1.8 Sequences and series

Definition 1.8.1 (Sequence)**.** *A sequence is a list of numbers in a defined order with a rule for obtaining each of the numbers.*

The elements of a sequence are called *terms* of the sequence, where the nth term is denoted by u_n. For instance,

i) $5, 10, 15, 20, \cdots$ (multiples of 5; general term $5n,\ n \in \mathbb{N}$).

ii) $1, -3, 9, -27, \cdots$ (powers of -3; general term $(-3)^{n-1},\ n \in \mathbb{N}$).

iii) $27, 64, 125, 216, \cdots$ (cubes of consecutive integers in ascending order starting with 3^3; general term $(n+2)^3,\ n \in \mathbb{N}$).

Definition 1.8.2 (Finite and Infinite sequence)**.** *A sequence that is made up of a finite number of terms is called a finite sequence; otherwise, it is called an infinite sequence.*

For example,

i) the sequence $10, 20, 30, \cdots, 100$ is finite while

ii) the sequence $10, 20, 30, \cdots$ is infinite

Definition 1.8.3 (Series)**.** *A series is obtained by adding the terms of a sequence.*

A finite series is one that corresponds to a finite sequence while an infinite series is one that corresponds to an infinite sequence. For example,

i) $10 + 20 + 30 + \cdots + 100 = \displaystyle\sum_{n=1}^{10} 10n$ and $3^3 + 4^3 + \cdots + 10^3 = \displaystyle\sum_{n=3}^{10} n^3$ are finite series (each of 10 terms) while

ii) $1 + \dfrac{1}{4} + \dfrac{1}{16} + \dfrac{1}{64} + \dfrac{1}{256} + \cdots = \displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^{n-1}$ is an in finite series (in this case, n has no upper limit).

1.8.1 Arithmetic Progression (A.P.)

An AP is a sequence in which any term differs from the previous by a constant number. In this case, the constant number is called the common difference. If an AP has its first term a and its common difference d, then its nth term is given by

$$\boxed{u_n = a + (n-1)d},$$

where n is the number of terms in the AP.

Example(s):

1. Determine the first six terms of the AP whose first term is 18 and common difference is -5.

   Solution

   Given that $a = 18, d = -5$. The nth term is given by $u_n = a + (n-1)d = 18 - 5(n-1)$. Thus,

   $$u_1 = 18,\ \ u_2 = 13,\ \ u_3 = 8,\ \ u_4 = 3,\ \ u_5 = -2,\ \text{and}\ u_6 = -7$$

2. Determine the 4th and 12th terms of the AP whose first term is 19 and common difference is 6. [ans: $u_4 = 37,\ \ u_{12} = 85$]

The sum of an AP

Consider an AP whose first term is a, whose common difference is d and with n terms. That is,

$$\text{AP:}\quad a, (a+d), (a+2d), \cdots, a+(n-3)d, a+(n-2)d, a+(n-1)d$$

The method of first principles (Euler's method) can be used to determine a formula for the sum of the first n terms of this AP as follows: Let L be the last term of the AP, i.e., $L = a+(n-1)d$. Adding the terms of the AP yields

$$S_n = a + (a+d) + (a+2d) + \cdots + (L-2d) + (L-d) + L \qquad (i)$$

Similarly, adding the terms in reverse order yields

$$S_n = L + (L-d) + (L-2d) + \cdots + (a+2d) + (a+d) + a \qquad (ii)$$

Adding the two series (i) and (ii), we have

$$2S_n = (a+L) + (a+L) + (a+L) + \cdots + (a+L) + (a+L) + (a+L)$$

$$\Rightarrow \quad 2S_n = n(a+L) \quad \Rightarrow \quad S_n = \frac{n}{2}(a+L)$$

Substituting $L = a+(n-1)d$, we get the formula for finding the sum of the first n terms of an AP as

$$\boxed{S_n = \frac{n}{2}\Big[2a+(n-1)d\Big]}$$

Example(s):

1. The fourth term of an AP is 13 and the seventh term is 22. Determine:
   (a) the common difference
   (b) the value of n if the nth term is 100.
   (c) the sum of the first 20 terms
   (d) the value of m if the sum of the first m terms of the series 175.

