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de Campina Grande

# Processamento Adaptativo de Sinais

# Algoritmo RLS

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1. Mínimos quadrados.
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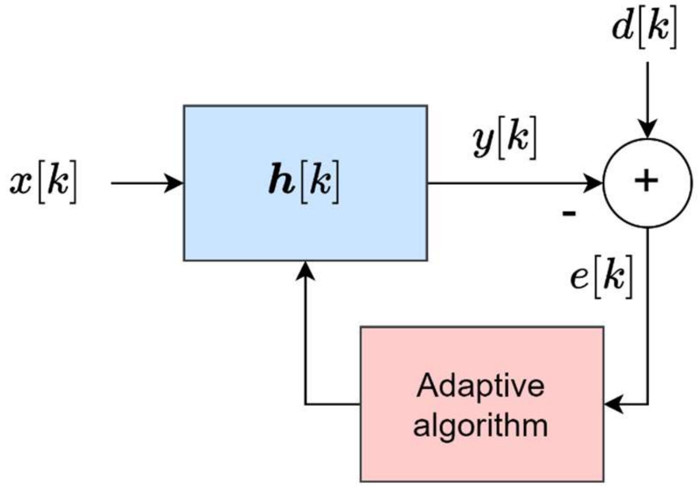
# Mínimos Quadrados

- Considere como função objetivo a soma dos quadrados dos erros de $M$ saídas consecutivas do filtro:

$$J(\boldsymbol{h}) \triangleq \sum_{i=0}^{M-1} |\epsilon[k-i]|^2$$



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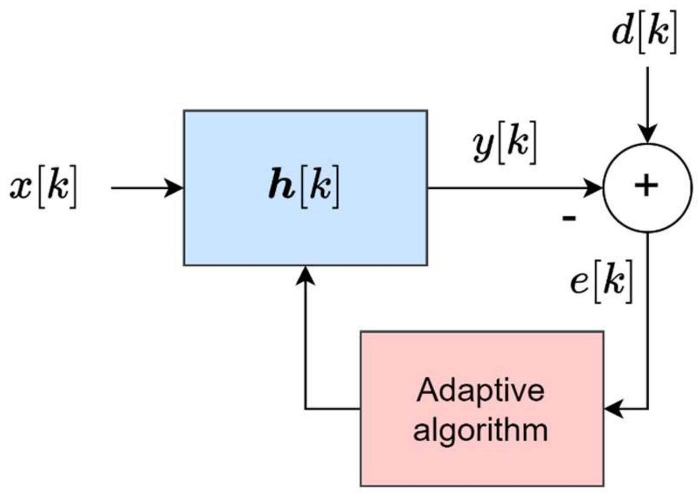
$$\boldsymbol{x}[k] = [x[k], x[k-1], \cdots, x[k-N+1]]^T$$

$$\boldsymbol{h} = [h_0, h_1, \cdots, h_{N-1}]^T$$

$$y[k] = \sum_{i=0}^{N-1} h_i x[k-i] = \boldsymbol{h}^T \boldsymbol{x}[k] = \boldsymbol{x}^T[k]\boldsymbol{h}$$



# Mínimos Quadrados

O erro em cada instante $k$ será dado por:

$$\begin{aligned}
\epsilon[k] &= d[k] - \boldsymbol{x}[k]^T\boldsymbol{h} \\
\epsilon[k-1] &= d[k-1] - \boldsymbol{x}[k-1]^T\boldsymbol{h} \\
&\vdots \\
\epsilon[k-M+1] &= d[k-M+1] - \boldsymbol{x}[k-M+1]^T\boldsymbol{h}
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&\vdots \\
\epsilon\left[k-M+1\right] &= d\left[k-M+1\right] - \boldsymbol{x}[k-M+1]^T\boldsymbol{h}
\end{aligned}$$

Na forma matricial:

$$\begin{bmatrix} \epsilon\left[k\right] \\ \epsilon\left[k-1\right] \\ \vdots \\ \epsilon\left[k-M+1\right] \end{bmatrix} = \begin{bmatrix} d\left[k\right] \\ d\left[k-1\right] \\ \vdots \\ d\left[k-M+1\right] \end{bmatrix} - \begin{bmatrix} x\left[k\right] & x\left[k-1\right] & \cdots & x\left[k-N+1\right] \\ x\left[k-1\right] & x\left[k-2\right] & \cdots & x\left[k-N\right] \\ \vdots & \vdots & \ddots & \vdots \\ x\left[k-M+1\right] & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{N-1} \end{bmatrix}$$

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Solução de mínimos quadrados (*Least Squares* - LS):

$$\boldsymbol{h}_{\mathrm{LS}} = \arg\min_{\boldsymbol{h}} J(\boldsymbol{h}) = \arg\min_{\boldsymbol{h}} \|\boldsymbol{d} - \boldsymbol{X}\boldsymbol{h}\|^2$$

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$$\begin{aligned} J(\boldsymbol{h}) &= \left(\boldsymbol{d}^T - \boldsymbol{h}^T\boldsymbol{X}^T\right)(\boldsymbol{d} - \boldsymbol{X}\boldsymbol{h}) \\ &= \boldsymbol{d}^T\boldsymbol{d} - \boldsymbol{h}^T\boldsymbol{X}^T\boldsymbol{d} - \boldsymbol{d}^T\boldsymbol{X}\boldsymbol{h} + \boldsymbol{h}^T\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{h} \\ &= \boldsymbol{d}^T\boldsymbol{d} - \boldsymbol{h}^T\boldsymbol{X}^T\boldsymbol{d} - \boldsymbol{h}^T\boldsymbol{X}^T\boldsymbol{d} + \boldsymbol{h}^T\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{h} \end{aligned}$$

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&= \boldsymbol{d}^T\boldsymbol{d} - \boldsymbol{h}^T\boldsymbol{X}^T\boldsymbol{d} - \boldsymbol{h}^T\boldsymbol{X}^T\boldsymbol{d} + \boldsymbol{h}^T\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{h}
\end{aligned}$$

Calculando o gradiente de $J(\boldsymbol{h})$ com relação ao vetor de coeficientes $\boldsymbol{h}$, temos:

$$\nabla J(\boldsymbol{h}) \triangleq \frac{\partial J(\boldsymbol{h})}{\partial \boldsymbol{h}} = 2\boldsymbol{X}^T\mathbf{X}\boldsymbol{h} - 2\boldsymbol{X}^T\boldsymbol{d}$$

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Logo:

$$\nabla J(\boldsymbol{h}) = \mathbf{0} \implies \boldsymbol{X}^T\mathbf{X}\boldsymbol{h} = \boldsymbol{X}^T\boldsymbol{d}$$

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Logo:

Solução de mínimos quadrados:

$$\nabla J(\boldsymbol{h}) = \mathbf{0} \implies \boldsymbol{X}^T\mathbf{X}\boldsymbol{h} = \boldsymbol{X}^T\boldsymbol{d} \implies \boldsymbol{h}_{\mathrm{LS}} = \left(\boldsymbol{X}^T\mathbf{X}\right)^{-1}\boldsymbol{X}^T\boldsymbol{d}$$

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Solução de mínimos quadrados: $\boldsymbol{h}_{\mathrm{LS}} = \left(\boldsymbol{X}^T \mathbf{X}\right)^{-1} \boldsymbol{X}^T \boldsymbol{d}$

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Observações:

- A solução existe apenas se $M > N$.

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- A solução $\boldsymbol{h}_{\text{LS}}$ será única se a matrix $\boldsymbol{X}$ tiver posto completo.

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- A solução $\boldsymbol{h}_{\text{LS}}$ será única se a matrix $\boldsymbol{X}$ tiver posto completo.
- Se a solução não for única, como no caso de um sistema sobredeterminado, é possível identificar uma solução apropriada (dentre as infinitas soluções) utilizando algum critério adicional de otimização. Em particular, a solução $\boldsymbol{h}_{\text{psi}}$ que minimiza a norma quadrática euclideana $\|\boldsymbol{h}\|^2$ é dada por

$$\boldsymbol{h}_{\text{psi}} = \boldsymbol{X}^T\left(\boldsymbol{X}\boldsymbol{X}^T\right)^{-1}\boldsymbol{d}$$

em que $\boldsymbol{X}^T\left(\boldsymbol{X}\boldsymbol{X}^T\right)^{-1}$ é conhecida como a matriz pseudoinversa de $\boldsymbol{X}$.

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Observações:

- A matrix $\boldsymbol{X}^T\boldsymbol{X}$ é uma estimativa temporal da matrix de correlação:

$$\boldsymbol{R}_{xx}[k] \propto \sum_{i=0}^{M-1} \boldsymbol{x}[k-i]\boldsymbol{x}^T[k-i] = \boldsymbol{X}^T\boldsymbol{X},$$

  em que o fator $1/M$ é omitido por simplicidade.

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- O vetor $\boldsymbol{X}^T\boldsymbol{d}$ é uma média temporal do vetor de correlação cruzada:

$$\boldsymbol{p}_{xd}[k] \propto \sum_{i=0}^{M-1} \boldsymbol{x}[k-i]d[k-i] = \boldsymbol{X}^T\boldsymbol{d},$$

  em que o fator $1/M$ é omitido por simplicidade.

# Mínimos Quadrados Ponderados

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- Considere a seguinte função objetivo que corresponde à soma ponderada dos quadrados dos erros de $M$ saídas consecutivas do filtro:

$$J(\boldsymbol{h}) = \sum_{i=0}^{M-1} \lambda^i |\epsilon[k-i]|^2 = \sum_{i=0}^{M-1} \lambda^i (d[k-i] - \boldsymbol{x}^T[k-i]\boldsymbol{h})^2$$

em que $0 \ll \lambda < 1$ é um *fator de esquecimento*, que reduz o peso de erros passados no somatório da função objetivo.

