Operations Research Letters 12 (1992) 1–2
North-Holland
July 1992

A projective algorithm for linear programming with no regularity condition *

J.-P. Vial
Département d'Economie Commerciale et Industrielle, Université de Genève, Geneva, Switzerland

Received March 1991
Revised November 1991

The combined phase I–phase II projective algorithm of de Ghellinck and Vial solves any linear programming problem in its primal–dual formulation, without assuming any regularity condition. The method does not require artificial variables and/or constraints. It consists simply in minimizing the sum of all primal and dual variables. The present note is an alternative to a recent proposal of Anstreicher.

linear programming; interior point algorithm; combined phase I–phase II; regularity condition

It is well-known that any interior point method for linear programming requires some kind of regularity condition to enforce convergence. The standard assumption is that the primal and the dual feasible sets have a nonempty interior. A less similar assumption is that there is no direction in the recession cone of the feasible set along which the objective function is constant. These assumptions are not restrictive in theory, in that the original problem can be reformulated into a polynomially equivalent one for which a suitable regularity condition holds. However the transformations involve the addition of dense rows and/or columns and also a large penalty term M associated with the artificial variables. Projective algorithms usually do not handle explicitly the added row. As to the additional column, the issue is not critical since the computations can easily be organized using rank-one updating to avoid explicitly adding the column. However a more direct approach is still desirable.

In a recent paper Anstreicher [1] proposed an ingenious device to circumvent the need for a "big M" coefficient. His method operates on the reformulation of any linear programming problem as a primal–dual feasibility problem. His transformation further involves one additional dense column and two additional rows (one dense, one sparse).

The object of this not is to show that it is possible to apply directly the projective algorithm of the Ghellinck–Vial [2] to the primal–dual feasibility problem without introducing an additional row or column. This algorithm is an extension of the feasibility algorithm of [3]. It is a combined phase I–phase II projective method with no artificial variable. As in other projective algorithms, a regularity condition is required. We shall show it to be automatically satisfied by the primal–dual reformulation of the problem. For problems with a known interior feasible solution, the projective algorithm in [2] was proved in [6] to be equivalent (same search direction, same potential function) to Todd–Burell's extension [5] of Karmarkar's algorithm [4]. However our problem consists in finding a feasible point and not in assuming that one is known.

* Research supported by FNRS through Grant 12-26434.89.
Correspondence to: Prof. J.-P. Vial, Département d'Economie Commerciale et Industrielle, Université de Genève, Département COMIN, 2 rue de Candolle, CH-1211, Geneva 4, Swiss.

Let

(P) $\min\{c^{\mathrm{T}}x:\ Ax=b,\ x\geq 0\}$

be the prototype linear programming problem to solve. (P) has a solution if and only if

(PD) $\operatorname{find}\{y:\ Dy=h,\ y\geq 0\}$

has a solution. Here

$$D=\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A^{\mathrm{T}} & -A^{\mathrm{T}} & I \\ c^{\mathrm{T}} & -b^{\mathrm{T}} & b^{\mathrm{T}} & 0 \end{pmatrix}, \qquad h=\begin{pmatrix} b \\ c \\ 0 \end{pmatrix}.$$

The matrix D involves the original data of the problem and no artificial variable. The unrestricted dual variable is decomposed in the difference of two nonnegative variables.

The projective algorithm of the Ghellinck–Vial requires that the condition

$$Dy=0,\ y\geq 0 \ \Rightarrow\ y=0$$

be satisfied. This condition is essential, both in theory and in practice. Since D has two identical columns of opposite signs, (PD) does not satisfy the above regularity condition.

To overcome this difficulty we associate with (PD) the linear programming problem

(PD') $\min\{e^{\mathrm{T}}y:\ Dy=h,\ y\geq 0\}$,

where e is the vector of all ones of appropriate dimension. This new problem satisfies the regularity condition of the combined phase I–phase II algorithm of de Ghellinck–Vial [2], namely

$$Dy=0,\ e^{\mathrm{T}}y=0,\ y\geq 0 \ \Rightarrow\ y=0.$$

The combined phase I–phase I requires an initial lower bound for the objective function. In our case, zero is an obvious such bound. Hence the problem can be solved in polynomial time, either yielding an optimal solution or proving that there is none. Note that the simple device of minimizing the sum of all variables can be applied to any feasibility problem with nonnegative variables.

Instead of minimizing the sum of all variables (excluding the homogenizing variable) one could choose the weighted sum $w^{\mathrm{T}}y$, where w is any vector with strictly positive components. If w is selected at random, with coefficient $w_i>0$, say $0.5\leq w_i\leq 1$, the algorithm will converge with probability one to an optimal extreme point (if such a point exists). In this way, we discard convergence to the center of an optimal facet, a feature of standard projective algorithms which is sometimes deemed undesirable. Finally we would like to point out that it is not necessary to decompose the free variables into differences of nonnegative variables. A direct treatment of free variables is given in [6].

The idea of minimizing the sum of the variables is related to Anstreicher's device [1]. This author first introduces an artificial variable and an homogenizing variable. He also adds the constraints that the artificial variable equals zero and that the sum of all variables is N, where N is the total number of variables. Finally he applies his combined phase I–phase II projective algorithm to the maximization of the homogenizing variable. Under the constraint that the sum of all variables is constant, his problem is clearly equivalent to minimizing the sum of all variables less the homogenizing variable.

References

[1] K. Anstreicher, "On interior algorithms for linear programming with no regularity assumptions", Manuscript, Department of Operations Research, Yale University, New Have, February 1991.

[2] G. de Ghellinck and J.-P. Vial, "A polynomial Newton method for linear programming", *Algorithmica* **1** (1986) 425-453.

[3] G. de Ghellinck and J.-P. Vial, "An extension of Karmarkar's algorithm for solving a system of linear homogeneous equations on the simplex", *Math. Programming* **39** (1987) 79-92.

[4] N. Karmarkar, "A new polynomial-time algorithm for linear programming", *Combinatorica* **4** (1984) 373-395.

[5] M.J. Todd and B.P. Burrell, "An extension of Karmarkar's algorithm for linear programming using dual variables", *Algorithmica* **1** (1986) 409-424.

[6] J.-P. Vial, "A unified approach to projective algorithms for linear programming", in: S. Dolecki, (ed.), *Optimization*, Proceedings of the Fifth French–German Conference held in Castel-Novel 1988, Lecture Notes in Mathematics No. 1405, Springer-Verlag, Berlin, 1989, 191-220.