   Solution

   (a) $u_4 = 13 \quad\Rightarrow\quad a+3d = 13 - - - (i)$ and $u_7 = 22 \quad\Rightarrow\quad a+6d = 22 - - - (ii)$. Solving equations (i) and (ii) simultaneously yields $a = 4, d = 3$. Thus, the common difference is 3.
   (b) $u_n = a+(n-1)d = 100 \quad\Rightarrow\quad 4+3(n-1) = 100 \quad\Rightarrow\quad n = 33$.
   (c) $S_{20} = \frac{20}{2}\Big[2(4)+3(20-1)\Big] = 10[8+3\times 19] = 650$.
   (d) $S_m = \frac{m}{2}\Big[2a+(m-1)d\Big] = 175 \quad\Rightarrow\quad m\Big[8+3(m-1)\Big] = 350 \quad\Rightarrow\quad 3m^2+5m-350 = 0$. Solving this quadratic equation yields the roots $m = -\frac{35}{3}$ or $m = 10$. Since $m \in \mathbb{N}$, then $m = 10$.

2. The sum of the first n terms of a series is given by $S_n = 2n(n+6)$. Find the fifth term of the series and show that the terms are in AP.

   Solution

   (i) Fifth term: $u_5 = S_5 - S_4 = 2\times 5(5+6) - 2\times 4(4+6) = 110 - 80 = 30$.

(ii) We know that $u_n = S_n - S_{n-1}$. Now,

$$\begin{aligned} S_n &= u_1 + u_2 + u_3 + \cdots + u_{n-1} + u_n = 2n(n+6) \\ S_{n-1} &= u_1 + u_2 + u_3 + \cdots + u_{n-1} = 2(n-1)\left[(n-1)+6\right] = 2(n-1)(n+5) \end{aligned}$$

Therefore,

$$u_n = 2n(n+6) - 2(n-1)(n+5) = 2n^2 + 12n - 2n^2 - 8n + 10 = 4n + 10$$

$$u_{n-1} = 4(n-1) + 10 = 4n + 6$$

Now, $u_n - u_{n-1} = (4n+10) - (4n+6) = 4$. Since 4 is the common difference regardless of the value of n, then the terms of the series form an AP.

3. Determine the sum of all integers between 1 and 1000 (inclusive) which are multiples of 5 or 8 or both.

Solution

- Let S be the sum of a series with multiples of 5 i.e., $S = 5 + 10 + 15 + \cdots + 1000$. This is an AP whose $a = 5, d = 5$ and with nth term given by $[5 + (n-1) \times 5] = 1000 \quad \Rightarrow \quad n = 200$. Thus,
$$S = \frac{200}{2}\Big[2(5) + (200-1) \times 5\Big] = 100\Big[10 + 199 \times 5\Big] = 100500$$
- Let T be the sum of a series with multiples of 8 i.e., $T = 8 + 16 + 24 + \cdots + 1000$. This is an AP whose $a = 8, d = 8$ and with nth term given by $[8 + (n-1) \times 8] = 1000 \quad \Rightarrow \quad n = 125$. Thus,
$$T = \frac{125}{2}\Big[2(8) + (125-1) \times 8\Big] = 62.5\Big[16 + 124 \times 8\Big] = 63000$$
- Let U be the sum of a series with multiples of both 5 and 8 i.e., $U = 40 + 80 + 120 + \cdots + 1000$. This is an AP whose $a = 40, d = 40$ and with nth term given by $[40 + (n-1) \times 40] = 1000 \quad \Rightarrow \quad n = 25$. Thus,
$$U = \frac{25}{2}\Big[2(40) + (25-1) \times 40\Big] = 12.5\Big[80 + 24 \times 40\Big] = 13000$$

Therefore, the sum of all integers between 1 and 1000 (inclusive) which are multiples of 5 or 8 or both is given by $S + T - U = 100500 + 63000 - 13000 = 150500$.