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$$\boldsymbol{x}[k] = [x[k], x[k-1], \cdots, x[k-N+1]]^T$$

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$$y[k] = \sum_{i=0}^{N-1} h_i[k]x[k-i] = \boldsymbol{h}^T\boldsymbol{x}[k] = \boldsymbol{x}^T[k]\boldsymbol{h}$$

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$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon\left[k\right] \\ \epsilon\left[k-1\right] \\ \vdots \\ \epsilon\left[k-M+1\right] \end{bmatrix}_{M\times 1} \qquad \boldsymbol{X} = \begin{bmatrix} \boldsymbol{x}^T\left[k\right] \\ \boldsymbol{x}^T\left[k-1\right] \\ \vdots \\ \boldsymbol{x}^T\left[k-M+1\right] \end{bmatrix}_{M\times N}$$

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# Mínimos Quadrados Ponderados

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$$\boldsymbol{d} = \begin{bmatrix} d[k] \\ d[k-1] \\ \vdots \\ d[k-M+1] \end{bmatrix}_{M\times 1} \quad \boldsymbol{\Lambda}_M = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \lambda^{M-2} & \vdots \\ 0 & 0 & \cdots & \lambda^{M-1} \end{bmatrix}$$

# Mínimos Quadrados Ponderados

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$$\boldsymbol{\epsilon} = \boldsymbol{d} - \boldsymbol{X}\boldsymbol{h}$$

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# Mínimos Quadrados Ponderados

$$\boldsymbol{\epsilon} = \boldsymbol{d} - \boldsymbol{X}\boldsymbol{h}$$

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$$J(\boldsymbol{h}) = \sum_{i=0}^{M-1} \lambda^i |\epsilon[k-i]|^2$$
$$= \boldsymbol{\epsilon}^T \boldsymbol{\Lambda}_M \boldsymbol{\epsilon}$$

$$\boldsymbol{d} = \begin{bmatrix} d\left[k\right] \\ d\left[k-1\right] \\ \vdots \\ d\left[k-M+1\right] \end{bmatrix}_{M\times 1} \quad \boldsymbol{\Lambda}_M = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \lambda^{M-2} & \vdots \\ 0 & 0 & \cdots & \lambda^{M-1} \end{bmatrix}$$

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$$\boldsymbol{\epsilon} = \boldsymbol{d} - \boldsymbol{X}\boldsymbol{h}$$

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$$\boldsymbol{d} = \begin{bmatrix} d\,[k] \\ d\,[k-1] \\ \vdots \\ d\,[k-M+1] \end{bmatrix}_{M\times 1} \quad \boldsymbol{\Lambda}_M = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \lambda^{M-2} & \vdots \\ 0 & 0 & \cdots & \lambda^{M-1} \end{bmatrix}$$

$$\begin{aligned} J(\boldsymbol{h}) &= \sum_{i=0}^{M-1} \lambda^i |\epsilon[k-i]|^2 \\ &= \boldsymbol{\epsilon}^T \boldsymbol{\Lambda}_M \boldsymbol{\epsilon} \\ &= (\boldsymbol{d} - \boldsymbol{X}\boldsymbol{h})^T \boldsymbol{\Lambda}_M (\boldsymbol{d} - \boldsymbol{X}\boldsymbol{h}) \end{aligned}$$

# Mínimos Quadrados Ponderados

$$J(\boldsymbol{h}) \propto (\boldsymbol{d} - \boldsymbol{X}\boldsymbol{h})^T \boldsymbol{\Lambda}_M (\boldsymbol{d} - \boldsymbol{X}\boldsymbol{h})$$

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$$J(\boldsymbol{h}) \propto (\boldsymbol{d} - \boldsymbol{X}\boldsymbol{h})^T \boldsymbol{\Lambda}_M (\boldsymbol{d} - \boldsymbol{X}\boldsymbol{h})$$

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Calculando o gradiente de $J(\boldsymbol{h})$ com relação ao vetor de coeficientes $\boldsymbol{h}$, temos:

$$\nabla J(\boldsymbol{h}) \triangleq \frac{\partial J(\boldsymbol{h})}{\partial \boldsymbol{h}} = 2\boldsymbol{X}^T \boldsymbol{\Lambda}_M \mathbf{X}\boldsymbol{h} - 2\boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{d}$$

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Calculando o gradiente de $J(\boldsymbol{h})$ com relação ao vetor de coeficientes $\boldsymbol{h}$, temos:

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Logo:

$$\nabla J(\boldsymbol{h}) = \mathbf{0} \implies \boldsymbol{X}^T \boldsymbol{\Lambda}_M \mathbf{X}\boldsymbol{h} = \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{d}$$

# Mínimos Quadrados Ponderados

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$$J(\boldsymbol{h}) \propto (\boldsymbol{d} - \boldsymbol{X}\boldsymbol{h})^T \boldsymbol{\Lambda}_M (\boldsymbol{d} - \boldsymbol{X}\boldsymbol{h})$$

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$$= \boldsymbol{d}^T \boldsymbol{\Lambda}_M \boldsymbol{d} - \boldsymbol{h}^T \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{d} - \boldsymbol{h}^T \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{d} + \boldsymbol{h}^T \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{X}\boldsymbol{h}$$

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Solução de mínimos quadrados ponderados:

$$\boldsymbol{h}_{\mathrm{LS}} = \left(\boldsymbol{X}^T \boldsymbol{\Lambda}_M \mathbf{X}\right)^{-1} \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{d}$$

# Mínimos Quadrados Ponderados

Solução de mínimos quadrados: $\boldsymbol{h}_{\mathrm{LS}} = \left(\boldsymbol{X}^T \boldsymbol{\Lambda}_M \mathbf{X}\right)^{-1} \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{d}$

Observações:

# Mínimos Quadrados Ponderados

Solução de mínimos quadrados: $\boldsymbol{h}_{\mathrm{LS}} = \left(\boldsymbol{X}^T \boldsymbol{\Lambda}_M \mathbf{X}\right)^{-1} \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{d}$

Observações:

- A matrix $\boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{X}$ é uma média temporal ponderada da matrix de correlação :

$$\boldsymbol{R}_{xx}[k] \propto \sum_{i=0}^{M-1} \lambda^i \boldsymbol{x}[k-i]\boldsymbol{x}^T[k-i] = \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{X}.$$

# Mínimos Quadrados Ponderados

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- O vetor $\boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{d}$ é uma média temporal ponderada do vetor de correlação cruzada:

$$\boldsymbol{p}_{xd}[k] \propto \sum_{i=0}^{M-1} \lambda^i \boldsymbol{x}[k-i] d[k-i] = \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{d}$$

# Mínimos Quadrados Ponderados

Solução de mínimos quadrados: $\boldsymbol{h}_{\text{LS}} = \left(\boldsymbol{X}^T \boldsymbol{\Lambda}_M \mathbf{X}\right)^{-1} \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{d}$

Observações:

- A matrix $\boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{X}$ é uma média temporal ponderada da matrix de correlação :

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- Logo, a solução de mínimos quadrados é dada por: $\boldsymbol{h}_{\text{LS}} = \boldsymbol{R}_{xx}^{-1}[k]\boldsymbol{p}_{xd}[k]$

# Mínimos Quadrados Recursivos

Definição do problema de filtragem adaptativa: dada a função objetivo somatório dos erros *a posteriori* ao quadrado com ponderamento exponencial

$$J(\boldsymbol{h}[k]) = \sum_{i=0}^{k} \lambda^i |\epsilon[k-i]|^2 = \sum_{i=0}^{k} \lambda^i (d[k-i] - \boldsymbol{x}^T[k-i]\boldsymbol{h}[k])^2$$

determine $\boldsymbol{h}[k]$ de tal forma que $J(\boldsymbol{h}[k])$ é minimizada.

# Mínimos Quadrados Recursivos

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A solução de mínimos quadrados é dada por:

$$\boldsymbol{h}_{\mathrm{LS}}[k] = \left(\boldsymbol{X}^T \boldsymbol{\Lambda}_M \mathbf{X}\right)^{-1} \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{d}$$

# Mínimos Quadrados Recursivos

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determine $\boldsymbol{h}[k]$ de tal forma que $J(\boldsymbol{h}[k])$ é minimizada.

A solução de mínimos quadrados é dada por:

$$\begin{aligned} \boldsymbol{h}_{\mathrm{LS}}[k] &= \left(\boldsymbol{X}^T \boldsymbol{\Lambda}_M \mathbf{X}\right)^{-1} \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{d} \\ &= \boldsymbol{R}_{xx}^{-1}[k] \boldsymbol{p}_{xd}[k] \end{aligned}$$

em que $\boldsymbol{R}_{xx}[k]$ e $\boldsymbol{p}_{xd}[k]$ são médias temporais da matriz de autocorrelação das entradas e do vetor de correlação cruzada da entrada para a saída.

# Mínimos Quadrados Recursivos

- O cálculo de $\boldsymbol{R}_{xx}[k]$ pode ser feito de maneira recursiva:

$$\begin{aligned}\boldsymbol{R}_{xx}[k] &= \boldsymbol{x}[k]\boldsymbol{x}^T[k] + \lambda \boldsymbol{R}_{xx}[k-1] \\ &= \boldsymbol{x}[k]\boldsymbol{x}^T[k] + \sum_{i=0}^{k-1} \lambda^{k-i} \boldsymbol{x}[i]\boldsymbol{x}^T[i]\end{aligned}$$

# Mínimos Quadrados Recursivos

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- O cálculo da inversa da matrix $\boldsymbol{R}_{xx}[k]$ possui custo computacional $O(N^3)$.

# Mínimos Quadrados Recursivos

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- O cálculo de $\boldsymbol{R}_{xx}[k]$ pode ser feito de maneira recursiva:

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- O cálculo da inversa da matrix $\boldsymbol{R}_{xx}[k]$ possui custo computacional $O(N^3)$.

- A complexidade no cálculo de $\boldsymbol{R}_{xx}^{-1}[k]$ pode ser reduzida a $O(N^2)$, utilizando-se o lema da inversão de matrizes:

$$[\boldsymbol{A} + \boldsymbol{B}\boldsymbol{C}\boldsymbol{D}]^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}\left[\boldsymbol{D}\boldsymbol{A}^{-1}\boldsymbol{B} + \boldsymbol{C}^{-1}\right]^{-1}\boldsymbol{D}\boldsymbol{A}^{-1}$$

em que $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$ são matrizes de dimensões apropriadas, sendo $\boldsymbol{A}$ e $\boldsymbol{C}$ não-singulares.

