relation

$$1 - y = 2(3 - x), \qquad \text{i.e.} \qquad y = 2x - 5.$$

In geometrical terms, this is a straight line on the plane, with intercepts $(5/2, 0)$ and $(0, -5)$.

**3. Let P be the regular pentagon below.



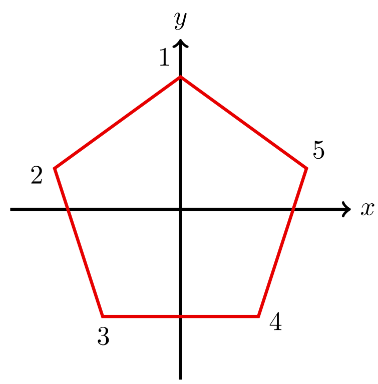
Let R be the rotation of P by $\frac{2\pi}{5}$ anticlockwise, that is by 72° or by one fifth of a full turn counterclockwise. Let F be the reflection of P in the vertical line of symmetry. Represent R and F by permutations and hence calculate

$$F^{-1}R^4FR^3FR^2F,$$

expressing this first as a permutation and then as a symmetry of P.

(3 marks)

Solution. Label the vertices counterclockwise from 1 to 5. There is more than one way to do this, depending on which vertex gets to be 1. For example:



With this labelling $R = (1\ 2\ 3\ 4\ 5)$ and $F = (2\ 5)(3\ 4)$. Thus

$$R^2 = (1\ 3\ 5\ 2\ 4), \quad R^3 = (1\ 4\ 2\ 5\ 3), \quad R^4 = (1\ 5\ 4\ 3\ 2), \quad F^{-1} = F.$$

Hence

$$
\begin{aligned}
&F^{-1}R^4FR^3FR^2F \\
&= \Big((2\ 5)(3\ 4)\Big)\Big((1\ 5\ 4\ 3\ 2)\Big)\Big((2\ 5)(3\ 4)\Big)\Big((1\ 4\ 2\ 5\ 3)\Big)\Big((2\ 5)(3\ 4)\Big)\Big((1\ 3\ 5\ 2\ 4)\Big)\Big((2\ 5)(3\ 4)\Big) \\
&= \Big((2\ 5)(3\ 4)(1\ 5\ 4\ 3\ 2)\Big)\Big((2\ 5)(3\ 4)(1\ 4\ 2\ 5\ 3)\Big)\Big((2\ 5)(3\ 4)(1\ 3\ 5\ 2\ 4)\Big)\Big((2\ 5)(3\ 4)\Big) \\
&= \Big((1\ 2)(3\ 5)\Big)\Big((1\ 3)(4\ 5)\Big)\Big((1\ 4)(2\ 3)\Big)\Big((2\ 5)(3\ 4)\Big) \\
&= (1\ 3\ 5\ 2\ 4) = R^2.
\end{aligned}
$$

Alternatively, one can note that since $F^2 = \mathrm{id}$ we have $FR^nF = (FRF)^n = (R{-}1)^n = R^{-n}$ and so

$$F^{-1}R^4FR^3FR^2F = FR^4FR^3FR^2F = R^{-4}R^3R^{-2} = R^{-3} = R^2.$$

4. Let $(G_1, \triangle)$ and $(G_2, \square)$ be groups. Then $\triangle : G_1 \times G_1 \to G_1$ is a binary operation on G_1, and $\square : G_2 \times G_2 \to G_2$ is a binary operation on G_2.

   Define a binary operation $*$ on the set $G_1 \times G_2$ by $(g_1, g_2) * (h_1, h_2) = (g_1 \triangle h_1, g_2 \square h_2)$.

   (a) Show that $(G_1 \times G_2, *)$ is a group.

   Solution. $*$ is a binary operation: since $g_1 \triangle h_1 \in G_1$ and $g_2 \square h_2 \in G_2$ we have $(g_1, g_2) * (h_1, h_2) \in G_1 \times G_2$.

