3. A person wants to invite 8 friends but there is only room for 4 of them. In how many ways can the four to be invited be chosen if two of the eight are twins and must not be separated?

   Solution

   There are two cases to consider

   i) The twins are invited: thus, we have to choose 2 out of the remaining 6 in ${}^6C_2 = 15$ ways
   ii) The twins are left out: thus, we have to choose 4 out of the remaining 6 in ${}^6C_4 = 15$ ways

   Therefore, the total number of ways is $15 + 15 = 30$.

4. Nine people are going to travel in two taxi. The larger has 5 seats and the smaller has 4 seats. In how many ways can the party be split?

   Solution

   Once the group of 5 has been selected, then the remaining 4 people will automatically comprise the other group. Thus, we have to select 5 from 9 in ${}^9C_5 = 126$ ways.

5. Five books are to be selected from 20 books of which 8 are paperback and 12 are hardback. How many selections are possible if at least one paperback book has to be included?

   Solution

   - If there is no restriction, we have ${}^{20}C_5 = 15,504$ ways of selecting the 5 books
   - If only hardback books are selected, we have ${}^{12}C_5 = 792$ ways of selecting the 5 books

   Therefore, the number of ways of selecting at least one paperback book is $15,504 - 792 = 14,712$.

Exercise:

1. A committee of six is to be formed from nine women and three men. In how many ways can the members be chosen so as to include at least one man? [ans: 840 ways]
2. A committee of ten is to be formed from nine men and six women. In how many ways can it be formed if at least four women are to be in the committee? [ans: 2142 ways]
3. In how many ways can a class of 20 children be split into two groups of 8 members and 12 members, respectively if there are two twins in the class who must not be separated. [ans: ${}^{18}C_6 + {}^{18}C_8 = 62,322$ ways]
4. Nine players are available to play for a table tennis team of 4 players. In how many ways can the team be selected if 2 of the players are brothers and must either both be included or both be excluded and if 2 other players have recently quarreled and should not both play in a team? [ans: 45 ways]

LECTURE 6

1.7 Binomial expansion

A binomial is the sum or difference of two terms. For example,

$$
\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4
\end{aligned}
$$

If the coefficients are written alone, we have the following triangle called the Pascal's triangle.

$n=0:$							1						
$n=1:$						1		1					
$n=2:$					1		2		1				
$n=3:$				1		3		3		1			
$n=4:$			1		4		6		4		1		
$n=5:$		1		5		10		10		5		1	
$n=6:$	1		6		15		20		15		6		1

Example(s):

1. Expand $(a+b)^7$.

   Solution

   From the Pascal's triangle, the coefficients are 1,7,21,35,35,21,7,1. Therefore,

$$(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

2. Obtain the expansion of $\left(2x - \frac{1}{2}\right)^4$ in descending powers of x.

   Solution

   From the Pascal's triangle, the coefficients are 1,4,6,4,1. Therefore,

$$\begin{aligned}\left(2x-\frac{1}{2}\right)^4 &= (2x)^4 + 4(2x)^3\left(-\frac{1}{2}\right) + 6(2x)^2\left(-\frac{1}{2}\right)^2 + 4(2x)\left(-\frac{1}{2}\right)^3 + \left(-\frac{1}{2}\right)^4 \\ &= 16x^4 - 16x^3 + 6x^2 - x + \frac{1}{16}\end{aligned}$$

Exercise:
Use the Pascal's triangle to expand the following binomials

(a) $(1-x)^3$

(b) $(x+\frac{1}{x})^4$

(c) $(2x-1)^5$

(d) $(x+y)^7$

1.7.1 Binomial theorem

If n is a positive integer, then

$$\begin{aligned}(a+b)^n &= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n-1}an^{n-1} + \binom{n}{n}b^n \\ &= \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r,\end{aligned}$$

where $\binom{n}{r} =^n C_r = \dfrac{n!}{(n-r)!r!}$. From the statement of the theorem, the term containing $a^{n-r}b^r$ is $\binom{n}{r}a^{n-r}b^r$.