Exercise:

1. Find the sum of an AP of 10 terms whose first term is 7 and whose last term is 10.
2. Show that the terms of the series $\sum_{r=1}^{n} \log 5^r$ are in AP. Hence, find the sum of the first twenty terms of the series and also the least value of n for which the sum to n terms exceeds 400. [ans: $S_{20} = 146.78, \quad n = 34$]
3. Find the difference between the sums of the first ten terms of the AP whose first terms are 12 and 8, and whose common differences are 2 and 3, respectively.
4. The first term of an AP is -12 and the last term is 40. If the sum of the progression is 196, find the number of terms and the common difference.
5. Find the sum of the odd numbers between 100 and 200.
6. Find the sum of the even numbers which are divisible by 3 and lie between 400 and 500.
7. The twenty-first term of an AP is $5\frac{1}{2}$, and the sum of the first twenty-one terms is $94\frac{1}{2}$. Find the first term, the common difference and the sum of the first 30 terms.

8. In an AP, the thirteenth term is 27, and the seventh term is three times the second term. Find the first term, the common difference and the sum of the first ten terms. [ans: $a = 3, d = 2$ and $S_{10} = 120$]

9. [***Assignment 2***] Determine the sum of all natural numbers between 1 and 200 (inclusive) which are not divisible by 9 or 12. [ans: = 16731]

10. Evaluate $60 + 64 + 68 + 72 + \cdots + 120$ [ans: =1440]

1.8.2 Geometric Progression (G.P.)

A GP is a sequence in which the ration between two consecutive terms is the same. In this case, the constant multiplying factor is called the common ratio. In general, if a GP has its first term a and common ratio r, then the nth term is given by

$$\boxed{u_n = ar^{n-1}},$$

where n is the number of terms in the GP. Thus, a GP having n terms is given as

$$a, ar, ar^2, ar^3, \cdots, ar^{n-1}$$

For example, the sequence $36, 12, 4, \frac{4}{3}, \frac{4}{9}, \frac{4}{27}, \cdots,$ is a GP with $a = 36$ and $r = \frac{1}{3}$.

The sum of a GP

Consider a GP whose first term is a and common ratio is r, and with n terms. That is,

$$\text{GP:} \quad a, ar, ar^2, \cdots, ar^{n-3}, ar^{n-2}, ar^{n-1}$$

The method of first principles (Euler's method) can be used to determine a formula for the sum of the first n terms of a GP as follows: Adding the terms of the GP yields

$$S_n = a + ar + ar^2 + \cdots + ar^{n-3} + ar^{n-2} + ar^{n-1} \qquad (i)$$

Multiplying through by r, we get

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} + ar^n \qquad (ii)$$

Subtracting the two series (i) and (ii) yields

$$S_n - rS_n = a - ar^n \quad \Rightarrow \quad (1-r)S_n = a\,(1 - r^n)$$

Making S_n the subject yields the formula for finding the sum of the first n terms of a GP as

$$\boxed{S_n = \frac{a\,(1-r^n)}{1-r}, \quad |r| \neq 1}$$

→ Note: if $|r| > 1$, we write $S_n = \dfrac{a\,(r^n - 1)}{r - 1}$.

Example(s):

1. In a GP, the sum of the second and third terms is 6, and the sum of the third and fourth terms is -12. Find the first term, the common ratio and the sum of the first ten terms.

Solution

Let the first term be a and the common ration be r. Then, we have

$$ar + ar^2 = 6 \quad \Rightarrow \quad ar(1+r) = 6 \quad \text{and} \quad ar^2 + ar^3 = -12 \quad \Rightarrow \quad ar^2(1+r) = -12$$

Thus,

$$\frac{ar^2(1+r)}{ar(1+r)} = \frac{-12}{6} \quad \Rightarrow \quad r = -2$$

Substituting $r = -2$ in $ar(1+r) = 6$ yields $a = 3$. Since $|r| > 1$, we have

$$S_{10} = \frac{a\left(r^{10}-1\right)}{r-1} = \frac{3\left[(-2)^{10}-1\right]}{-2-1} = -\left[(-2)^{10}-1\right] = -1023$$

2. Determine the smallest number of terms of the GP $8+24+72+\cdots$ whose sum exceeds 6,000,000.

Solution

Here, $a = 8$ and $r = \frac{24}{8} = \frac{72}{24} = 3$. Since $|r| > 1$, we have

$$S_n = \frac{a\left(r^n-1\right)}{r-1} = \frac{8\left(3^n-1\right)}{3-1} = 4\left(3^n-1\right)$$

We require that $S_n > 6000000$. Thus, we have

$$4\left(3^n-1\right) > 6000000 \quad \Rightarrow \quad 3^n - 1 > 1500000 \quad \Rightarrow \quad 3^n > 1500001$$

From which we obtain

$$n > \frac{\log 1500001}{\log 3} = 12.94 \text{ to 4 s.f}$$

For the sum to exceed 6000000, then the smallest value of n is 13.