# Mínimos Quadrados Recursivos

$$[\boldsymbol{A}+\boldsymbol{B}\boldsymbol{C}\boldsymbol{D}]^{-1}=\boldsymbol{A}^{-1}-\boldsymbol{A}^{-1}\boldsymbol{B}\left[\boldsymbol{D}\boldsymbol{A}^{-1}\boldsymbol{B}+\boldsymbol{C}^{-1}\right]^{-1}\boldsymbol{D}\boldsymbol{A}^{-1}$$

# Mínimos Quadrados Recursivos

$$[\boldsymbol{A}+\boldsymbol{BCD}]^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}\left[\boldsymbol{D}\boldsymbol{A}^{-1}\boldsymbol{B}+\boldsymbol{C}^{-1}\right]^{-1}\boldsymbol{D}\boldsymbol{A}^{-1}$$

$$\boldsymbol{A} = \lambda \boldsymbol{R}_{xx}[k-1]$$

# Mínimos Quadrados Recursivos

$$[\boldsymbol{A}+\boldsymbol{B}\boldsymbol{C}\boldsymbol{D}]^{-1}=\boldsymbol{A}^{-1}-\boldsymbol{A}^{-1}\boldsymbol{B}\left[\boldsymbol{D}\boldsymbol{A}^{-1}\boldsymbol{B}+\boldsymbol{C}^{-1}\right]^{-1}\boldsymbol{D}\boldsymbol{A}^{-1}$$

$$\boldsymbol{A}=\lambda\boldsymbol{R}_{xx}[k-1] \qquad \boldsymbol{B}=\boldsymbol{D}^{T}=\boldsymbol{x}[k]$$

# Mínimos Quadrados Recursivos

$$[\boldsymbol{A} + \boldsymbol{B}\boldsymbol{C}\boldsymbol{D}]^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}\left[\boldsymbol{D}\boldsymbol{A}^{-1}\boldsymbol{B} + \boldsymbol{C}^{-1}\right]^{-1}\boldsymbol{D}\boldsymbol{A}^{-1}$$

$$\boldsymbol{A} = \lambda\boldsymbol{R}_{xx}[k-1] \qquad \boldsymbol{B} = \boldsymbol{D}^T = \boldsymbol{x}[k] \qquad \boldsymbol{C} = 1$$

# Mínimos Quadrados Recursivos

$$[\boldsymbol{A}+\boldsymbol{B}\boldsymbol{C}\boldsymbol{D}]^{-1}=\boldsymbol{A}^{-1}-\boldsymbol{A}^{-1}\boldsymbol{B}\left[\boldsymbol{D}\boldsymbol{A}^{-1}\boldsymbol{B}+\boldsymbol{C}^{-1}\right]^{-1}\boldsymbol{D}\boldsymbol{A}^{-1}$$

$$\boldsymbol{A}=\lambda\boldsymbol{R}_{xx}[k-1] \qquad \boldsymbol{B}=\boldsymbol{D}^{T}=\boldsymbol{x}[k] \qquad \boldsymbol{C}=1$$

$$[\boldsymbol{A}+\boldsymbol{B}\boldsymbol{C}\boldsymbol{D}]^{-1}=\left[\lambda\boldsymbol{R}_{xx}[k-1]+\boldsymbol{x}[k]\boldsymbol{x}^{T}[k]\right]^{-1}=\boldsymbol{R}_{xx}^{-1}[k]$$

# Mínimos Quadrados Recursivos

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$$[\boldsymbol{A}+\boldsymbol{B}\boldsymbol{C}\boldsymbol{D}]^{-1}=\boldsymbol{A}^{-1}-\boldsymbol{A}^{-1}\boldsymbol{B}\left[\boldsymbol{D}\boldsymbol{A}^{-1}\boldsymbol{B}+\boldsymbol{C}^{-1}\right]^{-1}\boldsymbol{D}\boldsymbol{A}^{-1}$$

$$\boldsymbol{A}=\lambda\boldsymbol{R}_{xx}[k-1] \qquad \boldsymbol{B}=\boldsymbol{D}^{T}=\boldsymbol{x}[k] \qquad \boldsymbol{C}=1$$

$$[\boldsymbol{A}+\boldsymbol{B}\boldsymbol{C}\boldsymbol{D}]^{-1}=\left[\lambda\boldsymbol{R}_{xx}[k-1]+\boldsymbol{x}[k]\boldsymbol{x}^{T}[k]\right]^{-1}=\boldsymbol{R}_{xx}^{-1}[k]$$

$$\boldsymbol{A}^{-1}=\frac{1}{\lambda}\boldsymbol{R}^{-1}[k-1]$$

# Mínimos Quadrados Recursivos

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$$[\boldsymbol{A}+\boldsymbol{B}\boldsymbol{C}\boldsymbol{D}]^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}\left[\boldsymbol{D}\boldsymbol{A}^{-1}\boldsymbol{B}+\boldsymbol{C}^{-1}\right]^{-1}\boldsymbol{D}\boldsymbol{A}^{-1}$$

$$\boldsymbol{A} = \lambda \boldsymbol{R}_{xx}[k-1] \qquad \boldsymbol{B} = \boldsymbol{D}^T = \boldsymbol{x}[k] \qquad \boldsymbol{C} = 1$$

$$[\boldsymbol{A}+\boldsymbol{B}\boldsymbol{C}\boldsymbol{D}]^{-1} = \left[\lambda \boldsymbol{R}_{xx}[k-1] + \boldsymbol{x}[k]\boldsymbol{x}^T[k]\right]^{-1} = \boldsymbol{R}_{xx}^{-1}[k]$$

$$\boldsymbol{A}^{-1} = \frac{1}{\lambda}\boldsymbol{R}^{-1}[k-1]$$

$$\left[\boldsymbol{D}\boldsymbol{A}^{-1}\boldsymbol{B}+\boldsymbol{C}^{-1}\right]^{-1} = \left[\frac{1}{\lambda}\boldsymbol{x}^T[k]\boldsymbol{R}^{-1}[k-1]\boldsymbol{x}[k]+1\right]^{-1}$$

# Mínimos Quadrados Recursivos

$$\left[\boldsymbol{A}+\boldsymbol{BCD}\right]^{-1}=\boldsymbol{A}^{-1}-\boldsymbol{A}^{-1}\boldsymbol{B}\left[\boldsymbol{DA}^{-1}\boldsymbol{B}+\boldsymbol{C}^{-1}\right]^{-1}\boldsymbol{DA}^{-1}$$

# Mínimos Quadrados Recursivos

$$[\boldsymbol{A} + \boldsymbol{B}\boldsymbol{C}\boldsymbol{D}]^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}\left[\boldsymbol{D}\boldsymbol{A}^{-1}\boldsymbol{B} + \boldsymbol{C}^{-1}\right]^{-1}\boldsymbol{D}\boldsymbol{A}^{-1}$$

Logo, temos a seguinte expressão para o cálculo recursivo de $\boldsymbol{R}_{xx}^{-1}[k]$:

$$\boldsymbol{R}_{xx}^{-1}[k] = \frac{1}{\lambda}\left[\boldsymbol{R}_{xx}^{-1}[k-1] - \frac{\boldsymbol{R}_{xx}^{-1}[k-1]\boldsymbol{x}[k]\boldsymbol{x}^T[k]\boldsymbol{R}_{xx}^{-1}[k-1]}{\lambda + \boldsymbol{x}^T[k]\boldsymbol{R}_{xx}^{-1}[k-1]\boldsymbol{x}[k]}\right]$$

# Mínimos Quadrados Recursivos

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$$[\boldsymbol{A}+\boldsymbol{B}\boldsymbol{C}\boldsymbol{D}]^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}\left[\boldsymbol{D}\boldsymbol{A}^{-1}\boldsymbol{B}+\boldsymbol{C}^{-1}\right]^{-1}\boldsymbol{D}\boldsymbol{A}^{-1}$$

Logo, temos a seguinte expressão para o cálculo recursivo de $\boldsymbol{R}_{xx}^{-1}[k]$:

$$\boldsymbol{R}_{xx}^{-1}[k] = \frac{1}{\lambda}\left[\boldsymbol{R}_{xx}^{-1}[k-1] - \frac{\boldsymbol{R}_{xx}^{-1}[k-1]\boldsymbol{x}[k]\boldsymbol{x}^T[k]\boldsymbol{R}_{xx}^{-1}[k-1]}{\lambda+\boldsymbol{x}^T[k]\boldsymbol{R}_{xx}^{-1}[k-1]\boldsymbol{x}[k]}\right]$$

- O cálculo do vetor $\boldsymbol{p}_{xd}[k]$ pode ser feito de maneira recursiva:

$$\begin{aligned}\boldsymbol{p}_{xd}[k] &= \boldsymbol{x}[k]d[k] + \lambda\boldsymbol{p}_{xd}[k-1]\\ &= \boldsymbol{x}[k]d[k] + \sum_{i=0}^{k-1}\lambda^{k-i}\boldsymbol{x}[i]d[i]\end{aligned}$$

# Algoritmo RLS



**Algorithm 1** Algoritmo RLS (*Recursive Least Squares*)

**Entrada:** Sinal desejado $d[k]$, sinal de entrada $x[k]$, fator de esquecimento $\lambda$, ordem do filtro $N$.
**Inicializar:** Coeficientes do filtro $\boldsymbol{h} = [0, 0, ..., 0]^T$ (comprimento $N$)
▷ Podem ser inicializados de maneira arbitrária.
**Inicializar:** Vetor de correlação cruzada $\boldsymbol{p}_{xd} = [0, 0, ..., 0]^T$ (comprimento $N$)

**Inicializar:** Inversa da matrix de correlação determinística $\boldsymbol{R}_{xx}^{-1} = \delta \boldsymbol{I}$
▷ Em que $\delta$ é um número muito pequeno.
**for** $k = N$ até $M - 1$ **do** ▷ Onde $M$ é o comprimento do sinal
    Construir o vetor de entrada $\boldsymbol{x}[k] = [x[k], x[k-1], ..., x[k-N+1]]^T$
    Atualizar $\boldsymbol{R}_{xx}^{-1}$:

$$\boldsymbol{R}_{xx}^{-1} = \frac{1}{\lambda}\left[\boldsymbol{R}_{xx}^{-1} - \frac{\boldsymbol{R}_{xx}^{-1}\boldsymbol{x}[k]\boldsymbol{x}^T[k]\boldsymbol{R}_{xx}^{-1}}{\lambda + \boldsymbol{x}^T[k]\boldsymbol{R}_{xx}^{-1}\boldsymbol{x}[k]}\right]$$

    Atualizar $\boldsymbol{p}_{xd}$:

$$\boldsymbol{p}_{xd} = \boldsymbol{x}[k]d[k] + \lambda \boldsymbol{p}_{xd}$$

    Atualizar os coeficientes do filtro:

$$\boldsymbol{h} = \boldsymbol{R}_{xx}^{-1}\boldsymbol{p}_{xd}$$

    Calcular o sinal de erro:

$$\epsilon[k] = d[k] - \boldsymbol{h}^T\boldsymbol{x}[k]$$

**end for**
**Saída:** Coeficientes finais do filtro $\boldsymbol{h}$

# Algoritmo RLS

Formulação alternativa do algoritmo RLS: considere $\boldsymbol{P}[k] = \boldsymbol{R}_{xx}^{-1}[k]$.