Example(s):

1. Find the coefficients of x^{10} in the expansion of $(2x-3)^{14}$.

   Solution

   The general term in the expansion is given by

$$\binom{14}{r}(2x)^{14-r}(-3)^r = \binom{14}{r}2^{14-r}(-3)^r \cdot x^{14-r}$$

   The term in x^{10} is obtained when $14-r=10 \quad \Rightarrow \quad r=4$. Therefore, the required coefficient is $\binom{14}{4}2^{14-4}(-3)^4 = \dfrac{14!}{10!4!}2^{10}3^4$.

2. Find the coefficient of x in the expansion of $\left(2x^2-\frac{1}{x^2}\right)^{50}$.

   Solution

   The general term in the expansion is given by

$$\binom{50}{r}(2x^2)^{50-r}\left(-\frac{1}{x^2}\right)^r = \binom{50}{r}2^{50-r}(-1)^r \cdot \frac{x^{100-2r}}{x^{2r}} = \binom{50}{r}2^{50-r}(-1)^r x^{100-4r}$$

   The term in x is obtained when $100-4r=1 \quad \Rightarrow \quad r=\dfrac{99}{4}$. But r is not a whole number. Thus, the expansion doesn't have a term in x.

3. Obtain the first four terms of the expansion $\left(1+\frac{1}{2}x\right)^{10}$ in ascending powers of x. Hence, find the value of $(1.005)^{10}$, correct to 4dp.

   Solution

$$\begin{aligned}\left(1+\frac{1}{2}x\right)^{10} &= \binom{10}{0}(1)^{10}+\binom{10}{1}(1)^9\left(\frac{1}{2}x\right)+\binom{10}{2}(1)^8\left(\frac{1}{2}x\right)^2+\binom{10}{3}(1)^7\left(\frac{1}{2}x\right)^3+\cdots \\ &= 1+5x+\frac{45}{4}x^2+15x^3+\cdots\end{aligned}$$

   Now, $(1.005)^{10}=(1+0.005)^{10}$ and comparing with $\left(1+\frac{1}{2}x\right)^{10}$, we get

$$\frac{1}{2}x = 0.005 \quad \Rightarrow \quad x = 0.01$$

   Substituting this value of x in the above expansion, we obtain:

$$\begin{aligned}(1+0.005)^{10} &= 1+5(0.01)+\frac{45}{4}(0.01)^2+15(0.01)^3+\cdots \\ &= 1+0.05+0.001125+0.000015+\cdots \\ &= 1.0511 \quad \text{(to 4dp)}\end{aligned}$$

4. Obtain the expansion of $(1+x-2x^2)^8$ up to the term in x^3.

   Solution

$$\begin{aligned}\left(1+x-2x^2\right)^8 &= \left[1+(x-2x^2)\right]^8 \\ &= 1+\binom{8}{1}(x-2x^2)+\binom{8}{2}(x-2x^2)^2+\binom{8}{3}(x-2x^2)^3+\cdots \\ &= 1+8(x-2x^2)+28(x^2-4x^3+\cdots)+56(x^3+\cdots)+\cdots \\ &= 1+8x+12x^2-56x^3+\cdots\end{aligned}$$

Exercise:

1. Write down the coefficients of the terms indicated in the expansions of the following:
   (a) $(1+x)^{16}$, 3rd term.
   (b) $(3+2x)^6$, 4th term.
   (c) $(2-x)^{20}$, 18th term.
   (d) $(2+\frac{3}{2}x)^8$, 5th term.
2. In the expansion of $(1-2x+ax^2)^4$ as a series of powers of x, the coefficient of x^3 is zero. Show that $a=-\dfrac{4}{3}$ and find the coefficient of x^4. [ans: $=-37\frac{1}{3}$]
3. Use Binomial theorem to find the values of
   (a) $(1.01)^{10}$, correct to 3dp.
   (b) $(2.001)^{10}$, correct to 6 sf.
   (c) $(0.997)^{12}$, correct to 3dp.
   (d) $(1.998)^8$, correct to 2dp.
4. Expand the following as far as the term in x^3.
   (a) $(1+x+x^2)^3$
   (b) $(1+2x-x^2)^6$
   (c) $(2+x-2x^2)^7$
   (d) $(3-2x+x^2)^4$
5. Find the ratio of the term in x^7 to the term in x^8 in the expansion of $(3x+\frac{2}{3})^{17}$. [ans: $=\dfrac{8}{45x}$]
6. (a) Find, in factorial form, the coefficient of x in the expansion of $(2x^2-\frac{1}{x})^{50}$. [ans: $=-\dfrac{50!}{17!33!}2^{17}$]
   (b) Find the coefficient of y^8 in the expansion of $(2y+3)^{10}$. [ans: $=\dfrac{10!}{8!2!}2^8\cdot 3^2$]

