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Classical Coupled Kerr

0.1 Classical system

$$\begin{aligned}\mathcal{H}=\frac{H}{\hbar K N_e^2}=&-\frac{\delta}{2}\left(q_1^2+p_1^2\right)-\frac{\delta}{2}\left(q_2^2+p_2^2\right)+\frac{1}{4}\left(q_1^2+p_1^2\right)^2+\frac{1}{4}\left(q_2^2+p_2^2\right)^2\\&-\xi_2\left(q_1^2-p_1^2\right)-\xi_2\left(q_2^2-p_2^2\right)-\epsilon\left(q_1q_2+p_1p_2\right)\end{aligned} \tag{1}$$

Ecuaciones de movimiento

$$\begin{aligned}\frac{dq_1}{dt}&=\frac{\partial H}{\partial p_1}=-\delta p_1+p_1\left(q_1^2+p_1^2\right)+2\xi_2p_1-\epsilon p_2\\\frac{dp_1}{dt}&=-\frac{\partial H}{\partial q_1}=-\left[-\delta q_1+q_1\left(q_1^2+p_1^2\right)-2\xi_2q_1-\epsilon q_2\right]\\\frac{dq_2}{dt}&=\frac{\partial H}{\partial p_2}=-\delta p_2+p_2\left(q_2^2+p_2^2\right)+2\xi_2p_2-\epsilon p_1\\\frac{dp_2}{dt}&=-\frac{\partial H}{\partial q_2}=-\left[-\delta q_2+q_2\left(q_2^2+p_2^2\right)-2\xi_2q_2-\epsilon q_1\right]\end{aligned} \tag{2}$$

Tenemos el Jacobiano

$$\mathbb{J}=\begin{bmatrix}2q_1p_1 & -\delta+q_1^2+3p_1^2+2\xi_2 & 0 & -\epsilon\\ \delta-3q_1^2-p_1^2+2\xi_2 & -2q_1p_1 & \epsilon & 0\\ 0 & -\epsilon & 2q_2p_2 & -\delta+q_2^2+3p_2^2+2\xi_2\\ \epsilon & 0 & \delta-3q_2^2-p_2^2+2\xi_2 & -2q_2p_2\end{bmatrix} \tag{3}$$

Igualando a cero para encontrar los puntos fijos:

$$\begin{aligned}0&=-\delta p_1+p_1\left(q_1^2+p_1^2\right)+2\xi_2p_1-\epsilon p_2\\0&=-\delta q_1+q_1\left(q_1^2+p_1^2\right)-2\xi_2q_1-\epsilon q_2\\0&=-\delta p_2+p_2\left(q_2^2+p_2^2\right)+2\xi_2p_2-\epsilon p_1\\0&=-\delta q_2+q_2\left(q_2^2+p_2^2\right)-2\xi_2q_2-\epsilon q_1\end{aligned} \tag{4}$$

0.1.1 Caso $p_1=0$

$$\begin{aligned}0&=-\epsilon p_2\\0&=-\delta q_1+q_1^3-2\xi_2q_1-\epsilon q_2\\0&=-\delta p_2+p_2\left(q_2^2+p_2^2\right)+2\xi_2p_2\\0&=-\delta q_2+q_2\left(q_2^2+p_2^2\right)-2\xi_{2,2}q_2-\epsilon q_1\end{aligned} \tag{5}$$

p_2 debe ser cero:

$$\begin{aligned}0&=-\delta q_1+q_1^3-2\xi_2q_1-\epsilon q_2\\0&=-\delta q_2+q_2^3-2\xi_{2,2}q_2-\epsilon q_1\end{aligned} \tag{6}$$

Acomodando terminos

$$\begin{aligned}0&=q_1^3-\left(\delta+2\xi_2\right)q_1-\epsilon q_2\\0&=q_2^3-\left(\delta+2\xi_2\right)q_2-\epsilon q_1\end{aligned} \tag{7}$$

Por simetria $q_1 = q_2 = q$

$$0 = q^3 - (\delta + 2\xi_2)\, q - \epsilon q = q\left[q^2 - (\delta + 2\xi_2 + \epsilon)\right] = 0 \tag{8}$$

Puntos fijos

$$q = 0, \qquad q = \pm\sqrt{\delta + 2\xi_2 + \epsilon} \tag{9}$$

La transicion es

$$\epsilon = -(2\xi_2 + \delta) \tag{10}$$

ecuacion linealen el espacio (ξ_2, ϵ).