# Algoritmo RLS

Formulação alternativa do algoritmo RLS: considere $\boldsymbol{P}[k] = \boldsymbol{R}_{xx}^{-1}[k]$.

$$\boldsymbol{P}[k] = \frac{1}{\lambda}\left[\boldsymbol{P}[k-1] - \frac{\boldsymbol{P}[k-1]\boldsymbol{x}[k]\boldsymbol{x}^T[k]\boldsymbol{P}[k-1]}{\lambda + \boldsymbol{x}^T[k]\boldsymbol{P}[k-1]\boldsymbol{x}[k]}\right]$$

# Algoritmo RLS

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Formulação alternativa do algoritmo RLS: considere $\boldsymbol{P}[k] = \boldsymbol{R}_{xx}^{-1}[k]$.

$$\boldsymbol{P}[k] = \frac{1}{\lambda}\left[\boldsymbol{P}[k-1] - \frac{\boldsymbol{P}[k-1]\boldsymbol{x}[k]\boldsymbol{x}^T[k]\boldsymbol{P}[k-1]}{\lambda + \boldsymbol{x}^T[k]\boldsymbol{P}[k-1]\boldsymbol{x}[k]}\right]$$

$$\boldsymbol{P}[k] = \lambda^{-1}\boldsymbol{P}[k-1] - \frac{\lambda^{-1}\boldsymbol{P}[k-1]\boldsymbol{x}[k]\lambda^{-1}\boldsymbol{x}^T[k]\boldsymbol{P}[k-1]}{1 + \lambda^{-1}\boldsymbol{x}^T[k]\boldsymbol{P}[k-1]\boldsymbol{x}[k]}$$

# Algoritmo RLS

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Ganho de Kalman: $\boldsymbol{g}[k] = \dfrac{\lambda^{-1}\boldsymbol{P}[k-1]\boldsymbol{x}[k]}{1 + \lambda^{-1}\boldsymbol{x}^T[k]\boldsymbol{P}[k-1]\boldsymbol{x}[k]}$

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Equação de Riccati: $\boldsymbol{P}[k] = \lambda^{-1}\boldsymbol{P}[k-1] - \lambda^{-1}\boldsymbol{g}[k]\boldsymbol{x}^T[k]\boldsymbol{P}[k-1]$

# Algoritmo RLS

Temos, então: $$\boldsymbol{g}[k] = \frac{\lambda^{-1}\boldsymbol{P}^{-1}[k-1]\boldsymbol{x}[k]}{1+\lambda^{-1}\boldsymbol{x}^T[k]\boldsymbol{P}^{-1}[k-1]\boldsymbol{x}[k]}$$

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$$\boldsymbol{g}[k] + \lambda^{-1}\boldsymbol{g}[k]\boldsymbol{x}^T[k]\boldsymbol{P}^{-1}[k-1]\boldsymbol{x}[k] = \lambda^{-1}\boldsymbol{P}^{-1}[k-1]\boldsymbol{x}[k]$$

# Algoritmo RLS

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$$\boldsymbol{g}[k] = \left(\lambda^{-1}\boldsymbol{P}^{-1}[k-1] - \lambda^{-1}\boldsymbol{g}[k]\boldsymbol{x}^T[k]\boldsymbol{P}^{-1}[k-1]\right)\boldsymbol{x}[k]$$

# Algoritmo RLS

Temos, então: $$\boldsymbol{g}[k] = \frac{\lambda^{-1}\boldsymbol{P}^{-1}[k-1]\boldsymbol{x}[k]}{1+\lambda^{-1}\boldsymbol{x}^T[k]\boldsymbol{P}^{-1}[k-1]\boldsymbol{x}[k]}$$

$$\boldsymbol{g}[k] + \lambda^{-1}\boldsymbol{g}[k]\boldsymbol{x}^T[k]\boldsymbol{P}^{-1}[k-1]\boldsymbol{x}[k] = \lambda^{-1}\boldsymbol{P}^{-1}[k-1]\boldsymbol{x}[k]$$

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$$\boldsymbol{g}[k] = \underbrace{\left(\lambda^{-1}\boldsymbol{P}^{-1}[k-1] - \lambda^{-1}\boldsymbol{g}[k]\boldsymbol{x}^T[k]\boldsymbol{P}^{-1}[k-1]\right)}_{\boldsymbol{P}[k] = \lambda^{-1}\boldsymbol{P}^{-1}[k-1] - \lambda^{-1}\boldsymbol{g}[k]\boldsymbol{x}^T[k]\boldsymbol{P}^{-1}[k-1]}\boldsymbol{x}[k]$$

# Algoritmo RLS

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Temos, então: $$\boldsymbol{g}[k] = \frac{\lambda^{-1}\boldsymbol{P}^{-1}[k-1]\boldsymbol{x}[k]}{1+\lambda^{-1}\boldsymbol{x}^T[k]\boldsymbol{P}^{-1}[k-1]\boldsymbol{x}[k]}$$

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Portanto: $\boldsymbol{g}[k] = \boldsymbol{P}[k]\boldsymbol{x}[k]$

# Algoritmo RLS com atualização em função do erro *a priori*

Atualização dos coeficientes com erro *a priori* $e[k] = d[k] - \boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k-1]$:

$$\boldsymbol{R}_{xx}[k] = \boldsymbol{x}[k]\boldsymbol{x}^T[k] + \lambda \boldsymbol{R}_{xx}[k-1]$$

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$$(\boldsymbol{R}_{xx}[k] - \boldsymbol{x}[k]\boldsymbol{x}^T[k])\boldsymbol{h}_{\mathrm{LS}}[k-1] = (\boldsymbol{p}_{xd}[k] - \boldsymbol{x}[k]d[k])$$

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$$(\boldsymbol{R}_{xx}[k] - \boldsymbol{x}[k]\boldsymbol{x}^T[k])\boldsymbol{h}_{\mathrm{LS}}[k-1] = (\boldsymbol{p}_{xd}[k] - \boldsymbol{x}[k]d[k])$$

$$\boldsymbol{R}_{xx}[k]\boldsymbol{h}_{\mathrm{LS}}[k-1] - \boldsymbol{x}[k]\boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k-1] + \boldsymbol{x}[k]d[k] = \boldsymbol{p}_{xd}[k]$$

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$$\boldsymbol{R}_{xx}[k]\boldsymbol{h}_{\mathrm{LS}}[k-1] - \boldsymbol{x}[k]\boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k-1] + \boldsymbol{x}[k]d[k] = \boldsymbol{p}_{xd}[k]$$

$$\boldsymbol{R}_{xx}[k]\boldsymbol{h}_{\mathrm{LS}}[k-1] - \boldsymbol{x}[k](y[k] - d[k]) = \boldsymbol{p}_{xd}[k]$$

# Algoritmo RLS com atualização em função do erro *a priori*

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$$(\boldsymbol{R}_{xx}[k] - \boldsymbol{x}[k]\boldsymbol{x}^T[k])\boldsymbol{h}_{\mathrm{LS}}[k-1] = (\boldsymbol{p}_{xd}[k] - \boldsymbol{x}[k]d[k])$$

$$\boldsymbol{R}_{xx}[k]\boldsymbol{h}_{\mathrm{LS}}[k-1] - \boldsymbol{x}[k]\boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k-1] + \boldsymbol{x}[k]d[k] = \boldsymbol{p}_{xd}[k]$$

$$\boldsymbol{R}_{xx}[k]\boldsymbol{h}_{\mathrm{LS}}[k-1] - \boldsymbol{x}[k](y[k] - d[k]) = \boldsymbol{p}_{xd}[k]$$

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Atualização dos coeficientes com erro *a priori* $e[k] = d[k] - \boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k-1]$:

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$$\boldsymbol{h}_{\mathrm{LS}}[k-1] + \boldsymbol{R}_{xx}^{-1}[k]\boldsymbol{x}[k]e[k] = \boldsymbol{R}_{xx}^{-1}[k]\boldsymbol{p}_{xd}[k]$$

# Algoritmo RLS com atualização em função do erro *a priori*

Atualização dos coeficientes com erro *a priori* $e[k] = d[k] - \boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k-1]$:

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$$\boldsymbol{h}_{\mathrm{LS}}[k-1] + \boldsymbol{R}_{xx}^{-1}[k]\boldsymbol{x}[k]e[k] = \boldsymbol{R}_{xx}^{-1}[k]\boldsymbol{p}_{xd}[k]$$

$$\boldsymbol{h}_{\mathrm{LS}}[k-1] + \boldsymbol{R}_{xx}^{-1}[k]\boldsymbol{x}[k]e[k] = \boldsymbol{h}_{\mathrm{LS}}[k]$$

# Algoritmo RLS com atualização em função do erro *a priori*

Atualização dos coeficientes com erro *a priori* $e[k] = d[k] - \boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k-1]$:

$$\boldsymbol{R}_{xx}[k]\boldsymbol{h}_{\mathrm{LS}}[k-1] + \boldsymbol{x}[k]e[k] = \boldsymbol{p}_{xd}[k]$$

$$\boldsymbol{h}_{\mathrm{LS}}[k-1] + \boldsymbol{R}_{xx}^{-1}[k]\boldsymbol{x}[k]e[k] = \boldsymbol{R}_{xx}^{-1}[k]\boldsymbol{p}_{xd}[k]$$

$$\boldsymbol{h}_{\mathrm{LS}}[k-1] + \boldsymbol{R}_{xx}^{-1}[k]\boldsymbol{x}[k]e[k] = \boldsymbol{h}_{\mathrm{LS}}[k]$$

$$\boldsymbol{h}_{\mathrm{LS}}[k] = \boldsymbol{h}_{\mathrm{LS}}[k-1] + \boldsymbol{R}_{xx}^{-1}[k]\boldsymbol{x}[k]e[k]$$