1.7.2 Binomial theorem for any Index

For any rational number n, where n is not a positive integer, the binomial theorem is given by

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots,$$

provided $-1<x<1$, i.e., $|x|<1$. To expand $(a+x)^n$ (where $a\neq 1$), we first need to express the binomial as $(a+x)^n=\left[a(1+\frac{x}{a})\right]^n=a^n\left(1+\frac{x}{a}\right)^n$ and expand the expression in terms of $\dfrac{x}{a}$. Since the expansion holds only for certain values of x, this set of values must be stated.

Example(s):

1. Expand the following in ascending powers of x as far as the term in x^3, stating the range of values of x for which the expansion is valid.
   (a) $(1+x)^{\frac{1}{3}}$
   (b) $(1-4x)^{-3}$
   (c) $(3-x)^{-2}$

Solution

(a)

$$\begin{aligned}(1+x)^{\frac{1}{3}} &= 1+\left(\frac{1}{3}\right)x+\frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)}{2!}x^2+\frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}x^3+\cdots\\ &= 1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3+\cdots \quad (\text{provided } -1<x<1 \text{ i.e., } |x|<1)\end{aligned}$$

(b)

$$\begin{aligned}(1-4x)^{-3} &= 1+(-3)(-4x)+\frac{(-3)(-3-1)}{2!}(-4x)^2+\frac{(-3)(-3-1)(-3-2)}{3!}(-4x)^3+\cdots\\ &= 1+12x+96x^2+640x^3+\cdots \quad (\text{provided } -1<-4x<1 \text{ or } -\frac{1}{4}<x<\frac{1}{4})\end{aligned}$$

(c) $(3-x)^{-2}=3^{-2}(1-\frac{x}{3})^{-2}$. Thus,

$$\begin{aligned}3^{-2}(1-\frac{x}{3})^{-2} &= \frac{1}{9}\left[1+(-2)\left(-\frac{x}{3}\right)+\frac{(-2)(-2-1)}{2!}\left(-\frac{x}{3}\right)^2+\frac{(-2)(-2-1)(-2-2)}{3!}\left(-\frac{x}{3}\right)^3+\cdots\right]\\ &= \frac{1}{9}\left[1+\frac{2}{3}x+\frac{1}{3}x^2+\frac{4}{27}x^3+\cdots\right]\\ &= \frac{1}{9}+\frac{2}{27}x+\frac{1}{27}x^2+\frac{4}{243}x^3+\cdots \quad (\text{provided } -1<-\frac{x}{3}<1 \text{ or } -3<x<3)\end{aligned}$$

2. Expand $\dfrac{4}{(1+4x)(1-2x)}$ as far as the term in x^3, stating the range of values of x for which the expansion is valid.

