8. Projective Algorithms

'Αεὶ ὁ Θεός ὁ μέγας γεωμετρεῖ ...
Disciples of Euclid (c. 300 B.C.)

If the above quotation is *Greek* to you, you are quite right. In any case, counting the letters in each word separately you get 3.14159... which is a start on the number Pi (π) that was found in a sandbox of the Mediterranean well over 2,000 years ago and which got you all excited about mathematics in grade school already. And, of course, "God the Almighty always does geometry" is what it freely translates to.

So let us get into the analytic sandbox and do some geometry as well. We consider the linear program in standard form

$$\text{(LP)} \qquad \min\{\boldsymbol{cx} : \boldsymbol{Ax} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\} ,$$

where $\boldsymbol{A}$ is a $m \times n$ matrix of rank m, $\boldsymbol{c}$ a row vector of length n and $\boldsymbol{b}$ a column vector of length m and all data are integers. We denote as usually by

$$\mathcal{X} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{Ax} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$$

the feasible set of (LP). We will make the **assumptions** that

- there exists $\boldsymbol{x}^0 \in \mathcal{X}$ such that $\boldsymbol{x}^0 > \boldsymbol{0}$, i.e. the relative interior of $\mathcal{X}$ is nonempty, and
- the objective function $\boldsymbol{cx}$ over $\mathcal{X}$ is *not* a constant, i.e. in particular $\boldsymbol{c} \neq \boldsymbol{0}$.

While the second assumption – like the rank assumption about $\boldsymbol{A}$ – is a natural one to make, the existence of an *interior* feasible point $\boldsymbol{x}^0$ is at first sight restrictive. However, using an additional artificial variable x_{n+1} with the column

$$\boldsymbol{a}_{n+1} = -\sum_{j=1}^{n} \boldsymbol{a}_j + \boldsymbol{b}$$

and the Big M-method, i.e. setting $c_{n+1} = M$ in the objective function where M is a suitably chosen large constant, we can ensure that any linear program (LP) satisfies this assumption and we will thus assume for the sake of notational simplicity that (LP) is already of the desired form.

The basic idea of projective algorithms for (LP) is to generate a sequence of interior feasible points that converge towards an optimal solution of (LP). This is done – in essence – by solving a sequence of *restricted* problems that involve no inequalities and which are thus amenable to solution by the classical methods of calculus. The projective transformations that are employed correspond to an iterated "change of variables" that permits to implement this idea in several ways. The algorithms that we give are stated, however, in the original space of the variables of the problem (LP).

Let us rewrite the problem (LP) as follows

$$(\mathrm{LP}^*) \qquad \min\{\boldsymbol{cx} : \boldsymbol{Ax} - \boldsymbol{b}x_{n+1} = \boldsymbol{0},\ x_{n+1} = 1,\ \boldsymbol{x} \geq \boldsymbol{0},\ x_{n+1} \geq 0\}$$

where x_{n+1} is a "new" variable. Geometrically, we are embedding $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ by identifying a point $\boldsymbol{x} \in \mathbb{R}^n$ with the point $(\boldsymbol{x}, 1) \in \mathbb{R}^{n+1}$ where we write $(\boldsymbol{x}, 1)$ rather than $\begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix}$ if no confusion is possible (see Figure 8.1).

Formally, we can interpret the $(n+1)$-tuples $(x_1, \ldots, x_n, x_{n+1})$ also as the *homogeneous coordinates* of points $\widetilde{\boldsymbol{x}}$, say, of the n-dimensional projective space $\mathcal{P}_n$. Two $(n+1)$-tuples $\widetilde{\boldsymbol{x}}$ and $\widehat{\boldsymbol{x}}$ are *equal* in $\mathcal{P}_n$ if there exists a $\lambda \neq 0$ such that $\widetilde{\boldsymbol{x}} = \lambda\widehat{\boldsymbol{x}}$, which explains why the coordinates are called homogeneous and which makes $\mathcal{P}_n$ different from $\mathbb{R}^{n+1}$. Points $\widetilde{\boldsymbol{x}}$ with $x_{n+1} \neq 0$ are "proper" points of $\mathcal{P}_n$. They are those points of $\mathcal{P}_n$ for which $(1/x_{n+1})\boldsymbol{x}$ is a point of the n-dimensional affine space $\mathbb{R}^n$. Evidently, any two proper points of $\mathcal{P}_n$ that are equal define exactly one point of $\mathbb{R}^n$. Points $\widetilde{\boldsymbol{x}}$ with $x_{n+1} = 0$ and $\boldsymbol{x} \neq \boldsymbol{0}$ are "improper" points of $\mathcal{P}_n$. They are those points of $\mathcal{P}_n$ that correspond to some sort of "points at infinity" of $\mathbb{R}^n$; the point $\widetilde{\boldsymbol{x}} = \boldsymbol{0}$ has no meaning in $\mathcal{P}_n$, i.e. it simply does not exist. Lines, planes and hyperplanes are defined *mutatis mutandis* in $\mathcal{P}_n$ analogously to the objects in $\mathbb{R}^n$ through homogenization. In the first part of our development there is no need for the formalism of $\mathcal{P}_n$. Later on, however, we will make some use of the n-dimensional projective space $\mathcal{P}_n$.

By assumption we have a feasible $(\boldsymbol{x}^0, 1) \in \mathbb{R}^{n+1}$ to (LP^*) such that $\boldsymbol{x}^0 > \boldsymbol{0}$. Consider the projective transformation T_0

$$y_j = \frac{x_j/x_j^0}{1 + \sum_{j=1}^n x_j/x_j^0} \quad \text{for } j = 1, \ldots, n, \qquad y_{n+1} = \frac{1}{1 + \sum_{j=1}^n x_j/x_j^0}\,,$$

the "center" of which is the origin of $\mathbb{R}^{n+1}$. Since $(\boldsymbol{x}^0, 1) > \boldsymbol{0}$, it is well defined and maps the nonnegative "orthant" $\{(\boldsymbol{x}, 1) \in \mathbb{R}^{n+1} : \boldsymbol{x} \geq \boldsymbol{0}\}$ into the n-dimensional simplex

$$S^{n+1} = \left\{ \boldsymbol{y} \in \mathbb{R}^{n+1} : \sum_{j=1}^{n+1} y_j = 1,\ \boldsymbol{y} \geq \boldsymbol{0} \right\}.$$

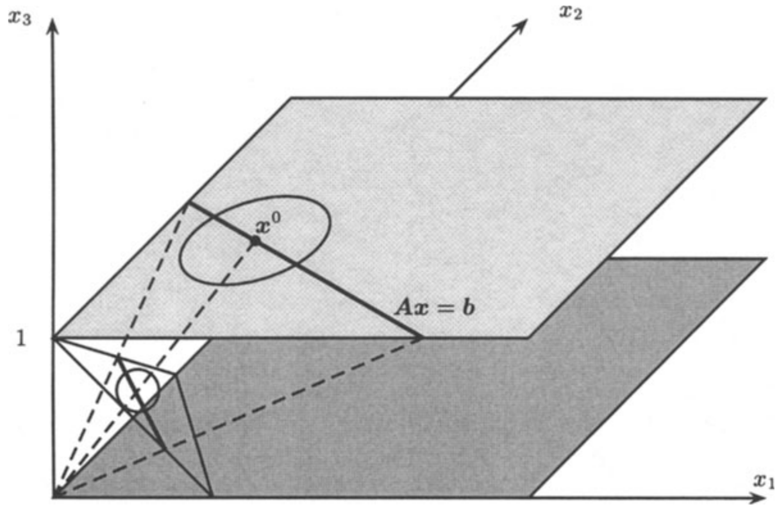

Fig. 8.1. Embedding of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ or $\mathcal{P}_n$ for $n = 2$.

Moreover, the point $(\boldsymbol{x}^0, 1) \in \mathbb{R}^{n+1}$ is mapped into the "center"

$$\boldsymbol{y}^0 = [1/(n+1)]\boldsymbol{f}$$

of the simplex S^{n+1} where $\boldsymbol{f}^T = (1, \ldots, 1)$ is the vector with $n+1$ components equal to one. We write $\boldsymbol{y} = T_0(\boldsymbol{x})$ to denote the image $\boldsymbol{y} \in \mathbb{R}^{n+1}$ of the point $(\boldsymbol{x}, 1) \in \mathbb{R}^{n+1}$ where $\boldsymbol{x} \in \mathbb{R}^n$ and thus T_0 is interpreted as a mapping from $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ in the natural way. The projective transformation possesses an inverse

$$x_j = x_j^0 y_j / y_{n+1} \text{ for } j = 1, \ldots, n \text{ and } x_{n+1} = 1,$$

when we consider the embedding of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$.

Denote $\boldsymbol{D} = diag(x_1^0, \ldots, x_n^0)$ the $n \times n$ matrix with diagonal elements x_i^0 for $i = 1, \ldots, n$ and zeroes elsewhere. The transformation T_0 in matrix form is then written as follows

$$(T_0) \qquad \begin{pmatrix} \boldsymbol{y}_N \\ y_{n+1} \end{pmatrix} = \frac{1}{1 + \boldsymbol{e}^T \boldsymbol{D}^{-1} \boldsymbol{x}} \begin{pmatrix} \boldsymbol{D}^{-1}\boldsymbol{x} \\ 1 \end{pmatrix},$$

where $\boldsymbol{e}^T = (1, \ldots, 1)$ is the vector with n components equal to one and $\boldsymbol{y}_N^T = (y_1, \ldots, y_n)$ is the subvector of the n first components of the vector $\boldsymbol{y} \in \mathbb{R}^{n+1}$. The inverse of the projective transformation in matrix form is

$$(T_0^{-1}) \qquad \boldsymbol{x} = (1/y_{n+1})\boldsymbol{D}\boldsymbol{y}_N$$

and $x_{n+1} = 1$. It follows that the image $T_0(\mathcal{X})$ of the set $\mathcal{X}$ under T_0 is

$$T_0(\mathcal{X}) = \left\{ \boldsymbol{y} \in \mathbb{R}^{n+1} : (\boldsymbol{AD},\ -\boldsymbol{b})\boldsymbol{y} = \mathbf{0},\ \boldsymbol{f}^T\boldsymbol{y} = 1,\ \boldsymbol{y} \geq \mathbf{0} \right\} ,$$

that $T_0(\mathcal{X}) \subseteq S^{n+1}$, $\boldsymbol{y}^0 \in T_0(\mathcal{X})$ and the linear program (LP) becomes the nonlinear programming problem

$$\text{(FLP)} \qquad \min\left\{ \frac{\boldsymbol{cDy}_N}{y_{n+1}} : \boldsymbol{y} \in T_0(\mathcal{X}) \right\} .$$

Let us denote by

$$B_\rho^{n+1} = \left\{ \boldsymbol{y} \in \mathbb{R}^{n+1} : \sum_{j=1}^{n+1} y_j = 1,\ \sum_{j=1}^{n+1} \left(y_j - \frac{1}{n+1} \right)^2 \leq \rho^2 \right\}$$

the intersection of the $(n+1)$-dimensional ball with radius ρ and center $\boldsymbol{y}^0$ with the affine subspace $\boldsymbol{f}^T\boldsymbol{y} = 1$ which gives an n-dimensional ball in the simplex S^{n+1} if the radius ρ is "small" enough. Now *you* show the following exercise where

$$r^2 = 1/n(n+1) . \tag{8.1}$$

Exercise 8.1 *(i) Prove that* $B_\rho^{n+1} \subseteq S^{n+1}$ *if and only if* $0 \leq \rho \leq r$. *(ii) Prove that* $B_\rho^{n+1} \supseteq S^{n+1}$ *if and only if* $\rho \geq \sqrt{n/(n+1)}$. (Hint*: Use the Lagrangean multiplier technique to show that* $y_i = -n\rho r + 1/(n+1)$, $y_j = \rho r + 1/(n+1)$ *for all* $j \neq i$ *solves the minimization problem* $\min\{y_i : \boldsymbol{y} \in B_\rho^{n+1}\}$ *and that* $y_i = n\rho r + 1/(n+1)$, $y_j = -\rho r + 1/(n+1)$ *for all* $j \neq i$ *solves the corresponding maximization problem.*)

So if $0 \leq \rho \leq r$ then B_ρ^{n+1} is a subset of the simplex S^{n+1}. We can thus replace (FLP) by

$$(\text{FLP}_\rho) \qquad \min\left\{ \frac{(\boldsymbol{cD}\,,\,0)\boldsymbol{y}}{y_{n+1}} : (\boldsymbol{AD},\ -\boldsymbol{b})\boldsymbol{y} = \mathbf{0},\ \boldsymbol{y} \in B_\rho^{n+1} \right\} ,$$

where $0 \leq \rho < r$ in order to ensure that $\boldsymbol{y} > \mathbf{0}$ and so we have for all $0 \leq \rho < r$

$$\min\{\boldsymbol{cx} : \boldsymbol{x} \in \mathcal{X}\} \leq \min\left\{ \frac{(\boldsymbol{cD}\,,\,0)\boldsymbol{y}}{y_{n+1}} : \boldsymbol{y} \in T_0(\mathcal{X}) \cap B_\rho^{n+1} \right\} . \tag{8.2}$$

The problem (FLP_ρ) is a *restriction* of (FLP) and it is a classical nonlinear optimization problem that can be approached by a variety of means. Of these we will discuss two different ones.

- In the first one we "linearize" the objective function of (FLP_ρ) which leads to a particularly simple solution.
- In the second one we solve (FLP_ρ) *exactly*.

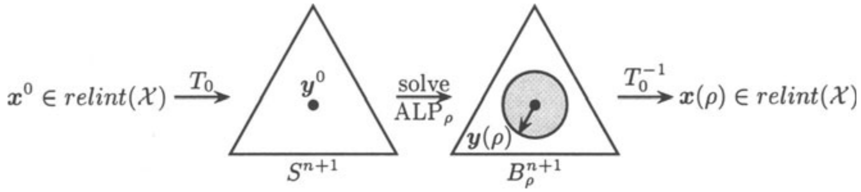

Fig. 8.2. The iterative step of projective algorithms.

In either case, once a solution to (FLP_ρ) or an approximation to it has been obtained, one can use the inverse transformation T_0^{-1} of the projective transformation T_0 to obtain a "new" interior point $\boldsymbol{x}^1 \in \mathcal{X}$ which gives rise to a new projective transformation T_1, etc, see Figure 8.2. Thus one can *iterate*, the basic algorithmic idea is clear and the question that ensues is, plainly, whether or not the idea "works". And it works rather well as we shall see.

8.1 A Basic Algorithm

Aller Anfang ist schwer.
German proverb

In the first approach to the problem we approximate (FLP) by the auxiliary linear programming problem

$$(ALP) \qquad \min\{(\boldsymbol{cD}\ ,\ 0)\boldsymbol{y} : \boldsymbol{y} \in T_0(\mathcal{X})\}$$

and make the **additional assumptions** that

- $\mathcal{X}$ is bounded and
- the optimal objective function value of (LP) equals **zero**.

We discuss later how to remove these assumptions. It follows that the optimal objective function value of (ALP) equals zero as well no matter what interior point $\boldsymbol{x}^0 \in \mathcal{X}$ is used in the projective transformation T_0. Like we did above, we replace the problem (FLP_ρ) by the auxiliary optimization problem

$$(\mathrm{ALP}_\rho) \qquad \min\{(\boldsymbol{cD}\ ,\ 0)\boldsymbol{y} : (\boldsymbol{AD},\ -\boldsymbol{b})\boldsymbol{y} = \boldsymbol{0},\ \boldsymbol{y} \in B_\rho^{n+1}\}\ .$$

By the additional assumption that we have made, an optimal solution to (ALP_ρ) exists and its optimal objective function value is nonnegative for all $0 \le \rho \le r$.

In the following remark – where we change momentarily our notation – the key facts about the solution of (ALP_ρ) are summarized. In the proof of the remark – as well as in similar situations below – we leave it to the reader to verify that the inequality in the definition of B_ρ^{n+1} can be replaced by an equality for the respective optimization problem. As there is only *one* nonlinear constraint we just need to argue the *two* possible cases: the constraint is either binding or not.

Remark 8.1 *Consider the linear program* $\min\{cz : Az = 0,\ e^T z = 1,\ z \geq 0\}$ *where* A *is an* $m \times n$ *matrix of rank* m, $z^0 = (1/n)e$ *is a* nonoptimal *feasible solution and the optimal objective function value equals zero, i.e.* $cz^0 > 0$. *Then for all* $\rho \geq 0$ *an optimal solution to*

$$(\mathrm{P}_\rho) \qquad \min\{cz : Az = 0,\ z \in B_\rho^n\}$$

is given by $z(\rho) = (1/n)e - \rho p/\|p\|$ *where*

$$p = (I_n - A^T(AA^T)^{-1}A - (1/n)ee^T)c^T$$

is the orthogonal projection of c *on the subspace*

$$\{z \in \mathbb{R}^n : Az = 0,\ e^T z = 0\} .$$

Moreover, for all $\rho \geq 0$ *the optimal solution* $z(\rho)$ *satisfies*

$$cz(\rho)/cz^0 \leq 1 - \rho\sqrt{n/(n-1)} .$$

Proof. Since the feasible set is compact an optimal solution to (P_ρ) exists and we can use the Lagrangean multiplier technique to calculate it. Forming the Lagrangean function and differentiating we find that any optimal solution to (P_ρ) satisfies

$$c + \mu A + \nu e^T + \lambda(z - (1/n)e)^T = 0 \qquad (i)$$

where $\mu \in \mathbb{R}^m$, $\nu \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. Since z^0 is a nonoptimal feasible solution it follows that $\lambda \neq 0$ since otherwise $cz = -\nu$ for all feasible z. Multiplying (i) by e we get $ce + n\nu = 0$ and thus $\nu = -cz^0$. Since A is of full row rank AA^T is positive definite and thus, in particular, nonsingular. Multiplying (i) by A^T we get $cA^T + \mu AA^T = 0$. Solving for μ and substituting μ and ν we get from (i) that $p^T + \lambda(z - (1/n)e)^T = 0$ or $z = (1/n)e - (1/\lambda)p$ since $\lambda \neq 0$. Consequently,

$$\lambda^2\rho^2 = p^T p \text{ or } \lambda = \pm\|p\|/\rho .$$

Since $\lambda \neq 0$ we have $\|p\| \neq 0$ and thus $z(\rho)$ is the minimizer for (P_ρ). Hence, $cz(\rho) = cz^0 - \rho\|p\|$ since $cp = \|p\|^2$ by the properties of orthogonal projections and the first part of Remark 8.1 follows. Since $S^n \subseteq B_{\rho^*}^n$ for $\rho^* = \sqrt{(n-1)/n}$ by part (ii) of Exercise 8.1 it follows from the assumption that the optimal objective function value of the linear program equals zero that $cz(\rho^*) \leq 0$. For, if not then $cz(\rho^*) > 0$ and since $S^n \subseteq B_{\rho^*}^n$ the optimal objective function of the linear program is positive which is a contradiction. Hence $cz^0 - \|p\|\sqrt{(n-1)/n} \leq 0$, i.e. $\|p\|/cz^0 \geq \sqrt{n/(n-1)}$. ■

8.1.1 The Solution of the Approximate Problem

It follows from Remark 8.1 that in order to solve the problem (ALP$_\rho$) we have to calculate the orthogonal projection of the vector $(\boldsymbol{cD},\ 0)$ on the subspace

$$\{\boldsymbol{y} \in \mathbb{R}^{n+1} : (\boldsymbol{AD},\ -\boldsymbol{b})\boldsymbol{y} = \mathbf{0},\ \boldsymbol{f}^T\boldsymbol{y} = 0\}\ . \tag{8.3}$$

To this end we need to find the inverse of the matrix

$$\widehat{\boldsymbol{G}} = \begin{pmatrix} \boldsymbol{AD} & -\boldsymbol{b} \\ \boldsymbol{e}^T & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{DA}^T & \boldsymbol{e} \\ -\boldsymbol{b}^T & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{AD}^2\boldsymbol{A}^T + \boldsymbol{bb}^T & \mathbf{0} \\ \mathbf{0} & n+1 \end{pmatrix}$$

since $\boldsymbol{ADe} = \boldsymbol{Ax}^0 = \boldsymbol{b}$ by assumption. Let us denote $\boldsymbol{G} = \boldsymbol{AD}^2\boldsymbol{A}^T$ and note that $\boldsymbol{G}$ is positive definite since $\boldsymbol{A}$ is of full row rank and $\boldsymbol{D}$ is nonsingular. Thus in particular, $\boldsymbol{G}^{-1}$ exists. Applying the formula of Chapter 4.4 we calculate the inverse of $\widehat{\boldsymbol{G}}$ to be

$$\widehat{\boldsymbol{G}}^{-1} = \begin{pmatrix} \boldsymbol{G}^{-1} - (1+\beta)^{-1}(\boldsymbol{G}^{-1}\boldsymbol{b})(\boldsymbol{b}^T\boldsymbol{G}^{-1}) & \mathbf{0} \\ \mathbf{0} & (n+1)^{-1} \end{pmatrix} ,$$

where $\beta = \boldsymbol{b}^T\boldsymbol{G}^{-1}\boldsymbol{b} \geq 0$ since $\boldsymbol{G}^{-1}$ is positive definite as well. Let us denote

$$\boldsymbol{P} = \boldsymbol{I}_n - \boldsymbol{DA}^T\boldsymbol{G}^{-1}\boldsymbol{AD}\ , \quad \boldsymbol{p} = \boldsymbol{PDc}^T\ , \quad \boldsymbol{d} = \boldsymbol{Pe}\ , \tag{8.4}$$

i.e. $\boldsymbol{p}$ is the orthogonal projection of the vector $\boldsymbol{Dc}^T$ and $\boldsymbol{d}$ the orthogonal projection of the vector $\boldsymbol{e}$ on the subspace

$$\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{ADx} = \mathbf{0}\}\ . \tag{8.5}$$

We calculate the projection operator $\boldsymbol{Q}$ on the subspace (8.3) to be

$$\begin{aligned} \boldsymbol{Q} &= \boldsymbol{I}_{n+1} - \begin{pmatrix} \boldsymbol{DA}^T & \boldsymbol{e} \\ -\boldsymbol{b}^T & 1 \end{pmatrix} \widehat{\boldsymbol{G}}^{-1} \begin{pmatrix} \boldsymbol{AD} & -\boldsymbol{b} \\ \boldsymbol{e}^T & 1 \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{P} & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix} + (1+\beta)^{-1} \begin{pmatrix} \boldsymbol{e} - \boldsymbol{d} \\ 1 \end{pmatrix} (\boldsymbol{e}^T - \boldsymbol{d}^T\ , 1) - (n+1)^{-1}\boldsymbol{ff}^T \end{aligned}$$

and consequently the orthogonal projection of $(\boldsymbol{cD}\ ,\ 0)$ is given by

$$\boldsymbol{q} = \boldsymbol{Q}\begin{pmatrix} \boldsymbol{Dc}^T \\ 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{p} \\ 0 \end{pmatrix} + \frac{z_0 - \gamma}{1+\beta}\begin{pmatrix} \boldsymbol{e} - \boldsymbol{d} \\ 1 \end{pmatrix} - \frac{z_0}{n+1}\begin{pmatrix} \boldsymbol{e} \\ 1 \end{pmatrix} , \tag{8.6}$$

where we have set $z_0 = \boldsymbol{cx}^0$ and $\gamma = \boldsymbol{p}^T\boldsymbol{d}$. For further reference we note that necessarily $\boldsymbol{p} \neq \mathbf{0}$ if $\boldsymbol{x}^0$ is a *nonoptimal* solution to (LP). Moreover,

$$\gamma = \boldsymbol{p}^T\boldsymbol{e} = \boldsymbol{cDd}$$

by the properties of orthogonal projections. By the definitions of $\boldsymbol{d}$ and β

$$\beta = n - ||\boldsymbol{d}||^2 = ||\boldsymbol{e} - \boldsymbol{d}||^2 \text{ and } ||\boldsymbol{d}||^2 \leq n$$

because $\beta \geq 0$. Likewise, $||\boldsymbol{q}||$ is calculated using the fact that

$$||\boldsymbol{q}||^2 = (\boldsymbol{cD}\ ,0)\ \boldsymbol{q}.$$

It follows from Remark 8.1 that the solution to (ALP$_\rho$) is given by

$$\boldsymbol{y}^K(\rho) = \boldsymbol{y}^0 - \rho\boldsymbol{q}/||\boldsymbol{q}||\ , \tag{8.7}$$

that $(\boldsymbol{cD}\ ,\ 0)\boldsymbol{y}^K(\rho) = (z_0/(n+1)) - \rho||\boldsymbol{q}||$ and that for all $\rho \geq 0$ we have

$$\frac{(\boldsymbol{cD}\ ,\ 0)\boldsymbol{y}^K(\rho)}{(\boldsymbol{cD}\ ,\ 0)\boldsymbol{y}^0} \leq 1 - \rho\sqrt{(n+1)/n}\ . \tag{8.8}$$

Reversing the projective transformation we find after some simplifications

$$\begin{aligned} \boldsymbol{x}^K(\rho) &= \boldsymbol{x}^0 - t(\rho)\boldsymbol{D}\left(\boldsymbol{p} - \frac{z_0 - \gamma}{1+\beta}\boldsymbol{d}\right)\ , \\ t(\rho) &= \frac{(1+\beta)(n+1)\rho}{(1+\beta)||\boldsymbol{q}|| + \rho(\gamma(n+1) - (n-\beta)z_0)} \end{aligned} \tag{8.9}$$

and that the objective function value of $\boldsymbol{x}^K(\rho)$ is given by

$$\boldsymbol{cx}^K(\rho) = z_0 - t(\rho)\left[||\boldsymbol{p}||^2 - \gamma(z_0 - \gamma)/(1+\beta)\right]\ . \tag{8.10}$$

It follows from its derivation that $t(\rho) = \rho(||\boldsymbol{q}||y^K_{n+1}(\rho))^{-1}$ where $y^K_{n+1}(\rho)$ is the last component of $\boldsymbol{y}^K(\rho) \in B^{n+1}_\rho$. Consequently, from $y^K_{n+1}(\rho) > 0$ we get $t(\rho) \geq 0$ for all $0 \leq \rho < r = 1/\sqrt{n(n+1)}$. From (8.9) we calculate $dt/d\rho > 0$ for all $0 \leq \rho < r$ and thus the change of parameters from ρ to $t(\rho)$ is a change that preserves strict monotonicity.

8.1.2 Convergence of the Approximate Iterates

Like the solution (8.7) to (ALP$_\rho$), the loci of $\boldsymbol{x}^K(\rho)$ given by (8.9) form a *line* in $\mathbb{R}^n$ (which is a consequence of the fact that projective transformations *always* map lines into lines). From (8.10) we infer that depending on the sign of the term in the square brackets the objective function value $\boldsymbol{cx}^K(\rho)$ may increase or decrease along this line contrary to the monotonic behavior of $(\boldsymbol{cD}\ ,\ 0)\boldsymbol{y}^K(\rho)$ which decreases linearly in ρ. *You* verify that

$$(\boldsymbol{cD}, -z_0)\boldsymbol{q} = ||\boldsymbol{p}||^2 - \gamma(z_0 - \gamma)/(1+\beta)$$

and thus the objective function increases if $(\boldsymbol{cD}, -z_0)^T$ and the orthogonal projection $\boldsymbol{q}$ of $(\boldsymbol{cD}, 0)^T$ onto the subspace (8.3) form an obtuse angle.

Nevertheless, by construction we have $\boldsymbol{x}^K(\rho) \in \mathcal{X}$ and $\boldsymbol{x}^K(\rho) > 0$ for all $0 \leq \rho < r$ and thus $\boldsymbol{x}^K(\rho)$ can serve as a new "iterate" in an algorithmic application of the basic idea.

To convince you that a "direct" estimation of the relative change just does not work we note that by construction and from (8.8)

$$\frac{\boldsymbol{c}\boldsymbol{x}^K(\rho)}{\boldsymbol{c}\boldsymbol{x}^0} = \frac{(\boldsymbol{c}\boldsymbol{D},0)\boldsymbol{y}^K(\rho)}{(\boldsymbol{c}\boldsymbol{D},0)\boldsymbol{y}^0}\frac{y^0_{n+1}}{y^K_{n+1}(\rho)} \leq \left(1-\rho\sqrt{\frac{n+1}{n}}\right)\frac{1}{(n+1)y^K_{n+1}(\rho)} .$$

To estimate the last term we need to know the minimum value of y_{n+1} for all $\boldsymbol{y} \in B^{n+1}_\rho$. Using the Lagrangean multiplier technique *you* calculate

$$(n+1)y_{n+1} \geq 1-\rho\sqrt{n(n+1)} \text{ for all } \boldsymbol{y} \in B^{n+1}_\rho$$

and thus from the above estimation

$$\frac{\boldsymbol{c}\boldsymbol{x}^K(\rho)}{\boldsymbol{c}\boldsymbol{x}^0} \leq \left(1-\rho\sqrt{\frac{n+1}{n}}\right)\left(1-\rho\sqrt{n(n+1)}\right)^{-1} .$$

This estimation is of no use to us since the term on the right-hand side is greater or equal to one for all $0 \leq \rho < 1/\sqrt{n(n+1)}$, but it shows that $\boldsymbol{c}\boldsymbol{x}^K(\rho)$ cannot *increase* by "too much" if ρ is small.

To prove convergence of the sequence of points generated by an iterative application of the basic idea it suffices, of course, to show that some measure other than the objective function gives a sufficiently large decrease. So consider the function

$$h(\boldsymbol{x}) = \boldsymbol{c}\boldsymbol{x}(\prod_{j=1}^n x_j)^{-1/(n+1)} , \tag{8.11}$$

which is the objective function divided by the *geometric mean* of the point $(\boldsymbol{x}\ ,\ 1) \in \mathbb{R}^{n+1}$. For all $\boldsymbol{x} > \boldsymbol{0}$ the function $h(\boldsymbol{x})$ is well defined and by the geometric/arithmetic mean inequality we have that

$$h(\boldsymbol{x}) \geq (n+1)\boldsymbol{c}\boldsymbol{x}(1+\sum_{j=1}^n x_j)^{-1} \geq 0 .$$

Since $\mathcal{X}$ is *bounded*, $\sum_{j=1}^n x_j \leq K$ for all $\boldsymbol{x} \in \mathcal{X}$ where $K > 0$ is some constant and thus minimizing $h(\boldsymbol{x})$ achieves the goal of minimizing $\boldsymbol{c}\boldsymbol{x}$. We calculate

$$\begin{aligned}\frac{h(\boldsymbol{x}^K(\rho))}{h(\boldsymbol{x}^0)} &= \frac{\boldsymbol{c}\boldsymbol{x}^K(\rho)}{\boldsymbol{c}\boldsymbol{x}^0}(\prod_{j=1}^n \frac{x^0_j}{x^K_j(\rho)})^{1/(n+1)} \\ &= \frac{(\boldsymbol{c}\boldsymbol{D}\ ,\ 0)\boldsymbol{y}^K(\rho)}{(\boldsymbol{c}\boldsymbol{D}\ ,\ 0)\boldsymbol{y}^0}(\prod_{j=1}^{n+1}\frac{1}{(n+1)y^K_j(\rho)})^{1/(n+1)} \\ &\leq (1-\rho\sqrt{(n+1)/n})(\prod_{j=1}^{n+1}(n+1)y^K_j(\rho))^{-1/(n+1)} \qquad (8.12)\end{aligned}$$

using (8.8) and, like in the unsuccessful attempt above, we are left with estimating the last term for $y \in B_\rho^{n+1}$. To do so we set $\rho = \alpha r$ where $0 \le \alpha \le 1$ and we change temporarily our notation to simplify the exposition.

Remark 8.2 *Let* $\rho = \alpha r$ *where* $r^2 = 1/n(n-1)$. *Then for all* $0 < \alpha < 1$

$$\max\{(\prod_{j=1}^{n} nz_j)^{-1/n} : z \in B_\rho^n\} = [1+\alpha/(n-1)]^{-1}[(1+\alpha/(n-1))/(1-\alpha)]^{1/n} .$$

Proof. Since B_ρ^n is a compact subset of $\mathbb{R}^n$ and $z > \mathbf{0}$ for all $z \in B_\rho^n$ and for all $0 < \rho < r$ the maximum exists and the assertion is equivalent to proving

$$\min\{\prod_{j=1}^{n} z_j : \sum_{j=1}^{n} z_j^2 = \alpha^2 r^2 + 1/n \,,\ \sum_{j=1}^{n} z_j = 1\} = \frac{1-\alpha}{n^n}\left(1+\frac{\alpha}{n-1}\right)^{n-1} .$$

Using the Lagrangean multiplier technique we get the equations $\mu z_j^2 + \nu z_j + 1 = 0$ for $j = 1, \dots, n$ after absorbing the product term into the multipliers μ and ν. It follows that $\mu \neq 0$ since otherwise $z_j = 1/n$ for all j which contradicts $\alpha > 0$. Consequently, the components of every solution z to the Lagrangean equations are of the form $z_j = a \pm b$ where a and b are scalars satisfying $0 < b < a$ since $z > \mathbf{0}$. Hence because of the symmetries in both the objective function and the constraints we can assume WROG that in the *natural* indexing there exists $\ell \in \{1, \dots, n-1\}$ such that $z_1 = \cdots = z_\ell < z_{\ell+1} = \cdots = z_n$. From the constraint $e^T z = 1$ we get $na + (n-2\ell)b = 1$, while from the other constraint we get $\ell(a-b)^2 + (n-\ell)(a+b)^2 = \alpha^2 r^2 + 1/n$. Squaring the first relation and simplifying the second one using (8.1) we obtain $b^2 = \alpha^2/4\ell(n-\ell)(n-1)$. Solving for a and b we find that *modulo* the indexing every solution to the Lagrangean equations is of the form

$$\begin{aligned} z_j &= \frac{1}{n}\left(1-\alpha\sqrt{\frac{n-\ell}{\ell(n-1)}}\right) && \text{for } 1 \le j \le \ell\,, \\ z_j &= \frac{1}{n}\left(1+\alpha\sqrt{\frac{\ell}{(n-\ell)(n-1)}}\right) && \text{for } j > \ell \end{aligned} .$$

We need to show that $\xi_1(\alpha) \le \xi_\ell(\alpha)$ for all $0 \le \alpha \le 1$ and $\ell \ge 2$ where

$$\xi_\ell(\alpha) = \left(1-\alpha\sqrt{(n-\ell)/\ell(n-1)}\right)^\ell \left(1+\alpha\sqrt{\ell/(n-\ell)(n-1)}\right)^{n-\ell} .$$

Suppose to the contrary that $\xi_\ell(\alpha) < \xi_1(\alpha)$ for some $0 < \alpha < 1$ and $\ell \ge 2$. Now $\xi_1(0) = \xi_\ell(0) = 1$, $\xi_1(1) = 0$ and $\xi_\ell(1) > 0$. So the function $\xi_\ell(\alpha) - \xi_1(\alpha)$ has a minimum in the interval $[0,1]$. We compute for $k \in \{1, \dots, n-1\}$

$$d\xi_k/d\alpha = -\frac{\alpha n \xi_k(\alpha)}{(n-1)\left(1-\alpha\sqrt{(n-k)/k(n-1)}\right)\left(1+\alpha\sqrt{k/(n-k)(n-1)}\right)}$$

and thus at a minimum of $\xi_\ell(\alpha)-\xi_1(\alpha)$ we have from $d(\xi_\ell(\alpha)-\xi_1(\alpha))/d\alpha=0$

$$\frac{\xi_1(\alpha)}{\xi_\ell(\alpha)}=\frac{(1-\alpha)(1+\alpha/(n-1))}{\left(1-\alpha\sqrt{\dfrac{(n-\ell)}{\ell(n-1)}}\right)\left(1+\alpha\sqrt{\dfrac{\ell}{(n-\ell)(n-1)}}\right)}\,.$$

So for $\ell\geq 2$ and $0<\alpha<1$ we have $\xi_1(\alpha)/\xi_\ell(\alpha)<1$ and thus $\xi_1(\alpha)<\xi_\ell(\alpha)$ since $\xi_\ell(\alpha)>0$, which is a contradiction. ■

From Remark 8.2 and (8.12) it follows that for all $0\leq\alpha<1$

$$\frac{h(\boldsymbol{x}^K(\alpha r))}{h(\boldsymbol{x}^0)}\leq\frac{1-\alpha/n}{1+\alpha/n}\left(\frac{1+\alpha/n}{1-\alpha}\right)^{1/(n+1)}=\tilde{g}(\alpha,n)\,. \tag{8.13}$$

To estimate $\tilde{g}(\alpha,n)$ more conveniently we remember that $1-x\leq e^{-x}$ for all x. (To prove the latter show that $(1-x)e^x\leq 1$ for all x.) Since $e^x\geq 1+x$ it follows that the function $f(x)=e^{-2x}-(1-x)/(1+x)$ is increasing and thus from $f(0)=0$ we have $(1-x)/(1+x)\leq e^{-2x}$ for all $x\geq 0$. Consequently,

$$\tilde{g}(\alpha,n)\leq\frac{e^{-\frac{2\alpha}{n}-\frac{\alpha}{n(n+1)}+\frac{\alpha}{n(n+1)}}}{(1-\alpha)^{1/n}}=\left(\frac{e^{-2\alpha}}{1-\alpha}\right)^{1/n}=g(\alpha,n)\,. \tag{8.14}$$

It follows that $g(\alpha,n)<1$ for all $0<\alpha<\alpha_0=0.7968\ldots$ and thus – despite the *nonmonotonicity* of $\boldsymbol{cx}^K(\rho)$ – we get a decrease in the auxiliary test function $h(\boldsymbol{x})$ that does not depend upon the initial interior point $\boldsymbol{x}^0$.