# Algoritmo RLS com atualização em função do erro *a priori*

Atualização dos coeficientes com erro *a priori* $e[k] = d[k] - \boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k-1]$:

$$\boldsymbol{R}_{xx}[k]\boldsymbol{h}_{\mathrm{LS}}[k-1] + \boldsymbol{x}[k]e[k] = \boldsymbol{p}_{xd}[k]$$

$$\boldsymbol{h}_{\mathrm{LS}}[k-1] + \boldsymbol{R}_{xx}^{-1}[k]\boldsymbol{x}[k]e[k] = \boldsymbol{R}_{xx}^{-1}[k]\boldsymbol{p}_{xd}[k]$$

$$\boldsymbol{h}_{\mathrm{LS}}[k-1] + \boldsymbol{R}_{xx}^{-1}[k]\boldsymbol{x}[k]e[k] = \boldsymbol{h}_{\mathrm{LS}}[k]$$

$$\boldsymbol{h}_{\mathrm{LS}}[k] = \boldsymbol{h}_{\mathrm{LS}}[k-1] + \boldsymbol{R}_{xx}^{-1}[k]\boldsymbol{x}[k]e[k]$$

$$\boldsymbol{h}_{\mathrm{LS}}[k] = \boldsymbol{h}_{\mathrm{LS}}[k-1] + \boldsymbol{g}[k]e[k]$$

# Algoritmo RLS com atualização em função do erro *a priori*

Atualização dos coeficientes com erro *a priori* $e[k] = d[k] - \boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k-1]$:

$$\boldsymbol{R}_{xx}[k]\boldsymbol{h}_{\mathrm{LS}}[k-1] + \boldsymbol{x}[k]e[k] = \boldsymbol{p}_{xd}[k]$$

$$\boldsymbol{h}_{\mathrm{LS}}[k-1] + \boldsymbol{R}_{xx}^{-1}[k]\boldsymbol{x}[k]e[k] = \boldsymbol{R}_{xx}^{-1}[k]\boldsymbol{p}_{xd}[k]$$

$$\boldsymbol{h}_{\mathrm{LS}}[k-1] + \boldsymbol{R}_{xx}^{-1}[k]\boldsymbol{x}[k]e[k] = \boldsymbol{h}_{\mathrm{LS}}[k]$$

$$\boldsymbol{h}_{\mathrm{LS}}[k] = \boldsymbol{h}_{\mathrm{LS}}[k-1] + \boldsymbol{R}_{xx}^{-1}[k]\boldsymbol{x}[k]e[k]$$

$$\boldsymbol{h}_{\mathrm{LS}}[k] = \boldsymbol{h}_{\mathrm{LS}}[k-1] + \boldsymbol{g}[k]e[k]$$

Lembrando que:

$$\boldsymbol{P}[k] = \boldsymbol{R}_{xx}^{-1}[k].$$

$$\boldsymbol{g}[k] = \boldsymbol{P}[k]\boldsymbol{x}[k]$$

# Algoritmo RLS com atualização em função do erro *a priori*

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**Algorithm 1** Algoritmo RLS com erro *a priori*

**Entrada:** Sinal desejado $d[k]$, sinal de entrada $x[k]$, fator de esquecimento $\lambda$, ordem do filtro $N$.
**Inicializar:** Coeficientes do filtro $\boldsymbol{h} = [0, 0, ..., 0]^T$ (comprimento $N$)
▷ Podem ser inicializados de maneira arbitrária.
**Inicializar:** Inversa da matrix de correlação determinística $\boldsymbol{P} = \delta \boldsymbol{I}$
▷ Em que $\delta$ é um número muito pequeno.
**for** $k = N$ até $M - 1$ **do** ▷ Onde $M$ é o comprimento do sinal
  Construir o vetor de entrada $\boldsymbol{x}[k] = [x[k], x[k-1], ..., x[k-N+1]]^T$
  Calcular ganho de Kalman:

$$\boldsymbol{g} = \frac{\lambda^{-1}\boldsymbol{P}\boldsymbol{x}[k]}{1 + \lambda^{-1}\boldsymbol{x}^T[k]\boldsymbol{P}\boldsymbol{x}[k]}$$

  Calcular erro a priori: $e[k] = d[k] - \boldsymbol{h}^T\boldsymbol{x}[k]$
  Atualizar os coeficientes do filtro: $\boldsymbol{h} = \boldsymbol{h} + \boldsymbol{g}e[k]$
  Atualizar a matriz $\boldsymbol{P}$: $\boldsymbol{P} = \lambda^{-1}\boldsymbol{P} - \lambda^{-1}\boldsymbol{g}\boldsymbol{x}^T[k]\boldsymbol{P}$
**end for**
**Saída:** Coeficientes finais do filtro $\boldsymbol{h}$

# Algoritmo RLS com atualização em função do erro *a posteriori*

Atualização dos coeficientes com erro *a posteriori* $\epsilon[k] = d[k] - \boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k]$:

# Algoritmo RLS com atualização em função do erro *a posteriori*

Atualização dos coeficientes com erro *a posteriori* $\epsilon[k] = d[k] - \boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k]$:

$$\boldsymbol{R}_{xx}[k]\boldsymbol{h}_{\mathrm{LS}}[k] = \boldsymbol{p}_{xd}[k]$$

# Algoritmo RLS com atualização em função do erro *a posteriori*

Atualização dos coeficientes com erro *a posteriori* $\epsilon[k] = d[k] - \boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k]$:

$$\boldsymbol{R}_{xx}[k]\boldsymbol{h}_{\mathrm{LS}}[k] = \boldsymbol{p}_{xd}[k]$$

$$(\lambda\boldsymbol{R}_{xx}[k-1] + \boldsymbol{x}[k]\boldsymbol{x}^T[k])\boldsymbol{h}_{\mathrm{LS}}[k] = (\lambda\boldsymbol{p}_{xd}[k-1] + \boldsymbol{x}[k]d[k])$$

# Algoritmo RLS com atualização em função do erro *a posteriori*

Atualização dos coeficientes com erro *a posteriori* $\epsilon[k] = d[k] - \boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k]$:

$$\boldsymbol{R}_{xx}[k]\boldsymbol{h}_{\mathrm{LS}}[k] = \boldsymbol{p}_{xd}[k]$$

$$(\lambda\boldsymbol{R}_{xx}[k-1] + \boldsymbol{x}[k]\boldsymbol{x}^T[k])\boldsymbol{h}_{\mathrm{LS}}[k] = (\lambda\boldsymbol{p}_{xd}[k-1] + \boldsymbol{x}[k]d[k])$$

$$\lambda\boldsymbol{R}_{xx}[k-1]\boldsymbol{h}_{\mathrm{LS}}[k] + \boldsymbol{x}[k]\boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{x}[k]d[k] = \lambda\boldsymbol{p}_{xd}[k-1]$$

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$$\lambda\boldsymbol{R}_{xx}[k-1]\boldsymbol{h}_{\mathrm{LS}}[k] + \boldsymbol{x}[k]\boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{x}[k]d[k] = \lambda\boldsymbol{p}_{xd}[k-1]$$

$$\lambda\boldsymbol{R}_{xx}[k-1]\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{x}[k]\epsilon[k] = \lambda\boldsymbol{p}_{xd}[k-1]$$

# Algoritmo RLS com atualização em função do erro *a posteriori*

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$$\lambda\boldsymbol{R}_{xx}[k-1]\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{x}[k]\epsilon[k] = \lambda\boldsymbol{p}_{xd}[k-1]$$

$$\boldsymbol{h}_{\mathrm{LS}}[k] = \boldsymbol{R}_{xx}^{-1}[k-1]\boldsymbol{p}_{xd}[k-1] + \lambda^{-1}\boldsymbol{R}_{xx}^{-1}[k-1]\boldsymbol{x}[k]\epsilon[k]$$

# Algoritmo RLS com atualização em função do erro *a posteriori*

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$$\lambda\boldsymbol{R}_{xx}[k-1]\boldsymbol{h}_{\mathrm{LS}}[k] + \boldsymbol{x}[k]\boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{x}[k]d[k] = \lambda\boldsymbol{p}_{xd}[k-1]$$

$$\lambda\boldsymbol{R}_{xx}[k-1]\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{x}[k]\epsilon[k] = \lambda\boldsymbol{p}_{xd}[k-1]$$

$$\boldsymbol{h}_{\mathrm{LS}}[k] = \boldsymbol{R}_{xx}^{-1}[k-1]\boldsymbol{p}_{xd}[k-1] + \lambda^{-1}\boldsymbol{R}_{xx}^{-1}[k-1]\boldsymbol{x}[k]\epsilon[k]$$

$$\boldsymbol{h}_{\mathrm{LS}}[k] = \boldsymbol{h}_{\mathrm{LS}}[k-1] + \lambda^{-1}\boldsymbol{R}_{xx}^{-1}[k-1]\boldsymbol{x}[k]\epsilon[k]$$

# Algoritmo RLS com atualização em função do erro *a posteriori*

Atualização dos coeficientes com erro *a posteriori* $\epsilon[k] = d[k] - \boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k]$:

$$\boldsymbol{R}_{xx}[k]\boldsymbol{h}_{\mathrm{LS}}[k] = \boldsymbol{p}_{xd}[k]$$

$$(\lambda\boldsymbol{R}_{xx}[k-1] + \boldsymbol{x}[k]\boldsymbol{x}^T[k])\boldsymbol{h}_{\mathrm{LS}}[k] = (\lambda\boldsymbol{p}_{xd}[k-1] + \boldsymbol{x}[k]d[k])$$

$$\lambda\boldsymbol{R}_{xx}[k-1]\boldsymbol{h}_{\mathrm{LS}}[k] + \boldsymbol{x}[k]\boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{x}[k]d[k] = \lambda\boldsymbol{p}_{xd}[k-1]$$

$$\lambda\boldsymbol{R}_{xx}[k-1]\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{x}[k]\epsilon[k] = \lambda\boldsymbol{p}_{xd}[k-1]$$

$$\boldsymbol{h}_{\mathrm{LS}}[k] = \boldsymbol{R}_{xx}^{-1}[k-1]\boldsymbol{p}_{xd}[k-1] + \lambda^{-1}\boldsymbol{R}_{xx}^{-1}[k-1]\boldsymbol{x}[k]\epsilon[k]$$

$$\boldsymbol{h}_{\mathrm{LS}}[k] = \boldsymbol{h}_{\mathrm{LS}}[k-1] + \lambda^{-1}\boldsymbol{R}_{xx}^{-1}[k-1]\boldsymbol{x}[k]\epsilon[k]$$