Solution

$$\begin{aligned}\frac{4}{(1+4x)(1-2x)} &= \frac{4}{1+2x-8x^2}=4\left[1+(2x-8x^2)\right]^{-1}\\ &= 4\left[1+(-1)(2x-8x^2)+\frac{(-1)(-2)}{2!}(2x-8x^2)^2+\frac{(-1)(-2)(-3)}{3!}(2x-8x^2)^3\cdots\right]\\ &= 4\left[1-2x+8x^2+(4x^2-32x^3+\cdots)-(8x^3+\cdots)\cdots\right]\\ &= 4\left[1-2x+12x^2-40x^3+\cdots\right]\\ &= 4-8x+48x^2-160x^3+\cdots\end{aligned}$$

Since we expanded $4(1+4x)^{-1}(1-2x)^{-1}$, the expansion is valid when $-1<4x<1$ and $-1<-2x<1$. That is, $-\frac{1}{4}<x<\frac{1}{4}$ and $-\frac{1}{2}<x<\frac{1}{2}$. This is so when $-\frac{1}{4}<x<\frac{1}{4}$.

3. Find the first four terms in the expansion of $\sqrt{1-8x}$ in ascending powers of x. Hence, substitute $x=0.01$ and obtain the value of $\sqrt{23}$ correct to 5 significant figures.

Solution

$$\begin{aligned}\sqrt{1-8x} &= (1-8x)^{\frac{1}{2}}=1+\frac{1}{2}(-8x)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(-8x)^2+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(-8x)^3+\cdots\\ &= 1-4x-8x^2-32x^3+\cdots\end{aligned}$$

Putting $x=0.01$ yields $\sqrt{1-8(0.01)}=\sqrt{0.92}=1-4(0.01)-8(0.01)^2-32(0.01)^3\approx 0.959168$. Now, expressing $\sqrt{23}$ in terms of $\sqrt{0.92}$ yields:

$$\sqrt{23}=\sqrt{\frac{92}{4}}=\sqrt{\frac{100\times 0.92}{4}}=5\sqrt{0.92}=5\times 0.959168=4.7958 \quad (\text{to 5 sf})$$

4. Expand $\sqrt{\dfrac{1+x}{1-x}}$ in ascending powers of x as far as the term in x^3.

Solution

$\sqrt{\dfrac{1+x}{1-x}} = (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$. Now,

$$\begin{aligned}
(1+x)^{\frac{1}{2}} &= 1+\frac{1}{2}x+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3+\cdots \\
&= 1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\cdots \\
(1-x)^{-\frac{1}{2}} &= 1-\frac{1}{2}(-x)+\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{2!}(-x)^2+\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x)^3+\cdots \\
&= 1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\cdots
\end{aligned}$$

Therefore,

$$\begin{aligned}
(1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}} &= \left(1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\cdots\right)\left(1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\cdots\right) \\
&= \left(1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3\right)+\frac{1}{2}x\left(1+\frac{1}{2}x+\frac{3}{8}x^2\right)-\frac{1}{8}x^2\left(1+\frac{1}{2}x\right)+\frac{1}{16}x^3(1)+\cdots \\
&= 1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\frac{1}{2}x+\frac{1}{4}x^2+\frac{3}{16}x^3-\frac{1}{8}x^2-\cancel{\frac{1}{16}x^3}+\cancel{\frac{1}{16}x^3}+\cdots \\
&= 1+x+\frac{1}{2}x^2+\frac{1}{2}x^3+\cdots
\end{aligned}$$

Exercise:

1. Expand the following as a series of ascending powers of x upto and including the term in x^3.
   (a) $(1-3x)^{-\frac{1}{2}}$
   (b) $\dfrac{1}{\sqrt{2+x^2}}$
   (c) $\dfrac{x+2}{(1+x)^2}$
   (d) $\dfrac{\sqrt{1+2x}}{1-x}$
   (e) $\dfrac{1-x}{\sqrt{1+x}}$
2. Show that if x is small enough for its cube and higher powers to be neglected, then $\sqrt{\dfrac{1-x}{1+x}} = 1-x+\dfrac{x^2}{2}$. By putting $x=\dfrac{1}{8}$, show that $\sqrt{7}\approx 2\frac{83}{128}$.
3. Expand $\sqrt{\dfrac{1+2x}{1-2x}}$ as a series of ascending powers of x upto and including the term in x^2. By putting $x=0.01$, find an approximation for $\sqrt{51}$ correct to 5 significant figures. [ans: $=1+2x+2x^2$, and $\sqrt{51}=7.1414$]

LECTURE 7