Consequently, the iterative application of the algorithmic idea produces a geometric convergence rate in terms of the test function $h(\boldsymbol{x})$ and for any fixed "step-size" α satisfying $0<\alpha<0.7968\ldots$ Bingo!

We are now ready to formulate a projective algorithm with input parameters α for the "step-size", p for the desired "precision" in terms of the relative error (remember: the optimum of (LP) is assumed to be zero!), the descriptive data for (LP) and a feasible interior starting point $\boldsymbol{x}^0>\boldsymbol{0}$ such that $\boldsymbol{cx}^0>0$.

Basic Algorithm (α, p, m, n, $\boldsymbol{A}$, $\boldsymbol{c}$, $\boldsymbol{x}^0$)

Step 0: Set $\boldsymbol{D}_0:=diag(x_1^0,\ldots,x_n^0)$, $z:=\boldsymbol{cx}^0$ and $k:=0$.
Step 1: Compute $\boldsymbol{G}:=\boldsymbol{AD}_k^2\boldsymbol{A}^T$, $\boldsymbol{G}^{-1}$ and $\boldsymbol{P}:=\boldsymbol{I}_n-\boldsymbol{D}_k\boldsymbol{A}^T\boldsymbol{G}^{-1}\boldsymbol{AD}_k$.
Step 2: Compute $\boldsymbol{p}:=\boldsymbol{PD}_k\boldsymbol{c}^T$, $\boldsymbol{d}:=\boldsymbol{Pe}$, $\gamma:=\boldsymbol{p}^T\boldsymbol{d}$, $\beta:=n-\|\boldsymbol{d}\|^2$,
$\|\boldsymbol{q}\|:=\sqrt{\|\boldsymbol{p}\|^2+(z-\gamma)^2/(1+\beta)-z^2/(n+1)}$ and

$$t:=\frac{\alpha(1+\beta)(n+1)}{(1+\beta)\sqrt{n(n+1)}\|\boldsymbol{q}\|+\alpha(\gamma(n+1)-(n-\beta)z)}\,.$$

Step 3: Set $\boldsymbol{x}^{k+1}:=\boldsymbol{x}^k-t\boldsymbol{D}_k\left(\boldsymbol{p}-\dfrac{z-\gamma}{1+\beta}\boldsymbol{d}\right)$, $\boldsymbol{D}_{k+1}:=diag(x_1^{k+1},\ldots,x_n^{k+1})$.

Step 4: **if** $\dfrac{\boldsymbol{c}\boldsymbol{x}^{k+1}}{\boldsymbol{c}\boldsymbol{x}^0} < 2^{-p}$ **stop** "$\boldsymbol{x}^{k+1}$ is a p-optimal solution to (LP)".
Set $z := \boldsymbol{c}\boldsymbol{x}^{k+1}$; replace $k+1$ by k; **go to** Step 1.

It is noteworthy that the right-hand side vector $\boldsymbol{b}$ of (LP) has "disappeared" from the input of the basic algorithm: given the matrix $\boldsymbol{A}$ its "informational content" has been absorbed into the starting value of $\boldsymbol{x}^0$ of the algorithm.

8.1.3 Correctness, Finiteness, Initialization

Remark 8.3 *For every* $0 < \alpha < 0.7968\ldots$ *and* $p \geq \log_2 K$ *the basic algorithm iterates at most* $\mathcal{O}(np)$ *times where* $K \geq 2$ *is such that* $\mathcal{X} \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : 0 \leq x_j \leq K \text{ for } j = 1, \ldots, n\}$.

Proof. By construction the sequence of points generated by the algorithm satisfies $\boldsymbol{x}^k > \boldsymbol{0}$ since $0 < \alpha < 1$ and $\boldsymbol{A}\boldsymbol{x}^k = \boldsymbol{b}$. Suppose that the algorithm executes $k \geq 1$ iterations. Then we estimate using (8.13), (8.14) and the geometric/arithmetic mean inequality

$$\frac{\boldsymbol{c}\boldsymbol{x}^k}{\boldsymbol{c}\boldsymbol{x}^0} = \frac{h(\boldsymbol{x}^k)}{h(\boldsymbol{x}^{k-1})}\cdots\frac{h(\boldsymbol{x}^1)}{h(\boldsymbol{x}^0)}\Big(\prod_{j=1}^n \frac{x_j^k}{x_j^0}\Big)^{1/(n+1)} \leq \frac{(g(\alpha,n))^k\left(1+\sum_{j=1}^n x_j^k\right)}{(n+1)\left(\prod_{j=1}^n x_j^0\right)^{1/(n+1)}}$$

$$\leq K\theta^{k/n}\Big(\prod_{j=1}^n x_j^0\Big)^{-1/(n+1)} = \kappa$$

where $0 < \theta = e^{-2\alpha}/(1-\alpha) < 1$ and thus if $k > n(-2p + (n+1)^{-1}\sum_{j=1}^n \log_2 x_j^0)/\log_2\theta$ then we have for $p \geq \log_2 K$ that $\kappa < 2^{-p}$. Consequently, the basic algorithm stops after at most $\mathcal{O}(np)$ iterations. ∎

The "best" step-size α that the analysis suggests is evidently the value of α that minimizes the function $\widetilde{g}(\alpha,n)$ of (8.13). *You* calculate that the minimum of $\widetilde{g}(\alpha,n)$ in the interval $[0,1]$ is attained for $\alpha = n/(2n-1)$, that $\widetilde{g}(\alpha,n) = 1$ for $\alpha = 0$, that $\widetilde{g}(\alpha,n)$ tends to $+\infty$ for all $n \geq 2$ as α is increased to 1 from below, $\lim_{n\to\infty}\widetilde{g}(\alpha,n) = 1$ for all $0 \leq \alpha < 1$ and that $\widetilde{g}(\alpha,n) < 1$ for all $0 < \alpha < \alpha_0 = 0.7968\ldots$. For $\alpha = n/(2n-1)$ we get

$$\widetilde{g}(n/(2n-1),n) = 2^{1/(n+1)}\,(1-1/n)^{n/(n+1)} \leq (2/e)^{1/(n+1)}$$

using $1-x \leq e^{-x}$ for all x. Thus – ignoring the dependency upon the constant K and the starting point $\boldsymbol{x}^0$ and estimating the (remaining) relative error directly from (8.13) – we get that about $3p(n+1)$ iterations suffice for the basic algorithm to come to a halt, using that $\ln 2 \leq 3/4$. This means that the *constant* in the $\mathcal{O}(np)$ estimation is reasonable for choices of α around the value 1/2. Of course, this estimation is *data-independent* and thus for given instances of (LP) "better" step-lengths may be found by e.g. analyzing (8.12) more closely for a given data set.

When the basic algorithm comes to a halt, we thus have a feasible solution $\boldsymbol{x} \in \mathcal{X}$ such that $0 \leq \boldsymbol{cx} \leq z_0 2^{-p}$. Denote by L the *digital size* of the linear program (LP) as defined in Chapter 1.1. Choosing a precision $p > \log_2 z_0 + L$ for the relative error we get $\boldsymbol{cx} < 2^{-L}$. By Chapter 7.4 we can in turn estimate $\log_2 z_0$ linearly in L. It follows that $\mathcal{O}(nL)$ *steps* are required by the basic algorithm to come to a halt, i.e. the *step complexity* of the basic algorithm is polynomially bounded in the size L of (LP) provided that its feasible region is bounded and its optimal objective function equals zero.

The *time complexity* of the basic algorithm is dominated by Step 1 and Step 2 where we have to find the projected vectors $\boldsymbol{p}$ and $\boldsymbol{d}$. Like in the case of simplex algorithms we have stated the basic algorithm using an "explicit" inverse $\boldsymbol{G}^{-1}$ whereas we *need* in reality to find only $\boldsymbol{p}$ and $\boldsymbol{d}$. The "inversion" of $\boldsymbol{G}$, i.e. the solution of the respective systems of equations, can be done in polynomial time for a given matrix $\boldsymbol{G}$. This matrix depends upon the current iterate $\boldsymbol{x}^k$ through scaling and we are not aware of an analysis of the basic algorithm in terms of *approximate arithmetic* that is as satisfactory for *combinatorial* optimization problems as the one that we know for the ellipsoid algorithm and which we shall treat in Chapter 9.

The practical experience with the basic algorithm and its derivatives has been good and indeed, in terms of the number of steps, it is far better than suggested by the theoretical analysis. As we shall see in Chapter 8.5 – after an analysis of the problem (FLP_ρ) – the step complexity of projective algorithms can be improved substantially.

Two issues remain to be settled before we can proceed. The first one concerns how to start the basic algorithm and the second one how to get a basic feasible solution to the linear program. Practice and theory diverge – as they do so often – on both of these points. We will be brief and discuss the "theoretical" side of the coin only.

Recall the proof of part (b) of Theorem 1. There we showed *constructively* how to find an optimal basic solution from any finite optimal solution $\boldsymbol{x} \in \mathcal{X}$. Clearly, the procedure given there requires at most n steps and every step has a time complexity that is polynomial in the input size of (LP). Since at the end of the basic algorithm we have a p-optimal solution and not necessarily an optimal one we need to change the construction of the proof of Theorem 1 somewhat so as to ensure that the objective function value does not increase when we change the vector $\boldsymbol{x}$ and that is all since $0 \leq \boldsymbol{cx} < 2^{-L}$. This is not difficult, the details are left as an exercise and thus we know how to deal with the second point raised above.

To show how to get started suppose that the *original* linear program that we wish to solve is in *canonical* form. Changing our notation, let the original linear program be given by $\max\{\boldsymbol{cx} : \boldsymbol{Ax} \le \boldsymbol{b},\ \boldsymbol{x} \ge \boldsymbol{0}\}$. From Remark 6.5 it follows that we can find an optimum solution *or* conclude that none exists, i.e. that either the feasible set is empty or that the optimum solution is unbounded, by solving the linear program

$$\min\{-\boldsymbol{cx} + \boldsymbol{b}^T\boldsymbol{u} : \boldsymbol{Ax} \le \boldsymbol{b},\ -\boldsymbol{A}^T\boldsymbol{u} \le -\boldsymbol{c}^T,\ \boldsymbol{u} \ge \boldsymbol{0},\ \boldsymbol{x} \ge \boldsymbol{0}\},$$

where we have written the vector $\boldsymbol{u}$ of dual variables as a column vector rather than a row vector. By Remark 6.5 the optimum objective function value of the latter linear program equals zero – if it exists. Let us denote $\boldsymbol{s} \in \mathbb{R}^m$ and $\boldsymbol{v} \in \mathbb{R}^n$ the vectors of the respective slack variables for the linear inequalities. Now choose any $\widehat{\boldsymbol{x}} > \boldsymbol{0}$, $\widehat{\boldsymbol{s}} > \boldsymbol{0}$, $\widehat{\boldsymbol{u}} > \boldsymbol{0}$ and $\widehat{\boldsymbol{v}} > \boldsymbol{0}$. Then $\boldsymbol{x} = \widehat{\boldsymbol{x}}$, $\boldsymbol{s} = \widehat{\boldsymbol{s}}$, $\boldsymbol{u} = \widehat{\boldsymbol{u}}$, $\boldsymbol{v} = \widehat{\boldsymbol{v}}$ and $\lambda = 1$ is a solution to the linear program

$$\begin{array}{rll} \min & -\boldsymbol{cx} + \boldsymbol{b}^T\boldsymbol{u} + M\lambda & \\ \text{subject to} & \boldsymbol{Ax} + \boldsymbol{s} + (\boldsymbol{b} - \boldsymbol{A}\widehat{\boldsymbol{x}} - \widehat{\boldsymbol{s}})\lambda & = \boldsymbol{b} \\ & \boldsymbol{A}^T\boldsymbol{u} - \boldsymbol{v} + (\boldsymbol{c}^T - \boldsymbol{A}^T\widehat{\boldsymbol{u}} + \widehat{\boldsymbol{v}})\lambda & = \boldsymbol{c}^T \\ & \boldsymbol{x},\ \boldsymbol{s},\ \boldsymbol{u},\ \lambda & \ge \boldsymbol{0} \end{array}$$

that is in the relative interior of the feasible set of this (bigger) linear program as required by the basic algorithm. M must be a sufficiently large number to ensure that $\lambda = 0$ in any optimal solution – provided that it exists. Such M can be estimated from the original data and its *digital size* is polynomial in the input size L, see Chapter 7.4. Finally, to ensure the boundedness of the feasible region we need to intersect it e.g. with a constraint that bounds the sum of $\boldsymbol{x}$, $\boldsymbol{s}$, $\boldsymbol{u}$, $\boldsymbol{v}$ and λ from above by a suitably large constant K. Such K can be found so that its *digital size* is polynomially bounded in L. Adding this constraint to the linear program with a slack variable we get a problem that satisfies all of the assumptions that we have made to prove Remark 8.3. Clearly, the above "trick" that gets us started doubles the size of the input which means that the digital size of the problem to be solved remains polynomial in terms of the input size of the original problem.

A different way of using the basic algorithm consists of utilizing the objective function as a constraint, i.e. by adding a constraint of the form $\boldsymbol{cx} \le z$ where z is a parameter, and minimizing its slack. As we know from Chapter 7.4 we can *bound* the objective function value of any linear program from both below and above by some finite numbers whose *digital sizes* are polynomial in the input size of the linear program. Since the basic algorithm can evidently be used to decide whether or not there exists $\boldsymbol{x} \in \mathcal{X}$ with $\boldsymbol{cx} \le 0$

one can use the *binary search algorithm* of Chapter 7.4 to find an optimal solution to (LP). The details of how this is done are not difficult to work out and left as an exercise. And there are other ways to implement the basic algorithm by a Two-Phase method and/or artificial variables. So we can bring any linear program into a form that fits the assumptions that we have made in Remark 8.3 and moreover, we can find a basic feasible optimal solution to the linear program provided there exists one with a time complexity that is polynomial in L.

Exercise 8.2 *(i) Write a computer program for the basic algorithm using any subroutine for inverting a matrix. Use as test problem e.g. anyone of the following class:* $\min\{\sum_{j=1}^{n} c_j x_j : \sum_{j=1}^{n} a_j x_j = a_0 \ , \ x_j \geq 0\}$ *where* $a_0 > 0$, $a_j > 0$, $c_j \geq 0$ *for all* $1 \leq j \leq n$ *and* $c_k = 0$ *for some* k. *To initialize you may use either* $x_j^0 = a_0/na_j$ *or* $x_j = a_0/\sum_{k=1}^{n} a_k$ *for* $1 \leq j \leq n$. *(ii) Use your program to solve the problem* $\min\{x_2 : x_1 + 5x_2 \leq 250,\ x_1 + x_2 \leq 80,\ 3x_1 + x_2 \leq 180,\ 2x_1 - 3x_2 \leq 40,\ x_1 \geq 0,\ x_2 \geq 0\}$, *after bringing it into standard form, with* $x_1^0 = 30$, $x_2^0 = 40$ *as a starting point.*

8.2 Analysis, Algebra, Geometry

Μὴ μου τοὺς κύκλους τάραττε!
Archimedes of Syracuse (c. 287–212 B.C.)

"Don't touch my circles!" is probably what Archimedes barked in his mother tongue at that *Roman* barbarian who had dared to break into the privacy of his Syracusan villa – while this great Greek mathematician was maybe about to approximate the number Pi(π) to an even greater accuracy than he had done previously or – maybe he was engineering one of his *devilish* inventions such as the catapult that was used for two millennia for aggression and defense alike. Indeed, with another one of his devilish inventions Archimedes had set the sails of all enemy ships blocking the harbor of Syracuse aflame. So we will never know with certainty. The Second Punic War (218–201 B.C.) was on, the city of Syracuse had been conquered and the barbarian was a Roman soldier who – disregarding orders to capture the famous scientist and engineer alive – used his sword. Had Archimedes said in Latin "Noli perturbare circulos meos!" who knows? According to Roman historians it was a misunderstanding and it was fatal. The circles in Archimedes' sandbox were in all likelihood destroyed during his decapitation – and most probably, the barbarian was crucified or flogged for disobedience, ignorance or both.

Circles, lines, triangles – already in times long since past *geometry* has dealt with such objects. But before we come back to these delightful constructs of the human mind (see Figures 8.4, 8.5 and 8.6 below), we must do some *analysis* and a lot of *algebra.*

To see what the first approximation means in terms of the problem (LP) we calculate from (ALP) using the projective transformation T_0

$$\min\{(\boldsymbol{cD}\ ,\ 0)\boldsymbol{y} : \boldsymbol{y} \in T_0(\mathcal{X})\} = \min\left\{\frac{\boldsymbol{cx}}{1+\boldsymbol{e}^T\boldsymbol{D}^{-1}\boldsymbol{x}} : \boldsymbol{x} \in \mathcal{X}\right\}$$

and likewise for (ALP$_\rho$). Since $1 + \boldsymbol{e}^T\boldsymbol{D}^{-1}\boldsymbol{x} \geq 1$ for all $\boldsymbol{x} \in \mathcal{X}$ the objective function of the re-transformed problem underestimates the "true" value of (LP). The fact that we are *not* minimizing $\boldsymbol{cx}$ explains *inter alia* the non-monotonic behavior of the objective function values of the basic algorithm.

In the second approach one solves the restriction (FLP$_\rho$) of the problem (FLP) exactly. (So you have to replace ALP$_\rho$ by FLP$_\rho$ in Figure 8.2.) We will do so in two steps.

- First, we derive the solution to (FLP$_\rho$) in the space of the variables of (LP) and for values of ρ less than or equal to r, see (8.1), which lends itself to direct interpretation in that space, see Figure 8.3.
- Then we recast the problem in the "transformed" space, derive the solution of (FLP$_\rho$) for values $\rho \geq r$ and arrive this way at a nice geometric interpretation of the solution.

As we shall see in Chapter 8.5, this leads to an algorithm for (LP) with monotonically decreasing objective function values, without the restrictive assumptions of the basic algorithm and with a step complexity of $\mathcal{O}(p\sqrt{n})$ – which is asymptotically far better than the bound $\mathcal{O}(pn)$ for the basic algorithm.

8.2.1 Solution to the Problem in the Original Space

Reversing the projective transformation the nonlinear optimization problem (FLP$_\rho$) becomes the problem

$$(\mathrm{LP}_\rho) \qquad \min\left\{\boldsymbol{cx} : \boldsymbol{x} \in \mathcal{X} \cap T_0^{-1}(B_\rho^{n+1})\right\}$$

and the first task is thus to find $T_0^{-1}(B_\rho^{n+1})$. We define a new parameter $R = R(\rho)$ by

$$R = \rho\sqrt{(n+1)/n(r^2-\rho^2)}\ . \tag{8.15}$$

It follows that $R \in [0,\infty)$ for all $\rho \in [0,r)$ and since

$$dR/d\rho = (n+1)\left(1-\rho^2/r^2\right)^{-3/2} > 0\ ,$$

the parameter change preserves strict monotonicity, i.e. $\rho < \rho'$ if and only if $R(\rho) < R(\rho')$. We denote the pre-image of B_ρ^{n+1} by $E(\boldsymbol{x}^0, R)$, i.e.

$$E(\boldsymbol{x}^0, R) = T_0^{-1}(B_\rho^{n+1}) \ ,$$

and by (LP_R) the optimization problem (LP_ρ) in the parameter R, i.e.

$$(\mathrm{LP}_R) \qquad \min\{\boldsymbol{cx} : \boldsymbol{x} \in \mathcal{X} \cap E(\boldsymbol{x}^0, R)\} \ .$$

Remark 8.4 *For $0 \le \rho < r$ the set $T_0^{-1}(B_\rho^{n+1})$ is the ellipsoid*

$$E(\boldsymbol{x}^0, R) = \left\{\boldsymbol{x} \in \mathbb{R}^n : (\boldsymbol{x} - \boldsymbol{x}_C)^T \boldsymbol{H}(\boldsymbol{x} - \boldsymbol{x}_C) \le R^2\right\}$$

where $\boldsymbol{x}_C = (1 + R^2)\boldsymbol{x}^0$ is its center, $R = R(\rho)$ is defined in (8.15) and

$$\boldsymbol{H} = \boldsymbol{D}^{-1}\left(\boldsymbol{I}_n - (1 + n + nR^2)^{-1}(1 + R^2)\boldsymbol{ee}^T\right)\boldsymbol{D}^{-1} \ .$$

Proof. Using the transformation T_0 we find that the pre-image of B_ρ^{n+1} is the set of points in $\mathbb{R}^n$ satisfying the inequality

$$\left(\frac{\boldsymbol{D}^{-1}\boldsymbol{x}}{1 + \boldsymbol{e}^T\boldsymbol{D}^{-1}\boldsymbol{x}} - \frac{1}{n+1}\boldsymbol{e}\right)^T \left(\frac{\boldsymbol{D}^{-1}\boldsymbol{x}}{1 + \boldsymbol{e}^T\boldsymbol{D}^{-1}\boldsymbol{x}} - \frac{1}{n+1}\boldsymbol{e}\right) + \left(\frac{1}{1 + \boldsymbol{e}^T\boldsymbol{D}^{-1}\boldsymbol{x}} - \frac{1}{n+1}\right)^2 \le \rho^2 \ .$$

Since $1 + \boldsymbol{e}^T\boldsymbol{D}^{-1}\boldsymbol{x} > 0$ for all $\boldsymbol{x} \ge \boldsymbol{0}$ we can simplify and obtain the inequality

$$\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x} - \frac{2(1 + R^2)}{1 + n + nR^2}\boldsymbol{e}^T\boldsymbol{D}^{-1}\boldsymbol{x} \le -1 + \frac{1 + R^2}{1 + n + nR^2}$$

in the new parameter R where $\boldsymbol{H}$ is defined above. Thus $T_0^{-1}(B_\rho^{n+1})$ is the solution set to a quadratic inequality and calculating the *center* $\boldsymbol{x}_C$ of the quadric we find

$$\boldsymbol{x}_C = (1 + n + nR^2)^{-1}(1 + R^2)\boldsymbol{H}^{-1}\boldsymbol{D}^{-1}\boldsymbol{e} = (1 + R^2)\boldsymbol{x}^0 \ ,$$

where we have used the formula of Chapter 4.4 to compute

$$\boldsymbol{H}^{-1} = \boldsymbol{D}\left(\boldsymbol{I}_n + (1 + R^2)\,\boldsymbol{ee}^T\right)\boldsymbol{D} \ . \tag{8.16}$$

Thus $E(\boldsymbol{x}^0, R)$ is of the stated form. For $0 \le R < \infty$ we have $\boldsymbol{H} = \boldsymbol{F}^T\boldsymbol{F}$ where

$$\boldsymbol{F} = \left(\boldsymbol{I}_n - n^{-1}\left(1 - (1 + n + nR^2)^{-1/2}\right)\boldsymbol{ee}^T\right)\boldsymbol{D}^{-1} \ .$$

Consequently, $\boldsymbol{H}$ is positive definite and hence $E(\boldsymbol{x}^0, R)$ is an ellipsoid. ∎

So we know that for $0 \le \rho < r$ the sets $T_0^{-1}(B_\rho^{n+1})$ are ellipsoids $E(\boldsymbol{x}^0, R)$ and satisfy

$$E(\boldsymbol{x}^0, R) \subset \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} > \mathbf{0}\} \text{ for all } 0 \leq R < \infty \;;$$

see Figure 8.1. For $\rho \geq r$ the sets $T_0^{-1}(B_\rho^{n+1})$ may be paraboloids, hyperboloids or they may simply not exist in *real* terms – which is why we have worked with the parameter R rather than with ρ in this subsection.

Exercise 8.3 *(i) Show that* $E(\boldsymbol{x}^0, R) \subset E(\boldsymbol{x}^0, R')$ *for all* $0 \leq R < R'$. (Hint: *It suffices to show that* $B_\rho^{n+1} \subset B_{\rho'}^{n+1}$ *for* $\rho < \rho'$.) *(ii) Show that* $\boldsymbol{x} = \left(1 \pm R/\sqrt{2+R^2}\right)\boldsymbol{x}_C \pm (Rx_i^0/\sqrt{2+R^2})\boldsymbol{u}_i$ *solves the problem* $\max\{x_i : \boldsymbol{x} \in E(\boldsymbol{x}^0, R)\}$ *if both plus signs are used and the corresponding minimization problem in the opposite case where* $\boldsymbol{u}_i \in \mathbb{R}^n$ *is the* i^{th} *unit vector. Moreover, for* $R \to +\infty$ *the minimizing point exists and is given by* $\boldsymbol{x} = \boldsymbol{x}^0 - x_i^0\boldsymbol{u}_i$ *where* $1 \leq i \leq n$. *(iii) Show that* $det(\boldsymbol{I}_n + (1+R^2)\boldsymbol{e}\boldsymbol{e}^T) = 1 + n + nR^2$ *and that the volume of* $E(\boldsymbol{x}^0, R)$ *is given by* $vol(E(\boldsymbol{x}^0, R)) = g^n(\boldsymbol{x}^0)R^n\pi^{n/2}\sqrt{1+n+nR^2}/\Gamma(1+n/2)$ *where* $g(\boldsymbol{x}^0)$ *is the geometric mean of* $\boldsymbol{x}^0$. *(iv) Let* λ_1 *be the smallest eigenvalue of* $\boldsymbol{H}^{-1}$*; see (8.16). Show that the length of the* smallest *principal axis* $R\sqrt{\lambda_1}$ *of* $E(\boldsymbol{x}^0, R)$ *satisfies* $R\sqrt{\lambda_1} \leq R(1+n+nR^2)^{1/2n}g(\boldsymbol{x}^0)$. *(v)* (Optional) *Show that for the data of Exercise 8.2 (ii) the ellipsoids* $E(\boldsymbol{x}^0, R)$ *in the* space of the variables x_1 *and* x_2 *are given by* $(0.7614 + 0.6029R^2)x_1^2 + (3.79575 + 3.125925R^2)x_2^2 + (1.7955+1.3797R^2)x_1x_2 - (117.504+87.822R^2)x_1 - (357.525+285.795R^2)x_2 \leq -(8913.06 + 6803.73R^2)$.

Using the same notation as in the development of the basic projective algorithm we are now ready to solve the optimization problem (LP_R).

Remark 8.5 *If* $\boldsymbol{x}^0 > \mathbf{0}$ *is a nonoptimal feasible solution to* (LP) *then*

$$\boldsymbol{x}(R) = \boldsymbol{x}^0 - (R/W)\boldsymbol{D}\left[\boldsymbol{p} - (1+\beta)^{-1}(z_0 - z(R) - \gamma)\boldsymbol{d}\right] \tag{8.17}$$

solves the optimization problem (LP_R) *for all* $0 \leq R < \infty$ *where*

$$z(R) = z_0 - R((1+\beta)W - \gamma R)/(1+\beta+\beta R^2) \tag{8.18}$$

is the objective function value of (LP_R),

$$W = W(R) = (1+\beta)^{-1/2}\sqrt{(1+\beta)||\boldsymbol{p}||^2 + \gamma^2 + (\beta||\boldsymbol{p}||^2 + \gamma^2)R^2} \tag{8.19}$$

and z_0, β, γ, $\boldsymbol{p}$, $\boldsymbol{d}$ *are defined in Chapter 8.1.1. Moreover,* $z(R)$ *is a strictly decreasing function of* R.

Proof. Since $E(\boldsymbol{x}^0, R)$ is a compact set in $\mathbb{R}^n$ we can use the Lagrangean multiplier technique to compute an optimal solution to (LP_R). The Lagrangean equations are

$$\boldsymbol{c} + \mu\boldsymbol{A} + \lambda(\boldsymbol{x} - \boldsymbol{x}_C)^T\boldsymbol{H} = \mathbf{0} \tag{i}$$

where $\boldsymbol{\mu} \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}$ are the multipliers. Since $\boldsymbol{x}^0$ is a nonoptimal solution to (LP) it follows that $\lambda \neq 0$ since otherwise $\boldsymbol{cx}$ is constant for all $\boldsymbol{x} \in \mathcal{X}$. Multiplying (i) by $\boldsymbol{D}^2\boldsymbol{A}^T$ and solving for $\boldsymbol{\mu}$ we obtain after substituting for $\boldsymbol{\mu}$ and multiplying the result by $\boldsymbol{D}$

$$\boldsymbol{p}^T - \lambda(\boldsymbol{x} - \boldsymbol{x}_C)^T \boldsymbol{H}\boldsymbol{D}^2\boldsymbol{A}^T\boldsymbol{G}^{-1}\boldsymbol{A}\boldsymbol{D} + \lambda(\boldsymbol{x} - \boldsymbol{x}_C)^T\boldsymbol{H}\boldsymbol{D} = 0 \ .$$

From the definition of $\boldsymbol{H}$, $\boldsymbol{x}_C$ and $\boldsymbol{A}\boldsymbol{x}^0 = \boldsymbol{b}$ we get for all $\boldsymbol{x}$ satisfying $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$

$$\begin{aligned}
&(\boldsymbol{x} - \boldsymbol{x}_C)^T\boldsymbol{H}\boldsymbol{D}^2\boldsymbol{A}^T\boldsymbol{G}^{-1}\boldsymbol{A}\boldsymbol{D} \\
&= (\boldsymbol{x} - \boldsymbol{x}_C)^T\boldsymbol{A}^T\boldsymbol{G}^{-1}\boldsymbol{A}\boldsymbol{D} - \frac{1+R^2}{1+n+nR^2}(\boldsymbol{x} - \boldsymbol{x}_C)^T\boldsymbol{D}^{-1}\boldsymbol{e}\boldsymbol{b}^T\boldsymbol{G}^{-1}\boldsymbol{A}\boldsymbol{D} \\
&= \left(-R^2 - \frac{1+R^2}{1+n+nR^2}\boldsymbol{x}^T\boldsymbol{D}^{-1}\boldsymbol{e} + \frac{n(1+R^2)^2}{1+n+nR^2}\right)\boldsymbol{b}^T\boldsymbol{G}^{-1}\boldsymbol{A}\boldsymbol{D} \ .
\end{aligned}$$

From $\boldsymbol{b}^T = \boldsymbol{e}^T\boldsymbol{D}\boldsymbol{A}^T$ we get $\boldsymbol{b}^T\boldsymbol{G}^{-1}\boldsymbol{A}\boldsymbol{D} = (\boldsymbol{e} - \boldsymbol{d})^T$ and thus (i) simplifies to

$$\boldsymbol{p}^T - \lambda\left(1 - (1+R^2)\frac{1+\boldsymbol{e}^T\boldsymbol{D}^{-1}\boldsymbol{x}}{1+n+nR^2}\right)(\boldsymbol{e} - \boldsymbol{d})^T + \lambda(\boldsymbol{x} - \boldsymbol{x}_C)^T\boldsymbol{H}\boldsymbol{D} = \boldsymbol{0} \ . \quad (ii)$$

To determine the term $1+\boldsymbol{e}^T\boldsymbol{D}^{-1}\boldsymbol{x}$ we multiply (ii) by the vector $\boldsymbol{e}$ and find

$$1 + \boldsymbol{e}^T\boldsymbol{D}^{-1}\boldsymbol{x} = (1+n+nR^2)\,(1+\beta-\gamma/\lambda)\,/(1+\beta+\beta R^2) \ . \qquad (iii)$$

Consequently, (ii) simplifies to

$$\boldsymbol{p}^T + (1+\beta+\beta R^2)^{-1}(\lambda R^2 - \gamma(1+R^2))(\boldsymbol{e}-\boldsymbol{d})^T + \lambda(\boldsymbol{x}-\boldsymbol{x}_C)^T\boldsymbol{H}\boldsymbol{D} = \boldsymbol{0} \ . \quad (iv)$$

Solving (iv) for $\boldsymbol{x} - \boldsymbol{x}_C$ using the inverse $\boldsymbol{H}^{-1}$ of Remark 8.4 yields

$$\boldsymbol{x} = \boldsymbol{x}^0 - (1/\lambda)\boldsymbol{D}\left[\boldsymbol{p} - (1+\beta+\beta R^2)^{-1}(\lambda R^2 - \gamma(1+R^2))\boldsymbol{d}\right] \ . \qquad (v)$$

Multiplying (iv) by $\boldsymbol{D}^{-1}(\boldsymbol{x} - \boldsymbol{x}_C)$ we get a quadratic equation for λ

$$\lambda^2 = [(1+\beta)||\boldsymbol{p}||^2 + \gamma^2 + (\beta||\boldsymbol{p}||^2 + \gamma^2)R^2]/(1+\beta)R^2 \ . \qquad (vi)$$

Hence $\lambda = \pm W/R$. Using (v) and (vi) we get for the objective function value

$$z(R) = z_0 - [(1+\beta)R^2\lambda - R^2\gamma]/(1+\beta+\beta R^2)$$

and thus the positive square root of the quadratic equation (vi) applies for the minimization while the negative root applies to the maximization problem. Consequently, (8.17), (8.18) and (8.19) follow. To prove the strict monotonicity of $z(R)$ we calculate

$$\frac{dz}{dR} = -\frac{((1+\beta)W - \gamma R)^2}{W(1+\beta+\beta R^2)^2} < 0 \qquad (8.20)$$

for all $0 \leq R < \infty$. This follows because $|\gamma|R < (1+\beta)W$ for all R and no other term vanishes since, e.g. $||\boldsymbol{p}|| > 0$ by the nonoptimality of $\boldsymbol{x}^0$. ■

Exercise 8.4 (Optional) *(i) Show that the second derivative of* $z(R)$ *is*

$$\frac{d^2z}{dR^2} = \frac{((1+\beta)W - \gamma R)^2}{W^3(1+\beta+\beta R^2)^3}\left\{2\gamma W + R\left[\frac{(\beta\|\boldsymbol{p}\|^2+\gamma^2)(1+\beta+\beta R^2)}{1+\beta} + 2\beta W^2\right]\right\}.$$

(ii) Show that $z(R)$ *is a convex function of* $R \in [0,\infty)$ *if* $\gamma \geq 0$, $z(R)$ *is concave for all* $R \in [0, R_0)$ *and convex for all* $R \in [R_0, \infty)$ *if* $\gamma < 0$ *and* $\beta \neq 0$ *where* R_0 *satisfies*

$$R_0^2 = \frac{1}{6\beta(\beta\|\boldsymbol{p}\|^2+\gamma^2)}\left\{4\gamma^2 - 3(1+\beta)(\beta\|\boldsymbol{p}\|^2+\gamma^2) + \sqrt{(1+\beta)(\beta\|\boldsymbol{p}\|^2+\gamma^2)(9(1+\beta)(\beta\|\boldsymbol{p}\|^2+\gamma^2)-8\gamma^2)}\right\},$$

and $z(R)$ *is concave for all* $R \in [0,\infty)$ *if* $\gamma < 0$ *and* $\beta = 0$. *(iii) Let* $z_\infty = \lim_{R\to\infty} z(R)$. *Show that* $z_\infty = z_0 - (\sqrt{(1+\beta)(\beta\|\boldsymbol{p}\|^2+\gamma^2)} - \gamma)/\beta$ *if* $\beta \neq 0$, *that* $z_\infty = z_0 - (\|\boldsymbol{p}\|^2+\gamma^2)/2\gamma$ *if* $\gamma > 0$ *and* $\beta = 0$ *and that* $z_\infty = -\infty$ *if* $\gamma \leq 0$ *and* $\beta = 0$.