Para $q_1 = -q_2 = q$ tenemos

$$\begin{aligned} 0 &= q^3 - (\delta + 2\xi_2)\, q + \epsilon q \\ 0 &= -q^3 + (\delta + 2\xi_2)\, q - \epsilon q \end{aligned} \qquad \rightarrow \qquad q^3 + (\epsilon - \delta - 2\xi_2) q = 0 \tag{11}$$

Puntos fijos

$$q = 0, \qquad q = \pm\sqrt{\delta + 2\xi_2 - \epsilon} \tag{12}$$

La transicion es

$$\epsilon = 2\xi_2 + \delta \tag{13}$$

0.1.2 COn $q_1 = 0$

Las ecuaciones

$$\begin{aligned} 0 &= -\delta p_1 + p_1\left(q_1^2 + p_1^2\right) + 2\xi_2 p_1 - \epsilon p_2 \\ 0 &= -\delta q_1 + q_1\left(q_1^2 + p_1^2\right) - 2\xi_2 q_1 - \epsilon q_2 \\ 0 &= -\delta p_2 + p_2\left(q_2^2 + p_2^2\right) + 2\xi_2 p_2 - \epsilon p_1 \\ 0 &= -\delta q_2 + q_2\left(q_2^2 + p_2^2\right) - 2\xi_2 q_2 - \epsilon q_1 \end{aligned} \tag{14}$$

Quedan como:

$$\begin{aligned} 0 &= -\delta p_1 + p_1^3 + 2\xi_2 p_1 - \epsilon p_2 \\ 0 &= -\epsilon q_2 \\ 0 &= -\delta p_2 + p_2\left(q_2^2 + p_2^2\right) + 2\xi_2 p_2 - \epsilon p_1 \\ 0 &= -\delta q_2 + q_2\left(q_2^2 + p_2^2\right) - 2\xi_2 q_2 \end{aligned} \tag{15}$$

q_2 debe de ser cero

$$\begin{aligned} 0 &= -\delta p_1 + p_1^3 + 2\xi_2 p_1 - \epsilon p_2 \\ 0 &= -\delta p_2 + p_2^3 + 2\xi_2 p_2 - \epsilon p_1 \end{aligned} \tag{16}$$

Por simetria $p_1 = p_2 = p$

$$\begin{aligned} 0 &= -\delta p + p^3 + 2\xi_2 p - \epsilon p \\ 0 &= -\delta p + p^3 + 2\xi_2 p - \epsilon p \end{aligned} \qquad \rightarrow \qquad p^3 + (-\epsilon - \delta + 2\xi_2) p = 0 \tag{17}$$

Los puntos fijos son

$$p = 0, \qquad p = \pm\sqrt{\epsilon + \delta - 2\xi_2} \tag{18}$$

La transicion es

$$\epsilon = 2\xi_2 - \delta \tag{19}$$

Para $p_1 = -p_2 = p$, las ecuaciones

$$\begin{aligned} 0 &= -\delta p_1 + p_1^3 + 2\xi_2 p_1 - \epsilon p_2 \\ 0 &= -\delta p_2 + p_2^3 + 2\xi_2 p_2 - \epsilon p_1 \end{aligned} \tag{20}$$

quedan

$$\begin{aligned} 0 &= -\delta p + p^3 + 2\xi_2 p + \epsilon p \\ 0 &= \delta p - p^3 - 2\xi_2 p - \epsilon p \end{aligned} \qquad \rightarrow \qquad p^3 + (\epsilon - \delta + 2\xi_2)p = 0 \tag{21}$$

los puntos son:

$$p = 0, \qquad p = \pm\sqrt{-\epsilon + \delta - 2\xi_2} \tag{22}$$

La transicion es

$$\epsilon = -2\xi_2 + \delta \tag{23}$$

en resumen, los 9 puntos son los siguientes:

$$\begin{array}{lcccc}
r_0 = & 0 & 0 & 0 & 0 \\
\\
r_1 = & \sqrt{\delta + 2\xi_2 + \epsilon} & \sqrt{\delta + 2\xi_2 + \epsilon} & 0 & 0 \\
r_2 = & -\sqrt{\delta + 2\xi_2 + \epsilon} & -\sqrt{\delta + 2\xi_2 + \epsilon} & 0 & 0 \\
r_3 = & \sqrt{\delta + 2\xi_2 - \epsilon} & -\sqrt{\delta + 2\xi_2 - \epsilon} & 0 & 0 \\
r_4 = & -\sqrt{\delta + 2\xi_2 - \epsilon} & \sqrt{\delta + 2\xi_2 - \epsilon} & 0 & 0 \\
\\
r_5 = & 0 & 0 & \sqrt{\delta - 2\xi_2 + \epsilon} & \sqrt{\delta - 2\xi_2 + \epsilon} \\
r_6 = & 0 & 0 & -\sqrt{\delta - 2\xi_2 + \epsilon} & -\sqrt{\delta - 2\xi_2 + \epsilon} \\
r_5 = & 0 & 0 & \sqrt{\delta - 2\xi_2 - \epsilon} & -\sqrt{\delta - 2\xi_2 - \epsilon} \\
r_6 = & 0 & 0 & -\sqrt{\delta - 2\xi_2 - \epsilon} & \sqrt{\delta - 2\xi_2 - \epsilon}
\end{array} \tag{24}$$

cuyas energias son

$$E_0 = 0, \qquad E_{1,2} = -\frac{(\delta + 2\xi_2 + \epsilon)^2}{2} \qquad E_{3,4} = -\frac{(\delta + 2\xi_2 - \epsilon)^2}{2} \tag{25}$$

$$E_{5,6} = -\frac{(\delta - 2\xi_2 + \epsilon)^2}{2} \qquad E_{7,8} = -\frac{(\delta - 2\xi_2 - \epsilon)^2}{2} \tag{26}$$