Exercise:

1. The fourth term of a GP is -6 and the seventh term is 48. Find the first three terms of the progression.
2. Find the difference between the sums to ten terms of the AP and GP whose first two terms are -2 and 4.
3. In a GP, the sum of the second and third terms is 9. The seventh term is eight times the fourth. Find the first term, the common ratio, and the fifth term.
4. If S_n is the sum of the first n terms of a GP whose first term is a and whose common ration is r, show that $\dfrac{S_{3n}-S_{2n}}{S_n} = r^{2n}$.

1.8.3 Applications of series

Example(s):

1. A customer makes a single deposit of Sh. 16000 in an account which pays compound interest at a rate of 6% p.a. Determine:
   (a) how much the investment is worth after 12 years
   (b) after how many years the investment will be worth three times its original value.

Solution

(a) Amount after

$$\begin{aligned}
\text{1 year:} \quad & 16000 + \frac{6}{100} \times 16000 = 16000(1.06) \\
\text{2 years:} \quad & 16000(1.06) + \frac{6}{100} \times 16000(1.06) = 16000(1.06)^2 \\
\text{3 years:} \quad & 16000(1.06)^2 + \frac{6}{100} \times 16000(1.06)^2 = 16000(1.06)^3
\end{aligned}$$

This pattern shows that the investment after n years will be

$$A = 16000(1.06)^n = 16000(1+\frac{6}{100})^n$$

So, the investment after 12 years will be $A = 16000(1.06)^{12} =$ Sh. $32,195$ (to the nearest shillings).

(b) We determine the value of n (in years) for which $16000(1.06)^n = 3 \times 16000$. Thus,

$$(1.06)^n = 3 \quad \Rightarrow \quad n = \frac{\log 3}{\log 1.06} = 18.85$$

Therefore, $n = 19$ complete years.

→ Note: in compound interest problems, the formula for the amount (value) A of a single investment (principal amount) P at a rate of $r\%$ per annum/unit time is

$$\boxed{A = P\left(1+\frac{r}{100}\right)^n}$$

This formula will be used in all cases where the value of an item appreciates/depreciates at a constant rate per unit time. However, in the case of depreciation r will be negative.

2. Sh. 100,000 was invested on 1st January 1990. An additional Sh. 6,000 was added to the investment at the beginning of each subsequent year. The investment earns a compound interest of 8% p.a. Find the value of the investment on 31st December 2000.

Solution

There are a total of 11 years between 1st Jan 1990 and 31st Dec 2000. Now, amount on

31st December,

$$\begin{aligned}
1990 \text{ (1 year)}: &\quad 100000 + \frac{8}{100} \times 100000 = 100000(1.08)\\
1991 \text{ (2 years)}: &\quad [100000(1.08)+6000] + \frac{8}{100} \times [100000(1.08)+6000] = 100000(1.08)^2 + 6000(1.08)\\
1992 \text{ (3 years)}: &\quad [100000(1.08)^2 + 6000(1.08) + 6000] \times 1.08 = 100000(1.08)^3 + 6000\left[1.08 + (1.08)^2\right]
\end{aligned}$$

This pattern shows that the investment after n years will be

$$A = 100000(1.08)^n + 6000\sum_{k=1}^{n-1}(1.08)^k$$

So, the investment after 11 years will be

$$\begin{aligned}
A &= 100000(1.08)^{11} + 6000\sum_{k=1}^{10}(1.08)^k\\
&= 100000(1.08)^{11} + 6000\left[1.08 + 1.08^2 + 1.08^3 + \cdots + 1.08^{10}\right]\\
&= 100000(1.08)^{11} + 6000\left\{\frac{1.08\,(1.08^{10}-1)}{1.08-1}\right\} = 100000(1.08)^{11} + 81000\left(1.08^{10}-1\right)\\
&= \text{Sh. } 327,037 \quad \text{(to the nearest shillings) after 11 years.}
\end{aligned}$$

3. A ball is dropped from a height of 9m. It hits the ground and bounces to a height of 6m. It continues to bounce up and down. On each bounce, it rises $\frac{2}{3}$ of the height of the previous bounce. How far has the ball traveled (both up and down) when it hits the ground for the seventh time?