It follows from part (iii) of Exercise 8.4 that for $R \to \infty$ the problem (LP_R) and hence the original problem (LP) have an *unbounded* optimum solution if $\beta = 0$ and $\gamma \leq 0$.

From formula (8.17) we see that the set of points $\boldsymbol{x}(R)$ obtained by varying R do *not* form a straight line in $\mathbb{R}^n$ since $z(R)$ is **not** a constant. Rather the loci of the solution vectors to (LP_R) form a *curve* in $\mathbb{R}^n$ that starts at the point $\boldsymbol{x}^0$.

This is a substantial difference to the solution $\boldsymbol{x}^K(\rho)$, see (8.9), that was obtained via an approximation and whose loci do form a straight line in $\mathbb{R}^n$. The *curvature* of (8.17) can be interpreted as some sort of "*adjustment to the perspective*" that the projective transformation T_0 achieves as the radius ρ of the ball B_ρ^{n+1} changes which is lost in the approximation (ALP_ρ) to (FLP_ρ).

Moreover, the line $\boldsymbol{x}^K(\rho)$ is certainly *not* the tangent (nor the normal) to the curve $\boldsymbol{x}(R)$ at the point $\boldsymbol{x}^0$ which one might expect to find if one compares formulas (8.9) and (8.17).

While we do not compute the exact "shape" of the curve $\boldsymbol{x}(R)$ we shall do so for its image under the projective transformation: it forms a semi-circle in some 2-dimensional plane of $\mathbb{R}^{n+1}$ if the orthoprojections $\boldsymbol{p}$ of $\boldsymbol{D}\boldsymbol{c}^T$ and $\boldsymbol{d}$ of $\boldsymbol{e}$ on the subspace (8.5) are linearly independent. If $\boldsymbol{p}$ and $\boldsymbol{d}$ are linearly *dependent* and only then, $\boldsymbol{x}(R)$ and $\boldsymbol{x}^K(\rho)$ coincide.

In the left part of Figure 8.3 we display both $\boldsymbol{x}^K(\rho)$ and $\boldsymbol{x}(R)$ when parameterized by α for the data of Exercise 8.2 (ii) where $\rho = \alpha r$, $r = 1/\sqrt{n(n+1)}$ and we first expressed $\boldsymbol{x}(R)$ in terms of ρ using (8.15) to get $\boldsymbol{x}(\rho)$. $\boldsymbol{x}^0$ is the

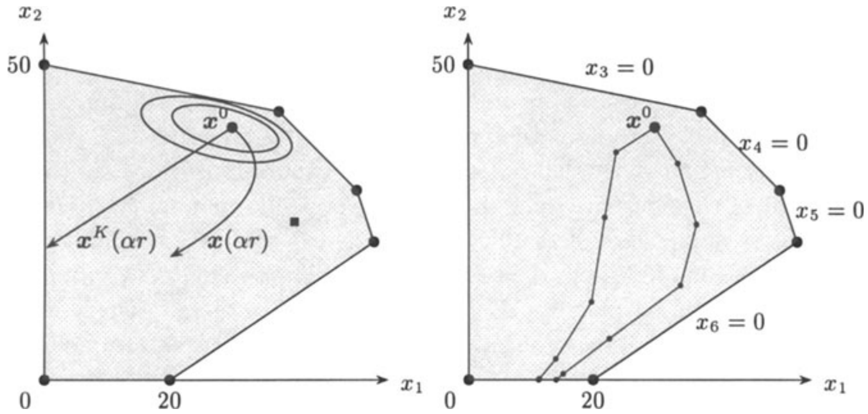

Fig. 8.3. The line (8.9), the projective curve (8.17) and interior paths to optimality.

point in $\mathbb{R}^2$ with coordinates $x_1 = 30$, $x_2 = 40$ like in Exercise 8.2 (ii) and $\boldsymbol{x}(\alpha r)$ is the truncation of $\boldsymbol{x}(\rho) \in \mathbb{R}^6$ to $\mathbb{R}^2$. The point indicated by ▪ has the coordinates $x_1 = 40$, $x_2 = 25$. $x_3, \ldots, x_6$ are the slack variables that correspond to the inequalities of our problem and define the respective line segments when they are equal to zero. Remember that r is the radius of the biggest ball B_ρ^{n+1} with center $\boldsymbol{y}^0$ that is inscribable into the simplex S^{n+1} where $n = 6$ in our example. Thus for values of $\alpha \geq 1$ we can not guarantee positivity of the solution to (FLP$_\rho$). We simply let the parameter α exceed the value of 1 to emphasize the difference between the two constructions.

The ellipsoids $E(\boldsymbol{x}^0, R)$ that we show in Figure 8.3 in (x_1, x_2)-space were computed exactly from the data of the problem of Exercise 8.2 (ii) for the values of $\alpha = 0.7$ and $\alpha = 0.99$ which corresponds to $R \approx 1.06$ and $R \approx 7.58$, respectively; see also Exercise 8.3 (iii). We do not show the centers of the nested ellipsoids which vary with α, see Remark 8.4. For $\alpha = 1$ the ellipsoid *almost* "hits" the boundary of the polytope which signals the fact that a corresponding slack variable gets "close" to zero, i.e. that "somewhere" in our construction we are about to leave the *relative interior* of the feasible set $\mathcal{X} \subset \mathbb{R}^6$ of our problem. (For the ellipsoid $E(\boldsymbol{x}^0, R)$ that we get in the "limit" for $R \to +\infty$, the point that maximizes $5x_1 + x_2$ over the ellipse is given by $x_1 = 26.7901$, $x_2 = 44.4749$ yielding $5x_1 + x_2 = 249.1646 < 250$ which shows that the boundary is not attained.) This occurrence may or may not happen along the path $\boldsymbol{x}(\rho)$ and thus it does not necessarily mean that the solution vector $\boldsymbol{x}(\rho)$ is no longer positive when α is increased beyond one. It is positive in our example and moving away from the boundary in question. Indeed, we shall show below that $\boldsymbol{x}(\rho)$ exists and is well-defined for all $0 \leq \rho < \rho_\infty = \sqrt{\|\boldsymbol{d}\|^{-2} - (n+1)^{-1}}$ which can be much bigger than r. Positivity of some components of $\boldsymbol{x}(\rho)$ may, however, be lost if ρ is increased to ρ_∞. In our case, $\rho_\infty \approx 0.55$, $r \approx 0.15$ and the positivity of $x_1(\rho)$ is lost for $\rho \approx 0.42$; see also Figure 8.7.

As we are minimizing x_2 in our example the intersection of the curve $\boldsymbol{x}(\rho)$ with the ellipsoid occurs at the point (x_1, x_2) of the ellipsoid with smallest x_2 value. *You* can verify that one on Figure 8.3 yourself using a ruler. In the right part of the figure we show the sequence of points that are generated by an iterative application of the basic idea. The left path was computed with $\alpha = 1$ by formula (8.9), the right one by (8.17). The two paths to optimality, which were both calculated in $\mathbb{R}^6$, are "qualitatively" quite different in $\mathbb{R}^2$. Moreover, as we shall see, *every* point of the projective curve $\boldsymbol{x}(\rho)$ defines a direction of ***descent*** in which we can "shoot" for the optimum of the linear program (LP) – which is remarkably different from the *single* direction given by the approximate solution (8.9).

8.2.2 The Solution in the Transformed Space

Applying the transformation T_0 and expressing the result in terms of the parameter ρ rather than R, see (8.15), we can calculate the solution $\boldsymbol{y}(\rho)$ to (FLP_ρ) for all $0 \le \rho < r$ where r is defined in (8.1). The algebra to do so is somewhat awkward, but the formulas turn out to be "nice". In principle, however, you can just take the following formulas (8.21), ... , (8.26), check their consistency and proceed from there as we will derive, *ex post factum*, the optimal solution to (FLP_ρ) for values $\rho \ge r$ as well using just these formulas. So we find after some algebra and simplifications that the solution to (FLP_ρ) for $0 \le \rho \le r$ is of the form

$$\boldsymbol{y}(\rho) = \boldsymbol{y}^0 - \rho \boldsymbol{q}(\rho)/\|\boldsymbol{q}(\rho)\| \,, \tag{8.21}$$

where $\boldsymbol{y}^0$ is the center of S^{n+1} and in the previous notation we have set

$$\begin{aligned} \boldsymbol{q}(\rho) &= \boldsymbol{Q}\begin{pmatrix} \boldsymbol{D}\boldsymbol{c}^T \\ -z(\rho) \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{p} \\ 0 \end{pmatrix} + \frac{z_0 - z(\rho) - \gamma}{1+\beta}\begin{pmatrix} \boldsymbol{e}-\boldsymbol{d} \\ 1 \end{pmatrix} - \frac{z_0 - z(\rho)}{n+1}\begin{pmatrix} \boldsymbol{e} \\ 1 \end{pmatrix} , \end{aligned} \tag{8.22}$$

$$z(\rho) = \frac{(\boldsymbol{c}\boldsymbol{D}\,,\,0)\boldsymbol{y}(\rho)}{y_{n+1}(\rho)} = z_0 - (n+1)\rho\frac{(1+\beta)W(\rho) - (n+1)\gamma\rho}{1+\beta-(n+1)(n-\beta)\rho^2} \,, \tag{8.23}$$

$$W(\rho) = (1+\beta)^{-1/2}\sqrt{(1+\beta)\|\boldsymbol{p}\|^2 + \gamma^2 - (\|\boldsymbol{p}\|^2\|\boldsymbol{d}\|^2 - \gamma^2)(n+1)\rho^2} \,. \tag{8.24}$$

After changing the parameter from R to ρ the quantity (8.19) becomes the quantity (8.24) multiplied by $(1 - n(n+1)\rho^2)^{-1/2}$ and likewise (8.18) transforms to (8.23). Thus they both exist for $0 \leq \rho < r$. The objective function (8.23) can also be written as

$$z(\rho) = z_0 - (n+1)\rho \frac{(1+\beta)\|\boldsymbol{p}\|^2 + \gamma^2}{(1+\beta)W(\rho) + (n+1)\gamma\rho} , \tag{8.25}$$

whereas the norm of $\boldsymbol{q}(\rho)$ satisfies the relation

$$\|\boldsymbol{q}(\rho)\|^2 = \|\boldsymbol{p}\|^2 + \frac{(z_0 - z(\rho) - \gamma)^2}{(1+\beta)} - \frac{(z_0 - z(\rho))^2}{(n+1)} = \frac{(z_0 - z(\rho))^2}{(n+1)^2\rho^2} \tag{8.26}$$

and thus, $z(\rho) = z_0 - (n+1)\rho\|\boldsymbol{q}(\rho)\|$. Formally we obtain now the approximate solution (8.7) from the exact solution (8.21) by setting $z(\rho)$ equal to zero in (8.22) which accentuates the difference between the two approaches to (FLP$_\rho$) because $z(\rho)$ is neither zero nor a constant. More precisely, we have from (8.22) and the definition (8.6) of $\boldsymbol{q}$ that

$$\boldsymbol{q}(\rho) = \boldsymbol{q} - z(\rho)\boldsymbol{r} \ , \ \text{where } \boldsymbol{r} = \frac{1}{1+\beta}\begin{pmatrix} \boldsymbol{e} - \boldsymbol{d} \\ 1 \end{pmatrix} - \frac{1}{n+1}\begin{pmatrix} \boldsymbol{e} \\ 1 \end{pmatrix} \tag{8.27}$$

and $\boldsymbol{r}$ is the orthoprojection of the $(n+1)^{st}$ unit vector of $\mathbb{R}^{n+1}$ on the subspace (8.3). The "direction vectors" of (8.21) are the normalized orthogonal projections of $(\boldsymbol{cD}\ , -z(\rho))$ on the subspace (8.3) that change as ρ varies and $\boldsymbol{q}(\rho) = \boldsymbol{q}$ if and only if $\beta = n$, i.e. $\boldsymbol{d} = \boldsymbol{0}$. Asking the more general question under what conditions the curves (8.17) and (8.21) are straight lines *you* find that this is the case if and only if $\boldsymbol{p}$ and $\boldsymbol{d}$ are linearly dependent, i.e. by the Cauchy-Schwarz inequality if and only if $\|\boldsymbol{p}\|\|\boldsymbol{d}\| = |\gamma|$. Moreover, in this case the two lines (8.9) and (8.17) are identical. To verify these statements you show that the direction vector $\frac{\boldsymbol{q}(\rho)}{\|\boldsymbol{q}(\rho)\|}$ does not change with ρ, e.g. that $\left\| \frac{\boldsymbol{q}(\rho)}{\|\boldsymbol{q}(\rho)\|} - \frac{\boldsymbol{q}(\rho')}{\|\boldsymbol{q}(\rho')\|} \right\|^2 = 0$ for any $0 \leq \rho < \rho'$, if and only if $\|\boldsymbol{p}\|\|\boldsymbol{d}\| = |\gamma|$. Note that the linear dependence of $\boldsymbol{p}$ and $\boldsymbol{d}$ includes the case $\boldsymbol{d} = \boldsymbol{0}$ and moreover, $\boldsymbol{q}$ and $\boldsymbol{r}$ are linearly independent if and only if $\boldsymbol{p}$ and $\boldsymbol{d}$ are linearly independent. So if $\boldsymbol{p}$ and $\boldsymbol{d}$ are linearly independent, the solutions that result from the two different approaches to (FLP$_\rho$) differ and so do, naturally, the sequences of points that are generated by an iterative application of the basic algorithmic idea.

To get a more appealing form of the solution $\boldsymbol{y}(\rho)$ to (FLP$_\rho$) we proceed as follows. From formula (8.21) we get using (8.22), (8.25), the second equation of (8.26) and by collecting the terms with ρ and ρ^2, respectively,

$$\boldsymbol{y}(\rho) = \boldsymbol{y}^0 + \frac{(1+\beta)\rho W(\rho)}{(1+\beta)\|\boldsymbol{p}\|^2 + \gamma^2}\boldsymbol{u} + \frac{(n+1)\rho^2}{(1+\beta)\|\boldsymbol{p}\|^2 + \gamma^2}\boldsymbol{v} , \tag{8.28}$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ and their respective norms are given by

$$
\begin{aligned}
\boldsymbol{u} &= -\begin{pmatrix} \boldsymbol{p} \\ 0 \end{pmatrix} + \frac{\gamma}{1+\beta}\begin{pmatrix} \boldsymbol{e}-\boldsymbol{d} \\ 1 \end{pmatrix}, \\
\boldsymbol{v} &= -\frac{\|\boldsymbol{p}\|^2\|\boldsymbol{d}\|^2-\gamma^2}{n+1}\begin{pmatrix} \boldsymbol{e} \\ 1 \end{pmatrix} - \begin{pmatrix} \gamma\boldsymbol{p}-\|\boldsymbol{p}\|^2\boldsymbol{d} \\ 0 \end{pmatrix},
\end{aligned}
$$

$$\|\boldsymbol{u}\| = \sqrt{\|\boldsymbol{p}\|^2+\gamma^2/(1+\beta)}\ ,\ \|\boldsymbol{v}\| = \|\boldsymbol{u}\|\sqrt{(1+\beta)(\|\boldsymbol{p}\|^2\|\boldsymbol{d}\|^2-\gamma^2)/(n+1)}\ .$$

Since $\beta \geq 0$ and $\|\boldsymbol{p}\| \neq 0$ we have $\boldsymbol{u} \neq \mathbf{0}$, while $\boldsymbol{v} = \mathbf{0}$ if and only if $\boldsymbol{p}$ and $\boldsymbol{d}$ are linearly dependent which follows from the Cauchy-Schwarz inequality. Moreover, one verifies by calculation that $\boldsymbol{u}^T\boldsymbol{v} = 0$, i.e. $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal to each other. Let us define two lines

$$\boldsymbol{u}(s) = \boldsymbol{y}^0 + s\boldsymbol{u}/\|\boldsymbol{u}\|\ , \quad \boldsymbol{v}(s) = \boldsymbol{y}^0 + s\boldsymbol{v}/\|\boldsymbol{v}\|\ , \tag{8.29}$$

where $s \in \mathbb{R}$ and we make the tacit assumption that $\|\boldsymbol{v}\| \neq 0$. Note that both lines depend *only* on the center of S^{n+1} and the orthogonal projections $\boldsymbol{p}$ and $\boldsymbol{d}$ of $\boldsymbol{D}\boldsymbol{c}^T$ and $\boldsymbol{e}$, respectively, but *not* on the objective function value z_0 of the starting point $\boldsymbol{x}^0$. In terms of $\boldsymbol{q}$ and $\boldsymbol{r}$, see (8.6) and (8.27), we find

$$\boldsymbol{u} = -\boldsymbol{q} + z_0\boldsymbol{r} \text{ and } \boldsymbol{v} = -\gamma\boldsymbol{q} + (\gamma(z_0-\gamma) - (1+\beta)\|\boldsymbol{p}\|^2)\boldsymbol{r}\ .$$

Thus from Remark 8.1 it follows, in particular, that $\boldsymbol{u}(s)$ solves the problem

$$\min\{(\boldsymbol{c}\boldsymbol{D}\ ,\ -z_0)\boldsymbol{y} : \boldsymbol{y} \in T_0(\mathcal{X}) \cap B_s^{n+1}\}$$

for all $s \geq 0$ since $\boldsymbol{u} = \boldsymbol{Q}\,(-\boldsymbol{c}\boldsymbol{D}\ ,\ z_0)^T$ where $\boldsymbol{Q}$ is the projection operator on the subspace (8.3). Using the proof of Remark 8.1 with a minor modification one proves also part (i) of the following exercise.

Exercise 8.5 *Assume that* $\boldsymbol{p}$ *and* $\boldsymbol{d}$ *are linearly independent and* $s \geq 0$. *(i) Show that* $\boldsymbol{v}(s)$ *solves the problem* $\min\{y_{n+1} : (\boldsymbol{c}\boldsymbol{D}\ ,\ -z_0)\boldsymbol{y} = 0\ ,\ \ \boldsymbol{y} \in T_0(\mathcal{X}) \cap B_s^{n+1}\}$. *(ii) Show that* $\boldsymbol{u}(s)$ *solves the problem* $\min\{(\boldsymbol{c}\boldsymbol{D}, -z_0)\boldsymbol{y} : \boldsymbol{y} \in T_0(\mathcal{X}) \cap B_s^{n+1}\}$.

From (8.28) and (8.29) it follows that the solution $\boldsymbol{y}(\rho)$ to (FLP$_\rho$) is a combination of the vectors $\boldsymbol{u}(s)$ and $\boldsymbol{v}(s)$ for some $s = s(\rho) \in \mathbb{R}$ and hence the solution $\boldsymbol{x}(R)$ to (LP$_R$) is a combination of the respective images of these lines under the transformation T_0^{-1}. We leave it as an exercise to calculate the images $\boldsymbol{x}^{\boldsymbol{u}}(R)$ and $\boldsymbol{x}^{\boldsymbol{v}}(R)$, say, in the parameter (8.15) which by the preceding are thus the optimal solutions to

$$\min\left\{\frac{\boldsymbol{c}\boldsymbol{x}-z_0}{1+\boldsymbol{e}^T\boldsymbol{D}^{-1}\boldsymbol{x}} : \boldsymbol{x} \in \mathcal{X}_R\right\} \text{ and } \min\left\{\frac{1}{1+\boldsymbol{e}^T\boldsymbol{D}^{-1}\boldsymbol{x}} : \boldsymbol{c}\boldsymbol{x} = z_0,\ \boldsymbol{x} \in \mathcal{X}_R\right\}$$

respectively, where

$$\mathcal{X}_R = \mathcal{X} \cap E(\boldsymbol{x}^0, R)\ .$$

Just what does this combination for $\boldsymbol{y}(\rho)$ look like? Let us first look at the line $\boldsymbol{v}(s)$ a little closer. From Exercise 8.5 it follows that

$$v_{n+1}(s) = (n+1)^{-1} - s\sqrt{(\|\boldsymbol{p}\|^2\|\boldsymbol{d}\|^2 - \gamma^2)/(n+1)((1+\beta)\|\boldsymbol{p}\|^2 + \gamma^2)} \geq 0$$

for all $0 \leq s \leq r = 1/\sqrt{n(n+1)}$, but one can prove it also directly. More precisely, one sees immediately that $v_{n+1}(s)$ is a decreasing function of s and proves that $v_{n+1}(r) = 0$ if and only if $\boldsymbol{d} = \boldsymbol{e}$ and $\gamma = 0$, i.e. in particular $\boldsymbol{p}$ and $\boldsymbol{d}$ are orthogonal. But then $\beta = n - \|\boldsymbol{d}\|^2 = 0$ and thus by Exercise 8.4 part (iii) we have an unbounded optimal solution to (LP). On the other hand, from the fact that $v_{n+1}(s)$ decreases monotonically it follows that $v_{n+1}(s_0) = 0$ for some $s_0 \geq r$ and solving the equation we get

$$s_0 = \sqrt{((1+\beta)\|\boldsymbol{p}\|^2 + \gamma^2)/(n+1)(\|\boldsymbol{p}\|^2\|\boldsymbol{d}\|^2 - \gamma^2)}\,.$$

Clearly, s_0 exists if and only if $\boldsymbol{p}$ and $\boldsymbol{d}$ are linearly independent which is an assumption that we make implicitly for the moment. Let z be *any* real number. Then we compute

$$\begin{aligned}(\boldsymbol{cD}\ , -z)\boldsymbol{v}(s) &= \frac{z_0 - z}{n+1} + \frac{s}{\|\boldsymbol{v}\|}\left(-\frac{\|\boldsymbol{p}\|^2\|\boldsymbol{d}\|^2 - \gamma^2}{n+1}(z_0 - z)\right) \\ &= (z_0 - z)(1 - s/s_0)/(n+1)\end{aligned}$$

and thus for $s = s_0$ we get $(\boldsymbol{cD}\ , -z)\boldsymbol{v}(s_0) = 0$, no matter what value z assumes. It follows that, $\boldsymbol{v}(s_0)$ is a "distinguished" point since *all hyperplanes* $(\boldsymbol{cD}\ , -z)\boldsymbol{y} = 0$ *obtained by varying* z *meet in this point.* Reversing the projective transformation we find that $(\boldsymbol{cD}\ , -z)\boldsymbol{y} = 0$ goes over into the hyperplane $\boldsymbol{cx} = z$ of $\mathbb{R}^n$ which yields a set of *parallel* hyperplanes in $\mathbb{R}^n$ if we vary the value of z. Thus after the projective transformation, these parallel hyperplanes of $\mathbb{R}^n$ have the distinguished point $\boldsymbol{v}(s_0)$ in common!

To resolve the mystery, remember that $v_{n+1}(s_0) = 0$ as well and so $\boldsymbol{v}(s_0)$ is an **improper** point of the n-dimensional projective space $\mathcal{P}_n$ which corresponds to a "point at infinity" of $\mathbb{R}^n$ if it corresponds to anything at all.

What we have here is a manifestation of the fact that Euclid's fifth axiom that *two parallel lines do not intersect* no longer applies in the projective space $\mathcal{P}_n$. In fact, attempts to *justify* Euclid's axiom of parallel lines have led historically to its "downfall" and the subsequent development of noneuclidean geometry of which projective geometry is but a small part. For an excellent overview of the historical aspects of the early developments of projective and noneuclidean geometry we refer to the book by the famous German mathematician Christian Felix Klein (1849-1925).

So we are now in $\mathcal{P}_n$. Denote $\boldsymbol{v}(s_0)$ by $\boldsymbol{w}^\infty$ and thus $\boldsymbol{w}^\infty = \boldsymbol{y}^0 + \boldsymbol{w}$ where

$$w = \frac{-1}{\|p\|^2\|d\|^2 - \gamma^2}\left\{\frac{\|p\|^2\|d\|^2 - \gamma^2}{n+1}\begin{pmatrix} e \\ 1 \end{pmatrix} + \begin{pmatrix} \gamma p - \|p\|^2 d \\ 0 \end{pmatrix}\right\} \tag{8.30}$$

$$\|w\|^2 = [(1+\beta)\|p\|^2 + \gamma^2]/(n+1)(\|p\|^2\|d\|^2 - \gamma^2) .$$

Now, $\|w^\infty\|^2 = \|w\|^2 + 1/(n+1)$ and taking a sort of a limit, the point w^∞ "slides off to infinity" as the vectors p and d "become" linearly dependent because $w = (\|p\|^2\|d\|^2 - \gamma^2)^{-1}v$. In this case, the line $v(s)$ degenerates into the point y^0 and $y(\rho)$ degenerates into the line $u(s)$, see (8.32) below.

It is entirely correct to write $\|w\| = \infty$ to indicate the linear dependence of p and d and that is what we are going to do from now on. Note that by the definition of equality of points in $\mathcal{P}_n$ linear dependence of $p \neq 0 \neq d$ simply means *equality* of the two improper points $(p, 0)$ and $(d, 0)$ of $\mathcal{P}_n$. We note for further reference that

$$z(\rho) = z_0 + (n+1)q^T(\rho)w , \tag{8.31}$$

which yields another expression for the optimal objective function value of (FLP_ρ). Rewriting formula (8.28) using w and formulas (8.24) and (8.30) we get for the solution $y(\rho)$ to (FLP_ρ) the expression

$$y(\rho) = y^0 + g(\rho) \text{ , where } g(\rho) = \frac{\rho}{\|u\|}\sqrt{1 - \frac{\rho^2}{\|w\|^2}}u + \frac{\rho^2}{\|w\|^2}w , \tag{8.32}$$

which has a very nice geometric interpretation. Denoting $\phi(\rho)$ the *angle* between $g(\rho)$ and w we get $\cos\phi(\rho) = \rho/\|w\|$ and using the formula $\sin^2\alpha + \cos^2\alpha = 1$ we find

$$y(\rho) = y^0 + \rho(\sin\phi(\rho)u/\|u\| + \cos\phi(\rho)w/\|w\|) .$$

8.2.3 Geometric Interpretations and Properties

In Figure 8.4 we give an illustration of the solution $y(\rho)$ to the problem (FLP_ρ) where we have assumed that $\rho < \sigma$ and $u(s(\rho))$, $u(s(\sigma))$ denote the points of intersection of the line $u(s)$ with the hyperplanes $(cD\ , -z(\rho))y = 0$ and $(cD\ , -z(\sigma))y = 0$, respectively. $z(\rho) = (cD, 0)y(\rho)/y_{n+1}(\rho)$ is the objective function value of $y(\rho)$ and $z(\sigma)$ is defined likewise. It is not difficult to show that the line $u(s)$ is the *tangent* and, thus by the orthogonality of u and w, the line $v(s)$ is the *normal* to the curve $y(\rho)$ in the point y^0. Formula (8.32) is well defined for $0 \leq \rho \leq \|w\|$ and we calculate

$$\|y(\rho) - (y^0 + w^\infty)/2\| = \|w\|/2 .$$

The loci of $y(\rho)$ form thus a semi-circle with center $(y^0 + w^\infty)/2$ and radius $\|w\|/2$ as ρ varies from 0 to $\|w\|$. The other half of the circle corresponds

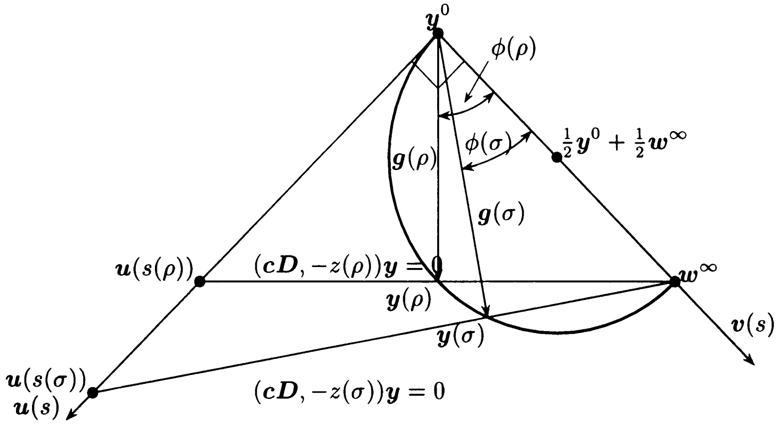

Fig. 8.4. The *semi-circle* determined by $y(\rho)$.

to the problem of *maximizing* the objective function of (FLP_ρ); see Exercise 8.6 (ii) below. Applying the inverse transformation the exact "shape" of the curve $x(\rho) = T_0^{-1}(y(\rho))$ and thereby of $x(R)$, see (8.17), in $\mathbb{R}^n$ is given by the image of a segment of a circle under the transformation T_0^{-1}.

By the preceding we know that $y(\rho)$ is the optimal solution to (FLP_ρ) for all $0 \le \rho < r$ where $r = 1/\sqrt{n(n+1)}$ is the radius of the largest ball B_ρ^{n+1} that can be *inscribed* into the simplex S^{n+1}.

More precisely, starting from (8.21), ... , (8.26) we obtained formula (8.32) by algebraic manipulations and thus comparing (8.21) and (8.32) we find

$$g(\rho) = -\rho q(\rho)/\|q(\rho)\| \,. \tag{8.33}$$

Thus in particular, (8.21) exists wherever $g(\rho)$ is well defined. Now since $\|d\|^2 \le \|e\|^2 = n$ by the properties of orthogonal projections, it follows that $\|w\|^2 \ge r^2$ and equality holds if and only if $d = e$ and $\gamma = 0$, which is the trivial case that we have encountered already when we defined w^∞. So in general we have $\|w\| > r$ and in the case that $\|w\| = \infty$ formula (8.32) becomes the line $u(s)$ defined in (8.29).

The question that ensues is whether or not $y(\rho)$ remains an optimal solution to (FLP_ρ) for values of ρ greater than r. To answer the question we need to know more about $y(\rho)$. Since u and w determine $y(\rho)$ we denote by L_{uw} the (two-dimensional) plane spanned by u and w and that contains y^0 and $y(\rho)$, i.e.

$$\begin{aligned} L_{uw} &= \{y \in \mathbb{R}^{n+1} : y = y^0 + su + tw \ \text{ for } s,t \in \mathbb{R}\} \\ &= \{y \in \mathbb{R}^{n+1} : y = y^0 + sq + tr \ \text{ for } s,t \in \mathbb{R}\} \ , \end{aligned}$$

where q is defined in (8.6) and r in (8.27). The second equality follows because u and w are linearly independent if and only if q and r are linearly independent and more precisely,

$$\boldsymbol{q} = -\left(1 - \frac{z_0\gamma}{(1+\beta)\|\boldsymbol{u}\|^2}\right)\boldsymbol{u} - \frac{z_0}{n+1}\cdot\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|^2},\ \boldsymbol{r} = \frac{\gamma}{1+\beta}\cdot\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|^2} - \frac{1}{n+1}\cdot\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|^2}\ .$$

If $\|\boldsymbol{w}\| = \infty$, i.e. if $\boldsymbol{u}$ and $\boldsymbol{w}$ as well as $\boldsymbol{q}$ and $\boldsymbol{r}$ are linearly dependent, then L_{uw} is simply the line $\boldsymbol{u}(s)$ and we drop $\boldsymbol{w}$ from the definition of L_{uw}.

Let us now take a closer look at the line $\boldsymbol{u}(s)$. The last component of $\boldsymbol{u}(s)$ is given by

$$u_{n+1}(s) = \frac{1}{n+1} + \frac{\gamma}{1+\beta}\frac{s}{\|\boldsymbol{u}\|}$$

and thus for $s = -(1+\beta)\|\boldsymbol{u}\|/\gamma(n+1)$ we get an improper point of $\mathcal{P}_n$ that we will call $\boldsymbol{u}^\infty$ for short.

Clearly, $\boldsymbol{u}^\infty$ exists if and only if $\gamma \neq 0$, i.e. if $\boldsymbol{p}$ and $\boldsymbol{d}$ are *nonorthogonal.* So let us make this assumption and write $\boldsymbol{u}^\infty = \boldsymbol{y}^0 + \boldsymbol{u}^0$ where

$$\boldsymbol{u}^0 = -[(1+\beta)/\gamma(n+1)]\boldsymbol{u}\ , \quad \|\boldsymbol{u}^0\| = (1+\beta)\|\boldsymbol{u}\|/|\gamma|(n+1)\ . \tag{8.34}$$

Thus $\|\boldsymbol{u}^\infty\|^2 = \|\boldsymbol{u}^0\|^2 + 1/(n+1)$ and taking again a sort of limit, the point $\boldsymbol{u}^\infty$ "slides off to infinity" as the vectors $\boldsymbol{p}$ and $\boldsymbol{d}$ "become" orthogonal, but the line $\boldsymbol{u}(s)$ still exists. As in the case of $\boldsymbol{w}^\infty$ we shall use $\|\boldsymbol{u}^\infty\| = \infty$ to indicate the orthogonality of $\boldsymbol{p}$ and $\boldsymbol{d}$.