Ganho de Kalman alternativo:

$$\tilde{\boldsymbol{g}}[k] = \lambda^{-1}\boldsymbol{P}[k-1]\boldsymbol{x}[k]$$

# Algoritmo RLS com atualização em função do erro *a posteriori*

Atualização dos coeficientes com erro *a posteriori* $\epsilon[k] = d[k] - \boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k]$:

$$\boldsymbol{R}_{xx}[k]\boldsymbol{h}_{\mathrm{LS}}[k] = \boldsymbol{p}_{xd}[k]$$

$$(\lambda\boldsymbol{R}_{xx}[k-1] + \boldsymbol{x}[k]\boldsymbol{x}^T[k])\boldsymbol{h}_{\mathrm{LS}}[k] = (\lambda\boldsymbol{p}_{xd}[k-1] + \boldsymbol{x}[k]d[k])$$

$$\lambda\boldsymbol{R}_{xx}[k-1]\boldsymbol{h}_{\mathrm{LS}}[k] + \boldsymbol{x}[k]\boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{x}[k]d[k] = \lambda\boldsymbol{p}_{xd}[k-1]$$

$$\lambda\boldsymbol{R}_{xx}[k-1]\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{x}[k]\epsilon[k] = \lambda\boldsymbol{p}_{xd}[k-1]$$

$$\boldsymbol{h}_{\mathrm{LS}}[k] = \boldsymbol{R}_{xx}^{-1}[k-1]\boldsymbol{p}_{xd}[k-1] + \lambda^{-1}\boldsymbol{R}_{xx}^{-1}[k-1]\boldsymbol{x}[k]\epsilon[k]$$

$$\boldsymbol{h}_{\mathrm{LS}}[k] = \boldsymbol{h}_{\mathrm{LS}}[k-1] + \lambda^{-1}\boldsymbol{R}_{xx}^{-1}[k-1]\boldsymbol{x}[k]\epsilon[k]$$

$$\boldsymbol{h}_{\mathrm{LS}}[k] = \boldsymbol{h}_{\mathrm{LS}}[k-1] + \tilde{\boldsymbol{g}}[k]\epsilon[k]$$

Ganho de Kalman alternativo:

$$\tilde{\boldsymbol{g}}[k] = \lambda^{-1}\boldsymbol{P}[k-1]\boldsymbol{x}[k]$$

# Algoritmo RLS com atualização em função do erro *a posteriori*

Perceba que as seguintes relações podem ser derivadas:

$$\epsilon[k] = d[k] - \boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k]$$

# Algoritmo RLS com atualização em função do erro *a posteriori*

Perceba que as seguintes relações podem ser derivadas:

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$$\epsilon[k] = d[k] - \boldsymbol{x}^T[k](\boldsymbol{h}_{\mathrm{LS}}[k-1] + \tilde{\boldsymbol{g}}[k]\epsilon[k]):$$

# Algoritmo RLS com atualização em função do erro *a posteriori*

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$$\epsilon[k] = d[k] - \boldsymbol{x}^T[k](\boldsymbol{h}_{\mathrm{LS}}[k-1] + \tilde{\boldsymbol{g}}[k]\epsilon[k]):$$

$$\epsilon[k] = e[k] - \boldsymbol{x}^T[k]\tilde{\boldsymbol{g}}[k]\epsilon[k]$$

# Algoritmo RLS com atualização em função do erro *a posteriori*

Perceba que as seguintes relações podem ser derivadas:

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$$\epsilon[k] = d[k] - \boldsymbol{x}^T[k](\boldsymbol{h}_{\mathrm{LS}}[k-1] + \tilde{\boldsymbol{g}}[k]\epsilon[k]):$$

$$\epsilon[k] = e[k] - \boldsymbol{x}^T[k]\tilde{\boldsymbol{g}}[k]\epsilon[k]$$

$$\epsilon[k] = \frac{e[k]}{1 + \boldsymbol{x}^T[k]\tilde{\boldsymbol{g}}[k]}$$

# Algoritmo RLS com atualização em função do erro *a posteriori*

Perceba que as seguintes relações podem ser derivadas:

$$\epsilon[k] = d[k] - \boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k]$$

$$\epsilon[k] = d[k] - \boldsymbol{x}^T[k](\boldsymbol{h}_{\mathrm{LS}}[k-1] + \tilde{\boldsymbol{g}}[k]\epsilon[k]):$$

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$$\epsilon[k] = \frac{e[k]}{1 + \boldsymbol{x}^T[k]\tilde{\boldsymbol{g}}[k]}$$

$$\epsilon[k] = \frac{e[k]}{1 + \lambda^{-1}\boldsymbol{x}^T[k]\boldsymbol{P}[k-1]\boldsymbol{x}[k]}$$

# Algoritmo RLS com atualização em função do erro *a posteriori*

Perceba que as seguintes relações podem ser derivadas:

$$\epsilon[k] = d[k] - \boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k]$$

$$\epsilon[k] = d[k] - \boldsymbol{x}^T[k](\boldsymbol{h}_{\mathrm{LS}}[k-1] + \tilde{\boldsymbol{g}}[k]\epsilon[k]):$$

$$\epsilon[k] = e[k] - \boldsymbol{x}^T[k]\tilde{\boldsymbol{g}}[k]\epsilon[k]$$

$$\epsilon[k] = \frac{e[k]}{1 + \boldsymbol{x}^T[k]\tilde{\boldsymbol{g}}[k]}$$

$$\epsilon[k] = \frac{e[k]}{1 + \lambda^{-1}\boldsymbol{x}^T[k]\boldsymbol{P}[k-1]\boldsymbol{x}[k]}$$

Fator de conversão entre erros:

$$\tilde{\alpha}[k] = 1 + \lambda^{-1}\boldsymbol{x}^T[k]\boldsymbol{P}[k-1]\boldsymbol{x}[k]$$

# Algoritmo RLS com atualização em função do erro *a posteriori*

Perceba que as seguintes relações podem ser derivadas:

$$\epsilon[k] = d[k] - \boldsymbol{x}^T[k]\boldsymbol{h}_{\mathrm{LS}}[k]$$

$$\epsilon[k] = d[k] - \boldsymbol{x}^T[k](\boldsymbol{h}_{\mathrm{LS}}[k-1] + \tilde{\boldsymbol{g}}[k]\epsilon[k]):$$

$$\epsilon[k] = e[k] - \boldsymbol{x}^T[k]\tilde{\boldsymbol{g}}[k]\epsilon[k]$$

$$\epsilon[k] = \frac{e[k]}{1 + \boldsymbol{x}^T[k]\tilde{\boldsymbol{g}}[k]}$$

$$\epsilon[k] = \frac{e[k]}{1 + \lambda^{-1}\boldsymbol{x}^T[k]\boldsymbol{P}[k-1]\boldsymbol{x}[k]} \Longrightarrow \epsilon[k] = \frac{e[k]}{\tilde{\alpha}[k]}$$

Fator de conversão entre erros:

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$$\epsilon[k] = d[k] - \boldsymbol{x}^T[k](\boldsymbol{h}_{\mathrm{LS}}[k-1] + \tilde{\boldsymbol{g}}[k]\epsilon[k]):$$

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Fator de conversão entre erros:

$$\tilde{\alpha}[k] = 1 + \lambda^{-1}\boldsymbol{x}^T[k]\boldsymbol{P}[k-1]\boldsymbol{x}[k]$$

$$\epsilon[k] = \frac{e[k]}{1 + \boldsymbol{x}^T[k]\tilde{\boldsymbol{g}}[k]}$$

$$\epsilon[k] = \frac{e[k]}{1 + \lambda^{-1}\boldsymbol{x}^T[k]\boldsymbol{P}[k-1]\boldsymbol{x}[k]} \Longrightarrow \epsilon[k] = \frac{e[k]}{\tilde{\alpha}[k]} \Longrightarrow \boldsymbol{g}[k] = \frac{\tilde{\boldsymbol{g}}[k]}{\tilde{\alpha}[k]}$$

# Algoritmo RLS com atualização em função do erro *a posteriori*

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**Algorithm 1** Algoritmo RLS com erro *a posteriori*

**Entrada:** Sinal desejado $d[k]$, sinal de entrada $x[k]$, fator de esquecimento $\lambda$, ordem do filtro $N$.
**Inicializar:** Coeficientes do filtro $\boldsymbol{h} = [0, 0, ..., 0]^T$ (comprimento $N$)
    ▷ Podem ser inicializados de maneira arbitrária.
**Inicializar:** Inversa da matrix de correlação determinística $\boldsymbol{P} = \delta \boldsymbol{I}$
    ▷ Em que $\delta$ é um número muito pequeno.
**for** $k = N$ até $M - 1$ **do**    ▷ Onde $M$ é o comprimento do sinal
    Construir o vetor de entrada $\boldsymbol{x}[k] = [x[k], x[k-1], ..., x[k-N+1]]^T$
    Calcular ganho de Kalman modificado: $\tilde{\boldsymbol{g}} = \lambda^{-1} \boldsymbol{P} \boldsymbol{x}[k]$
    Calcular erro a priori: $e[k] = d[k] - \boldsymbol{h}^T \boldsymbol{x}[k]$
    Calcular fator de conversão: $\tilde{\alpha} = 1 + \tilde{\boldsymbol{g}}^T \boldsymbol{x}[k]$
    Calcular erro a posteriori: $\epsilon[k] = e[k] / \tilde{\alpha}$
    Atualizar os coeficientes do filtro: $\boldsymbol{h} = \boldsymbol{h} + \tilde{\boldsymbol{g}} \epsilon[k]$
    Atualizar a matriz $\boldsymbol{P}$: $\boldsymbol{P} = \lambda^{-1} \boldsymbol{P} - \lambda^{-1} (\tilde{\boldsymbol{g}} / \tilde{\alpha}) \boldsymbol{x}^T[k] \boldsymbol{P}$
**end for**
**Saída:** Coeficientes finais do filtro $\boldsymbol{h}$