   Associative:

$$
\begin{aligned}
\big((g_1, g_2) * (g_1', g_2')\big) * (h_1, h_2) &= (g_1 \triangle g_1', g_2 \square g_2') * (h_1, h_2) \\
&= \big((g_1 \triangle g_1') \triangle h_1, (g_2 \square g_2') \square h_2\big) \\
&= \big(g_1 \triangle (g_1' \triangle h_1), g_2 \square (g_2' \square h_2)\big) \\
&= (g_1, g_2) * (g_1' \triangle h_1, g_2' \square h_2) \\
&= (g_1, g_2) * \big((g_1', g_2') * (h_1, h_2)\big).
\end{aligned}
$$

   Identity: $(1_{G_1}, 1_{G_2})$ is the identity:

$$
\begin{aligned}
(1_{G_1}, 1_{G_2}) * (g_1, g_2) = (1_{G_1} \triangle g_1, 1_{G_2} \square g_2) = (g_1, g_2), \\
(g_1, g_2) * (1_{G_1}, 1_{G_2}) = (g_1 \triangle 1_{G_1}, g_2 \square 1_{G_2}) = (g_1, g_2).
\end{aligned}
$$

   Inverses: (g_1^{-1}, g_2^{-1}) is the inverse of (g_1, g_2):

$$
\begin{aligned}
(g_1^{-1}, g_2^{-1}) * (g_1, g_2) = (g_1^{-1} \triangle g_1, g_2^{-1} \square g_2) = (1_{G_1}, 1_{G_2}), \\
(g_1, g_2) * (g_1^{-1}, g_2^{-1}) = (g_1 \triangle g_1^{-1}, g_2 \square g_2^{-1}) = (1_{G_1}, 1_{G_2}).
\end{aligned}
$$

   **(b) Let H_1 be a subgroup of $(G_1, \triangle)$. Let H_2 be a subgroup of $(G_2, \square)$. Show that $H_1 \times H_2$ is a subgroup of $(G_1 \times G_2, *)$. (2 marks)

Solution. Let $(g_1, g_2) \in H_1 \times H_2$ and $(h_1, h_2) \in H_1 \times H_2$. Then

$$(g_1, g_2) * (h_1, h_2) = (g_1 \triangle h_1, g_2 \square h_2) \in H_1 \times H_2$$

since $g_1 \triangle h_1 \in H_1$ and $g_2 \square h_2 \in H_2$.
Let $(g_1, g_2) \in H_1 \times H_2$. By part (a), the inverse is (g_1^{-1}, g_2^{-1}). We have $g_1^{-1} \in H_1$ and $g_2^{-1} \in H_2$ so $(g_1^{-1}, g_2^{-1}) \in H_1 \times H_2$.

5. Let $(G, *)$ be a group with 4 elements, and let $G = \{1_G, g, h, k\}$. Suppose G is not cyclic. Using Lagrange's Theorem show that g, h and k all have order 2. What is $g * h$?

   Solution. By Lagrange's Theorem, the orders of the elements g, h, k divide 4. So the orders $o(g), o(h), o(k)$ are either 1, 2 or 4. Only the identity element has order 1. Suppose for the sake or argument that G has an element of order 4. Then $\langle x \rangle$ would be a subgroup of G with 4 elements. This would mean $G = \langle x \rangle$. However we know that G is not cyclic. Therefore no element of G has order 4, so g, h and k all have order 2. In particular $g * g = 1_G$, $h * h = 1_G$ and $k * k = 1_G$.

   Suppose $g * h = 1_G$. Then $g * h * h = 1_G * h$ and so $g = h$, which is false.

   Suppose $g * h = g$. Then $g * g * h = g * g$ and so $h = 1_G$ which is false. In the same way, if $g * h = h$ then $g = 1_G$ which is false.

   So $g * h = k$ is the only possibility.

6. Let $(G, *)$ be a finite group with no subgroups apart from $\{1_G\}$ and G.

   (a) Show that G is cyclic.

   Solution. Suppose G has at least 2 elements and let g be an element which is not 1_G. The cyclic subgroup $\langle g \rangle$ is a subgroup of G. So it is either $\{1_G\}$ or G. We assumed $g \neq 1_G$ and so we have $\langle g \rangle = 1_G$. Thus G is cyclic.

   **(b) Show that the number of elements in G is either 1 or a prime number. (2 marks)

   Solution. Let g be an arbitrary element different from 1_G, so that $\langle g \rangle = G$. Now suppose that $o(g)$ is not prime, i.e. $o(g) = a \times b$ with $a, b > 1$. Let $h = g^a$. Since $a < o(g)$we have $h \neq 1_G$. On the other hand $h^b = g^{ab} = 1_G$,so $o(h) \leq b$. Thus the cyclic subgroup generated by h has fewer elements than G, so it must be $\{1_G\}$. This gives a contradiction because $h \neq 1_G$. Therefore $o(g)$ is prime.