It follows that $\|\boldsymbol{w}^\infty\| = \infty = \|\boldsymbol{u}^\infty\|$ if and only if $\boldsymbol{d} = \boldsymbol{0}$. Unlike the point $\boldsymbol{w}^\infty$, whose position is "fixed", the point $\boldsymbol{u}^\infty$ lies on the halfline $\boldsymbol{u}(s)$ for $s \geq 0$ if $\gamma < 0$ while $\boldsymbol{u}^\infty$ lies on the halfline $\boldsymbol{u}(s)$ for $s \leq 0$ if $\gamma > 0$.

In any case, assuming $\|\boldsymbol{u}^\infty\| < \infty$ and $\|\boldsymbol{w}^\infty\| < \infty$ the three points $\boldsymbol{y}^0$, $\boldsymbol{w}^\infty$ and $\boldsymbol{u}^\infty$ determine a triangle in the plane L_{uw} to which $\boldsymbol{y}(\rho)$ belongs if $\gamma < 0$ (see Figure 8.5). *You* are encouraged to supply the illustrations for the cases when $\gamma = 0$ and $\gamma > 0$, respectively, yourself.

Not just only out of curiosity, let us ask ourselves what the *perpendicular* from $\boldsymbol{y}^0$ on the line defined by $\boldsymbol{u}^\infty$ and $\boldsymbol{w}^\infty$, i.e. the hypotenuse of the triangle, looks like. Denote that point by $\boldsymbol{y}^\infty$. We thus have

$$\boldsymbol{y}^\infty = \mu\boldsymbol{u}^\infty + (1-\mu)\boldsymbol{w}^\infty \text{ for some } 0 < \mu < 1$$

since $\boldsymbol{u}$ and $\boldsymbol{w}$ form a right angle in the plane L_{uw}. From the condition of orthogonality we get the equation $(\boldsymbol{y}^0 - \boldsymbol{y}^\infty)^T(\boldsymbol{u}^\infty - \boldsymbol{w}^\infty) = 0$. Solving the equation for μ and simplifying we find

$$\boldsymbol{y}^\infty = \frac{1}{\|\boldsymbol{d}\|^2}\begin{pmatrix}\boldsymbol{d}\\ 0\end{pmatrix}\ , \quad \|\boldsymbol{y}^0 - \boldsymbol{y}^\infty\|^2 = (1+\beta)/(n+1)(n-\beta)\ . \tag{8.35}$$

You verify that $r^2 \leq \|\boldsymbol{y}^0 - \boldsymbol{y}^\infty\|^2 = \|\boldsymbol{d}\|^{-2} - (n+1)^{-1} \leq \|\boldsymbol{w}\|^2$. Equality holds in the first inequality if and only if $\boldsymbol{d} = \boldsymbol{e}$, i.e. $\beta = 0$, and in the second

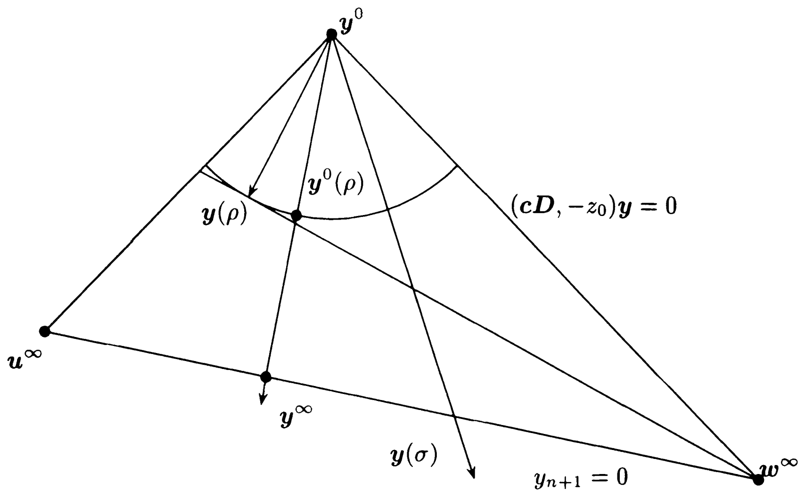

Fig. 8.5. The *triangle* determined by y^0, u^∞ and w^∞ if $\gamma < 0$.

one if and only if $\gamma = 0$ and $||w^\infty|| < \infty$. So in general we will have strict inequality on both sides.

To see why we are interested in y^∞ consider the problem

$$\min\{y_{n+1} : y \in T_0(\mathcal{X}) \cap B_\rho^{n+1}\} .$$

The solution exists for all $\rho \geq 0$ and let us denote it by $y^0(\rho)$. Using Remark 8.1 again we calculate $y^0(\rho) = y^0 - \rho r/||r||$ where r is defined in (8.27) and thus we get

$$y^0(\rho) = y^0 + \rho(y^\infty - y^0)/||y^\infty - y^0||$$

for all $\rho \geq 0$. Denote $\rho_\infty = ||y^\infty - y^0||$. Hence we have e.g. from (8.35)

$$(n+1)y_{n+1} \geq 1 - \rho/\rho_\infty > 0 \tag{8.36}$$

for all $y \in T_0(\mathcal{X}) \cap B_\rho^{n+1}$ and $0 \leq \rho < \rho_\infty$. Moreover,

$$y^\infty = y^0 - (n+1)\rho_\infty r = y^0(\rho_\infty)$$

and if $\gamma < 0$ then we have $y(\rho_\infty) = y^\infty$ as *you* verify using (8.32). In the case that $\gamma = 0$ we get $w^\infty = y^\infty$ and the triangle degenerates into a semi-open infinite rectangle.

Now if $\gamma < 0$ and $\rho > \rho_\infty$ then there exist $y \in T_0(\mathcal{X}) \cap B_\rho^{n+1}$ such that $y_{n+1} < 0$. But "crossing the line" defined by the hypotenuse of the triangle corresponds to "passing through infinity" in $\mathbb{R}^n$ and "coming back from infinity" which is what the *sign change* for y_{n+1} entails. Keeping in mind that we wish to apply the inverse T_0^{-1} of the projective transformation, it makes no sense to permit "solutions" to (FLP$_\rho$) having $y_{n+1} < 0$. We are thus led to consider the following restriction

$$(\mathrm{FLP}_\rho^+) \qquad \min\left\{\frac{(\boldsymbol{cD}\,,\,0)\boldsymbol{y}}{y_{n+1}} : \boldsymbol{y} \in T_0(\mathcal{X}) \cap B_\rho^{n+1}\,,\; y_{n+1} > 0\right\}$$

of (FLP$_\rho$) which is exactly (FLP$_\rho$) for all radii $0 \le \rho < \rho_\infty$ and thus in particular, for all radii $0 \le \rho < r$. But as we shall see this "restriction" is not really a restriction.

8.2.4 Extending the Exact Solution and Proofs

Remark 8.6 *(i) If $\boldsymbol{x}^0$ is a nonoptimal solution to* (LP)*, then the vector $\boldsymbol{y}(\rho)$ given by (8.32) solves the problems* (FLP$_\rho$) *and* (FLP$_\rho^+$) *for all $0 \le \rho < \rho_\infty = \sqrt{\|\boldsymbol{d}\|^{-2} - (n+1)^{-1}} \ge r$. Moreover, if $\gamma = \boldsymbol{p}^T\boldsymbol{d} \ge 0$ then the statement remains correct for all $0 \le \rho < \|\boldsymbol{w}\|$. If $\gamma < 0$, then a finite optimal solution to* (FLP$_\rho$) *and* (FLP$_\rho^+$) *does not exist for $\rho_\infty \le \rho < \|\boldsymbol{w}\|$.*
(ii) If for some $\rho > 0$ a finite optimal solution to (FLP$_\rho$) *with an objective function value less than $z_0 = \boldsymbol{cx}^0$ exists, then the vector $\boldsymbol{y}(\rho)$ given by (8.32) solves the problems* (FLP$_\rho$) *and* (FLP$_\rho^+$) *and its corresponding objective function value $z(\rho)$ satisfies $z(\rho) < z_0$. Moreover, for every $\sigma \in (0, \rho)$ both problems have the same finite optimal solution with objective function value $z(\sigma) > z(\rho)$, $z(\sigma) < z_0$.*

Proof. (i) From (8.32) we have that $\boldsymbol{y}(\rho)$ is well defined in *real* terms for all $0 \le \rho \le \|\boldsymbol{w}\|$ and by construction it is a feasible solution to (FLP$_\rho$). This includes the cases when $\|\boldsymbol{w}^\infty\| = \infty$ or $\|\boldsymbol{u}^\infty\| = \infty$. To derive the objective function value of $\boldsymbol{y}(\rho)$ we write the last component of $\boldsymbol{y}(\rho)$ as $y_{n+1}(\rho) = (n+1)^{-1}\sqrt{1-\rho^2/\|\boldsymbol{w}\|^2}\left(\sqrt{1-\rho^2/\|\boldsymbol{w}\|^2} + (n+1)\gamma\rho/(1+\beta)\|\boldsymbol{u}\|\right)$. So if $\gamma \ge 0$ then $y_{n+1}(\rho) > 0$ for all $0 \le \rho < \|\boldsymbol{w}\|$, while $y_{n+1}(\rho) > 0$ for all $0 \le \rho < \rho_\infty$, $y_{n+1}(\rho) < 0$ for all $\rho_\infty < \rho < \|\boldsymbol{w}\|$ otherwise. Thus $\boldsymbol{y}(\rho)$ is a feasible solution to (FLP$_\rho^+$) for $0 \le \rho < \rho_\infty$ as well, the objective function value of $\boldsymbol{y}(\rho)$ is well defined and we get

$$\begin{aligned} z(\rho) &= \frac{(\boldsymbol{cD}\,,\,0)\boldsymbol{y}(\rho)}{y_{n+1}(\rho)} = z_0 - \frac{(n+1)\|\boldsymbol{u}\|\rho}{\sqrt{1-\frac{\rho^2}{\|\boldsymbol{w}\|^2}} + \frac{(n+1)\gamma\rho}{(1+\beta)\|\boldsymbol{u}\|}} \\ &= z_0 - (n+1)\rho\frac{(1+\beta)\|\boldsymbol{u}\|\sqrt{1-\rho^2/\|\boldsymbol{w}\|^2} - (n+1)\gamma\rho}{(1+\beta) - (n-\beta)(n+1)\rho^2}\,. \end{aligned} \tag{8.37}$$

Calculating the derivative of $z(\rho)$ from (8.37) we find

$$dz/d\rho = -\frac{(n+1)\|\boldsymbol{u}\|}{\sqrt{1-\rho^2/\|\boldsymbol{w}\|^2}}\left(\sqrt{1-\frac{\rho^2}{\|\boldsymbol{w}\|^2}} + \frac{(n+1)\gamma\rho}{(1+\beta)\|\boldsymbol{u}\|}\right)^{-2} < 0\,,$$

since by the nonoptimality of $\boldsymbol{x}^0$ we have $\|\boldsymbol{u}\| \neq 0$ and all other quantities are well defined, i.e. they do not vanish for $0 \leq \rho < \rho_\infty$, and thus $z(\rho)$ decreases monotonically for all $0 \leq \rho < \rho_\infty$. Moreover, from (8.32) we find using $(\boldsymbol{cD}\ , -z(\rho))\boldsymbol{y}(\rho) = 0$ that

$$z(\rho) = z_0 - (n+1)\rho\|\boldsymbol{q}(\rho)\|\ . \tag{8.38}$$

To prove that $\boldsymbol{y}(\rho)$ is an optimal solution to (FLP$_\rho$) for all $0 \leq \rho < \rho_\infty$ let us denote by $\boldsymbol{B}$ any $(n+1) \times (n-m)$ matrix corresponding to an orthonormal basis of the subspace (8.3). Thus $\boldsymbol{B}^T\boldsymbol{B} = \boldsymbol{I}_{n-m}$ and $\boldsymbol{B}\boldsymbol{B}^T = \boldsymbol{Q}$ is the orthogonal projection operator on the subspace (8.3) since $\boldsymbol{B}\boldsymbol{B}^T$ projects $\mathbb{R}^{n+1}$ orthogonally on the subspace (8.3) and the orthogonal projection operator on a proper subspace of any finite-dimensional Euclidean space is unique. Consequently, every $\boldsymbol{y} \in T_0(\mathcal{X})$ can be written as $\boldsymbol{y} = \boldsymbol{y}^0 + \boldsymbol{B}\boldsymbol{\xi}$ where $\boldsymbol{\xi} \in \mathbb{R}^{n-m}$ is arbitrary. For all $\boldsymbol{y} \in T_0(\mathcal{X}) \cap B_\rho^{n+1}$ we find $\|\boldsymbol{y} - \boldsymbol{y}^0\|^2 = \boldsymbol{\xi}^T\boldsymbol{B}^T\boldsymbol{B}\boldsymbol{\xi} = \|\boldsymbol{\xi}\|^2 \leq \rho^2$. From (8.36) it follows that an optimal solution to (FLP$_\rho$) exists and that it has a finite objective function value for every ρ in the semi-open interval $[0, \rho_\infty)$. So suppose that $\boldsymbol{y}^* \in T_0(\mathcal{X}) \cap B_\rho^{n+1}$ is an optimal solution to (FLP$_\rho$) for $0 \leq \rho < \rho_\infty$ and that $\boldsymbol{y}(\rho)$ is not optimal. By (8.36) $y^*_{n+1} > 0$ and thus the objective function value z_* of $\boldsymbol{y}^*$ satisfies $(\boldsymbol{cD}\ , -z(\rho))\boldsymbol{y}^* < 0$. But $\boldsymbol{y}^* = \boldsymbol{y}^0 + \boldsymbol{B}\xi^*$ for some $\boldsymbol{\xi}^* \in \mathbb{R}^{n-m}$ and necessarily $\|\boldsymbol{\xi}^*\|^2 \leq \rho^2$. We get

$$\begin{aligned}(\boldsymbol{cD}\ , -z(\rho))\boldsymbol{y}^* &= (n+1)^{-1}(z_0 - z(\rho)) + (\boldsymbol{cD}\ , -z(\rho))\boldsymbol{B}\boldsymbol{\xi}^* \\ &= (n+1)^{-1}(z_0 - z(\rho)) + \boldsymbol{q}^T(\rho)\boldsymbol{B}\boldsymbol{\xi}^*\ ,\end{aligned}$$

since the columns of $\boldsymbol{B}$ are elements of the subspace (8.3) and $\boldsymbol{q}(\rho)$ is the orthoprojection of $(\boldsymbol{cD}\ , -z(\rho))^T$ on that subspace, see (8.22). Solving the problem $\min\{\boldsymbol{a}^T\boldsymbol{\xi} : \|\boldsymbol{\xi}\|^2 \leq \rho^2\}$ where $\boldsymbol{a}^T = \boldsymbol{q}^T(\rho)\boldsymbol{B}$ we get the solution $\boldsymbol{\xi} = -\rho\boldsymbol{a}/\|\boldsymbol{a}\|$ with an objective function value of $\boldsymbol{a}^T\boldsymbol{\xi} = -\rho\|\boldsymbol{q}(\rho)\|$, since $\|\boldsymbol{q}(\rho)\| \neq 0$ by the strict monotonicity of $z(\rho)$ for all $0 \leq \rho < \rho_\infty$. Hence we get the contradiction $0 > (\boldsymbol{cD}\ , -z(\rho))\boldsymbol{y}^* \geq (n+1)^{-1}(z_0 - z(\rho)) - \rho\|\boldsymbol{q}(\rho)\| = 0$. If $\gamma \geq 0$, then our argument remains correct if we replace ρ_∞ by $\|\boldsymbol{w}\|$ and thus $\boldsymbol{y}(\rho)$ as defined in (8.32) is an optimal solution to (FLP$_\rho$) for all $0 \leq \rho < \|\boldsymbol{w}\|$ in this case. If $\gamma < 0$, then for $\rho_\infty < \rho < \|\boldsymbol{w}\|$ formula (8.32) is well defined, formulas (8.37) and (8.38) remain correct and $z(\rho) \geq z_0$. In this case $z(\rho)$ has a singularity for $\rho = \rho_\infty$, however. Moreover, for $\rho_\infty \leq \rho < \|\boldsymbol{w}\|$ a solution to (FLP$_\rho$) and thus to (FLP$^+_\rho$) does not exist, since y_{n+1} vanishes in the interior of B_ρ^{n+1} while $(\boldsymbol{cD}\, , 0)\boldsymbol{y}$ does not, if $\gamma < 0$. To prove this formally consider the point $\boldsymbol{h}(t) = \boldsymbol{y}^0 + t(\mu(\boldsymbol{y}^\infty - \boldsymbol{y}^0) + (1-\mu)\boldsymbol{w})$ for $t \geq 0$ which for $t = 1$ and every $0 \leq \mu \leq 1$ is an improper point on the hypotenuse of the triangle. Then $z_h(t) = (\boldsymbol{cD}\, , 0)\boldsymbol{h}(t)/h_{n+1}(t) = z_0 + t(n+1)\mu\gamma/(1-t)(n-\beta)$ and if $\gamma < 0$ then $z_h(t) \to -\infty$ for every $0 < \mu \leq 1$ if t is increased to 1.

(ii) If a solution to (FLP$_\rho$) with objective function $z < z_0$ exists, then it satisfies the equations $(\boldsymbol{cD}, -z)\boldsymbol{y} = 0$, $(\boldsymbol{AD}, -\boldsymbol{b})\boldsymbol{y} = 0$, $\boldsymbol{f}^T\boldsymbol{y} = 1$ and $\|\boldsymbol{y} - \boldsymbol{y}^0\| \leq \rho^2$. Like in the first part of the proof, we can write $\boldsymbol{y} = \boldsymbol{y}^0 + \boldsymbol{B\xi}$ where $\boldsymbol{\xi} \in \mathbb{R}^{n-m}$ is arbitrary and thus the system reduces to $z_0 - z + (n+1)\boldsymbol{q}^T(\rho)\boldsymbol{B\xi} = 0$, $\boldsymbol{\xi}^T\boldsymbol{\xi} \leq \rho^2$ as in part (i) where $\boldsymbol{q}(\rho)$ satisfies (8.22) with $z(\rho)$ replaced by z. Let $\boldsymbol{a}^T = \boldsymbol{q}^T(\rho)\boldsymbol{B}$. Since $z < z_0$ we have $\boldsymbol{a}^T \neq \boldsymbol{0}$ and thus $\min\{\boldsymbol{a}^T\boldsymbol{\xi} : \boldsymbol{\xi}^T\boldsymbol{\xi} \leq \rho^2\} = \min\{\boldsymbol{a}^T\boldsymbol{\xi} : \boldsymbol{\xi}^T\boldsymbol{\xi} = \rho^2\}$. The latter problem, however, has the *unique* solution $\boldsymbol{\xi} = -\rho\boldsymbol{a}/\|\boldsymbol{a}\|$. Consequently, $\boldsymbol{y}(\rho)$ is unique as well, satisfies (8.21) and $z = z(\rho)$ is given by (8.38). Hence, (8.26) follows and solving the quadratic equation given by the second part of (8.26) for $z_0 - z(\rho)$ we find (8.23), (8.24) and (8.25). Consequently, $\boldsymbol{y}(\rho)$ satisfies (8.32) and thus $\rho \leq \|\boldsymbol{w}\|$. The rest of the remark then follows from part (i). ∎

The plane L_{uw} is divided into four parts by the two hyperplanes $y_{n+1} = 0$ and $(\boldsymbol{cD}, -z_0)\boldsymbol{y} = 0$. The point $\boldsymbol{w}^\infty$ is the "origin" and every hyperplane $(\boldsymbol{cD}, -z)\boldsymbol{y} = 0$ defines a line in L_{uw} that contains $\boldsymbol{w}^\infty$.

Decreasing z we get a "bundle" of lines that turn counter-clockwise around the point $\boldsymbol{w}^\infty$. In Figure 8.6 we illustrate the situation for $\gamma < 0$. It follows from the analysis of this section that approaching the line given by $y_{n+1} = 0$ "from above" corresponds to z tending to $-\infty$, while "on the other side" of this line z is arbitrarily large. The values z_i shown in Figure 8.6 satisfy $+\infty > z_1 > z_2 > z_0 > z_3 > z_4 > -\infty$. Because $(\boldsymbol{cD}, -z)\boldsymbol{w}^\infty = 0$ for *all* z, the objective function value $z(\rho)$ of (8.37) for $\rho = \|\boldsymbol{w}\|$ simply does not exist: $z(\|\boldsymbol{w}\|)$ can be any real number.

Clearly, the point $\boldsymbol{u}^\infty$ plays a role similar to that of $\boldsymbol{w}^\infty$ in our development except that $\boldsymbol{u}^\infty$ changes its position depending on the sign of $\gamma = \boldsymbol{p}^T\boldsymbol{d}$. Since $(\boldsymbol{cD}, -z)\boldsymbol{w}^\infty = 0$ for all $z \in \mathbb{R}$ one can ask oneself: what is the family of hyperplanes of the form $(\boldsymbol{a}^T, -z)\boldsymbol{y} = 0$ where $\boldsymbol{a} \in \mathbb{R}^n$ is "fixed" and $z \in \mathbb{R}$ arbitrary that are tangent to the n-dimensional ball B_ρ^{n+1} and that meet all in the point $\boldsymbol{u}^\infty$? *You* show the following curiosity.

Exercise 8.6 *(i) Let* $\Gamma = \gamma(n+1)/((1+\beta)\|\boldsymbol{p}\|^2 + \gamma^2)$. *Show that* $(\Gamma\boldsymbol{cD} - \boldsymbol{e}^T, -z)\boldsymbol{u}^\infty = 0$ *for all* $z \in \mathbb{R}$. *Show that the optimal solution* $\widetilde{\boldsymbol{y}}(\rho)$ *and the corresponding objective function value* $\widetilde{z}(\rho)$ *of the problem* $\min\{(\Gamma\boldsymbol{cD} - \boldsymbol{e}^T, 0)\boldsymbol{y}/y_{n+1} : \boldsymbol{y} \in T_0(\mathcal{X}) \cap B_\rho^{n+1}\}$ *are given by*

$$\widetilde{\boldsymbol{y}}(\rho) = \boldsymbol{y}^0 + \rho\sqrt{1 - \frac{\rho^2}{\|\boldsymbol{u}^0\|^2}}\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} + \frac{\rho^2}{\|\boldsymbol{u}^0\|^2}\boldsymbol{u}^0 ,$$

$$\widetilde{z}(\rho) = \Gamma z_0 - n - \frac{(n+1)\rho}{\|\boldsymbol{w}\|\sqrt{1 - \rho^2/\|\boldsymbol{u}^0\|^2} - \rho}$$

for all $0 \leq \rho < \rho_\infty$. *Give a geometric interpretation of* $\widetilde{\boldsymbol{y}}(\rho)$ *similar to the one of* $\boldsymbol{y}(\rho)$, *see Figures 8.4 and 8.5. (ii) Show that* $\boldsymbol{y}^{max}(\rho) =$

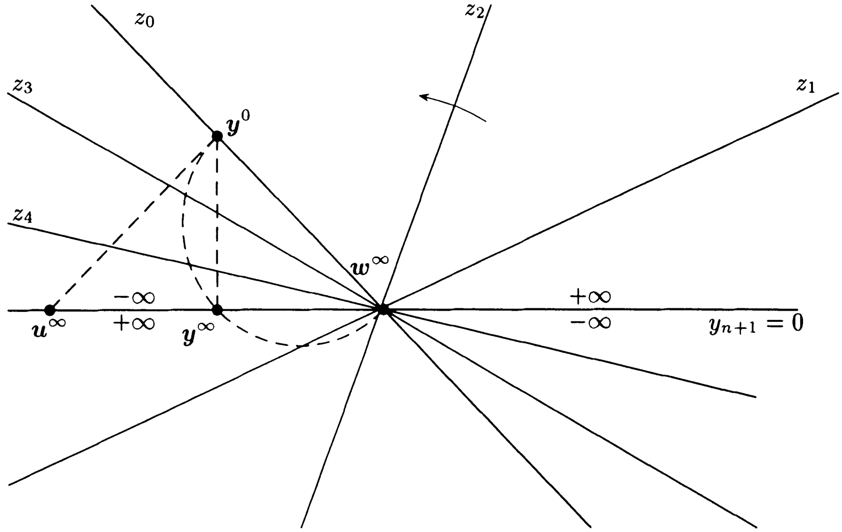

Fig. 8.6. *Lines* in the plane L_{uw} if $\gamma < 0$.

$\boldsymbol{y}^0 + \rho(-\sqrt{1-\rho^2/\|\boldsymbol{w}\|^2}\boldsymbol{u}/\|\boldsymbol{u}\| + \rho\boldsymbol{w}/\|\boldsymbol{w}\|^2)$ *with an objective function value of* $z^{max}(\rho) = z_0 + \rho(n+1)\|\boldsymbol{u}\|^2/(\|\boldsymbol{u}\|\sqrt{1-\rho^2/\|\boldsymbol{w}\|^2} - (n+1)\gamma\rho/(1+\beta))$ *is the* maximizer *for* (FLP_ρ). *Moreover,* $z^{max}(\rho) - z(\rho) = 2\|\boldsymbol{u}\|\rho(n+1)\sqrt{1-\rho^2/\|\boldsymbol{w}\|^2}/(1-\rho^2/\rho_\infty^2)$ *where* $z(\rho)$ *is given in (8.37).* (Hint: *Apply Remark 8.6.*)

8.2.5 Examples of Projective Images

In Figure 8.7 we show projective images of the polytope of the problem of Exercise 8.2 (ii) for two different points $\boldsymbol{x}^0$ that are used in the transformation T_0: in the left part we use the point $\boldsymbol{x}^0$ of Figure 8.3 with coordinates $x_1 = 30$, $x_2 = 40$, in the right part the point with coordinates $x_1 = 40$, $x_2 = 25$ for the transformation, i.e. the one indicated by ▪ in Figure 8.3.

We have carried out all calculations in $\mathbb{R}^7$ or $\mathcal{P}^6$, of course; so the pictures are *not* an "artist's rendering" of some projective transformation, even though both may have a certain artistic quality to your eye.

The respective points $\boldsymbol{x}^0$ we have chosen for the transformation T_0 are both rather "off-center" or "close" to the boundary of the polytope of Figure 8.3. Nevertheless, after the projective transformation, their respective images $\boldsymbol{y}^0$ appear in or close to some "center" of the feasible set, their immediate neighborhood appears to be "magnified" while the more distant parts loose more and more in proportion – which is typical of projective transformations. Just look up the picture of the view of a New Yorker, overlooking the world to his west from a penthouse on the eastside of the intersection of

9th Avenue and some street, by the celebrated Rumano-American artist Saul Steinberg which appeared on the cover of the magazine *The New Yorker* in 1976. Steinberg's *mensch* sees it all, and he sees it in a vision that borders on autism.

Our "pictures" are exact up to possible rounding errors in the calculation of the projections $\boldsymbol{p} \in \mathbb{R}^6$ and $\boldsymbol{d} \in \mathbb{R}^6$ of the corresponding scaled objective function vector $\boldsymbol{Dc}^T$ and the vector $\boldsymbol{e}$ as well as in the calculation of the various divisions, square roots, etc that are required and all of which we carried out using a computer and floating point operations. In the left picture we get $||\boldsymbol{p}|| \approx 4.76$, $||\boldsymbol{d}|| \approx 1.51$ and in the right one $||\boldsymbol{p}|| \approx 7.62$, $||\boldsymbol{d}|| \approx 1.56$.

The starting point $\boldsymbol{x}^0$ for the left gives $\gamma \approx -1.29$ which indicates that $\boldsymbol{p}$ and $\boldsymbol{d}$ form an obtuse angle of about $100.4°$, while the point $\boldsymbol{x}^0$ for the right picture we get $\gamma \approx 0.29$ and an acute angle of about $88.6°$. The "third" point $\boldsymbol{u}^\infty$ of the respective triangles thus lies to the left of $\boldsymbol{w}^\infty$ in the left picture and to the far right of $\boldsymbol{w}^\infty$ in the right one: γ is rather small and $\boldsymbol{p}$ and $\boldsymbol{d}$ are very close to being orthogonal. The lines $y_i = 0$ for $1 \le i \le 7$ of the figure are the lines that we get by intersecting the 2-dimensional plane L_{uw} with the hyperplanes $y_i = 0$ of $\mathbb{R}^7$.

They are calculated as follows: since $\boldsymbol{p}$ and $\boldsymbol{d}$ are in both cases linearly independent, the lines $\boldsymbol{u}(s)$ and $\boldsymbol{v}(s)$, see (8.29), together with the point $\boldsymbol{y}^0$ form a rectangular coordinate system for L_{uw}. Now all *you* need to do is to calculate the intersection points of the lines $L_{uw} \cap \{\boldsymbol{y} \in \mathbb{R}^7 : y_i = 0\}$ with the two lines $\boldsymbol{u}(s)$ and $\boldsymbol{v}(s)$ to get the desired result.

Since the feasible set of our example problem is a *polytope*, the line $y_7 = 0$ does not meet the transformed feasible set – there are no *feasible* points "at infinity". The indexing $y_1, \ldots, y_6$ corresponds to the listing of the constraint set in Exercise 8.2 (ii): y_1 and y_2 correspond to x_1 and x_2, respectively, $y_3, \ldots, y_6$ to the slack variables of the problem in the order of the constraints of Exercise 8.2 (ii). Thus the line $y_1 = 0$ corresponds to the x_2-axis, $y_2 = 0$ to the x_1-axis and $y_3 = 0$ to the line $5x_1 + x_2 = 250$ of Figure 8.3, etc, and the origin of Figure 8.3 is mapped to the respective points labelled O of Figure 8.7.

The projective transformation produces a "mirror-image" of the feasible set. The two polytopes of Figure 8.7 as well as the one of Figure 8.3 are, however, *projectively equivalent*: they all have the same dimension and the same number of extreme points. Angles and lengths are, however, "distorted" by comparison to the "original" of Figure 8.3 in the two images of Figure 8.7. These *metric quantities* of the affine space are invariant under affine transformations, but not under projective transformations. However, if you take any four points on a line of the "original" polytope, then their respective images

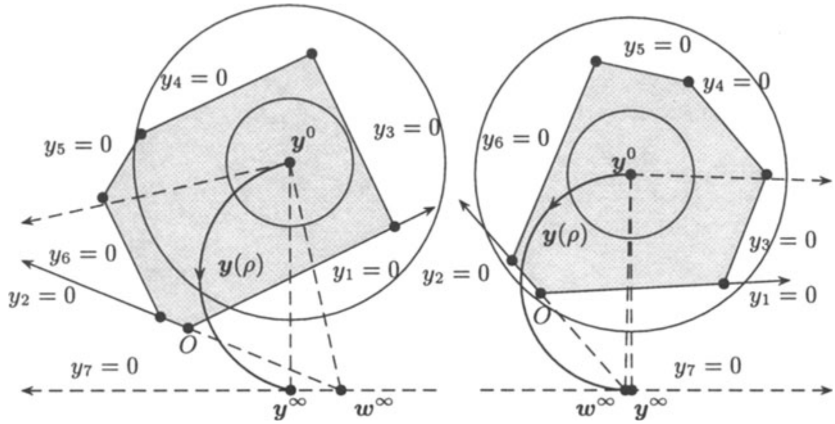

Fig. 8.7. Projective images of Figure 8.3 in the plane L_{uw} of $\mathcal{P}^6$.

lie on a line as well and forming the *cross ratio*, see Chapter 8.3 below, you will find that its value remains unchanged.

Caveat emptor : the "original" polytope of our example is really a two-dimensional object, the remaining four variables being slack variables. For higher-dimensional "original" polyhedra we get a two-dimensional "shadow" in the plane L_{uw} and the projective equivalence to the original is lost, of course. The boundary of that shadow corresponds to the intersection of L_{uv} with the boundary of S^{n+1}.

In Figure 8.7 we show also the projective curves $\boldsymbol{y}(\rho)$, see (8.21) or (8.32), which are the optimal solutions to the corresponding problems (FLP$_\rho$).

In the left part $\boldsymbol{y}(\rho)$ misses the optimal face $y_2 = 0$, while on the right it meets the optimal face of the polytope. The line $y_2 = 0$ corresponds to the line $(\boldsymbol{cD}, -z_*)\boldsymbol{y} = 0$ of L_{uw} where $z_* = 0$ is the optimal objective function value of our example linear program and $\boldsymbol{c} = (0, 1, 0, 0, 0, 0)$. Like the lines of Figure 8.6 it must meet the point $\boldsymbol{w}^\infty$ and, of course, it does.

The inner circle around $\boldsymbol{y}^0$ corresponds to the circle with radius $r = 1/\sqrt{n(n+1)}$ where $n = 6$. It is strictly contained in the transformed feasible set and this is typically the case: from Exercise 8.3 (ii) we find using T_0 that the only points of B_r^{n+1} that are on the boundary of S^{n+1} are precisely the $n+1$ points $(1/n)(\boldsymbol{f} - \boldsymbol{u}^i)$ for $1 \leq i \leq n+1$ where $\boldsymbol{u}^i \in \mathbb{R}^{n+1}$ is the i-th unit vector. Since $\boldsymbol{x}^0 > 0$ none of these points is in $T_0(\mathcal{X})$ unless the matrix $\boldsymbol{A}$ of (LP) contains a zero column or $\boldsymbol{b} = \boldsymbol{0}$, which is uninteresting; see Remark 3.1. Thus the domain of "guaranteed" positivity of $\boldsymbol{y}(\rho)$ and hence of the transformed curve $\boldsymbol{x}(\rho)$ is bigger than the set $T_0(\mathcal{X}) \cap B_r^{n+1}$.

The outer circle has a radius of $1/\sqrt{n+1}$: it shows you the extent of the distortion or contortion produced by the two different projective transformations. The "circumscribing" ball for S^{n+1} with radius $\sqrt{n/(n+1)}$ is relatively too big to be shown in the figures.

8.3 The Cross Ratio

Πᾶν μέτρον ἄριστον!
Kleoboulos of Lindos (c. 550 B.C.)

"Everything in good measure!" is what Kleoboulos recommended to his subjects whom he ruled gently while he was the *tyrant* of Lindos. Indeed, he was considered a great man in his time and by later generations, as one of the seven sages of Greece. As it goes with the "wisdom" of others, you have to know how to apply it. To apply this one you need to know, first and foremost, what constitutes a "measure".

To come to a measure of the progress of the objective function value that one makes along the curve $\boldsymbol{y}(\rho)$ or $\boldsymbol{x}(R)$ as defined by (8.32) and (8.17) we will first make the assumption that (LP) has a finite optimum. Denote by z_* the optimum objective function value.