# Comparativo

| RLS | Atualização *a priori* | Atualização *a posteriori* |
|---|---|---|
| Ganho de Kalman | $\mathbf{g}[k] = \boldsymbol{P}[k]\boldsymbol{x}[k]$ | $\tilde{\boldsymbol{g}}[k] = \lambda^{-1}\boldsymbol{P}[k-1]\boldsymbol{x}[k]$ |
| Erro *a priori* | $e[k] = d[k] - \boldsymbol{h}^T[k-1]\boldsymbol{x}[k]$ | $\epsilon[k] = d[k] - \boldsymbol{h}[k]^T\boldsymbol{x}[k]$ |
| Fator de conversão | $\alpha[k] = 1 - \boldsymbol{g}[k]^T\boldsymbol{x}[k]$ | $\tilde{\alpha}[k] = 1 + \tilde{\boldsymbol{g}}^T[k]\boldsymbol{x}[k]$ |
| Erro *a posteriori* | $\epsilon[k] = \alpha[k]e[k]$ | $\epsilon[k] = \tilde{\alpha}^{-1}[k]e[k]$ |
| Atualização de $\boldsymbol{h}$ | $\boldsymbol{h}[k] = \boldsymbol{h}[k-1] + \boldsymbol{g}[k]e[k]$ | $\boldsymbol{h}[k] = \boldsymbol{h}[k-1] + \tilde{\boldsymbol{g}}[k]\epsilon[k]$ |

# Comparativo

| RLS | Atualização *a priori* | Atualização *a posteriori* |
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| Ganho de Kalman | $\mathbf{g}[k] = \boldsymbol{P}[k]\boldsymbol{x}[k]$ | $\tilde{\boldsymbol{g}}[k] = \lambda^{-1}\boldsymbol{P}[k-1]\boldsymbol{x}[k]$ |
| Erro *a priori* | $e[k] = d[k] - \boldsymbol{h}^T[k-1]\boldsymbol{x}[k]$ | $\epsilon[k] = d[k] - \boldsymbol{h}[k]^T\boldsymbol{x}[k]$ |
| Fator de conversão | $\alpha[k] = 1 - \boldsymbol{g}[k]^T\boldsymbol{x}[k]$ | $\tilde{\alpha}[k] = 1 + \tilde{\boldsymbol{g}}^T[k]\boldsymbol{x}[k]$ |
| Erro *a posteriori* | $\epsilon[k] = \alpha[k]e[k]$ | $\epsilon[k] = \tilde{\alpha}^{-1}[k]e[k]$ |
| Atualização de $\boldsymbol{h}$ | $\boldsymbol{h}[k] = \boldsymbol{h}[k-1] + \boldsymbol{g}[k]e[k]$ | $\boldsymbol{h}[k] = \boldsymbol{h}[k-1] + \tilde{\boldsymbol{g}}[k]\epsilon[k]$ |

Observações:

# Comparativo

| RLS | Atualização *a priori* | Atualização *a posteriori* |
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| Ganho de Kalman | $\mathbf{g}[k] = \boldsymbol{P}[k]\boldsymbol{x}[k]$ | $\tilde{\boldsymbol{g}}[k] = \lambda^{-1}\boldsymbol{P}[k-1]\boldsymbol{x}[k]$ |
| Erro *a priori* | $e[k] = d[k] - \boldsymbol{h}^T[k-1]\boldsymbol{x}[k]$ | $\epsilon[k] = d[k] - \boldsymbol{h}[k]^T\boldsymbol{x}[k]$ |
| Fator de conversão | $\alpha[k] = 1 - \boldsymbol{g}[k]^T\boldsymbol{x}[k]$ | $\tilde{\alpha}[k] = 1 + \tilde{\boldsymbol{g}}^T[k]\boldsymbol{x}[k]$ |
| Erro *a posteriori* | $\epsilon[k] = \alpha[k]e[k]$ | $\epsilon[k] = \tilde{\alpha}^{-1}[k]e[k]$ |
| Atualização de $\boldsymbol{h}$ | $\boldsymbol{h}[k] = \boldsymbol{h}[k-1] + \boldsymbol{g}[k]e[k]$ | $\boldsymbol{h}[k] = \boldsymbol{h}[k-1] + \tilde{\boldsymbol{g}}[k]\epsilon[k]$ |

Observações:

- A complexidade computacional do algoritmo RLS é $O(N^2)$.

# Comparativo

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| Ganho de Kalman | $\mathbf{g}[k] = \boldsymbol{P}[k]\boldsymbol{x}[k]$ | $\tilde{\boldsymbol{g}}[k] = \lambda^{-1}\boldsymbol{P}[k-1]\boldsymbol{x}[k]$ |
| Erro *a priori* | $e[k] = d[k] - \boldsymbol{h}^T[k-1]\boldsymbol{x}[k]$ | $\epsilon[k] = d[k] - \boldsymbol{h}[k]^T\boldsymbol{x}[k]$ |
| Fator de conversão | $\alpha[k] = 1 - \boldsymbol{g}[k]^T\boldsymbol{x}[k]$ | $\tilde{\alpha}[k] = 1 + \tilde{\boldsymbol{g}}^T[k]\boldsymbol{x}[k]$ |
| Erro *a posteriori* | $\epsilon[k] = \alpha[k]e[k]$ | $\epsilon[k] = \tilde{\alpha}^{-1}[k]e[k]$ |
| Atualização de $\boldsymbol{h}$ | $\boldsymbol{h}[k] = \boldsymbol{h}[k-1] + \boldsymbol{g}[k]e[k]$ | $\boldsymbol{h}[k] = \boldsymbol{h}[k-1] + \tilde{\boldsymbol{g}}[k]\epsilon[k]$ |

Observações:

- A complexidade computacional do algoritmo RLS é $O(N^2)$.
- Existem versões rápidas do algoritmo RLS (fast RLS), nas quais as propriedades de simetria e redundância são utilizadas para reduzir a complexidade a $O(7N)$.

# Propriedades da solução de mínimos quadrados

- Princípio da ortogonalidade: o vetor de erros e os vetores de entrada $\boldsymbol{x}[k]$ são ortogonais.

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$$\boldsymbol{X}^T\boldsymbol{\Lambda}_M\left(\boldsymbol{X}\boldsymbol{h} - \boldsymbol{d}\right) = \boldsymbol{0}$$

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Vetor de erros *a posteriori*:

$$\boldsymbol{\epsilon} = \boldsymbol{d} - \boldsymbol{X}\boldsymbol{h}$$

$$\boldsymbol{d} = \begin{bmatrix} d[k] \\ d[k-1] \\ \vdots \\ d[k-M] \end{bmatrix}_{M\times 1} \qquad \boldsymbol{\Lambda}_M = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \lambda^{M-2} & \vdots \\ 0 & 0 & \cdots & \lambda^{M-1} \end{bmatrix}$$

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$$\boldsymbol{X}^T\boldsymbol{\Lambda}_M(\boldsymbol{X}\boldsymbol{h} - \boldsymbol{d}) = \boldsymbol{0} \quad \boldsymbol{X}^T\boldsymbol{\Lambda}_M(-\boldsymbol{\epsilon}) = \boldsymbol{0} \implies \boldsymbol{X}^T\boldsymbol{\epsilon} = \boldsymbol{0}$$

# Propriedades da solução de mínimos quadrados

- Relação entre a solução de mínimos quadrados e a solução de Wiener:

$$\boldsymbol{h}_{\mathrm{LS}} = \left(\boldsymbol{X}^T \boldsymbol{\Lambda}_M \mathbf{X}\right)^{-1} \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{d} \qquad \boldsymbol{h}_{\mathrm{LS}}[k] = \boldsymbol{R}_{xx}^{-1}[k]\boldsymbol{p}_{xd}[k]$$

# Propriedades da solução de mínimos quadrados

- Relação entre a solução de mínimos quadrados e a solução de Wiener:

$$\boldsymbol{h}_{\mathrm{LS}} = \left(\boldsymbol{X}^T \boldsymbol{\Lambda}_M \mathbf{X}\right)^{-1} \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{d} \qquad \boldsymbol{h}_{\mathrm{LS}}[k] = \boldsymbol{R}_{xx}^{-1}[k]\boldsymbol{p}_{xd}[k]$$

- A matrix $\boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{X}$ é uma média temporal ponderada da matrix de correlação:

$$\boldsymbol{R}_{xx}[k] \propto \sum_{i=0}^{M-1} \lambda^i \boldsymbol{x}[k-i]\boldsymbol{x}^T[k-i] = \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{X}.$$

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- O vetor $\boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{d}$ é uma média temporal ponderada do vetor de correlação cruzada:

$$\boldsymbol{p}_{xd}[k] \propto \sum_{i=0}^{M-1} \lambda^i \boldsymbol{x}[k-i] d[k-i] = \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{d}$$

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- Portanto, a solução de mínimos quadrados converge para a solução de Wiener se os sinais envolvidos forem ergódicos e estacionários:

$$\lim_{k\to\infty} \boldsymbol{h}_{\mathrm{LS}}[k] = \lim_{k\to\infty} \boldsymbol{R}_{xx}^{-1}[k]\boldsymbol{p}_{xd}[k] = \boldsymbol{R}_{xx}^{-1}\boldsymbol{p}_{xd} = \boldsymbol{h}_{\mathrm{opt}}$$

# Propriedades da solução de mínimos quadrados

- Considere que o sinal de referência $d[n]$ pode ser representado por um processo média móvel, perturbado por ruído aditivo gaussiano:

$$d[k] = \boldsymbol{h}_0^T \boldsymbol{x}[k] + n[k] = \boldsymbol{x}^T[k]\boldsymbol{h}_0 + n[k]$$

em que $\boldsymbol{h}_0$ é um vetor de parâmetros fixo e $n[k]$ é uma variável aleatória gaussiana com média nula e variância $\sigma_n^2$.

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- Desse modo, se $\boldsymbol{n}$ possui média nula e é descorrelacionado de $\boldsymbol{X}$, então $\boldsymbol{h}_{\mathrm{LS}}$ é um estimador não enviesado de $\boldsymbol{h}_0$.

# Propriedades da solução de mínimos quadrados

- Considere o erro de estimação $\Delta \boldsymbol{h}[k]$ no instante $k$, dado por,

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- $\frac{\lambda^{k+1}}{\delta}\mathbb{E}\left[\boldsymbol{P}[k]\right]\Delta\boldsymbol{h}[-1] \rightarrow 0$, $k \rightarrow \infty$.