The intersection of the hyperplane $(\boldsymbol{cD}\ , -z_*)\boldsymbol{y} = 0$ with the plane L_{uw} of the previous section is a line that – by the preceding analysis – contains the point $\boldsymbol{w}^\infty$. Since in the n-dimensional projective space $\mathcal{P}_n$ *any two lines that belong to a plane either are identical or have a nonempty intersection* it follows that for fixed $\rho \in [0, \rho_\infty)$ the line

$$\boldsymbol{y}(\rho,\tau) = \boldsymbol{y}^0 + (\tau/\rho)(\boldsymbol{y}(\rho) - \boldsymbol{y}^0) \tag{8.39}$$

intersects the line $L_{uw} \cap \{\boldsymbol{y} \in \mathbb{R}^{n+1} : (\boldsymbol{cD}\ , -z_*)\boldsymbol{y} = 0\}$ somewhere. The parameter $\tau \in \mathbb{R}$ satisfies $\|\boldsymbol{y}(\rho,\tau) - \boldsymbol{y}^0\| = |\tau|$ because $\|\boldsymbol{g}(\rho)\| = \rho$. Clearly, the two lines are not identical because $\boldsymbol{y}^0$ belongs to $\boldsymbol{y}(\rho,\tau)$, but not to the second line since the original point $\boldsymbol{x}^0$ is, by assumption, nonoptimal.

So denote by $\tau(\rho) \in \mathbb{R}$ the value of τ where the two lines intersect. We now have three points $\boldsymbol{y}^0$, $\boldsymbol{y}(\rho)$ and $\boldsymbol{y}(\rho, \tau(\rho))$ and to measure "distance relations" in $\mathcal{P}_n$ in a meaningful way we need an additional point of reference.

Consider the hyperplane $\{\boldsymbol{y} \in \mathbb{R}^{n+1} : y_{n+1} = 0\}$. Its intersection with L_{uw} is the line that contains the points $\boldsymbol{w}^\infty$, $\boldsymbol{y}^\infty$ and $\boldsymbol{u}^\infty$ of the previous section and this line is also different from the line $\boldsymbol{y}(\rho,\tau)$ since $y^0_{n+1} = 1/(n+1) > 0$. Denote by $\boldsymbol{y}^\infty_\rho$ the point of intersection of $\boldsymbol{y}(\rho,\tau)$ and the hyperplane $y_{n+1} = 0$. $\boldsymbol{y}^\infty_\rho$ is, of course, in general different from the point $\boldsymbol{y}^\infty$; see Figure 8.9 below for an illustration.

We can use now the *cross ratio* of the resulting four points as a measure of progress and, as we shall see, this yields the usual *relative error measure*

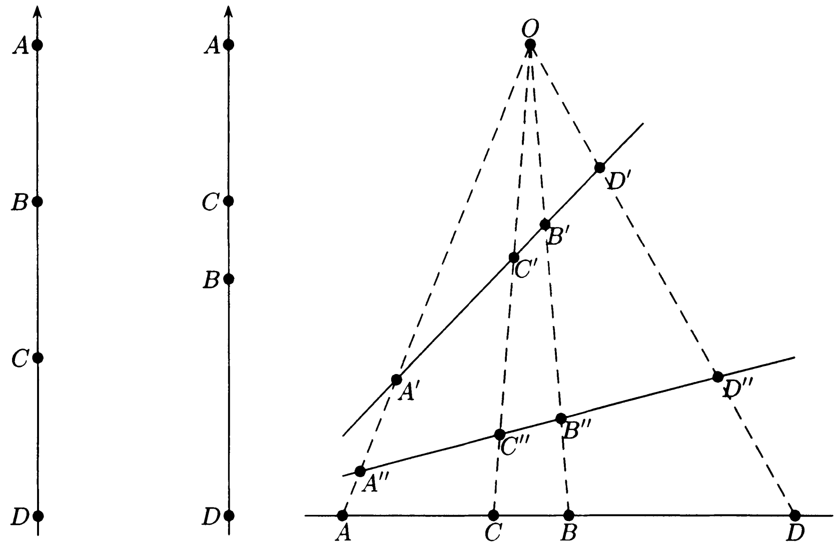

Fig. 8.8. The cross ratio of four points on a line.

since we have deliberately chosen the "improper" hyperplane $y_{n+1} = 0$ to construct the fourth point.

Let us first recall the definition of the cross ratio. In elementary geometry the cross ratio of four points A, B, C, D that lie on a line is the double ratio

$$Dv(A, B; C, D) = \frac{AC}{BC} : \frac{AD}{BD} ,$$

where AC, BC, AD, BD are the "lengths" of the respective line segments with their respective signs according to the relative positioning of the four points; see Figure 8.8. The abbreviation Dv stands for the German *Doppelverhältnis* which translates to cross ratio and as our notation suggests, the first two points A and B are set in relation to the third and the fourth as follows: first we relate A and B to C by a ratio, we then relate A and B to D in the same fashion and finally, we form the double ratio.

"Relative positioning" of the points on the line means that we choose an orientation for the line and give the length of, say, the line segment AC a negative sign if C occurs "before" A on the line. Thus the order of the points matters in the definition of $Dv(A, B; C, D)$ which can be positive or negative. From the definition it follows that $Dv(A, B; D, C) = Dv(A, B; C, D)^{-1}$, i.e. $Dv(A, B; D, C)Dv(A, B; C, D) = 1$. If the four points are such that the two cross ratios are *equal* to each other, then from this equation we get $Dv(A, B; C, D) = Dv(A, B; D, C) = \pm 1$. But $Dv(A, B; C, D) = 1$ implies that $D = C$ (or $A = B$). Thus if we have four distinct points satisfying $Dv(A, B; C, D) = Dv(A, B; D, C)$ then $Dv(A, B; C, D) = -1$. On the other hand, any four points satisfying $Dv(A,B;C,D) = -1$ satisfy $Dv(A,B;D,C) = -1$ as well and in this case the point C (or D) is said to divide the line segment AB *harmonically.* Such points C and D are called *harmonic conjugates* with respect to A and B. Assuming distinctness it follows that $Dv(A,B;C,D) = Dv(B,A;C,D) = -1$ as well for any pair of harmonic conjugates C, D and thus the two couples of points A, B and C, D live in perfect harmony because assuming commutativity, $Dv(A,B;C,D) = Dv(C,D;A,B)$. The second vertical line in Figure 8.8 shows two pairs of harmonic conjugates.

It is a fundamental property of the cross ratio – which was probably known already to the Greek geometers of antiquity – that it is *invariant* under central projections. In the right-hand side part of Figure 8.8 we show a projection with center O and by the invariance of the cross ratio we have $Dv(A,B;C,D) = Dv(A',B';C',D') = Dv(A'',B'';C'',D'')$ which can be proven by an elementary geometric construction. Eminent mathematicians like Jean-Victor Poncelet, whom we have encountered already in Chapter 6.4, August Ferdinand Möbius (1790-1868), Julius Plücker (1801-1868), Edmond Nicolas Laguerre (1834-1886) and others studied many aspects of projective geometry in the early eighteen hundreds and in particular, the properties of the cross ratio which led to a concise concept of "distance" and thus to a *metric* in the projective space of n dimensions.

The intuitive 2-dimensional concept outlined above is generalized as follows to the n-dimensional projective space $\mathcal{P}_n$. Given any four points $\boldsymbol{y}^1$, $\boldsymbol{y}^2$, $\boldsymbol{y}^3$, $\boldsymbol{y}^4$ of $\mathcal{P}_n$ with $\boldsymbol{y}^1 \neq \boldsymbol{y}^4$ and $\boldsymbol{y}^2 \neq \boldsymbol{y}^3$ that lie on a line of $\mathcal{P}_n$ and any two projective hyperplanes $\boldsymbol{d}_3\boldsymbol{y} = 0$ and $\boldsymbol{d}_4\boldsymbol{y} = 0$, say, such that $\boldsymbol{d}_3\boldsymbol{y}^3 = \boldsymbol{d}_4\boldsymbol{y}^4 = 0$ and $\boldsymbol{d}_3\boldsymbol{y}^2 \neq 0 \neq \boldsymbol{d}_4\boldsymbol{y}^1$, the cross ratio of the four points $\boldsymbol{y}^1$, $\boldsymbol{y}^2$, $\boldsymbol{y}^3$, $\boldsymbol{y}^4$ is

$$Dv(\boldsymbol{y}^1,\boldsymbol{y}^2;\boldsymbol{y}^3,\boldsymbol{y}^4) = \frac{(\boldsymbol{d}_3\boldsymbol{y}^1)(\boldsymbol{d}_4\boldsymbol{y}^2)}{(\boldsymbol{d}_3\boldsymbol{y}^2)(\boldsymbol{d}_4\boldsymbol{y}^1)} \; .$$

"Projective hyperplanes" just means that like $\boldsymbol{y}$ the $\boldsymbol{d}_i$ are *nonzero* $(n+1)$-tuples for $i = 3,4$. Like in the elementary case, the cross ratio depends on the order in which we write the points, i.e. $Dv(\boldsymbol{y}^2,\boldsymbol{y}^1;\boldsymbol{y}^3,\boldsymbol{y}^4) \neq Dv(\boldsymbol{y}^1,\boldsymbol{y}^2;\boldsymbol{y}^3,\boldsymbol{y}^4)$, and the "relative positioning of the four points on the line" translates into signs of the quantities $\boldsymbol{d}_i\boldsymbol{y}^k$ used in the definition of Dv. Just like in the affine space $\mathbb{R}^{n+1}$ where we can change the coordinate system, one can introduce different coordinate systems in $\mathcal{P}_n$. Indeed, the homogeneous coordinates that we consider are only a special case of more general *projective coordinate systems* of $\mathcal{P}_n$. Without going into the details of how this is done, a fundamental result of projective geometry states that *the cross ratio is invariant* under changes of the coordinate system for $\mathcal{P}_n$.

You verify that our four points $\boldsymbol{y}(\rho)$, $\boldsymbol{y}^0$, $\boldsymbol{y}(\rho,\tau(\rho))$, $\boldsymbol{y}^\infty_\rho$, see also Figure 8.9, together with the hyperplanes $(\boldsymbol{cD}\ \ ,-z_*)\boldsymbol{y} = 0$ and $y_{n+1} = 0$,

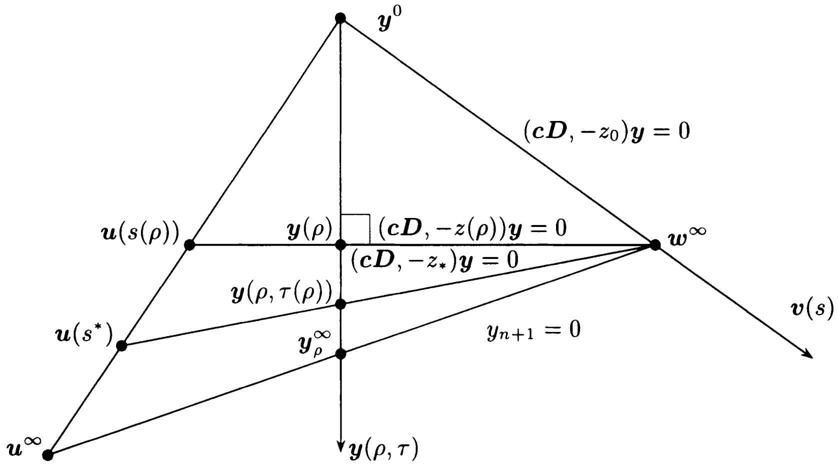

Fig. 8.9. Cross ratios for the problem (FLP$_\rho$) if $\gamma < 0$.

respectively, satisfy the assumptions of the definition of Dv. Calculating the cross ratio in the stated order we find

$$\begin{aligned} Dv(\boldsymbol{y}(\rho), \boldsymbol{y}^0; \boldsymbol{y}(\rho, \tau(\rho)), \boldsymbol{y}_\rho^\infty) &= \frac{(\boldsymbol{cD}, -z_* + z(\rho) - z(\rho))\boldsymbol{y}(\rho)}{(\boldsymbol{cD}, -z_* + z_0 - z_0)\boldsymbol{y}^0} \cdot \frac{(n+1)^{-1}}{y_{n+1}(\rho)} \\ &= \frac{z(\rho) - z_*}{z_0 - z_*}, \end{aligned}$$

which is indeed the relative error as we have indicated. To use the cross ratio for our purpose of estimation, we need a second way of calculating it. We will do this first for the general case of points $\boldsymbol{y}^1$, $\boldsymbol{y}^2$, $\boldsymbol{y}^3$, $\boldsymbol{y}^4$ of $\mathcal{P}_n$ used in the definition. Since $\boldsymbol{y}^1$, $\boldsymbol{y}^2$, $\boldsymbol{y}^3$, $\boldsymbol{y}^4$ lie on a line we can find μ_1, $\mu_2 \in \mathbb{R}$ and λ_1, $\lambda_2 \in \mathbb{R}$ such that $\boldsymbol{y}^3 = \mu_1\boldsymbol{y}^1 + \mu_2\boldsymbol{y}^2$ and $\boldsymbol{y}^4 = \lambda_1\boldsymbol{y}^1 + \lambda_2\boldsymbol{y}^2$. Consequently, from $\boldsymbol{d}_3\boldsymbol{y}^3 = 0$ we get $\mu_1\boldsymbol{d}_3\boldsymbol{y}^1 + \mu_2\boldsymbol{d}_3\boldsymbol{y}^2 = 0$ and likewise, $\lambda_1\boldsymbol{d}_4\boldsymbol{y}^1 + \lambda_2\boldsymbol{d}_4\boldsymbol{y}^2 = 0$. Now $\boldsymbol{y}^1 \neq \boldsymbol{y}^4$, $\boldsymbol{y}^2 \neq \boldsymbol{y}^3$ imply $\lambda_2 \neq 0$ and $\mu_1 \neq 0$ and thus $\boldsymbol{d}_3\boldsymbol{y}^1/\boldsymbol{d}_3\boldsymbol{y}^2 = -\mu_2/\mu_1$ and $\boldsymbol{d}_4\boldsymbol{y}^2/\boldsymbol{d}_4\boldsymbol{y}^1 = -\lambda_1/\lambda_2$. Hence we have

$$Dv(\boldsymbol{y}^1, \boldsymbol{y}^2; \boldsymbol{y}^3, \boldsymbol{y}^4) = \lambda_1\mu_2/\lambda_2\mu_1 .$$

To apply this formula for the cross ratio to our situation denote by τ_ρ^∞ the value of τ such that $\boldsymbol{y}(\rho, \tau) = \boldsymbol{y}_\rho^\infty$. From the definitions of $\tau(\rho)$ and of τ_ρ^∞ it now follows that

$$Dv(\boldsymbol{y}(\rho), \boldsymbol{y}^0; \boldsymbol{y}(\rho, \tau(\rho)), \boldsymbol{y}_\rho^\infty) = \frac{z(\rho) - z_*}{z_0 - z_*} = \frac{1 - \rho/\tau(\rho)}{1 - \rho/\tau_\rho^\infty}, \tag{8.40}$$

and the question is simply whether or not we can estimate the last expression appropriately. In Figure 8.9 we have illustrated the case where $\gamma = \boldsymbol{p}^T\boldsymbol{d} < 0$. To get a different way of estimating the progress of the objective function value of (LP) we can also use e.g. four points on the line $\boldsymbol{u}(s)$. In Figure 8.9 we have indicated four such points and we leave it as an exercise to verify that $Dv(\boldsymbol{u}(s(\rho)), \boldsymbol{y}^0; \boldsymbol{u}(s^*), \boldsymbol{u}^\infty) = (z(\rho) - z_*)/(z_0 - z_*)$ as well.

Remember now from Chapter 8.2 that $(\boldsymbol{cD}, -z)\boldsymbol{w}^\infty = 0$ for all $z \in \mathbb{R}$. The line $\boldsymbol{y}(\rho, \tau)$ intersects the hyperplane $(\boldsymbol{cD}, -z)\boldsymbol{y} = 0$ somewhere in the plane L_{uw} and clearly, the point of intersection lies on the line segment between $\boldsymbol{y}^0$ and $\boldsymbol{y}^\infty_\rho$ for all $z_0 \geq z > -\infty$ if $\gamma < 0$. Indeed, taking the limit $z \to -\infty$ the hyperplane $(\boldsymbol{cD}, -z)\boldsymbol{y} = 0$ becomes the hyperplane $y_{n+1} = 0$ since $(\boldsymbol{cD}, -z)\boldsymbol{y} = 0 = ((1/z)\boldsymbol{cD}, -1)\boldsymbol{y}$ for all $z \neq 0$ and one reasons likewise when $\gamma \geq 0$; see also Figure 8.6. In case that (LP) has an unbounded optimum the points $\boldsymbol{y}(\rho, \tau(\rho))$ and $\boldsymbol{y}^\infty_\rho$ thus coincide and the cross ratio (8.40) is still defined as we have not assumed that $\boldsymbol{y}^3 \neq \boldsymbol{y}^4$ in the definition of it and its value equals one. So in the following we will always work with some finite lower bound – *fictive or real* – on the optimum objective function value of (LP) when we employ the cross ratio.

8.4 Reflection on a Circle and Sandwiching

Γνῶθι σ' αὐτόν!
Thales of Militos (643-548 B.C.)

Surely you have heard "Know thyself!" – which is what Thales of Militos recommended his fellow human beings already some 2, 500 years ago. Like Kleoboulos of Lindos, Thales was considered a great man in his time – you probably know his name from trigonometry – and by later generations, as one of the seven sages of Greece. Stretching Thales' advice a bit, to start knowing yourself you might begin by looking at your own reflection in a mirror. But beware of its trappings: the mythological Narkissos apparently was so enamoured by his own reflection in a fountain that he could not get away from it. He just languished – admiring his own reflection – and starved himself slowly to death. To avoid this from happening to us we will avoid flat mirrors and look into a "circular" mirror instead.

The analysis of the problem (FLP_ρ) settles the existence and uniqueness of its solution and tells us how to move from a given solution $\boldsymbol{x}^0$ to a new solution $\boldsymbol{x}^1$, say, such that $z_1 = \boldsymbol{cx}^1 < \boldsymbol{cx}^0 = z_0$ provided that $\boldsymbol{x}^0$ is nonoptimal. The cross ratio tells us how to measure the progress that we make towards the solution of (LP) by solving the problem (FLP_ρ) for some $\rho \geq 0$. For the basic algorithm of Chapter 8.1 we assumed that the optimum value of (LP) is known and equals zero. Clearly, we wish to remove this assumption, i.e. we want to solve any linear program (LP) with a nonempty relative interior. Fortunately, using the material of Chapter 7.4, we can *always* find *some* finite lower bound on the optimum objective function value of (LP) provided it is finite or at least give a finite bound on it that *if exceeded* lets us conclude that the linear program (LP) is unbounded. In what follows we construct, starting from suitable *initial* upper and lower bounds, a sequence of lower bounds $v^1, v^2, \ldots$ *concurrently* with the upper bounds $z_1, z_2, \ldots$ that permit us to conclude faster polynomial step complexity of a substantially different realization of the basic algorithmic idea for the solution of linear programs having a nonempty relative interior.

From the analysis of Chapter 8.2 it follows also that the optimal solution $\boldsymbol{y}(\rho)$ to (FLP$_\rho$) exists at least for all

$$0 \leq \rho < \rho_\infty = \sqrt{\|\boldsymbol{d}\|^{-2} - (n+1)^{-1}}\ .$$

Thus if the orthoprojection $\boldsymbol{d}$ of $\boldsymbol{e}$ onto the subspace (8.5) has a "short" Euclidean length, then ρ_∞ becomes "big" and thus $\boldsymbol{y}(\rho)$ as defined in (8.21) or (8.32) is an optimal solution to (FLP$_\rho$) for $\rho \gg r = 1/\sqrt{n(n+1)}$. The length of $\boldsymbol{d}$ depends, of course, on the "initial" interior point $\boldsymbol{x}^0 \in \mathcal{X}$ and thus we assume *initially* that $\boldsymbol{x}^0$ is chosen such that all assumptions are met.

A technique – which dates to antiquity – to perform geometrical constructions is the *inversion of a point in a circle*, or the reflection on a circle for short, and it goes as follows.

Given a circle with center O and radius ρ, say, take any point $P \neq O$ in the circle and construct a point P' on the line determined by O and P by requiring that $OP \cdot OP' = \rho^2$ where OP and OP' are the Euclidean lengths of the corresponding line segments. To every point inside the circle there corresponds exactly one point outside of it and it is not difficult to see that the reverse holds as well. The mapping is thus bi-unique and to complete it, let us think of the center as being mapped into some (unique) "point at infinity". Let us now apply this construction to B_ρ^{n+1}. By the prescription

$$\boldsymbol{y}^{\text{inv}} = \boldsymbol{y}^0 + \frac{\rho^2}{\|\boldsymbol{y} - \boldsymbol{y}^0\|^2}(\boldsymbol{y} - \boldsymbol{y}^0)$$

we define for every point $\boldsymbol{y} \neq \boldsymbol{y}^0$ inside (outside) of B_ρ^{n+1} its "inverse" point $\boldsymbol{y}^{\text{inv}}$ outside (inside) of B_ρ^{n+1}. It follows that for $\rho = 1/\sqrt{n+1}$ the set B_r^{n+1} is mapped into the set

$$\left\{ \boldsymbol{y} \in \mathbb{R}^{n+1} : \sum_{j=1}^{n+1} y_j = 1,\ \sum_{j=1}^{n+1} (y_j - \frac{1}{n+1})^2 \geq \frac{n}{n+1} \right\}$$

and vice versa. Remember that $r = 1/\sqrt{n(n+1)}$ is the largest radius of a ball that can be inscribed into S^{n+1}, whereas $\sqrt{n/(n+1)}$ is the smallest radius of a ball that can be circumscribed to S^{n+1}; see Exercise 8.1 and the left part of Figure 8.10 below where $\rho = (n+1)^{-1/2}$, $R = \sqrt{n/(n+1)}$ and reflection on the circle with radius ρ is illustrated for $n = 2$.

This "inversion" or reflection interchanges thus the subset of $\mathbb{R}^{n+1}$ where positivity of the solution $\boldsymbol{y}(\rho)$ to (FLP$_\rho$) can be guaranteed with an "outside" where the positivity of $\boldsymbol{y}(\rho)$ is certain to be lost on some component of $\boldsymbol{y}(\rho)$.

For every radius ρ with $0 < \rho < r$ we get by reflection on the boundary of the ball with radius $(n+1)^{-1/2}$ a "twin" problem (FLP$_\sigma$) to the problem (FLP$_\rho$) where $\sigma = 1/(n+1)\rho > \sqrt{n/(n+1)}$. (FLP$_\sigma$) can be thought of as a *relaxation* of the problem (FLP) of the introduction and thus of the original (LP) – provided that the solution to (FLP$_\sigma$) *exists.* But if ρ is not "too small", then σ is not "too big" and – as we know from Remark 8.6 – a solution to (FLP$_\sigma$) may very well exist in this case.

Remark 8.7 *Let $\boldsymbol{x}^0 \in \mathcal{X}$ satisfying $\boldsymbol{x}^0 > \boldsymbol{0}$ and $z_0 = \boldsymbol{c}\boldsymbol{x}^0$ be a nonoptimal solution to* (LP) *such that for some $\sigma > \sqrt{n/(n+1)}$ the optimal solution $\boldsymbol{y}(\sigma)$ to the problem (FLP$_\sigma$) exists and denote its objective function value by $z(\sigma)$. Then $z(\sigma) < z_0$ and there exists an "interior" solution to the dual of the linear program* (LP*) *of the introduction to this chapter with objective function value equal to $z(\sigma)$, i.e. $z(\sigma)$ is a lower bound on the optimal objective function value of* (LP).

Proof. Let $\boldsymbol{y}(\sigma) = \boldsymbol{y}^0 + \boldsymbol{g}(\sigma)$ be the optimal solution to (FLP$_\sigma$) with objective function value $z(\sigma)$ where $\boldsymbol{g}(\sigma)$ is defined e.g. in (8.32) or (8.33) and thus $\|\boldsymbol{g}(\sigma)\| = \sigma$. From the nonoptimality of $\boldsymbol{x}^0$ we have $z(\sigma) < z_0$ by Remark 8.6. Inverting $\boldsymbol{y}(\sigma)$ on the ball with radius $1/\sqrt{n+1}$ and defining $\rho = 1/(n+1)\sigma$ we get the point

$$\boldsymbol{y}(\sigma,\rho) = \boldsymbol{y}^0 + (\rho/\sigma)\boldsymbol{g}(\sigma) \in T_0(\mathcal{X}) \cap B_\rho^{n+1} , \tag{8.41}$$

which lies on the line connecting $\boldsymbol{y}^0$ to $\boldsymbol{y}(\sigma)$ and satisfies $\|\boldsymbol{y}(\sigma,\rho)\| = \rho$ and $\boldsymbol{y}(\sigma,\rho) > 0$ since $0 < \rho < r$. Consequently, the point $\boldsymbol{y}^-(\sigma,\rho)$ given by

$$\boldsymbol{y}^-(\sigma,\rho) = \boldsymbol{y}^0 - (\rho/\sigma)\boldsymbol{g}(\sigma) > \boldsymbol{0} , \tag{8.42}$$

i.e. it is positive because $\boldsymbol{y}^-(\sigma,\rho) \in T_0(\mathcal{X}) \cap B_\rho^{n+1}$. Consider the subspace

$$N_{AD} = \{\boldsymbol{y} \in \mathbb{R}^{n+1} : (\boldsymbol{AD}, -\boldsymbol{b})\boldsymbol{y} = \boldsymbol{0}\} .$$

Calculating the orthogonal projection $\boldsymbol{Q}_{AD}$ on N_{AD} we get

$$\boldsymbol{Q}_{AD} = \boldsymbol{I}_{n+1} - \begin{pmatrix} \boldsymbol{D}\boldsymbol{A}^T \\ -\boldsymbol{b}^T \end{pmatrix} (\boldsymbol{A}\boldsymbol{D}^2\boldsymbol{A}^T + \boldsymbol{b}\boldsymbol{b}^T)^{-1}(\boldsymbol{AD}, -\boldsymbol{b}) , \tag{8.43}$$

see Remark 8.1, and thus $\boldsymbol{Q}_{AD} = \boldsymbol{Q} + (n+1)^{-1}\boldsymbol{f}\boldsymbol{f}^T$ where $\boldsymbol{Q}$ is the orthogonal projection on the subspace (8.3) of Chapter 8.1.1. It follows from (8.22) that the projection $\boldsymbol{q}(\sigma)$ of $(\boldsymbol{cD}, -z(\sigma))^T$ on the subspace (8.3) satisfies

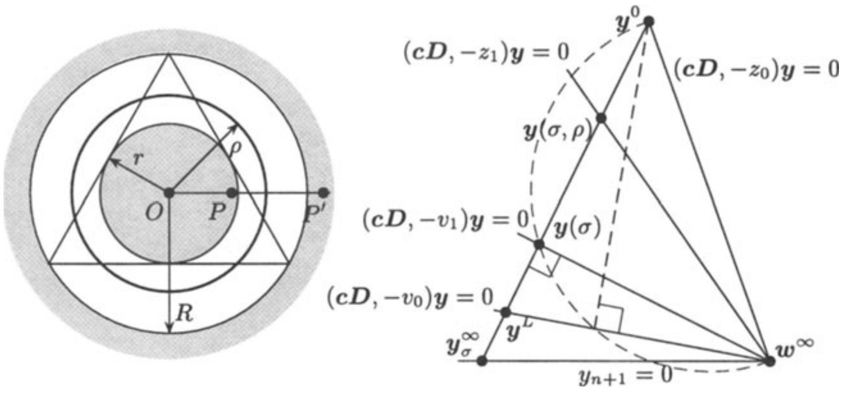

Fig. 8.10. Reflection on a circle and cross ratios for sandwiching if $\gamma < 0$.

$$q(\sigma) = Q_{AD}\begin{pmatrix} Dc^T \\ -z(\sigma) \end{pmatrix} - (n+1)^{-1}(z_0 - z(\sigma))\begin{pmatrix} e \\ 1 \end{pmatrix} .$$

From the formula for Q_{AD} we get $Q_{AD}(cD, -z(\sigma))^T = (cD, -z(\sigma))^T - (\mu^T AD, -\mu^T b)^T$ where $\mu = (AD^2A^T + bb^T)^{-1}(AD^2c^T + z(\sigma)b) \in \mathbb{R}^m$. Consequently we have established that

$$\begin{pmatrix} Dc^T \\ -z(\sigma) \end{pmatrix} - \begin{pmatrix} DA^T \\ -b^T \end{pmatrix}\mu = q(\sigma) + (z_0 - z(\sigma))y^0 \quad \text{for some } \mu \in \mathbb{R}^m . \tag{8.44}$$

From the expression (8.38) for the optimal objective function value $z(\sigma)$ we know that $z(\sigma) = z_0 - (n+1)\sigma\|q(\sigma)\|$. Hence we get by dividing and multiplying $q(\sigma)$ by $\|q(\sigma)\|$ from relation (8.33) with ρ replaced by σ that

$$\begin{pmatrix} D(c^T - A^T\mu) \\ -z(\sigma) + b^T\mu \end{pmatrix} = (z_0 - z(\sigma))\left(y^0 - \frac{\rho}{\sigma}g(\sigma)\right) > \mathbf{0} , \tag{8.45}$$

where we have used (8.42). Since $x^0 > \mathbf{0}$ and $D = diag(x_1^0, \ldots, x_n^0)$ it follows that $c^T - A^T\mu > \mathbf{0}$ and $-z(\sigma) + b^T\mu > 0$. Consequently, $(\mu, z(\sigma))$ is a feasible *interior* solution to the dual of (LP*) which is the linear program $\max\{\mu_0 : \mu^T A \le c,\ -\mu^T b + \mu_0 \le 0\}$. ■

Exercise 8.7 *Let $x^0 \in \mathcal{X}$, $x^0 > 0$ be nonoptimal and $c \in \mathbb{R}^n$ be such that $\gamma = p^T d > 0$ and $\|w\| > \rho_* = \sqrt{n/(n+1)}$ where w is defined in (8.30). Show that $cx \ge z_0 - (1+\beta)\|u\|^2/\gamma$ for all $x \in \mathcal{X}$.* (Hint: *Use the construction of the proof of Remark 8.7 e.g. with $\sigma = (\|w\| + \rho_*)/2$ and $\rho = 1/\sigma(n+1)$.*)

The following remark shows that by inverting on the ball with radius $1/\sqrt{n+1}$ we get a sharper estimation of the relative error than for the basic algorithm. It spells out most of the assumptions on the radii ρ and σ that are needed for an iterative application.

Remark 8.8 *Let $x^0 \in \mathcal{X}$, $x^0 > 0$ and $z_0 = cx^0$, be such that for some $\omega \geq r$ the problem* (FLP$_\omega$) *has an optimal solution with objective function value $z(\omega) = v_0 < z_0$. Then for any ρ and σ satisfying $0 < \rho < r < \sigma < \omega$ and $\rho\sigma(n+1) = 1$ the following statements are correct: (i) The optimal objective function value $v_1 = z(\sigma)$ of* (FLP$_\sigma$) *is a lower bound for* (LP), *$v_1 > v_0$ and thus v_0 is a lower bound for* (LP) *as well. (ii) $x^1 = T_0^{-1}(y(\sigma,\rho))$ satisfies $x^1 > 0$, $x^1 \in \mathcal{X}$ and $z_1 = cx^1 < z_0$ where $y(\sigma,\rho)$ is given by (8.41). (iii) If in addition to the above $(n+1)\rho \leq 1$ and $\sigma^2 \leq \omega$, then setting $\alpha = (n+1)\rho$ it follows that*

$$\frac{z_1 - v_1}{z_0 - v_0} \leq \left(1 + \frac{\alpha(1-\alpha)}{\sqrt{n+1}}\right)^{-1} . \tag{8.46}$$

Proof. By Remark 8.6 (ii) we have $z(\sigma) > z(\omega)$ since $\sigma < \omega$ and thus part (i) follows from Remark 8.7. From (8.41) we have $y(\sigma,\rho) > 0$ and thus $x^1 > 0$ and $x^1 \in \mathcal{X}$. Computing $z_1 = cx^1$ we get for *fixed* σ and *any* ρ such that $y_{n+1}(\sigma,\rho) \neq 0$

$$\begin{aligned} z_1 &= \frac{(cD, -z(\sigma) + z(\sigma))y(\sigma,\rho)}{y_{n+1}(\sigma,\rho)} \\ &= z(\sigma) + \frac{1-\rho/\sigma}{(n+1)y_{n+1}(\sigma,\rho)}(z_0 - z(\sigma)) \end{aligned} \tag{8.47}$$

and thus the derivative $dz_1/d\rho = -y_{n+1}(\sigma)\|q(\sigma)\|/y_{n+1}^2(\sigma,\rho)$, where we have used (8.38). Since by Remark 8.6 (ii) $y_{n+1}(\sigma) > 0$ it follows that $dz_1/d\rho < 0$ and thus $v_1 = z(\sigma) < z_0$ and $z_1 < z_0$ since $0 < \rho < \sigma$. So part (ii) follows. To prove (iii) we write

$$\frac{z_1 - v_1}{z_0 - v_0} = \frac{z_0 - v_1}{z_0 - v_0} \cdot \frac{z_1 - v_1}{z_0 - v_1}$$

and denote by $y(\sigma,\tau)$ the line (8.39) through $y(\sigma)$. From (8.47) we know

$$\frac{z_1 - v_1}{z_0 - v_1} = \frac{1-\rho/\sigma}{(n+1)y_{n+1}(\sigma,\rho)} ,$$

which is the cross ratio $Dv(y(\sigma,\rho), y^0; y(\sigma), y_\sigma^\infty)$. $y_\sigma^\infty = y^0 + (\kappa/\sigma)g(\sigma)$ for some $\kappa \in \mathbb{R}$ is the improper point on the line $y(\sigma,\tau)$. Denote by $y^L = y^0 + (\lambda/\sigma)g(\sigma)$ for some $\lambda > 0$ the point of intersection of the line $y(\sigma,\tau)$ with the hyperplane $(cD\,, -v_0)y = 0$. We calculate

$$\frac{z_0 - v_1}{z_0 - v_0} = Dv(y(\sigma), y^L; y^0, y_\sigma^\infty) = \frac{\sigma(1-\lambda/\kappa)}{\lambda(1-\sigma/\kappa)} .$$

Since the $(n+1)$-st component of y_σ^∞ equals zero we get from $(n+1)^{-1} + (\kappa/\sigma)g_{n+1}(\sigma) = 0$ that $(n+1)\kappa q = -1$ where we have set $q = g_{n+1}(\sigma)/\sigma$ and $g_{n+1}(\sigma)$ is the $(n+1)$-st component of $g(\sigma)$. Thus $1 - \sigma/\kappa = 1 + \sigma(n+1)q =$

$(n+1)y_{n+1}(\sigma) > 0$. Since $v_0 < v_1 < z_0$ it follows that $\sigma(1-\lambda/\kappa) > 0$. From $(\boldsymbol{cD}, -v_0)\boldsymbol{y}^L = 0$ and $\sigma < \omega$ we have $\|\boldsymbol{y}^L - \boldsymbol{y}^0\| = \lambda > \omega$ and thus we get $(\sigma/\lambda)(1-\lambda/\kappa) = \sigma/\lambda + \sigma(n+1)q \le (n+1)\rho + \sigma(n+1)q$ because by assumption $\sigma^2 \le \omega$ and $(n+1)\rho\sigma = 1$. Combining the estimations we get

$$\frac{z_1 - v_1}{z_0 - v_0} \le \frac{(n+1)\rho + \sigma(n+1)q}{1+\sigma(n+1)q} \cdot \frac{1-\rho/\sigma}{1+\rho(n+1)q}\,.$$

The second term of the product is less than one since $z_1 < z_0$ by part (ii). The first term is less than or equal to one since by assumption $(n+1)\rho \le 1$. Thus it suffices to estimate each term individually. Suppose now that $q\sqrt{n+1} \ge 1-(n+1)\rho$. Then we estimate

$$\frac{1-\rho/\sigma}{1+(n+1)\rho q} < (1+(n+1)\rho q)^{-1} \le \left(1 + \frac{(1-(n+1)\rho)(n+1)\rho}{\sqrt{n+1}}\right)^{-1}$$

and thus setting $\alpha = (n+1)\rho$ formula (8.46) follows. If $q\sqrt{n+1} < 1-(n+1)\rho$ then from $(n+1)\sigma\rho = 1$ we get $(n+1)\sigma q < (1-(n+1)\rho)/(\rho\sqrt{n+1})$ and consequently we can estimate

$$\begin{aligned}
\frac{(n+1)(\rho+\sigma q)}{1+\sigma(n+1)q} &= \left(1 + \frac{1-(n+1)\rho}{(n+1)(\rho+\sigma q)}\right)^{-1} \\
&< \left(1 + \frac{1-(n+1)\rho}{(n+1)\rho + \dfrac{1-(n+1)\rho}{\rho\sqrt{n+1}}}\right)^{-1} \\
&< \left(1 + \frac{(1-(n+1)\rho)(n+1)\rho}{\sqrt{n+1}}\right)^{-1}.
\end{aligned}$$

Hence with $\alpha = (n+1)\rho$ we have formula (8.46) in both cases. ■

Exercise 8.8 *Show that for fixed $\rho \in (0, \rho_\infty)$ the pre-image $\boldsymbol{x}(\rho,\tau)$ of the line $\boldsymbol{y}(\rho,\tau)$ of (8.39) under the projective transformation T_0 defines a direction of descent for (LP), i.e. that $\boldsymbol{x}(\rho,\tau) \in relint\mathcal{X}$ for $0 \le \tau \le \tau(\rho)$ where $\tau(\rho) > 0$ and that the objective function $\boldsymbol{cx}(\rho,\tau)$ decreases strictly for $\tau \ge 0$.*

8.4.1 The Iterative Step

Suppose now that radii ρ, σ and ω with the required properties exist *initially* – how to get there we will see later. Comparing (8.46) to the estimation we made for the basic algorithm, see e.g. (8.8) where the right-hand side equals $1-\alpha/n$, we see that we now get a decrease in the error measure that depends upon $\sqrt{n+1}$. Note that the parameterization $\alpha = \rho\sqrt{n(n+1)}$ of Chapter 8.1 is different from the parameterization $\alpha = (n+1)\rho$ we use in this section. This change does, however, not affect the order of magnitude of the dependence on n. It follows that the iterative application of Remark 8.8 – if that is possible – produces a far better *asymptotic* behavior, i.e. more rapid convergence for large n, than the one we proved for the basic algorithm.

To apply Remark 8.8 iteratively we have to prove e.g. that the initially selected radii ρ and σ continue to satisfy the various assumptions of Remark 8.8 for some suitable radius ω *after* we have moved to a new point. If radii ρ and σ that do not depend upon the "current" solution to (LP) exist, then we get new upper and lower bounds such that (8.46) remains correct.

So like in Remark 8.8 let $\boldsymbol{x}^1$ be the point obtained from (8.41) under the projective transformation, $z_1 = \boldsymbol{c}\boldsymbol{x}^1$ its objective function value, $v_1 = z(\sigma) < z_1$ be the current lower bound and $\boldsymbol{D}_1 = diag(x_1^1, \dots, x_n^1)$ the diagonal matrix of the "next" transformation T_1, say, that maps $\boldsymbol{x}^1$ into the center of S^{n+1}. Denote by $\boldsymbol{q}^1(v_1)$ the orthogonal projection of $(\boldsymbol{c}\boldsymbol{D}_1, -v_1)^T$ on the subspace that we get if $\boldsymbol{D}$ is replaced by $\boldsymbol{D}_1$ in the definition (8.3).

Claim 1 *If $\boldsymbol{x}^1$ is a nonoptimal solution to* (LP), *then $\boldsymbol{q}^1(v_1) \neq \boldsymbol{0}$.*

Proof. If $\boldsymbol{q}^1(v_1) = \boldsymbol{0}$ then by the properties of orthogonal projections there exists $\boldsymbol{\lambda} \in \mathbb{R}^{m+1}$ such that

$$(\boldsymbol{c}\boldsymbol{D}_1, -v_1) = \boldsymbol{\lambda} \begin{pmatrix} \boldsymbol{A}\boldsymbol{D}_1 & -\boldsymbol{b} \\ \boldsymbol{e}^T & 1 \end{pmatrix}$$

and thus $(\boldsymbol{c}\boldsymbol{D}_1, -v_1)\boldsymbol{y} = (z_1 - v_1)/(n+1)$ for all

$$\boldsymbol{y} \in \mathcal{Y} = \{\boldsymbol{y} \in \mathbb{R}^{n+1} : (\boldsymbol{A}\boldsymbol{D}_1, -\boldsymbol{b})\boldsymbol{y} = 0,\ \boldsymbol{f}^T\boldsymbol{y} = 1\}$$

because $\boldsymbol{y}^0 \in \mathcal{Y}$. Now let $\widehat{\boldsymbol{w}}^\infty$ be the distinguished point that corresponds to the point $\boldsymbol{w}^\infty$ of Chapter 8.2, but in the changed coordinates. Then we have $(\boldsymbol{c}\boldsymbol{D}_1, -v_1)\widehat{\boldsymbol{w}}^\infty = 0$ and since $\widehat{\boldsymbol{w}}^\infty \in \mathcal{Y}$ we get $z_1 = v_1$ if $\|\widehat{\boldsymbol{w}}^\infty\| < \infty$, which is a contradiction. If $\|\widehat{\boldsymbol{w}}^\infty\| = \infty$ then the vector $\boldsymbol{q}^1(v_1)$ is *up to scaling* the vector $\widehat{\boldsymbol{u}}$ that corresponds to the vector $\boldsymbol{u}$ of Chapter 8.2 in the changed coordinates, see also the definition of the plane L_{uw} in that section. Consequently, unless $\boldsymbol{x}^1$ is an optimal solution to (LP) we get $\boldsymbol{q}^1(v_1) \neq \boldsymbol{0}$ in this case as well, since $\widehat{\boldsymbol{u}} \neq \boldsymbol{0}$. ∎

Let $\boldsymbol{x}^1$ be a nonoptimal solution for (LP), then by Claim 1 $\boldsymbol{q}^1(v_1) \neq \boldsymbol{0}$ and thus the line $\widehat{\boldsymbol{y}}(\tau) = \boldsymbol{y}^0 - \tau\boldsymbol{q}^1(v_1)/\|\boldsymbol{q}^1(v_1)\|$ is well defined and intersects the hyperplane $(\boldsymbol{c}\boldsymbol{D}_1, -v_1)\boldsymbol{y} = 0$ for some $\tau = \omega_1$, say, where

$$\omega_1 = (z_1 - v_1)/(n+1)\|\boldsymbol{q}^1(v_1)\| > 0 \tag{8.48}$$

since $z_1 - v_1 > 0$. Now we have a radius ω_1 corresponding to the radius ω of Remark 8.8. Denote by $(\mathrm{FLP}^1_{\omega_1})$ the nonlinear program that we get under the transformation T_1 using $\boldsymbol{D}_1$ rather than $\boldsymbol{D}$. Like in Chapter 8.2 we get a two-dimensional plane $L_{\widehat{u}\widehat{w}}$, say, where $\widehat{\boldsymbol{u}}$ corresponds to the vector $\boldsymbol{u}$ and $\widehat{\boldsymbol{w}}$ to the vector $\boldsymbol{w}$ of Chapter 8.2. The line $\widehat{\boldsymbol{y}}(\tau)$ then satisfies $\widehat{\boldsymbol{y}}(\tau) \in L_{\widehat{u}\widehat{w}}$ for all $\tau \geq 0$ and corresponds to the "broken" line of Figure 8.10.

Claim 2 *If* $\boldsymbol{q}^1(v_1) \neq \mathbf{0}$, *then* (FLP$^1_{\omega_1}$) *has a finite optimal solution.*

Proof. We first prove that $\omega_1 = \|\widehat{\boldsymbol{y}}(\omega_1) - \boldsymbol{y}^0\| \leq \|\widehat{\boldsymbol{w}}^\infty - \boldsymbol{y}^0\| = \|\widehat{\boldsymbol{w}}\|$. This follows because $(\boldsymbol{c}\boldsymbol{D}_1, -z)\widehat{\boldsymbol{w}}^\infty = 0$ for *all* $z \in \mathbb{R}$ and the *perpendicular* from $\boldsymbol{y}^0$ on the line formed by the intersection of $(\boldsymbol{c}\boldsymbol{D}_1, -v_1)\boldsymbol{y} = 0$ and the plane $L_{\widehat{u}\widehat{w}}$ is necessarily shorter than $\|\widehat{\boldsymbol{w}}\|$. So we calculate using (8.48) and the notation and formulas of Chapter 8.2

$$(\widehat{\boldsymbol{y}}(\omega_1) - \boldsymbol{y}^0)^T(\widehat{\boldsymbol{y}}(\omega_1) - \widehat{\boldsymbol{w}}^\infty) = \omega_1^2\left(1 + [(n+1)/(z_1 - v_1)](\boldsymbol{q}^1(v_1))^T\widehat{\boldsymbol{w}}\right) = 0 ,$$

where we have used in particular (8.31). Thus $\widehat{\boldsymbol{y}}(\omega_1)$ is the perpendicular point in question. According to the analysis of Chapter 8.2 we have to distinguish two cases depending upon the sign of $\widehat{\gamma}$, say, which corresponds to the quantity γ of that section. If $\widehat{\gamma} \leq 0$, then we get in the correspondingly changed quantities of Chapter 8.2 that $\omega_1 < \widehat{\rho}_\infty = \|\widehat{\boldsymbol{y}}^\infty - \boldsymbol{y}^0\|$ since $v_1 < z_1$ is a *finite* real number and the claim follows. If $\widehat{\gamma} > 0$ then by Remark 8.6 all we need is $\omega_1 \leq \|\widehat{\boldsymbol{w}}\|$ which we have proven already. ∎

It follows from Claim 2 that the problem (FLP$^1_\rho$) has an optimal solution for $\rho = \omega_1$ with an objective function value of $z^1(\omega_1)$, say, and unless $\boldsymbol{x}^1$ is an optimal solution to (LP) then we have necessarily that $z^1(\omega_1) < z_1$ as required for Remark 8.8. Note that $z^1(\omega_1) = v_1$ if $\widehat{\gamma} \leq 0$, but it is entirely *possible* that $z^1(\omega_1) > v_1$ if $\widehat{\gamma} > 0$. This latter possibility does, however, not change the validity of the argument used to prove Remark 8.8 since we conclude that $z^1(\omega_1)$ is a lower bound for (LP) if the assumptions of Remark 8.8 are met. The relative error estimation remains correct in this case, too.

So if $\boldsymbol{q}^1(v_1) \neq \mathbf{0}$, then the nonlinear program (FLP$^1_{\omega_1}$) that we get at the new point $\boldsymbol{x}^1 \in \mathcal{X}$ has a finite optimal solution. To prove that, for a suitable initial choice, the "original" radii ρ and σ of Remark 8.8 can be applied again we need to estimate the length of the vector $\boldsymbol{q}^1(v_1)$.

To facilitate the exposition we will use the *Hadamard product* or *H-product*, for short, so named after Jacques Salomon Hadamard whom we have encountered in Chapter 7.1 already.

For any two column vectors $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{y} \in \mathbb{R}^n$ let

$$\boldsymbol{x} * \boldsymbol{y} = (x_1y_1, \dots, x_ny_n)^T ,$$

i.e. the H-product $\boldsymbol{x} * \boldsymbol{y}$ is the column vector of length n defined by the products of each pair of components with the same index. Note that for $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z} \in \mathbb{R}^n$ the expressions $\boldsymbol{x} * \boldsymbol{y} * \boldsymbol{z} = (\boldsymbol{x} * \boldsymbol{y}) * \boldsymbol{z} = \boldsymbol{z} * \boldsymbol{x} * \boldsymbol{y}$ etc. are all equal, i.e. the order in which we multiply does not matter. Moreover, we have distributivity, i.e. $(\boldsymbol{x} + \boldsymbol{y}) * \boldsymbol{z} = \boldsymbol{x} * \boldsymbol{z} + \boldsymbol{y} * \boldsymbol{z}$ etc.

Claim 3 *If* $0 < (n+1)\rho < 1/\sqrt{2}$, $\sigma\rho(n+1) = 1$ *and* $\sigma^2 \le \omega$, *then*

$$\|\boldsymbol{q}^1(v_1)\| < (z_1 - v_1)(n+1)\rho^2$$

for all $0 < (n+1)\rho < \varepsilon_n$, *where*

$$\varepsilon_n = \sqrt{(n+1)(1 - \sqrt{n/(n+1)})}\ .$$

Moreover, $\varepsilon_n \ge \frac{1}{\sqrt{2}}$ *for all* $n \ge 1$, $\lim_{n\to\infty} \varepsilon_n = \frac{1}{\sqrt{2}}$ *and* $\sqrt{2} < \sigma < \sigma^2 \le \omega_1$.

Proof. Using the H-product we write the point $\boldsymbol{x}^1$ that we get under T_0^{-1} from $\boldsymbol{y}(\sigma,\rho)$ as

$$\begin{pmatrix} \boldsymbol{x}^1 \\ 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}^0 \\ 1 \end{pmatrix} * \frac{\boldsymbol{y}(\sigma,\rho)}{y_{n+1}(\sigma,\rho)}\ . \tag{8.49}$$

Since $0 < (n+1)\rho < 1/\sqrt{2} = 0.707\ldots < 1$ we conclude like in (8.44) when applied to $\boldsymbol{q}^1(v_1)$ that there exists $\boldsymbol{\mu}' \in \mathbb{R}^m$ such that

$$\begin{pmatrix} \boldsymbol{D}_1\boldsymbol{c}^T - \boldsymbol{D}_1\boldsymbol{A}^T\boldsymbol{\mu}' \\ -v_1 + \boldsymbol{b}^T\boldsymbol{\mu}' \end{pmatrix} = \boldsymbol{q}^1(v_1) + (z_1 - v_1)\boldsymbol{y}^0\ .$$

Now setting $\boldsymbol{\mu}'' = \boldsymbol{\mu} - \boldsymbol{\mu}'$ where $\boldsymbol{\mu} \in \mathbb{R}^m$ is such that (8.45) holds we get using $v_1 = z(\sigma)$ the following identities:

$$\begin{aligned}
\boldsymbol{q}^1(v_1) - \begin{pmatrix} \boldsymbol{D}_1\boldsymbol{A}^T\boldsymbol{\mu}'' \\ -\boldsymbol{b}^T\boldsymbol{\mu}'' \end{pmatrix} &= \begin{pmatrix} \boldsymbol{D}_1(\boldsymbol{c}^T - \boldsymbol{A}^T\boldsymbol{\mu}) \\ -z(\sigma) + \boldsymbol{b}^T\boldsymbol{\mu} \end{pmatrix} - (z_1 - v_1)\boldsymbol{y}^0 \\
&= \begin{pmatrix} \boldsymbol{x}^1 \\ 1 \end{pmatrix} * \begin{pmatrix} \boldsymbol{c}^T - \boldsymbol{A}^T\boldsymbol{\mu} \\ -z(\sigma) + \boldsymbol{b}^T\boldsymbol{\mu} \end{pmatrix} - (z_1 - v_1)\boldsymbol{y}^0 && \text{by the definition of the H-product,} \\
&= \begin{pmatrix} \boldsymbol{x}^0 \\ 1 \end{pmatrix} * \frac{\boldsymbol{y}(\sigma,\rho)}{y_{n+1}(\sigma,\rho)} * \begin{pmatrix} \boldsymbol{c}^T - \boldsymbol{A}^T\boldsymbol{\mu} \\ -z(\sigma) + \boldsymbol{b}^T\boldsymbol{\mu} \end{pmatrix} - (z_1 - v_1)\boldsymbol{y}^0 && \text{using (8.49),} \\
&= \frac{\boldsymbol{y}(\sigma,\rho)}{y_{n+1}(\sigma,\rho)} * \begin{pmatrix} \boldsymbol{x}^0 \\ 1 \end{pmatrix} * \begin{pmatrix} \boldsymbol{c}^T - \boldsymbol{A}^T\boldsymbol{\mu} \\ -z(\sigma) + \boldsymbol{b}^T\boldsymbol{\mu} \end{pmatrix} - (z_1 - v_1)\boldsymbol{y}^0 && \text{by the commutativity of the H-product,} \\
&= \frac{\boldsymbol{y}(\sigma,\rho)}{y_{n+1}(\sigma,\rho)} * \begin{pmatrix} \boldsymbol{D}\boldsymbol{c}^T - \boldsymbol{D}\boldsymbol{A}^T\boldsymbol{\mu} \\ -z(\sigma) + \boldsymbol{b}^T\boldsymbol{\mu} \end{pmatrix} - (z_1 - v_1)\boldsymbol{y}^0 && \text{by the definition of the H-product,} \\
&= \frac{z_0 - z(\sigma)}{y_{n+1}(\sigma,\rho)} \left(\boldsymbol{y}^0 + \frac{\rho}{\sigma}\boldsymbol{g}(\sigma)\right) * \left(\boldsymbol{y}^0 - \frac{\rho}{\sigma}\boldsymbol{g}(\sigma)\right) - (z_1 - v_1)\boldsymbol{y}^0 && \text{by (8.41) and (8.45),} \\
&= \frac{(n+1)(z_1 - v_1)\rho^2}{1 - (n+1)\rho^2} \left(\boldsymbol{y}^0 - \frac{1}{\sigma^2}\boldsymbol{g}(\sigma) * \boldsymbol{g}(\sigma)\right) && \text{from (8.47) and by simplification.}
\end{aligned}$$

Since $\boldsymbol{q}^1(v_1)$ is in the subspace (8.3) with $\boldsymbol{D}$ replaced by $\boldsymbol{D}_1$ the second vector of the left-hand side of the above first equation is orthogonal to $\boldsymbol{q}^1(v_1)$. We thus calculate

$$\|\boldsymbol{q}^1(v_1)\|^2 \leq \left(\frac{(n+1)(z_1-v_1)\rho^2}{1-(n+1)\rho^2}\right)^2 \left(\frac{1}{n+1} - \frac{2}{\sigma^2(n+1)}\sum_{j=1}^{n+1} g_j^2(\sigma) + \frac{1}{\sigma^4}\sum_{j=1}^{n+1} g_j^4(\sigma)\right) .$$

But $\|\boldsymbol{g}(\sigma)\|^2 = \sigma^2$ and since $\sum_{j=1}^n g_j^4(\sigma) \leq \left(\sum_{j=1}^n g_j^2(\sigma)\right)^2 = \sigma^4$ we estimate the last term in the parentheses to be at most one. Consequently, since $z_1 > v_1$ and $\rho^2(n+1) < 1$ we get

$$\|\boldsymbol{q}^1(v_1)\| \leq \frac{(n+1)(z_1-v_1)\rho^2}{1-(n+1)\rho^2}\sqrt{n/(n+1)} .$$

It follows that $\|\boldsymbol{q}^1(v_1)\| < (z_1-v_1)(n+1)\rho^2$ for all $0 < \rho < \varepsilon_n/(n+1)$ where the quantities

$$\varepsilon_n = \sqrt{(n+1)\left(1-\sqrt{n/(n+1)}\right)}$$

satisfy $\varepsilon_n \geq 1/\sqrt{2}$ for all $n \geq 1$ and $\lim_{n\to\infty} \varepsilon_n = 1/\sqrt{2}$. Multiplying the last inequality for $\|\boldsymbol{q}^1(v_1)\|$ by σ^2 we find that

$$\sigma^2\|\boldsymbol{q}^1(v_1)\| < (z_1-v_1)(n+1)\rho^2\sigma^2$$

and hence using $\rho\sigma(n+1) = 1$ and (8.48) we get $\sigma^2 \leq \omega_1$ as needed. ■

We have $\varepsilon_n < 1$ for all $n \geq 1$ and thus the above estimation of $\|\boldsymbol{q}^1(v_1)\|$ does not apply to *all* ρ satisfying $0 < \rho(n+1) < 1$, but it applies to all ρ satisfying $0 < \rho(n+1) < 1/\sqrt{2} = 0.707\ldots$

Consequently, if we can find an *initial* radius ρ such that

$$0 < (n+1)\rho < 1/\sqrt{2}$$

then Remark 8.8 applies *mutatis mutandis* for the new point $\boldsymbol{x}^1$, the radius ω_1, the same ρ and σ as used previously and all $n \geq 1$. The iterative application of Remark 8.8 is thus correct for any ρ in the stated bounds.

The iterative process that results produces a sequence of increasingly sharper upper bounds $z_1, z_2, \ldots$ and lower bounds $v_1, v_2, \ldots$ which is exactly what we need in order to "sandwich" the solution value of the linear program (LP) from "both sides". Inequality (8.46) ensures that the iterative process converges rather rapidly provided that we can start it somehow.

8.5 A Projective Algorithm

In fine initium.
Latin proverb

Given a nonoptimal interior point $\boldsymbol{x}^0 \in \mathcal{X}$ with objective function value z_0 and an initial lower bound v_0 on the optimal value of the objective function value of (LP) we are now ready to formulate a projective algorithm with input parameters α for the step-size, p for the desired precision in terms of the relative error and the descriptive data for (LP).

Projective Algorithm (α, p, m, n, $\boldsymbol{A}$, $\boldsymbol{c}$, $\boldsymbol{x}^0$, z_0, v_0)

Step 0: Set $\boldsymbol{D}_0 := diag(x_1^0, \dots, x_n^0)$, $z := z_0$ and $k := 0$.
Step 1: Compute $\boldsymbol{G} := \boldsymbol{A}\boldsymbol{D}_k^2\boldsymbol{A}^T$, $\boldsymbol{G}^{-1}$ and $\boldsymbol{P} := \boldsymbol{I}_n - \boldsymbol{D}_k\boldsymbol{A}^T\boldsymbol{G}^{-1}\boldsymbol{A}\boldsymbol{D}_k$.
Step 2: Compute $\boldsymbol{p} := \boldsymbol{P}\boldsymbol{D}_k\boldsymbol{c}^T$, $\boldsymbol{d} := \boldsymbol{P}\boldsymbol{e}$, $\gamma := \boldsymbol{p}^T\boldsymbol{d}$, $\beta := n - \|\boldsymbol{d}\|^2$,
$\Lambda := (1+\beta)\|\boldsymbol{p}\|^2 + \gamma^2$, $K := (n+1)(\|\boldsymbol{p}\|^2\|\boldsymbol{d}\|^2 - \gamma^2)$,
$v := z - \dfrac{\sqrt{(1+\beta)(\Lambda\alpha^2 - K)} - (n+1)\gamma}{(n+1)^{-1}(1+\beta)\alpha^2 - n + \beta}$ and
$t := (n+1)(1+\beta)[\gamma(n+1) + (z-v)(1+2\beta-n)]^{-1}$.
Step 3: Set $\boldsymbol{x}^{k+1} := \boldsymbol{x}^k - t\boldsymbol{D}_k\left(\boldsymbol{p} - \dfrac{z-v-\gamma}{1+\beta}\boldsymbol{d}\right)$ and
$\boldsymbol{D}_{k+1} := diag(x_1^{k+1}, \dots, x_n^{k+1})$.
Step 4: **if** $\dfrac{\boldsymbol{c}\boldsymbol{x}^{k+1} - v}{z_0 - v_0} < 2^{-p}$ **stop** "$\boldsymbol{x}^{k+1}$ is a p-optimal solution to (LP)".
Set $z := \boldsymbol{c}\boldsymbol{x}^{k+1}$; replace $k+1$ by k; **go to** Step 1.

To prove convergence of the projective algorithm we need, of course, to have an *initial* interior point $\boldsymbol{x}^0 \in \mathcal{X}$ and a lower bound v_0 such that the various assumptions of Remark 8.8 are satisfied.

More precisely, denote $\boldsymbol{q}(v_0)$ the orthoprojection of $(\boldsymbol{c}\boldsymbol{D}, -v_0)^T$ on the subspace (8.3) where

$$\boldsymbol{D} = diag(x_1^0, \dots, x_n^0)$$

is given by the starting point $\boldsymbol{x}^0$. Then like in Chapter 8.4, see (8.48), we conclude that $\boldsymbol{q}(v_0) \neq \boldsymbol{0}$, set

$$\omega_0 = (z_0 - v_0)/(n+1)\|\boldsymbol{q}(v_0)\|$$

and prove that (FLP$_{\omega_0}$) has a finite optimal solution.

For the iterative application of Remark 8.8 we need to find a step-size $\alpha = (n+1)\rho$, say, such that $\sigma^2 < \omega_0$ where $\sigma = 1/\rho(n+1) = 1/\alpha$ and thus we need

$$\alpha > \alpha_0 = \sqrt{(n+1)\|\boldsymbol{q}(v_0)\|/(z_0 - v_0)}\,. \tag{8.50}$$

On the other hand, from the analysis of Chapter 8.4 we have an upper bound of $1/\sqrt{2}$ on the step-size α. Thus for any pair of values z_0 and v_0 such that $\alpha_0\sqrt{2} < 1$ we get a nonempty interval for the step-size that permits us to get started. Since the projection $\boldsymbol{q}(v_0)$ depends upon $\boldsymbol{x}^0 \in \mathcal{X}$, of course, not *every* interior point will work.

Remark 8.9 (Correctness and finiteness) *For any step length α satisfying $\alpha_0 < \alpha < 1/\sqrt{2}$ the projective algorithm iterates at most $\mathcal{O}(p\sqrt{n+1})$ times, where α_0 is defined in (8.50) with respect to a suitable interior point $\boldsymbol{x}^0 \in \mathcal{X}$ and initial upper and lower bounds z_0 and v_0 for (LP), respectively.*

Proof. Let α be the step-size of the algorithm. Then in the notation of Chapter 8.4 $\rho = \alpha/(n+1)$ and $\sigma = 1/\alpha$ are the corresponding radii. The value v calculated in Step 2 is obtained from the second part of (8.37) evaluated for the parameter value $1/\alpha$ and thus gives the value $z(\sigma) = v_1$ of Remark 8.8. In the notation of Chapter 8.4 the value t calculated in Step 2 equals

$$[(z_0 - z(\sigma))y_{n+1}(\sigma,\rho)]^{-1}$$

and is obtained using (8.22), (8.33), (8.38), (8.41) and $(n+1)\rho\sigma = 1$. The new iterate $\boldsymbol{x}^{k+1}$ equals $T_0^{-1}(\boldsymbol{y}(\sigma,\rho))$ in the previous notation. So by construction we have $\boldsymbol{x}^k > \boldsymbol{0}$ and $\boldsymbol{x}^k \in \mathcal{X}$ for all $k \geq 0$. Suppose that the algorithm executes $k \geq 1$ iterations and denote by v_{k+1} the value assigned to v in Step 2 of iteration k of the algorithm. Since α is chosen such that Remark 8.8 applies at every iteration we compute and estimate

$$0 \leq \frac{\boldsymbol{c}\boldsymbol{x}^k - v_k}{\boldsymbol{c}\boldsymbol{x}^0 - v_0} = \frac{\boldsymbol{c}\boldsymbol{x}^1 - v_1}{\boldsymbol{c}\boldsymbol{x}^0 - v_0} \cdots \frac{\boldsymbol{c}\boldsymbol{x}^k - v_k}{\boldsymbol{c}\boldsymbol{x}^{k-1} - v_{k-1}} < \left(1 + \frac{\alpha(1-\alpha)}{\sqrt{n+1}}\right)^{-k} < 2^{-p}$$

for all $k \geq p\log 2 + p\log 2\sqrt{n+1}/\alpha(1-\alpha)$. Here we have used that $(1+x)^{-1} = 1 - x/(1+x)$ and $1 - x \leq e^{-x}$ for all x in the estimation. Thus the projective algorithm comes to a stop after at most $\mathcal{O}(p\sqrt{n+1})$ iterations. ■

The convergence rate for the projective algorithm is thus essentially $\mathcal{O}(p\sqrt{n})$ and asymptotically this bound is far better than $\mathcal{O}(pn)$, of course. Choosing the precision p such that

$$(z_0 - v_0)2^{-p} < 2^{-L}$$

where L is the *digital size* of the linear program (LP) we thus get $\mathcal{O}(\sqrt{n}L)$ convergence, since $\log_2(z_0 - v_0)$ can be estimated linearly in L.

Note that – different from the convergence proof of the basic algorithm – we do not use the auxiliary test function (8.11) at all. So it is the *approximation* (ALP_ρ) of the "true" problem (FLP_ρ) that renders the test function of the convergence proof of the basic algorithm a necessity.

The projective transformations and a "cautious" step-size apparently suffice to "stay away" from the boundary of the set $\mathcal{X}$ – which is exactly what the division of $\boldsymbol{cx}$ by the geometric mean in (8.11) accomplishes; just look at the function $\left(\prod_{j=1}^{n} x_j\right)^{-1}$ on the nonnegative orthant of $\mathbb{R}^n$ when x_j tends to zero for some $j \in \{1, \ldots, n\}$ and for more detail, see Chapter 8.6.

Not surprisingly, the projective algorithm also produces a sequence of *monotonically decreasing* objective function values for (LP) which is another difference to the basic algorithm.

Let us now briefly discuss how we can start the projective algorithm for a general linear program (LP) for which an interior point $\boldsymbol{x}^0 \in \mathcal{X}$ is known.

We need to ensure that $\alpha_0\sqrt{2} < 1$. Consider $\boldsymbol{q}(v_0)$ and write $\boldsymbol{q}(v_0) = \boldsymbol{q} - v_0\boldsymbol{r}$ where $\boldsymbol{q}$ is defined in (8.6) and $\boldsymbol{r}$ in (8.27). If $\boldsymbol{r} = \boldsymbol{0}$, then $\|\boldsymbol{q}(v_0)\| = \|\boldsymbol{q}\|$ is independent of the numerical value of v_0 and thus by choosing any finite $v_0 > -\infty$ that is "small enough" we can make $\alpha_0 \geq 0$ as small as we wish.

Now by (8.27) $\boldsymbol{r} = \boldsymbol{0}$ if and only if $\boldsymbol{d} = \boldsymbol{0}$ in which case the plane L_{uw} of Chapter 8.2 degenerates into a line and $\|\boldsymbol{w}^\infty\| = \infty$ since $\boldsymbol{p}$ and $\boldsymbol{d}$ are linearly dependent.

So any interior point $\boldsymbol{x}^0 \in \mathcal{X}$ with the property that the orthoprojection (8.4) of $\boldsymbol{e}$ on the subspace (8.5) equals zero will do.

To see how we can always "force" this to happen initially let $K > 0$ be any integer such that *every basic feasible solution* to (LP) satisfies $\sum_{j=1}^{n} x_j < K$ with *strict* inequality and denote by x_{n+1} the corresponding slack variable.

Now remember from the introduction to this chapter the "trick" involving the Big M-method to obtain a feasible interior starting point. We can do the same here except that we want $\boldsymbol{x}^0 = \kappa\boldsymbol{e}$, $x^0_{n+1} = x^0_{n+2} = \kappa$ to be the starting point where $\kappa = K/(n+2)$. So after clearing fractions we get the linear program

$$
(\mathrm{LP}') \qquad \begin{array}{lrcrcrcl}
\min & \boldsymbol{cx} & + & 0x_{n+1} & + & Mx_{n+2} & & \\
\textit{subject to} & K\boldsymbol{Ax} & + & \boldsymbol{0}x_{n+1} & + & \widehat{\boldsymbol{b}}x_{n+2} & = & K\boldsymbol{b} \\
 & \sum_{j=1}^{n} x_j & + & x_{n+1} & + & x_{n+2} & = & K \\
 & \boldsymbol{x} & , & x_{n+1} & , & x_{n+2} & \geq & \boldsymbol{0}\,,
\end{array}
$$

in $n+2$ variables where $\widehat{\boldsymbol{b}} = (n+2)\boldsymbol{b} - K\boldsymbol{Ae}$. (LP$'$) has integer data and its *digital size* remains polynomial in the size L of the original linear program (LP), see Chapter 7.4, even though the size and wordlength of the hypothetical computer to run (LP$'$) increase somewhat, but *polynomially* in L. Moreover, the point $\boldsymbol{x}^0 = \kappa\boldsymbol{e}$, $x^0_{n+1} = x^0_{n+2} = \kappa$ is a suitable starting point for (LP$'$) *and* for the projective algorithm.

To see this we need to show that the orthoprojection $\boldsymbol{d} \in \mathbb{R}^{n+2}$ of $\boldsymbol{f}^T = (1, \ldots, 1) \in \mathbb{R}^{n+2}$, say, on the correspondingly changed subspace (8.5) equals zero. This is indeed the case since the system of equations

$$f^T = \mu \begin{pmatrix} \kappa K A & \mathbf{0} & \kappa \hat{b} \\ \kappa e^T & \kappa & \kappa \end{pmatrix}$$

possesses the solution $\mu_i = 0$ for $i = 1, \ldots, m$, $\mu_{m+1} = 1/\kappa$ and thus the vector $\boldsymbol{f}$ is in the *orthogonal complement* of the correspondingly changed subspace (8.5) which implies that the projection $\boldsymbol{d} = \mathbf{0}$.

Now we can proceed as above and choose a sufficiently small $v_0 > -\infty$ and thus $\alpha_0 < 1/\sqrt{2} = 0.707\ldots$ to satisfy the starting condition for the projective algorithm when applied to (LP′).

At the end of the calculations we "round" the solution iteratively by applying the (slight) modification of part (b) of Theorem 1 that we discussed for the case of the basic algorithm.

If the basic feasible solution to (LP′) that is obtained this way satisfies $x_{n+2} > 0$, then we conclude that (LP) has no feasible solution. So assume that $x_{n+2} = 0$. If the basic feasible solution to (LP′) satisfies $x_{n+1} = 0$, then the original linear program (LP) possesses optimal solutions that are unbounded – which, however, does not preclude *a priori* the existence of *some* basic feasible solution that is optimal as well, see the examples of Chapter 6.2 for more detail on this subtle distinction. (The remaining pathology can be resolved in polynomial time by altering the "rounding" procedure so as to find a basic feasible solution that maximizes x_{n+1} while keeping the objective function value below the value produced by the projective algorithm.) Otherwise, the polynomial rounding procedure yields an optimal basic feasible solution to (LP′) satisfying $x_{n+1} > 0$ and thus an optimal solution to (LP).