# Propriedades da solução de mínimos quadrados

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$$\mathbb{E}[\Delta \boldsymbol{h}[k]] = \mathbb{E}\left[\lambda^{k+1}\boldsymbol{P}[k]\boldsymbol{R}_{xx}[-1]\Delta\boldsymbol{h}[-1]\right] + \mathbb{E}\left[\boldsymbol{P}[k]\sum_{i=0}^{k}\lambda^{k-i}\boldsymbol{x}[i]e_o[i]\right]$$

$$\mathbb{E}[\Delta \boldsymbol{h}[k]] = \lambda^{k+1}\mathbb{E}\left[\boldsymbol{P}[k]\right]\frac{1}{\delta}\boldsymbol{I}\Delta\boldsymbol{h}[-1] + \mathbb{E}\left[\boldsymbol{P}[k]\sum_{i=0}^{k}\lambda^{k-i}\boldsymbol{x}[i]e_o[i]\right]$$

Observações:

- A matrix $\boldsymbol{P}[k]$ tende a se tornar relativamente invariante quando o número de iterações $k$ aumenta, uma vez que $\boldsymbol{P}[k]$ depende de todas as entradas $\boldsymbol{x}[k]$ passadas; $\boldsymbol{x}[k]$ e $e_o[k]$ podem ser considerados descorrelacionados (princípio da ortogonalidade). Logo, $\mathbb{E}\left[\boldsymbol{P}[k]\sum_{i=0}^{k}\lambda^{k-i}\boldsymbol{x}[i]e_o[i]\right] \to 0$, $k \to \infty$.
- $\frac{\lambda^{k+1}}{\delta}\mathbb{E}\left[\boldsymbol{P}[k]\right]\Delta\boldsymbol{h}[-1] \to 0$, $k \to \infty$.
- Portanto, $\mathbb{E}[\Delta\boldsymbol{h}[k]] \to \boldsymbol{0}$ de maneira relativamente independente dos autovalores da matrix de correlação $\boldsymbol{R}_{xx}$.

# Propriedades da solução de mínimos quadrados

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- Covariância do o erro de estimação $\Delta\boldsymbol{h}[k]$:

$$\mathrm{cov}[\Delta\boldsymbol{h}[k]] = \mathbb{E}\left[\left(\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{h}_o\right)\left(\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{h}_o\right)^T\right]$$

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$$\boldsymbol{h}_{\mathrm{LS}}[k] = \boldsymbol{R}_{xx}^{-1}[k] \boldsymbol{p}_{xd}[k] \qquad \boldsymbol{P}[k] = \boldsymbol{R}_{xx}^{-1}[k].$$

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$$\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{h}_o = \boldsymbol{P}[k]\boldsymbol{p}_{\boldsymbol{x}d}[k] - \boldsymbol{P}[k]\boldsymbol{P}^{-1}[k]\boldsymbol{h}_o$$

# Propriedades da solução de mínimos quadrados

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- Covariância do o erro de estimação $\Delta \boldsymbol{h}[k]$:

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# Propriedades da solução de mínimos quadrados

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$$\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{h}_o = \left(\boldsymbol{X}^T\boldsymbol{\Lambda}_M\boldsymbol{X}\right)^{-1}(\boldsymbol{X}^T\boldsymbol{\Lambda}_M\boldsymbol{d}[k] - \boldsymbol{X}^T\boldsymbol{\Lambda}_M\boldsymbol{X}\boldsymbol{h}_o)$$

# Propriedades da solução de mínimos quadrados

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# Propriedades da solução de mínimos quadrados

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$$\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{h}_o = \left(\boldsymbol{X}^T\boldsymbol{\Lambda}_M\mathbf{X}\right)^{-1}\boldsymbol{X}^T\boldsymbol{\Lambda}_M(\boldsymbol{d}[k] - \mathrm{X}\boldsymbol{h}_o)$$

$$\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{h}_o = \left(\boldsymbol{X}^T\boldsymbol{\Lambda}_M\mathbf{X}\right)^{-1}\boldsymbol{X}^T\boldsymbol{\Lambda}_M\boldsymbol{n}[k]$$

# Propriedades da solução de mínimos quadrados

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$$\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{h}_o = \left(\boldsymbol{X}^T \boldsymbol{\Lambda}_M \mathbf{X}\right)^{-1} \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{n}[k]$$

# Propriedades da solução de mínimos quadrados

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$$\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{h}_o = \left(\boldsymbol{X}^T \boldsymbol{\Lambda}_M \mathbf{X}\right)^{-1} \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{n}[k]$$

$$\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{h}_o = \boldsymbol{P}[k] \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{n}[k]$$

# Propriedades da solução de mínimos quadrados

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$$\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{h}_o = \left(\boldsymbol{X}^T \boldsymbol{\Lambda}_M \mathbf{X}\right)^{-1} \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{n}[k]$$

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$$\mathrm{cov}[\Delta \boldsymbol{h}_{\mathrm{LS}}[k]] = \mathbb{E}\left[\left(\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{h}_o\right)\left(\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{h}_o\right)^T\right]$$

# Propriedades da solução de mínimos quadrados

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$$\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{h}_o = \left(\boldsymbol{X}^T \boldsymbol{\Lambda}_M \mathbf{X}\right)^{-1} \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{n}[k]$$

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$$\begin{aligned}
\operatorname{cov}[\Delta \boldsymbol{h}_{\mathrm{LS}}[k]] &= \mathbb{E}\left[(\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{h}_o)(\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{h}_o)^T\right] \\
&= \mathbb{E}\left[\left(\boldsymbol{P}[k] \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{n}[k]\right)\left(\boldsymbol{P}[k] \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{n}[k]\right)^T\right]
\end{aligned}$$

# Propriedades da solução de mínimos quadrados

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&= \mathbb{E}\left[\left(\boldsymbol{P}[k]\boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{n}[k]\right)\left(\boldsymbol{P}[k]\boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{n}[k]\right)^T\right] \\
&= \mathbb{E}\left[\boldsymbol{P}[k]\boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{n}[k] \boldsymbol{n}^T[k] \boldsymbol{\Lambda}_M \boldsymbol{X} \boldsymbol{P}^T[k]\right]
\end{aligned}$$

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&= \mathbb{E}\left[\left(\boldsymbol{P}[k] \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{n}[k]\right)\left(\boldsymbol{P}[k] \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{n}[k]\right)^T\right] \\
&= \mathbb{E}\left[\boldsymbol{P}[k] \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{n}[k] \boldsymbol{n}^T[k] \boldsymbol{\Lambda}_M \boldsymbol{X} \boldsymbol{P}^T[k]\right] \\
&= \mathbb{E}\left[\boldsymbol{P}[k] \boldsymbol{X}^T \boldsymbol{\Lambda}_M \mathbb{E}\left[\boldsymbol{n}[k] \boldsymbol{n}^T[k]\right] \boldsymbol{\Lambda}_M \boldsymbol{X} \boldsymbol{P}^T[k]\right]
\end{aligned}$$

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$$\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{h}_o = \left(\boldsymbol{X}^T \boldsymbol{\Lambda}_M \mathbf{X}\right)^{-1} \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{n}[k]$$

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&= \mathbb{E}\left[\boldsymbol{P}[k]\boldsymbol{X}^T\boldsymbol{\Lambda}_M \boldsymbol{n}[k]\boldsymbol{n}^T[k]\boldsymbol{\Lambda}_M \boldsymbol{X}\boldsymbol{P}^T[k]\right] \\
&= \mathbb{E}\left[\boldsymbol{P}[k]\boldsymbol{X}^T\boldsymbol{\Lambda}_M \mathbb{E}\left[\boldsymbol{n}[k]\boldsymbol{n}^T[k]\right]\boldsymbol{\Lambda}_M \boldsymbol{X}\boldsymbol{P}^T[k]\right] \\
&= \mathbb{E}\left[\boldsymbol{P}[k]\boldsymbol{X}^T\boldsymbol{\Lambda}_M \sigma_n^2 \boldsymbol{I}_M \boldsymbol{\Lambda}_M \boldsymbol{X}\boldsymbol{P}^T[k]\right]
\end{aligned}$$

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$$\boldsymbol{h}_{\mathrm{LS}}[k] - \boldsymbol{h}_o = \left(\boldsymbol{X}^T \boldsymbol{\Lambda}_M \mathbf{X}\right)^{-1} \boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{n}[k]$$

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&= \mathbb{E}\left[\boldsymbol{P}[k]\boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{n}[k]\boldsymbol{n}^T[k] \boldsymbol{\Lambda}_M \boldsymbol{X} \boldsymbol{P}^T[k]\right] \\
&= \mathbb{E}\left[\boldsymbol{P}[k]\boldsymbol{X}^T \boldsymbol{\Lambda}_M \mathbb{E}\left[\boldsymbol{n}[k]\boldsymbol{n}^T[k]\right] \boldsymbol{\Lambda}_M \boldsymbol{X} \boldsymbol{P}^T[k]\right] \\
&= \mathbb{E}\left[\boldsymbol{P}[k]\boldsymbol{X}^T \boldsymbol{\Lambda}_M \sigma_n^2 \boldsymbol{I}_M \boldsymbol{\Lambda}_M \boldsymbol{X} \boldsymbol{P}^T[k]\right] \\
&= \sigma_n^2 \mathbb{E}\left[\boldsymbol{P}[k]\boldsymbol{X}^T \boldsymbol{\Lambda}_M \boldsymbol{\Lambda}_M \boldsymbol{X} \boldsymbol{P}^T[k]\right]
\end{aligned}$$

# Propriedades da solução de mínimos quadrados

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\operatorname{cov}[\Delta \boldsymbol{h}_{\mathrm{LS}}[k]] &= \mathbb{E}\left[\left(\boldsymbol{h}_{\mathrm{LS}}[k]-\boldsymbol{h}_o\right)\left(\boldsymbol{h}_{\mathrm{LS}}[k]-\boldsymbol{h}_o\right)^T\right] \\
&= \sigma_n^2 \mathbb{E}\left[\boldsymbol{P}[k]\boldsymbol{P}[k]^{-1}\boldsymbol{\Lambda}_M \boldsymbol{P}^T[k]\right]
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&= \sigma_n^2 \mathbb{E}\left[\boldsymbol{\Lambda}_M \boldsymbol{P}[k]\right]
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&= \sigma_n^2 \mathbb{E}\left[\boldsymbol{\Lambda}_M \boldsymbol{P}[k]\right]
\end{aligned}
$$

Observações:

- Portanto, a matriz de covariância do vetor de erro dos coeficientes tende a diminuir sua norma à medida que o tempo avança, uma vez que $\boldsymbol{P}[k]$ também é normativamente decrescente.
- A variância do ruído aditivo $\boldsymbol{n}[k]$ influencia diretamente a norma da matriz de covariância.