The fact that we need an additional constraint that *bounds* the feasible set is not surprising – we have used exactly the same "trick" to start the *dual* simplex algorithm – since the projective algorithm works simultaneously with primal *and* dual information on the linear program (LP).

Like in the basic algorithm the *time complexity* of the projective algorithm is dominated by the calculation of the projections. This can, however, be done in a numerically stable and thus satisfactory way which – more than the complexity analysis of projective algorithms and their derivatives – has led to an increasing acceptance of the related basic algorithmic idea for the solution of large-scale linear programs and thereby to a viable alternative to simplex algorithms.

Exercise 8.9 *Write a computer program that converts any linear program into the form required by the projective algorithm. Write a computer program for the algorithm using any "canned" subroutine for inverting a square matrix and solve the numerical examples of Exercises 5.1, 6.8 and 8.2.*

8.6 Centers, Barriers, Newton Steps

Δός μοι ποῦ στῶ καὶ τὴν γῆν κινήσω!
Archimedes of Syracuse (c. 287-212 B.C.)

The two algorithms of this chapter use (at least) one notion that we have not made precise at all: the projective transformation in both cases maps the current interior point $x^o \in \mathcal{X}$ into the "center" of the simplex S^{n+1}. Because the simplex S^{n+1} is a truly simple polyhedron, we have used the notion of "centrality" intuitively and assumed that you can relate to it without difficulty. But what is a center generally? Clearly, an *unbounded* polyhedron does not have a "center" or a "central point" in the intuitive meaning of the word. So let us assume throughout this section that the feasible region $\mathcal{X} \subseteq \mathbb{R}^n_+$ is a polytope having an interior point $x^0 \in \mathcal{X}$ with $x^0 > 0$ – just like the simplex S^{n+1}. The notion of "centrality" can now be made precise in more than one way. Let $x^1, \dots, x^p$ be the vertices of $\mathcal{X}$. Then the **barycenter** or the *center of gravity* of $\mathcal{X}$ is the point

$$x^G = \frac{1}{p}\sum_{i=1}^{p} x^i \ , \tag{8.51}$$

which is obtained as the convex combination of all vertices with equal weights. Clearly, $x^G \in relint\mathcal{X}$ because otherwise an interior point $x^0 \in \mathcal{X}$, $x^0 > 0$ simply cannot exist. On the other hand, why choose all weights in this convex combination to be equal to each other? Indeed, we need not and we can define a "weighted" barycenter with respect to any set of positive weights $w_i > 0$, $\sum_{i=1}^{p} w_i = 1$ so that x^G becomes the special case where $w_i = 1/p$ for $1 \le i \le p$. Any such weighted barycenter is in the relative interior of $\mathcal{X}$, but what weights make the center "central?" What does "central" mean?

The situation is quite different e.g. for ellipsoids and balls in $\mathbb{R}^n$. These are compact convex sets S of $\mathbb{R}^n$ that are **centrally symmetric**, i.e. there exists $x^0 \in S$ such that for any $y \in \mathbb{R}^n$ with $x^0 + y \in S$ we also have $x^0 - y \in S$. The existence of such a point $x^0 \in S$ implies properties of the set S and of x^0 that agree with the intuitive notion of centrality and thus x^0 is a "natural" center for such sets. Polytopes are typically not centrally symmetric, but when they are then their barycenter is precisely the point x^0 with the previous "central" property. (Why?) For compact convex sets that are not centrally symmetric the notion of a "centroid" is used frequently to approximate some sort of a "center" of S. We shall assume that $S \subseteq \mathbb{R}^n$ is such a set of full dimension. Then S has a positive volume $vol(S)$ and let us define the **centroid** x^C of S componentwise by

$$x_j^C = \frac{1}{vol(S)} \int \cdots \int_S x_j dx_1 \cdots dx_n \quad \text{for } 1 \le j \le n \ . \tag{8.52}$$

So, borrowing some terminology from elementary probability theory, the centroid $\boldsymbol{x}^C$ is an "expected" value of $\boldsymbol{x} \in S$ which is calculated with respect to a "uniform distribution" of the points in the interior of the set S. Just like in the definition of a barycenter we could thus utilize arbitrary "probability distributions" over S to define "weighted" centroids for S – which we shall not do. Using integral calculus one proves that $\boldsymbol{x}^C = \boldsymbol{x}^0$ if S is centrally symmetric with respect to $\boldsymbol{x}^0$. For if $\boldsymbol{ax} = a_0$ with $\boldsymbol{a} \neq \boldsymbol{0}$ is any hyperplane containing $\boldsymbol{x}^0$, i.e. $\boldsymbol{ax}^0 = a_0$, then the sets $S_1 = S \cap \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{ax} \leq a_0\}$, $S_2 = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{ax} \geq a_0\}$ satisfy $dim S_i = n$, $vol(S_i) = \frac{1}{2}vol(S)$ for $i = 1, 2$ and

$$x_j^C = \frac{1}{vol(S)} \int \cdots \int_{S_1 \cup S_2} x_j dx_1 \cdots dx_n = \frac{2x_j^0}{vol(S)} \int \cdots \int_{S_1} dx_1 \cdots dx_n = x_j^0$$

for $1 \leq j \leq n$ because $dim S = n$ and $\boldsymbol{x} \in S_1$ if and only if $2\boldsymbol{x}^0 - \boldsymbol{x} \in S_2$ by the central symmetry of S with respect to $\boldsymbol{x}^0$. If S is an *arbitrary* compact convex set of full dimension in $\mathbb{R}^n$, then a hyperplane passing through the centroid $\boldsymbol{x}^C$ of S can, of course, not be expected to divide S into two parts of *equal* volume. However, defining S_1 and S_2 as above, the volumina of S_i and S satisfy the inequality

$$vol(S_i) \geq \left(\frac{n}{n+1}\right)^n vol(S) \quad \text{for } i = 1, 2 \ , \tag{8.53}$$

the proof of which is too complicated to be given here. This inequality implies that every hyperplane that passes through the *centroid* divides the full-dimensional compact and convex set $S \subseteq \mathbb{R}^n$ into two parts such that the ratio of the volume of *either* part to the volume of S is *at least* $e^{-1} \approx 0.368$ and *at most* $1 - e^{-1} \approx 0.632$ where e is Euler's number. The latter follows from (8.53) because $vol(S) = vol(S_1) + vol(S_2)$ and thus

$$vol(S_i) \leq \left(1 - \left(\frac{n}{n+1}\right)^n\right) vol(S) \leq (1 - e^{-1}) vol(S)$$

for $i = 1, 2$ and all $n \geq 1$. To prove the second inequality we set $y = n/(n+1)$. The assertion is equivalent to proving that $h(y) = 1 - y + y \log y \geq 0$ for all $0 < y < 1$ which follows because $h(y)$ is monotonically decreasing in $(0, 1)$ and $h(1) = 0$. So we have $(n/(n+1))^n \geq e^{-1}$ for all $n \geq 1$ and thus $vol(S_i)/vol(S) \geq e^{-1}$ for $i = 1, 2$ as well.

Returning to polytopes, let $\mathcal{X}^{\leq} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{Ax} \leq \boldsymbol{b}, \ \boldsymbol{x} \geq \boldsymbol{0}\}$ be the feasible set of a linear program in canonical form. The assumption of the existence of an interior point $\boldsymbol{x}^0 \in \mathcal{X}$ with $\boldsymbol{x}^0 > \boldsymbol{0}$ is equivalent to requiring that $dim \mathcal{X}^{\leq} = n$. In this case the calculation of the centroid $\boldsymbol{x}^C$ is (tedious, but) straightforward. To apply this notion to flat polytopes $\mathcal{X} = \{\boldsymbol{x} \in \mathbb{R}^n :$

$\boldsymbol{Ax} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}$ having an interior point $\boldsymbol{x}^0 \in \mathcal{X}$ with $\boldsymbol{x}^0 > 0$, remember that we assume that $r(\boldsymbol{A}) = m$ where $\boldsymbol{A}$ is an $m \times n$ matrix. Let $\boldsymbol{B}$ be any basis of $\boldsymbol{A}$, let $(\boldsymbol{B}, \boldsymbol{R})$ be the corresponding partitioning of $\boldsymbol{A}$ and consider

$$\mathcal{X}' = \{\boldsymbol{y} \in \mathbb{R}^{n-m} : \boldsymbol{B}^{-1}\boldsymbol{R}\boldsymbol{y} \leq \boldsymbol{B}^{-1}\boldsymbol{b},\ \boldsymbol{y} \geq \boldsymbol{0}\} .$$

The assumption that there exists $\boldsymbol{x}^0 \in \mathcal{X}$, $\boldsymbol{x}^0 > 0$ then implies that $dim\mathcal{X}' = n - m$. So we can calculate the centroid of $\mathcal{X}'$ and the centroid of $\mathcal{X}$ in a (tedious, but) straightforward way. Barycenters and centroids of polytopes do, of course, not always coincide.

Exercise 8.10 *(i) Show that the barycenter of the feasible set of the linear program of Exercise 8.2 (ii) is given by* $x_1^G = 1{,}175/44 \approx 26.705$, $x_2^G = 3{,}175/132 \approx 24.053$ *and that its centroid is given by* $x_1^C = 331{,}855/15{,}774 \approx 21.038$, $x_2^C = 2{,}719{,}075/110{,}418 \approx 24.625$. *(ii) Show that* $\boldsymbol{x}^0 = \frac{1}{2}\boldsymbol{e}$ *is the barycenter and centroid of the n-dimensional unit cube* C_n.

In the case of the simplex S^{n+1} the notions of a barycenter and of a centroid coincide and they do as well – at least in certain cases – with the following concept of centrality which takes some of the arbitrariness out of the definition by way of an "objective function."

Let $bar(\boldsymbol{x})$ be any continuous function that maps the polytope $\mathcal{X}$ into $\mathbb{R}$ and that satisfies the following two properties:

- $bar(\mu\boldsymbol{x}^1 + (1-\mu)\boldsymbol{x}^2) \leq \mu\, bar(\boldsymbol{x}^1) + (1-\mu)\, bar(\boldsymbol{x}^2)$ for all $0 \leq \mu \leq 1$ and $\boldsymbol{x}^1, \boldsymbol{x}^2 \in relint\mathcal{X}$ with strict inequality if $\boldsymbol{x}^1 \neq \boldsymbol{x}^2$ and $0 < \mu < 1$.
- $bar(\boldsymbol{x}) = +\infty$ for all $\boldsymbol{x} \in \mathcal{X} - relint\mathcal{X}$.

Such functions are called **barrier functions** with respect to $\mathcal{X}$.
Since every barrier function is continuous and strictly convex,

$$\min\{bar(\boldsymbol{x}) : \boldsymbol{x} \in \mathcal{X}\}$$

exists and the minimum is attained at a *unique* point $\boldsymbol{x}^{bar} \in \mathcal{X}$.
Since $bar(\boldsymbol{x}) = +\infty$ for all $\boldsymbol{x} \in \mathcal{X} - relint\mathcal{X}$ and $\boldsymbol{x}^0 \in \mathcal{X}$, $\boldsymbol{x}^0 > \boldsymbol{0}$ exists it follows that $\boldsymbol{x}^{bar} \in relint\mathcal{X}$.
We call $\boldsymbol{x}^{bar}$ the **center** of $\mathcal{X}$ with respect to the barrier function $bar(\boldsymbol{x})$.

Clearly, different barrier functions exist for $\mathcal{X}$ and thus different "centers" for $\mathcal{X}$ result. To be useful for computation, barrier functions must typically satisfy additional properties such as continuous differentiability.

Since $\mathcal{X} \subseteq \mathbb{R}^n_+$ it makes perfect sense to look at the reciprocal of the *geometric mean* as a candidate for a barrier function for $\mathcal{X}$. So let

$$gbar(\boldsymbol{x}) = \left(\prod_{j=1}^{n} x_j\right)^{-1/n} \tag{8.54}$$

be the *geometric barrier function*, which we have used multiplicatively in the convergence proof of the basic algorithm – see (8.11) where we worked in $\mathbb{R}^{n+1}$. Taking the logarithm and ignoring constants we get the *logarithmic barrier function*

$$\ell bar(\boldsymbol{x}) = -\sum_{j=1}^{n} \log x_j \ . \tag{8.55}$$

Evidently, $gbar(\boldsymbol{x})$ and $\ell bar(\boldsymbol{x})$ are continuous functions for $\boldsymbol{x} \in \mathcal{X}$ and $gbar(\boldsymbol{x}) = \ell bar(\boldsymbol{x}) = +\infty$ for all $\boldsymbol{x} \in \mathcal{X} - relint\mathcal{X}$. To show the strict convexity of $\ell bar(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathcal{X}$, we note that for all $\boldsymbol{x}^1$, $\boldsymbol{x}^2 \in \mathcal{X}$ and $0 \leq \mu \leq 1$

$$\begin{aligned} \ell bar(\mu\boldsymbol{x}^1 + (1-\mu)\boldsymbol{x}^2) &= -\sum_{j=1}^{n} \log(\mu x_j^1 + (1-\mu)x_j^2) \\ &\leq -\sum_{j=1}^{n} \mu \log x_j^1 + (1-\mu)\log x_j^2 \\ &= \mu\, \ell bar(\boldsymbol{x}^1) + (1-\mu)\, \ell bar(\boldsymbol{x}^2) \end{aligned}$$

by the concavity of the function $\log(x)$ for all $x \geq 0$. Since $\log(x)$ is strictly concave for $x \geq 0$, strict convexity of $\ell bar(\boldsymbol{x})$ follows. Logarithmizing $gbar(\boldsymbol{x})$ and delogarithmizing the expressions we get from the inequality for $\ell bar(\boldsymbol{x})$

$$\begin{aligned} gbar(\mu\boldsymbol{x}^1 + (1-\mu)\boldsymbol{x}^2) &\leq (gbar(\boldsymbol{x}^1))^{\mu}(gbar(\boldsymbol{x}^2))^{1-\mu} \\ &\leq \mu\, gbar(\boldsymbol{x}^1) + (1-\mu)\, gbar(\boldsymbol{x}^2) \end{aligned}$$

for all $0 \leq \mu \leq 1$, where the last inequality follows again from the concavity of $\log(x)$ for $x \geq 0$. Strict convexity of the function $gbar(\boldsymbol{x})$ follows as before. Consequently, the two functions $gbar(\boldsymbol{x})$ and $\ell bar(\boldsymbol{x})$ deserve their name, i.e. they are barrier functions for $\mathcal{X}$. Both define, of course, the same "center" of $\mathcal{X}$, which is sometimes called the *analytic* center. We may as well call it the "geometric" or the "logarithmic" center of $\mathcal{X}$.

To give an intuitively more appealing justification of this notion of centrality, suppose that the linear program is in the canonical form $\boldsymbol{Ax} \leq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$ and that the dimension of the corresponding feasible set $\mathcal{X}^{\leq}$ is full. For every point $\boldsymbol{x}^0 \in relint\mathcal{X}^{\leq}$ we can compute the *shortest distance* of $\boldsymbol{x}^0$ from each hyperplane defining $\mathcal{X}^{\leq}$. If $\boldsymbol{A}$ is of size $m \times n$ we thus get $m+n$ such numbers and the analytic or logarithmic or geometric center is the unique point of $relint\mathcal{X}^{\leq}$ for which the *geometric mean* of these $m+n$ numbers is *maximized* – which is a perfectly reasonable choice for the definition of a center of $\mathcal{X}^{\leq}$ if the linear description of $\mathcal{X}^{\leq}$ is *ideal*, i.e. minimal and complete. On the other hand, the *exact* computation of the center of the polytope $\mathcal{X}^{\leq}$ is prohibitive: it requires the solution of a system of equations of degree

$m+n-2$ in n variables in the case of an underlying *linear* program in n variables and m linear inequalities not counting the nonnegativities! Just try to compute the center exactly for the linear program of Exercise 8.2 (ii), where we have merely $n = 2$ and $m = 4$.

Exercise 8.11 *(i) Show that the analytic center of the simplex S^{n+1} is given by $x^{gbar} = \frac{1}{n+1} \boldsymbol{f}$ where $\boldsymbol{f}^T = (1, \ldots, 1) \in \mathbb{R}^{n+1}$.* (Hint: *Use the geometric/arithmetic mean inequality.) (ii) Show that the analytic center of the polytope of the linear program of Exercise 8.2 (ii) is given by the unique positive maximizer (x_1^0, x_2^0) of the function* $10^6 x_1 x_2 (144 - 11.976x_1 + 5.32x_2 + 0.2856x_1^2 - 0.349x_2^2) + 10^3 x_1 x_2 (50.2x_1x_2 - 3.97x_1^2 x_2 - 2.46x_1^3 + 6.14x_1x_2^2 + 4.45x_2^3) - 15x_1^3x_2^3 + 29x_1^4x_2^2 - 53x_1^2x_2^4 + 6x_1^5x_2 - 15x_1x_2^5$ *and that* $x_1^0 \approx 12.507$, $x_2^0 \approx 24.407$.

8.6.1 A Method of Centers

As the last exercise shows, centers can *sometimes* be found in closed form and so let us consider the following algorithmic idea that utilizes this notion of a center and that dates back to the early 1960's. We are minimizing $\boldsymbol{cx}$ over $\mathcal{X}$ and let $z \in \mathbb{R}$ be any real number. Let us assume that $\boldsymbol{x}^0 \in \mathcal{X}$ with $\boldsymbol{x}^0 > \boldsymbol{0}$ exists and define

$$\mathcal{X}_z = \mathcal{X} \cap \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{cx} \le z\} , \tag{8.56}$$

where $\mathcal{X}$ is a nonempty polytope. The set $\mathcal{X}_z$ is either empty or it has no $\boldsymbol{x}^1 \in \mathcal{X}_z$ with $\boldsymbol{x}^1 > \boldsymbol{0}$ and $\boldsymbol{cx}^1 < z$ or such a point exists.

By choosing z initially large enough we can always avoid the first possibility. In the second case, $\mathcal{X}_z$ is a face of $\mathcal{X}$ and all points of $\mathcal{X}_z$ are optimal for $\boldsymbol{cx}$. For suppose not. Then some point $\boldsymbol{x}^2 \in \mathcal{X}_z$ exists with $\boldsymbol{cx}^2 < z$. By assumption we have $\boldsymbol{x}^0 \in \mathcal{X}$ with $\boldsymbol{x}^0 > \boldsymbol{0}$ and clearly, $\boldsymbol{cx}^0 \ge z$. The line segment $\mu \boldsymbol{x}^0 + (1-\mu)\boldsymbol{x}^2$ intersects the hyperplane $\boldsymbol{cx} = z$ for some μ, $0 \le \mu \le 1$. We can thus construct $\boldsymbol{x}^1 \in \mathcal{X}_z$ with $\boldsymbol{x}^1 > \boldsymbol{0}$ and $\boldsymbol{cx}^1 < z$ by taking e.g. the midpoint of the line segment given by $\boldsymbol{x}^2$ and the point of intersection of $\mu \boldsymbol{x}^0 + (1-\mu)\boldsymbol{x}^2$ with the hyperplane $\boldsymbol{cx} = z$ – which gives a contradiction. So if z is *not* the optimum objective function value of $\boldsymbol{cx}$ over $\mathcal{X}$, then $\boldsymbol{x}^1 \in \mathcal{X}_z$ with $\boldsymbol{x}^1 > \boldsymbol{0}$ and $\boldsymbol{cx}^1 < z$ exists, i.e. we are in the third case of the three possibilities.

Given any barrier function $bar(\boldsymbol{x})$ for $\mathcal{X}$ let us denote its restriction to $\mathcal{X}_z$ by $bar_z(\boldsymbol{x})$. It is clear how to modify $gbar(\boldsymbol{x})$ or $\ell bar(\boldsymbol{x})$ to obtain the corresponding restrictions to $\mathcal{X}_z$ – we just include the inequality $\boldsymbol{cx} \le z$ as one of the "bounding" hyperplanes in the definition of the corresponding function, e.g.

$$\ell bar_z(\boldsymbol{x}) = -\sum_{j=1}^{n} \log x_j - \log(z - \boldsymbol{cx}) \ .$$

Now denote by $\boldsymbol{x}^{bar}(z)$ the center of $\mathcal{X}_z$ with respect to $bar_z(\boldsymbol{x})$. It follows that $\boldsymbol{cx}^{bar}(z) < z$. Thus starting e.g. initially at $\boldsymbol{x}^{bar} \in \mathcal{X}$ we can construct a sequence of points $\boldsymbol{x}^{bar}(z_k)$ and a corresponding sequence of z_k with $z_0 = \boldsymbol{cx}^{bar} > z_1 > z_2 > \cdots$. Since $\mathcal{X}$ and thus the $\mathcal{X}_{z_k}$ are polytopes, the sequence $\{z_k\}$ for $k = 0, 1, 2, \ldots$ is bounded from below and consequently, it has a point of accumulation z_∞, say. By the same argument that we have used above it follows that either z_∞ is the optimum objective function of $\boldsymbol{cx}$ over $\mathcal{X}$ or that there exists z_k in the infinite sequence with $z_k < z_\infty$. Thus an iterative application may "stall" temporarily, but it will get itself out of the "trap" eventually, i.e. there exists a subsequence $\{z_{k_i}\}$, say, of $\{z_k\}$ that converges to the optimum objective function value of $\boldsymbol{cx}$ over $\mathcal{X}$.

Under "suitable" assumptions – which we shall not discuss – this method of the centers can be extended to compact convex regions that are not necessarily polyhedral and to the minimization of convex functions over such sets. But what does a straightforward implementation of this algorithmic idea require? It requires the successive calculation of *centers* of "shrinking" convex sets and that is a formidable task.

Different from Archimedes who demanded "Give me a place to stand and I will unhinge the earth!" and who was thus merely asking for *some* point outside of the earth (of course!), we want to be right in the center of the sets $\mathcal{X}_z$. Thus unless we are in the lucky position that we can calculate the corresponding centers in closed form, the algorithm that we get for convex optimization is in its present form just about as useful or useless as the Universal Algorithm for Optimization published by *Anonymous* in the journal *Mathematical Programming* in 1972.

In the preceding method we try to essentially accomplish *two* objectives simultaneously: we want, of course, to minimize $\boldsymbol{cx}$ over the polytope $\mathcal{X}$ and at the same time we want to stay in the relative interior of $\mathcal{X}$. Combining these two objectives linearly into a single objective function, we are led to consider the family of problems

$$(\mathrm{P}_\mu^{bar}) \qquad \min\{\boldsymbol{cx} + \mu\, bar(\boldsymbol{x}) : \boldsymbol{Ax} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\} \ ,$$

where $\mu > 0$ is a parameter or some relative "weight". Since $\boldsymbol{cx}$ is linear, the function $\boldsymbol{cx} + \mu\, bar(\boldsymbol{x})$ is strictly convex on $\mathcal{X}$ for every $\mu > 0$. Since $\mathcal{X}$ is bounded and $\boldsymbol{x}^0 \in \mathcal{X}$, $\boldsymbol{x}^0 > \boldsymbol{0}$ exists, the minimum exists and the minimizer of (P_μ^{bar}) is unique and positive. We can thus ignore the nonnegativity constraints $\boldsymbol{x} \geq \boldsymbol{0}$ of the problem (P_μ^{bar}). Assuming continuous differentiability of $bar(\boldsymbol{x})$ we can determine the minimizer from the first order conditions for an extremum of the corresponding Lagrangean function for any fixed $\mu > 0$. By varying the parameter μ we obtain a family of solutions $\boldsymbol{x}^{bar}(\mu)$. For $\mu \to +\infty$ the solution $\boldsymbol{x}^{bar}(\mu)$ converges towards the center $\boldsymbol{x}^{bar} \in \mathcal{X}$, while for $\mu \to 0$ it converges towards $\boldsymbol{x}^* \in \mathcal{X}$, an optimal solution to our linear program (LP). This technique – in the more general context of constrained nonlinear optimization – was a hotbed of research in the 1950's and 1960's. It reduces a constrained optimization problem to sequences of unconstrained optimization problems which thus become amenable to the methods of calculus.

8.6.2 The Logarithmic Barrier Function

More concretely, let us now consider the logarithmic barrier function $\ell bar(\boldsymbol{x})$, which is more tractable than $gbar(\boldsymbol{x})$, and denote by (P_μ) the corresponding family of problems

$$(\mathrm{P}_\mu) \qquad \min\{\boldsymbol{cx} + \mu\, \ell bar(\boldsymbol{x}) : \boldsymbol{Ax} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}\ ,$$

where $\mu > 0$ is an arbitrary parameter. Since there exists a unique *positive* minimizer for (P_μ) we are permitted to ignore the nonnegativity constraints. So we form the Lagrangean function

$$L(\boldsymbol{x}, \boldsymbol{u}, \mu) = \boldsymbol{cx} + \mu\, \ell bar(\boldsymbol{x}) + \boldsymbol{u}^T(\boldsymbol{b} - \boldsymbol{Ax}),$$

where $\boldsymbol{u} \in \mathbb{R}^m$ are the Lagrangean multipliers. The minimizer must satisfy the first-order conditions for an extremum of $L(\boldsymbol{x}, \boldsymbol{u}, \mu)$. The first order conditions for $L(\boldsymbol{x}, \boldsymbol{u}, \mu)$ yield the nonlinear system of equations

$$c_j - \mu x_j^{-1} - \sum_{i=1}^{m} u_i a_j^i = 0 \text{ for } 1 \leq j \leq n,\ \sum_{j=1}^{n} a_j^i x_j - b_i = 0 \text{ for } 1 \leq i \leq m,$$

for which we seek the unique solution $(\boldsymbol{x}(\mu), \boldsymbol{u}(\mu))$ such that $\boldsymbol{x}(\mu) > \boldsymbol{0}$.

The parameter $\mu > 0$ is assumed to be fixed, but all that we really want is the unique solution for the "limiting" case where $\mu \to 0$. Define $r_j = \mu x_j^{-1}$ for $1 \leq j \leq n$ and $\boldsymbol{r}^T = (r_1, \ldots, r_n)$. Then the first order conditions for an extremum of $L(\boldsymbol{x}, \boldsymbol{u}, \mu)$ become in matrix form

$$\boldsymbol{Ax} - \boldsymbol{b} = \boldsymbol{0} \tag{8.57}$$

$$\boldsymbol{A}^T\boldsymbol{u} + \boldsymbol{r} - \boldsymbol{c}^T = \boldsymbol{0} \tag{8.58}$$

$$\boldsymbol{x} * \boldsymbol{r} - \mu\boldsymbol{e} = \boldsymbol{0}\ , \tag{8.59}$$

where $\boldsymbol{x} * \boldsymbol{r}$ is the H-product of Chapter 8.4.1 and $\boldsymbol{e}^T = (1, \ldots, 1) \in \mathbb{R}^n$. If $(\boldsymbol{x}(\mu), \boldsymbol{u}(\mu))$ is a feasible solution to the first order conditions with $\boldsymbol{x}(\mu) > \boldsymbol{0}$,

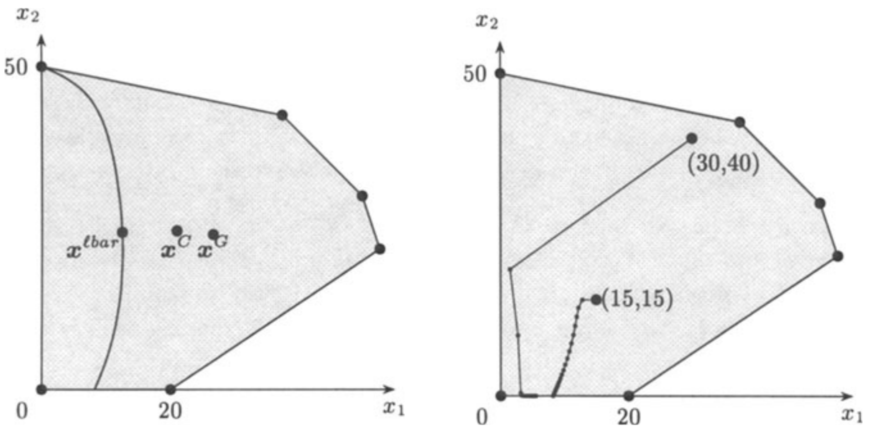

Fig. 8.11. Three "centers", the log-central path and paths to optimality.

then $\boldsymbol{r} = \boldsymbol{r}(\mu) > \mathbf{0}$ and thus $(\boldsymbol{u}(\mu), \boldsymbol{r}(\mu))$ is an *interior* feasible solution to the linear program

$$(d\text{LP}) \qquad \max\{\boldsymbol{b}^T\boldsymbol{u} : \boldsymbol{A}^T\boldsymbol{u} + \boldsymbol{r} = \boldsymbol{c}^T,\ \boldsymbol{r} \geq \mathbf{0}\}\ ,$$

which is the dual linear program to our primal linear program (LP) of the introduction to this chapter. From (8.59) we find $\boldsymbol{r}^T\boldsymbol{x} = n\mu$ and thus from (8.57) and (8.58) we have

$$\boldsymbol{r}^T\boldsymbol{x} = n\mu = \boldsymbol{c}\boldsymbol{x} - \boldsymbol{u}^T\boldsymbol{b}\ , \tag{8.60}$$

which is the *duality gap* for the primal-dual pair $(\boldsymbol{x}(\mu), \boldsymbol{u}(\mu), \boldsymbol{r}(\mu))$. Consequently, any primal-dual pair $(\boldsymbol{x}(0), \boldsymbol{u}(0), \boldsymbol{r}(0))$ with $\boldsymbol{x}(0) \geq \mathbf{0}$ and $\boldsymbol{r}(0) \geq \mathbf{0}$, i.e. any feasible solution to (8.57), (8.58), (8.59) for $\mu = 0$, yields a pair of optimal solutions to (LP) and (dLP).

For $0 \leq \mu < \infty$ the loci of $\boldsymbol{x}(\mu)$ form a path connecting the logarithmic center of $\mathcal{X}$ to some point in the optimal face of $\mathcal{X}$. If the optimal face is a singleton set, i.e. a vertex, then $\boldsymbol{x}(\mu)$ converges into the optimal vertex. Since we are assuming that $\mathcal{X}$ is bounded, we may consider the problem of maximization over $\mathcal{X}$ as well: forming the corresponding barrier problem $\max\{\boldsymbol{c}\boldsymbol{x} - \nu\, \ell bar(\boldsymbol{x}) : \boldsymbol{x} \in \mathcal{X}\}$ it follows that for every $\nu > 0$ a unique optimizer exists because the objective function is continuous and strictly concave over the compact set $\mathcal{X}$. Like before we can determine the first order conditions. These are again of the form (8.57), (8.58), (8.59) with the only difference that in (8.58) we replace the term $+\boldsymbol{r}$ by $-\boldsymbol{r}$ and in (8.59) we replace the parameter μ by ν. For $\nu \to +\infty$ the solution $\boldsymbol{x}(\nu)$ tends to the logarithmic center of $\mathcal{X}$ while for $\nu \to 0$ it tends towards a point in the optimal face of $\mathcal{X}$ with respect to maximization. The path connecting the optimal face of $\mathcal{X}$ with respect to maximization to the one for minimization is the **log-central path** or simply, the *central path* of $\mathcal{X}$, which – by construction – passes through the log-center $\boldsymbol{x}^{\ell bar}$ of $\mathcal{X}$ with respect to the barrier function $\ell bar(\boldsymbol{x})$; see the left part of Figure 8.11 where we display the barycenter $\boldsymbol{x}^G$, the centroid $\boldsymbol{x}^C$ and the log-central path for the polytope of Exercise 8.2 (ii).

Exercise 8.12 *(i) Suppose that* $\mathcal{X} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\} \neq \emptyset$ *is bounded and that* $\mathbf{c} \in \mathbb{R}^n$ *is arbitrary. Show that* $\mathcal{U}_{\mathbf{c}}$ *is* not *bounded and that* $relint\,\mathcal{U}_{\mathbf{c}} \neq \emptyset$ *where* $\mathcal{U}_{\mathbf{c}} = \{\boldsymbol{u} \in \mathbb{R}^m : \boldsymbol{u}^T\boldsymbol{A} \leq \boldsymbol{c}\}$. (Hint: *See Exercise 6.9 (ii).*)
(ii) Compute the log-center and the log-central path for the polytope of Exercise 8.2 (ii).

To find an approximatively optimal solution to (LP) we must solve the system of nonlinear equations (8.57), (8.58), (8.59) for $\mu \approx 0$. A closed form solution is not possible and thus we have to look for an iterative method for solving this system of nonlinear equations. This is done e.g. by a multivariate version of *Newton's* method for finding the root of a (nonlinear) equation, so named after the famous English physicist and mathematician Sir Isaac Newton (1642-1727), and which goes as follows. Let $\boldsymbol{F}(\boldsymbol{z})$ be any continuously differentiable function mapping $\mathbb{R}^t$ into $\mathbb{R}^q$. We wish to find $\boldsymbol{z}^0 \in \mathbb{R}^t$ such that $\boldsymbol{F}(\boldsymbol{z}^0) = \boldsymbol{0}$ or componentwise, such that $F_i(\boldsymbol{z}^0) = 0$ for $1 \leq i \leq q$. By the multivariate mean-value theorem of differential calculus we know that

$$F_i(\boldsymbol{z} + \Delta\boldsymbol{z}) = F_i(\boldsymbol{z}) + \nabla F_i(\boldsymbol{z} + \theta_i\Delta\boldsymbol{z})\Delta\boldsymbol{z}$$

for some $0 \leq \theta_i \leq 1$, where $\nabla F_i = \left(\frac{\partial F_i}{\partial z_j}\right)_{j=1,\dots,t}$ is the vector of the first derivatives of F_i and $1 \leq i \leq q$. $\Delta\boldsymbol{z} = (\Delta z_1, \dots, \Delta z_t)^T$ is a vector of "change" for the components of $\boldsymbol{z}$, e.g. $\Delta\boldsymbol{z} = \boldsymbol{z}' - \boldsymbol{z}$ for some $\boldsymbol{z}' \neq \boldsymbol{z} \in \mathbb{R}^t$. Given a "trial" solution $\boldsymbol{z} \in \mathbb{R}^t$ for the root $\boldsymbol{z}^0$ of $\boldsymbol{F}$ the "new" trial solution is $\boldsymbol{z} + \Delta\boldsymbol{z}$. As we wish to obtain a root, we set $\boldsymbol{F}(\boldsymbol{z} + \Delta\boldsymbol{z}) = \boldsymbol{0}$ and ignore the dependence of ∇F_i on $\Delta\boldsymbol{z}$ by boldly setting all $\theta_i = 0$. Denoting by $\nabla\boldsymbol{F} = \left(\frac{\partial F_i}{\partial z_j}\right)_{j=1,\dots,t}^{i=1,\dots,q}$ the $q \times t$ matrix of the first derivatives we get the system of linear equations

$$\boldsymbol{\nabla F}(\boldsymbol{z})\,\Delta\boldsymbol{z} = -\boldsymbol{F}(\boldsymbol{z}) \tag{8.61}$$

in the variables $\Delta\boldsymbol{z}$, where $\boldsymbol{\nabla F}(\boldsymbol{z})$ and $\boldsymbol{F}(\boldsymbol{z})$ are evaluated at the current iterate $\boldsymbol{z}$. Every solution $\Delta\boldsymbol{z}$ to this system gives a *Newton direction* or a *Newton step* . The "new" iterate is simply $\boldsymbol{z} + \Delta\boldsymbol{z}$ or more generally $\boldsymbol{z} + \alpha\,\Delta\boldsymbol{z}$ where $\alpha \geq 0$ is the *step length.* If started "close" to a root $\boldsymbol{z}^0$ of $\boldsymbol{F}$, then the resulting iterative scheme converges rather fast to the root $\boldsymbol{z}^0$, but in general it does not converge to $\boldsymbol{z}^0$. You will find much more detail on Newton methods in any decent book on nonlinear optimization.

For the nonlinear system (8.57), (8.58), (8.59) the mapping $\boldsymbol{F}(\boldsymbol{z})$ is

$$F(z) = \begin{pmatrix} Ax - b \\ A^T u + r - c^T \\ x * r - \mu e \end{pmatrix} ,$$

where $z = (x, u, r)$ is the vector of variables. Let now $x > 0$, $r > 0$ and u arbitrary be any fixed trial solution to our problem where $\mu > 0$ is arbitrary, but fixed as well. Then forming ∇F and evaluating ∇F at the point (x, u, r) we get from (8.61) the following system of linear equations for the variables $\Delta z = (\Delta x, \Delta u, \Delta r)$

$$\begin{pmatrix} A & O & O \\ O & A^T & I_n \\ R & O & D \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta u \\ \Delta r \end{pmatrix} = \begin{pmatrix} b - Ax \\ c^T - A^T u - r \\ \mu e - x * r \end{pmatrix} , \tag{8.62}$$

where we have set $D = diag(x_1, \dots, x_n)$ and $R = diag(r_1, \dots, r_n)$. Since $x > 0$ and $r > 0$ we have $r(R) = r(D) = n$. Since $r(A) = m$ by our blanket assumption, the $(2n + m) \times (2n + m)$ matrix of the linear system (8.62) is nonsingular and hence the direction vectors are determined uniquely. To prove the nonsingularity and to compute the inverse of the matrix of (8.62) you can use e.g. the formula for partitioned matrices of Chapter 2.2 where you let the matrix B of that formula correspond to the submatrix $\begin{pmatrix} R & D \\ O & I_n \end{pmatrix}$.

Given $x > 0$, $r > 0$, a vector $u \in \mathbb{R}^m$ and $\mu > 0$ let us denote

$$f = b - Ax \; , \quad g = c^T - A^T u - r \; , \quad h = \mu e - x * r$$

and $B = AR^{-1}DA^T$. Since $r(A) = m$ the inverse B^{-1} exists and solving (8.62) we get

$$\begin{aligned} \Delta x &= R^{-1}DA^TB^{-1}f + R^{-1}(I_n - DA^TB^{-1}AR^{-1})(h - Dg) \; , \\ \Delta u &= B^{-1}f + B^{-1}AR^{-1}(Dg - h) \; , \\ \Delta r &= -A^TB^{-1}f + (I_n - A^TB^{-1}AR^{-1}D)g + A^TB^{-1}AR^{-1}h \; . \end{aligned}$$

Since $r > 0$ and $x > 0$ the matrix $T = (R^{-1}D)^{1/2}$ is well defined. Let

$$S = I_n - TA^TB^{-1}AT \text{ where } T = (R^{-1}D)^{1/2} \text{ and } B = AT^2A^T \tag{8.63}$$

be the orthogonal projection operator on the subspace $\{x \in \mathbb{R}^n : ATx = 0\}$; see Remark 8.1. Note that $x * r = DRe$ and that e.g. $R^{-1}DR = D$ since D and R are diagonal matrices. So after some algebraic manipulations and simplifications we can bring the expressions for Δx, Δu and Δr into the following form which has a nice interpretation.

$$\Delta x = -TSTc^T + \mu TSTD^{-1}e + T^2A^TB^{-1}f \tag{8.64}$$

$$\Delta u = B^{-1}b - \mu B^{-1}AR^{-1}e + B^{-1}AT^2g \tag{8.65}$$

$$\Delta r = -A^TB^{-1}b + \mu A^TB^{-1}AR^{-1}e + T^{-1}STg \tag{8.66}$$

The first term $\Delta \boldsymbol{x}^s$, say, of $\Delta \boldsymbol{x}$ is a *steepest descent direction* in the "transformed" space, i.e. up to normalization it is the direction given by the solution to the problem

$$\min\{\boldsymbol{cTz} : \|\boldsymbol{z}\| = 1,\ \boldsymbol{ATz} = \boldsymbol{0}\} .$$

The second term $\Delta \boldsymbol{x}^c$, say, of $\Delta \boldsymbol{x}$ is up to normalization the direction given by the solution to the problem

$$\max\{\boldsymbol{e}^T \boldsymbol{D}^{-1} \boldsymbol{Tz} : \|\boldsymbol{z}\| = 1,\ \boldsymbol{ATz} = \boldsymbol{0}\}$$

and is called the *centering direction* because the objective function "points away" from the "walls" $x_j = 0$ of the polytope $\mathcal{X}$. Both the steepest descent and the centering directions lie in the nullspace of $\boldsymbol{A}$, i.e.

$$\boldsymbol{A}\,\Delta \boldsymbol{x}^s = \boldsymbol{A}\,\Delta \boldsymbol{x}^c = \boldsymbol{0} .$$

Thus they do not affect the feasibility of the new iterate $\boldsymbol{x} + \alpha\,\Delta \boldsymbol{x}$ as far as the system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ is concerned. The third term reduces the infeasibility in the system of equations to zero if $\alpha = 1$ and is therefore called the *feasibility direction.*

A similar interpretation can be given to the three terms of $(\Delta \boldsymbol{u}, \Delta \boldsymbol{r})$ for the dual linear program (dLP). The first term $(\Delta \boldsymbol{u}^s, \Delta \boldsymbol{r}^s)$, say, of $(\Delta \boldsymbol{u}, \Delta \boldsymbol{r})$ is a *steepest ascent direction*, i.e. up to scaling it is the direction vector given by the solution to the problem $\max\{\boldsymbol{b}^T \boldsymbol{z} : \|\boldsymbol{Ts}\| = 1,\ \boldsymbol{A}^T \boldsymbol{z} + \boldsymbol{s} = \boldsymbol{0}\}$. The second term $(\Delta \boldsymbol{u}^c, \Delta \boldsymbol{r}^c)$, say, is up to scaling the direction vector given by the solution to the problem $\max\{\boldsymbol{e}^T \boldsymbol{R}^{-1} \boldsymbol{s} : \|\boldsymbol{Ts}\| = 1,\ \boldsymbol{A}^T \boldsymbol{z} + \boldsymbol{s} = \boldsymbol{0}\}$ and thus some sort of a "centering" direction as well, because the equations $r_j = 0$ are the "walls" of the dual polyhedron. Both direction vectors annul the rows of the matrix $(\boldsymbol{A}^T\ \boldsymbol{I}_n)$ and do not affect the feasibility of $(\boldsymbol{u}, \boldsymbol{r})$ whereas the third term does. Since the positivity of $\boldsymbol{r}$ must be preserved we need a step length $\beta \geq 0$ so that $\boldsymbol{r} + \beta\,\Delta \boldsymbol{r} > 0$.

We are now ready to state an iterative scheme that is designed to find a solution $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r})$ with $\boldsymbol{x} \geq \boldsymbol{0}$ and $\boldsymbol{r} \geq \boldsymbol{0}$ to (8.57), (8.58), (8.59) for $\mu \approx 0$ and thereby an approximatively optimal primal-dual pair of solutions to (LP) and (dLP). We start with any triplet $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r})$ satisfying $\boldsymbol{x} > \boldsymbol{0}$ and $\boldsymbol{r} > \boldsymbol{0}$ and some $\mu > 0$, e.g. $\mu = 0.1(\boldsymbol{r}^T \boldsymbol{x}/n)$. We calculate the orthoprojection (8.63) to find the direction vectors (8.64), (8.65), (8.66). Then we update

$$\boldsymbol{x}^{new} = \boldsymbol{x} + \frac{1}{\alpha_p}\,\Delta \boldsymbol{x}\ , \quad \boldsymbol{u}^{new} = \boldsymbol{u} + \frac{1}{\alpha_d}\,\Delta \boldsymbol{u}\ , \quad \boldsymbol{r}^{new} = \boldsymbol{r} + \frac{1}{\alpha_d}\,\Delta \boldsymbol{r}\ ,$$

where α_p and α_d are step lengths that are chosen to maintain the positivity of $\boldsymbol{x}^{new}$ and $\boldsymbol{r}^{new}$. We reduce μ by setting $\mu^{new} = 0.1(\boldsymbol{r}^{new})^T \boldsymbol{x}^{new}/n$ if $\boldsymbol{cx}^{new} > \boldsymbol{b}^T \boldsymbol{u}^{new}$ and set $\mu^{new} = 2(\boldsymbol{r}^{new})^T \boldsymbol{x}^{new}/n$ otherwise. This is motivated by relation (8.60) and in the second case, designed to permit the correction of a possible error. We thus have a new triplet $(\boldsymbol{x}^{new}, \boldsymbol{u}^{new}, \boldsymbol{r}^{new})$ and $\mu^{new} > 0$ and we can iterate until primal and dual feasibility are attained and the duality gap is smaller than some tolerance, e.g. smaller than 10^{-6}. To ensure positivity of $\boldsymbol{x}^{new}$ and $\boldsymbol{r}^{new}$ one chooses α_p and α_d e.g. as follows:

$$\alpha_p = \max\{1, -\frac{\Delta x_1}{0.95x_1}, \ldots, -\frac{\Delta x_n}{0.95x_n}\},\ \alpha_d = \max\{1, -\frac{\Delta r_1}{0.95r_1}, \ldots, -\frac{\Delta r_n}{0.95r_n}\}$$

This iterative scheme works apparently well in computational practice. Note that for "fixed" μ this iterative scheme does not bother to find an "exact" root of the system (8.57), (8.58), (8.59): all that we are interested in is a solution to the nonlinear equation system "in the limit" $\mu \to 0$ and that is utilized in the iterative scheme. In the right part of Figure 8.11 we display the path to optimality when the algorithm is run with the data of Exercise 8.2 (ii) where $x_1^0 = 30$, $x_2^0 = 40$, $\boldsymbol{r}^0 = \boldsymbol{c}^T - \boldsymbol{A}^T\boldsymbol{u}^0$ and $u_j^0 = -0.1$ for $1 \le j \le 4$.

Exercise 8.13 *(i) Write a computer program for the iterative procedure discussed in this section using any subroutine for inverting a square matrix and solve the numerical examples of Exercises 5.1, 6.8 and 8.2. (ii) Derive a method to find a* **basic feasible solution** $\boldsymbol{x}^1$ *for the linear program (LP) given a near-optimal feasible interior point* $\boldsymbol{x}^0 \in \mathcal{X}$ *satisfying* $\boldsymbol{cx}^1 \le \boldsymbol{cx}^0$. *Generalize this construction so as to permit a practical way of "crossing over" to a simplex algorithm from any near-optimal interior point* $\boldsymbol{x}^0 \in \mathcal{X}$. (Hint: *Use the proof of Theorem 1.*)

8.6.3 A Newtonian Algorithm

To give a theoretical justification of the above algorithm we will make the assumption that $\boldsymbol{x}$ and $(\boldsymbol{u}, \boldsymbol{r})$ are feasible solutions to (LP) and (dLP), respectively, which are already – in some way to be specified – "not too far" from a root of the system of the nonlinear equations (8.57), (8.58), (8.59) for some $\mu > 0$. Like we have done before, we will then have to show that we can always find solutions with the assumed properties in order to start the algorithm correctly. This means, of course, that we will have to replace the above updating rules for α_p, α_d and μ by much more conservative rules that can be proven analytically to work; this is just another example of the splendid difference between mathematical theory and computational practice.

We call a triplet $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r}) \in \mathbb{R}^{2n+m}$ a *feasible triplet* if $\boldsymbol{x} \in \mathbb{R}^n$ is a feasible solution to (LP) with $\boldsymbol{x} > \boldsymbol{0}$ and $(\boldsymbol{u}, \boldsymbol{r}) \in \mathbb{R}^{m+n}$ a feasible solution to (dLP) with $\boldsymbol{r} > \boldsymbol{0}$, respectively. From (8.58) and (8.59) we know that every feasible triplet satisfies $\boldsymbol{r}^T\boldsymbol{x} = \boldsymbol{cx} - \boldsymbol{b}^T\boldsymbol{u}$. To satisfy (8.59) as well we need $\mu = \boldsymbol{r}^T\boldsymbol{x}/n$

since $e^T(x * r) = r^T x$. Consequently, a feasible triplet (x, u, r) belongs to the log-central path if and only if $||x * r - \mu e|| = 0$ for this value of μ.

This suggests to call a feasible triplet (x, u, r) "close" to the log-central path if for $\mu = r^T x/n$ we have $||x * r - \mu e|| \le \Theta\mu$ for some "small" $\Theta \ge 0$. As we shall see this notion of "proximity" does the job. We first prove an inequality.

$$||s * t|| \le \frac{1}{2}||s + t||^2 \quad \text{for all } s, t \in \mathbb{R}^n \text{ with } s^T t \ge 0. \tag{8.67}$$

Proof. Since $\left(\sum_{i=1}^n s_i^2\right)\left(\sum_{i=1}^n t_i^2\right) \ge \sum_{i=1}^n s_i^2 t_i^2$ we have $||s * t|| \le ||s||\,||t||$ for all $s, t \in \mathbb{R}^n$. Since by assumption $s^T t \ge 0$ we get $||s||^2 + ||t||^2 \le ||s||^2 + ||t||^2 + 2s^T t = ||s + t||^2$ and thus $0 \le (||s|| - ||t||)^2 \le ||s + t||^2 - 2||s||\,||t||$. ■

Remark 8.10 *Let $(x, u, r) \in \mathbb{R}^{2n+m}$ be a feasible triplet satisfying*

$$||x * r - \mu e|| \le \Theta\mu\ , \quad r^T x = n\mu \tag{8.68}$$

where Θ is a real number that satisfies

$$0 \le \Theta \le \frac{1}{2}\ , \quad \Theta^2 + \delta^2 \le 2\Theta(1 - \Theta)(1 - \frac{\delta}{\sqrt{n}}) \tag{8.69}$$

for some δ with $0 < \delta < \sqrt{n}$. Let $\widehat{\mu} = \mu(1 - \delta/\sqrt{n})$ and $(\widehat{x}, \widehat{u}, \widehat{r}) \in \mathbb{R}^{2n+m}$ be defined by $\widehat{x} = x + \Delta x$, $\widehat{u} = u + \Delta u$, $\widehat{r} = r + \Delta r$, where $(\Delta x, \Delta u, \Delta r)$ is a solution to (8.62) with μ replaced by $\widehat{\mu}$. Then (i) $c\widehat{x} - b^T\widehat{u} = \widehat{r}^T\widehat{x} = n\widehat{\mu}$, (ii) $||\widehat{x} * \widehat{r} - \widehat{\mu}e|| \le \Theta\widehat{\mu}$, and (iii) $(\widehat{x}, \widehat{u}, \widehat{r})$ is a feasible triplet.

Proof. Since by (8.62) we have $R\,\Delta x + D\,\Delta r = \widehat{\mu}e - x * r$ and since e.g. $R\,\Delta x = r * \Delta x$ we calculate

$$\widehat{x} * \widehat{r} = (x + \Delta x) * (r + \Delta r) = x * r + x * \Delta r + r * \Delta x + \Delta x * \Delta r = \widehat{\mu}e + \Delta x * \Delta r\ .$$

Also from (8.62) we get by multiplying $A\,\Delta x = 0$ with $(\Delta u)^T$ and using $A^T\,\Delta u + \Delta r = 0$ that $(\Delta r)^T \Delta x = 0$. Consequently, $e^T(\Delta x * \Delta r) = (\Delta r)^T \Delta x = 0$ and thus $\widehat{r}^T\widehat{x} = e^T(\widehat{x} * \widehat{r}) = n\widehat{\mu}$. From $A\widehat{x} = b$ and $\widehat{u}^T A + \widehat{r}^T = c$ we have $\widehat{r}^T\widehat{x} = c\widehat{x} - \widehat{u}^T b$ and thus (i) follows.
Multiplying $R\,\Delta x + D\,\Delta r = \widehat{\mu}e - x * r$ by $(DR)^{-1/2}$ we find

$$T^{-1}\,\Delta x + T\,\Delta r = (DR)^{-1/2}(\widehat{\mu}e - x * r)\ \text{, where } T = (R^{-1}D)^{1/2}\ .$$

Since $(T\,\Delta r)^T(T^{-1}\Delta x) = (\Delta r)^T\Delta x = 0$ we can use (8.67) with $s = T^{-1}\Delta x$, $t = T\,\Delta r$ and $s + t = (DR)^{-1/2}(\widehat{\mu}e - x * r)$. We then calculate

$$\begin{aligned} ||\widehat{x} * \widehat{r} - \widehat{\mu}e|| &= ||\Delta x * \Delta r|| = ||T^{-1}\Delta x * T\,\Delta r|| \\ &\le \frac{1}{2}||(DR)^{-1/2}(\widehat{\mu}e - x * r)||^2 \\ &= \frac{1}{2}\sum_{i=1}^n \frac{(\widehat{\mu} - x_i r_i)^2}{x_i r_i} \le \frac{||x * r - \widehat{\mu}e||^2}{2xr_{min}}\ , \end{aligned}$$

where $xr_{min} = \min\{x_i r_i : 1 \le i \le n\}$. xr_{min} satisfies $xr_{min} \ge \mu(1-\Theta)$ because $||\boldsymbol{x} * \boldsymbol{r} - \mu\boldsymbol{e}|| \le \Theta\mu$ by assumption. Calculating we find

$$||\boldsymbol{x} * \boldsymbol{r} - \widehat{\mu}\boldsymbol{e}||^2 = ||\boldsymbol{x} * \boldsymbol{r} - \mu\boldsymbol{e} + (\mu - \widehat{\mu})\boldsymbol{e}||^2 = (\widehat{\mu} - \mu)^2 n + ||\boldsymbol{x} * \boldsymbol{r} - \mu\boldsymbol{e}||^2$$

because $\boldsymbol{e}^T(\boldsymbol{x}*\boldsymbol{r} - \mu\boldsymbol{e}) = n\mu - \boldsymbol{r}^T\boldsymbol{x} = 0$. By (8.68) we have $||\boldsymbol{x}*\boldsymbol{r}-\mu\boldsymbol{e}||^2 \le \Theta\mu$ and from the definition of $\widehat{\mu}$ we have $n(\widehat{\mu} - \mu)^2 = \mu^2\delta^2$. Combining the estimations we find

$$||\widehat{\boldsymbol{x}} * \widehat{\boldsymbol{r}} - \widehat{\mu}\boldsymbol{e}|| \le \frac{(\Theta^2 + \delta^2)\mu^2}{2xr_{min}} \le \frac{(\Theta^2 + \delta^2)\mu^2}{2\mu(1-\Theta)} \le \Theta\mu\left(1 - \frac{\delta}{\sqrt{n}}\right) = \Theta\widehat{\mu} \ ,$$

where we have used (8.69) and thus (ii) follows.

To prove (iii) we note that $\boldsymbol{A}\widehat{\boldsymbol{x}} = \boldsymbol{b}$ and $\boldsymbol{A}^T\widehat{\boldsymbol{u}} + \widehat{\boldsymbol{r}} = \boldsymbol{c}^T$ by definition. So all we have to show is that $\widehat{\boldsymbol{x}} > \boldsymbol{0}$ and $\widehat{\boldsymbol{r}} > \boldsymbol{0}$. Suppose to the contrary that $\widehat{x_i} \le 0$ or $\widehat{r_i} \le 0$ for some $i \in \{1, \ldots, n\}$. From part (ii) it follows that $\widehat{x}_i\widehat{r}_i \ge \widehat{\mu}(1-\Theta) > 0$ and thus $\widehat{x}_i < 0$ and $\widehat{r}_i < 0$. Consequently, $r_i < -\Delta r_i$ and $x_i < -\Delta x_i$. From $r_i > 0$ and $x_i > 0$ it follows that $x_i r_i < \Delta x_i \, \Delta r_i$. From part (ii) we get $0 < \Delta x_i \, \Delta r_i \le ||\Delta\boldsymbol{x} * \Delta\boldsymbol{r}|| = ||\widehat{\boldsymbol{x}} * \widehat{\boldsymbol{r}} - \widehat{\mu}\boldsymbol{e}|| \le \Theta\widehat{\mu} < \Theta\mu$. Thus, $\Theta\mu > x_i r_i \ge (1-\Theta)\mu$ by (8.68) which contradicts $\Theta \le 1/2$. ■

The Newtonian algorithm takes the data of linear program (LP), a "reduction" parameter δ, a number p for the desired precision as well as a feasible triplet $(\boldsymbol{x}^0, \boldsymbol{u}^0, \boldsymbol{r}^0)$ as inputs and can be stated as follows. Note that the "informational content" of both $\boldsymbol{c}$ and $\boldsymbol{b}$ for the algorithm is passed on through the starting points $\boldsymbol{x}^0$, $\boldsymbol{u}^0$, $\boldsymbol{r}^0$.

Newtonian Algorithm (δ, p, m, n, $\boldsymbol{A}$, $\boldsymbol{x}^0$, $\boldsymbol{u}^0$, $\boldsymbol{r}^0$)

Step 0: Set $k := 0$, $\boldsymbol{R}_0 := diag(r_1^0, \ldots, r_n^0)$, $\boldsymbol{D}_0 := diag(x_1^0, \ldots, x_n^0)$, and $\mu_0 = (\boldsymbol{r}^0)^T\boldsymbol{x}^0/n$.

Step 1: **if** $\mu^k/\mu^0 \le 2^{-p}$ **stop** "$(\boldsymbol{x}^k, \boldsymbol{u}^k, \boldsymbol{r}^k)$ is a p-optimal triplet."

Step 2: Set $\mu_{k+1} = \mu_k(1 - \delta/\sqrt{n})$ and solve $\boldsymbol{R}_k\Delta\boldsymbol{x} + \boldsymbol{D}_k\Delta\boldsymbol{r} = \mu_{k+1}\boldsymbol{e} - \boldsymbol{r}^k * \boldsymbol{x}^k$, $\boldsymbol{A}\Delta\boldsymbol{x} = \boldsymbol{0}$, $\boldsymbol{A}^T\Delta\boldsymbol{u} + \Delta\boldsymbol{r} = \boldsymbol{0}$ for $(\Delta\boldsymbol{x}, \Delta\boldsymbol{u}, \Delta\boldsymbol{r})$.

Step 3: Set $\boldsymbol{x}^{k+1} = \boldsymbol{x}^k + \Delta\boldsymbol{x}$, $\boldsymbol{u}^{k+1} = \boldsymbol{u} + \Delta\boldsymbol{u}$, $\boldsymbol{r}^{k+1} = \boldsymbol{r} + \Delta\boldsymbol{r}$, $\boldsymbol{D}_{k+1} = diag(x_1^{k+1}, \ldots, x_n^{k+1})$, $\boldsymbol{R}_{k+1} = diag(r_1^{k+1}, \ldots, r_n^{k+1})$; replace $k+1$ by k; **go to** Step 1.

For the solution of the system of linear equations in Step 2 of the algorithm we use the formulas (8.64), (8.65), (8.66) with μ replaced by $\widehat{\mu}$ which simplify because $\boldsymbol{f} = \boldsymbol{0}$ and $\boldsymbol{g} = \boldsymbol{0}$ in the case of feasible triplets $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r})$.

Remark 8.11 (Correctness and finiteness) *For every $\delta \in \mathbb{R}$ with $0 < \delta < \sqrt{n}$ and $\Theta \in \mathbb{R}$ satisfying (8.69) the Newtonian algorithm iterates at most $O(p\sqrt{n})$ times where $p \ge 1$ is any integer and where $(\boldsymbol{x}^0, \boldsymbol{u}^0, \boldsymbol{r}^0)$ is any feasible triplet satisfying $||\boldsymbol{r}^0 * \boldsymbol{x}^0 - \mu_0\boldsymbol{e}|| \le \Theta\mu_0$ for $\mu_0 = (\boldsymbol{r}^0)^T\boldsymbol{x}^0/n$.*

Proof. From Remark 8.10 it follows by induction that $\boldsymbol{c}\boldsymbol{x}^k - \boldsymbol{b}^T\boldsymbol{u}^k = n\mu_k$, $\|\boldsymbol{r}^k * \boldsymbol{x}^k - \mu_k\boldsymbol{e}\| \le \Theta\mu_k$, that every triplet $(\boldsymbol{x}^k, \boldsymbol{u}^k, \boldsymbol{r}^k)$ is a feasible triplet and that $\mu_k = \mu_0(1 - \delta/\sqrt{n})^k$ for $k = 1, 2, \ldots$. Consequently

$$0 \le \frac{\boldsymbol{c}\boldsymbol{x}^k - \boldsymbol{b}^T\boldsymbol{u}^k}{\boldsymbol{c}\boldsymbol{x}^0 - \boldsymbol{b}^T\boldsymbol{u}^0} = \frac{n\mu_k}{n\mu_0} = \left(1 - \frac{\delta}{\sqrt{n}}\right)^k \le 2^{-p}$$

for all $k \ge p\sqrt{n}\log 2/\delta$ where we have used $1 - x \le e^{-x}$ for all x. Thus the Newtonian algorithm stops after at most $\mathcal{O}(p\sqrt{n})$ iterations. ■

Like in the case of the projective algorithm of Chapter 8.5 we conclude that we get $\mathcal{O}(\sqrt{n}L)$ convergence for a linear program of digital size L. Moreover, we can start the Newtonian algorithm by essentially the same trick that we have used to start the projective algorithm. Consider the linear program (LP′) of Chapter 8.5 in $n + 2$ variables and $m + 1$ equations. Then $x_j^0 = \kappa$ for $1 \le j \le n + 2$ is a feasible interior point $\boldsymbol{x}^0$ to (LP′). To obtain a suitable dual solution $(\boldsymbol{u}^0, \boldsymbol{r}^0)$ where $\boldsymbol{u}^0 \in \mathbb{R}^{m+1}$ we set $u_j^0 = 0$ for $1 \le j \le m$ and choose u_{m+1}^0 as follows: let $r_j^0 = c_j' - u_{m+1}^0$ for $1 \le j \le n+2$ where $c_j' = c_j$ for $1 \le j \le n$, $c_{n+1}' = 0$, $c_{n+2}' = M$. By making u_{m+1}^0 a small enough negative number we get $\boldsymbol{r}^0 > \boldsymbol{0}$. To satisfy the "closeness" criterion (8.68) we calculate $\mu_0 = (\boldsymbol{r}^0)^T\boldsymbol{x}^0/(n + 2) = \kappa(C - u_{m+1}^0)$ where $C = \sum_{j=1}^{n+2} c_j'/(n + 2)$. So

$$\|\boldsymbol{r}^0 * \boldsymbol{x}^0 - \mu^0\boldsymbol{e}\| = \kappa\sqrt{\sum_{j=1}^{n+2}(c_j' - C)^2} = \kappa C^* \ .$$

Choosing e.g. $\delta = \Theta = 0.35$ and u_{m+1}^0 such that $u_{m+1}^0 \le \min\{-1, C - C^*/\Theta\}$ the various assumptions that we have made are all satisfied for (LP′). Thus we can start the Newtonian algorithm. The remainder of the discussion of its validity and complexity goes just like in the case of the projective algorithm.

In the right part of Figure 8.11 we display the sequence of iterates of the Newtonian algorithm for the data of Exercise 8.2 (ii) when started with $x_1^0 = x_2^0 = 15$, $\boldsymbol{r}^0 = \boldsymbol{c}^T - \boldsymbol{A}^T\boldsymbol{u}^0$, $u_1^0 = -0.1$, $u_2^0 = -0.4$, $u_3^0 = -0.2$, $u_4^0 = -0.3$ and $\delta = 0.40$. For the resulting initial feasible triplet $(\boldsymbol{x}^0, \boldsymbol{u}^0, \boldsymbol{r}^0) \in \mathbb{R}^{16}$ the measure of proximity to the log-central path is satisfied with $\Theta = 0.45$. In this instance at least, the algorithm first "seeks" the log-central path and then follows it in very cautious step sizes to the optimal face which translates into slow convergence of the algorithm. This is in stark contrast to the bold steps – initially at least – of the "practical" iterative scheme, even when started at an off-center point.

Projective transformations apparently do not enter into the derivation of the Newtonian algorithm and you will find other algorithms in the literature that are similar to the one discussed here. So why did we treat this subject in the context of projective algorithms? To get a glimpse on the connection to projective algorithms let us take a closer look at the updating formulas (8.64), (8.65), (8.66) for a feasible triplet $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r})$. Remember that for a feasible triplet we have $\boldsymbol{x} > \boldsymbol{0}$, $\boldsymbol{r} > \boldsymbol{0}$, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{A}^T\boldsymbol{u} + \boldsymbol{r} = \boldsymbol{c}^T$. Thus – as far as the first order conditions (8.57), (8.58), (8.59) for the optimization problem (P_μ) are concerned – only the conditions (8.59) may be violated. So let us assume that our feasible triplet satisfies (8.59) as well: this means that we assume that the current iterate $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r})$ of the Newtonian algorithm lies on the log-central path of $\mathcal{X}$. In this case the matrix $\boldsymbol{T}$ of (8.63) becomes simply $\frac{1}{\sqrt{\mu}}\boldsymbol{D}$ and we calculate

$$\boldsymbol{S} = \boldsymbol{I}_n - \boldsymbol{D}\boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{D}^2\boldsymbol{A}^T)^{-1}\boldsymbol{A}\boldsymbol{D}\,,$$

which is exactly the orthogonal projection operator $\boldsymbol{P}$ of Chapter 8.1.1, see (8.4), where $\boldsymbol{x} = \boldsymbol{x}^0$ is the current iterate. From (8.64) for $\mu = \widehat{\mu}$ we find using $\boldsymbol{x}^{new} = \boldsymbol{x} + \Delta\boldsymbol{x}$ that

$$\boldsymbol{x}^{new} = \boldsymbol{x} - \frac{1}{\mu}\boldsymbol{D}\left(\boldsymbol{p} - \widehat{\mu}\boldsymbol{d}\right)\ , \tag{8.70}$$

where $\boldsymbol{p}$ and $\boldsymbol{d}$ are defined in relation (8.4) as well and $\widehat{\mu} = \mu(1 - \delta/\sqrt{n})$ as required in Step 2 of the algorithm. Comparing the formula for $\boldsymbol{x}^{new}$ with formulas (8.9) and (8.17) the similarity is evident. Thus if the current iterate $\boldsymbol{x}$ lies on the central path of $\mathcal{X}$ the updating formula for $\boldsymbol{x}^{new}$ can be interpreted as having being obtained via some approximation to the problem (FLP_ρ) utilizing the projective transformation T_0 of the introduction to this chapter. Thus, if we "skid along" the log-central path of $\mathcal{X}$, we execute some sort of a projective algorithm. The particular combination of the vectors $\boldsymbol{D}\boldsymbol{p}$ and $\boldsymbol{D}\boldsymbol{d}$ in (8.70) is clearly only one of many possible choices; every point of the projective curve $\boldsymbol{y}(\rho)$ yields a direction of *descent*, see Exercise 8.8, while the line defined by (8.70) does not guarantee monotonicity of successive objective function values – just like the initial line (8.9). The nonmonotonicity of the Newtonian algorithm follows because from (8.70) we calculate

$$\boldsymbol{c}\boldsymbol{x}^{new} = \boldsymbol{c}\boldsymbol{x} - \left[||\boldsymbol{p}||^2/\mu - (1 - \delta/\sqrt{n})\gamma\right]$$

where $\gamma = \boldsymbol{p}^T\boldsymbol{d}$. Thus $\boldsymbol{c}\boldsymbol{x}^{new} > \boldsymbol{c}\boldsymbol{x}$ if the term in the square brackets is negative. The nonmonotonicity can be observed e.g. when the algorithm of Chapter 8.6.2 is run for the starting point $x_1^0 = x_2^0 = 15$ as done above and 0.95 is replaced by 0.995 in the formulas for α_p and α_d. Whenever the current iterate $\boldsymbol{x}$ is off the central path a direct comparison is a bit more complicated: the projective transformation T_0 based on the matrix $\boldsymbol{T}$ of (8.63) does not put $\boldsymbol{x}$ into the center of the simplex S^{n+1}.

Exercise 8.14 *Write a computer program for the Newtonian algorithm using the formulas (8.64), (8.65), (8.66) and any subroutine for inverting a square matrix. Use your program to reproduce the path to optimality starting at* $x_1 = x_2 = 15$ *for Figure 8.11 and Exercise 8.2.*

8.7 Coda

Πάντα ῥεῖ, πάντα χωρεῖ καὶ οὐδὲν μένει.
Heraklitos of Ephesos (c. 480 B.C.)

If the above quotation is still *Greek* to you, don't be surprised. It *is* and it is not about *accelerant* change, mind you, just *change.* "Everything is in flux, everything occupies space and nothing stays" is what this one – freely – translates to. If you need a more elaborate translation, please do *not* call Heraklitos – he would not understand you. Consult a dictionary or some philosophical tract instead. And most certainly, the author would not recommend to call Robert Hooke (1635-1703), Sir Isaac Newton (1642-1727), Gottfried Wilhelm von Leibniz (1646-1716) nor *le comte* Joseph Louis de Lagrange (1736-1813) on the matter of interior point methods for large-scale linear programs either – *più o meno* for the same reason. Heraklit's dictum – though some 2,400 years old – should interest contemporary organization theorists, it applies very well to your children and yes – also to mathematics. Or should it not? Πάντα ῥεῖ – change has taken place since the basic algorithmic idea of projective algorithms was published in 1984 – see Chapter 1 – and indeed, a whole *flood* of articles has appeared in the scientific literature dealing with improvements, the acceleration, historical predecessors of and, most importantly, computational experiments with the *basic* algorithm and its derivatives. Πάντα χωρεῖ – just *count* the number of *printed* pages. (*Nota bene* : the references to this section show only the tip of an iceberg! A 1993 bibliographical compilation lists over 1,380 entries of *post* 1984 works related to interior point methods.) Comparatively little work has been done on the underlying *projective geometry* and this is what we have chosen to focus on in this chapter. It is entirely possible that studies along these lines will lead to a further acceleration of the solution methods for linear programming. *Variable*, rather than *fixed*, step-sizes for projective algorithms are a promising avenue to pursue, though their analysis does not appear to be easy. In any case, to give *you* an opportunity to know what the author did *not* treat in this chapter we have included some references that we believe to be representative of the contemporary *currents* in linear computation utilizing interior point methods which of course – like everything else or so it seems – are subject to change at a rapidly accelerating pace – καὶ οὐδὲν μένει.