Solution

The ball originally drops 9m and when the ball hits the ground for the 7th time, it has completed 6 bounces. So, the total distance traveled until the ball hits the ground for the seventh time is given by the series

$$9+\frac{2}{3}(9)(2)+\left(\frac{2}{3}\right)^2(9)(2)+\left(\frac{2}{3}\right)^3(9)(2)+\cdots+\left(\frac{2}{3}\right)^6(9)(2)$$

$$=9+\frac{a\left(1-r^6\right)}{1-r}, \quad \text{where} \quad a=\frac{2}{3}(9)(2) \quad \text{and} \quad r=\frac{2}{3}$$

$$=9+\frac{\frac{2}{3}(9)(2)\left[1-\left(\frac{2}{3}\right)^6\right]}{1-\frac{2}{3}}=9+36\left[1-\frac{64}{729}\right]=41.84\text{m}$$

Exercise:

1. A customer deposits Sh. 10,000 on first January every year for four years. How much is the investment worth at the end of the four years if it attracts a compound interest of 12% p.a.? [ans: Sh. 53,528]
2. A single deposit of Sh. 150,000 is invested for four years at compound interest. determine the rate at which the investment will be Sh. 182,326. [ans: $r=5\%$]
3. A car has an initial value of Sh. 3,000,000. If it depreciates at a rate of 14% p.a., determine after how many complete years the car will be worth Sh. 1,000,000. [ans: = 8 years]
4. A man deposits Sh. 2000 in his savings account on 1st January each year for 20 years. If the account gives 3% compound interest p.a. What will be the total value of his savings at the end of 20 years? (Answer correct to the nearest shillings). [ans: = Sh. 55353]
5. To save for her daughter's college education, Ms. Mirada decides to put aside $ 50 every month in a credit union account paying 10% interest compounded monthly. She begins this savings program when her daughter is 3 years old. How much will she have saved by the time she makes the 180th deposit? How old is her daughter at this time?
6. Given that an experiment culture has an initial population of 50 bacteria and the population increased by 80% every 20 minutes. Determine the time it will take to have a population of 1.2 million bacteria. [ans:]

1.8.4 Convergence of a series

If S_n is the sum of the first n terms of any series and if $\lim_{n\to\infty} S_n$ exists and is finite, then the series S_n is said to converge. Here, the sum to infinity is given by

$$S=S_\infty=\lim_{n\to\infty} S_n$$

If the series doesn't converge i.e., $\lim_{n\to\infty} S_n=\pm\infty$, then it is said to be divergent. We consider the AP and GP.

i) **Convergence of an AP:** any AP where $d>0$ will diverge i.e., $\lim_{n\to\infty}\frac{n}{2}[2a+(n-1)d]=\infty$ and if $d<0$, $\lim_{n\to\infty}\frac{n}{2}[2a+(n-1)d]=-\infty$ meaning it will diverge too.

ii) **Convergence of a GP:** recall that for a general GP, the sum to n terms is given by $S_n=\dfrac{a\left(1-r^n\right)}{1-r}$.

- If $|r|<1$, then $\lim_{n\to\infty} r^n=0$. So, $\lim_{n\to\infty} S_n=\lim_{n\to\infty}\dfrac{a\left(1-r^n\right)}{1-r}=\dfrac{a}{1-r}$. Hence, a GP converges to the sum $\dfrac{a}{1-r}$, provided $|r|<1$.

- If $|r| > 1$, then $\lim_{n\to\infty} r^n = \infty$ and the series diverges.

Example(s):

1. Consider the problem of the ball in example (3) above. Find the distance covered by the ball before coming to rest.

   Solution

   Theoretically, the ball bounces up and down indefinitely (practically, the ball comes to rest after a finite number of bounces). So, the distance traveled before coming to rest is

$$S_n = 9 + \frac{2}{3}(9)(2) + \left(\frac{2}{3}\right)^2 (9)(2) + \left(\frac{2}{3}\right)^3 (9)(2) + \cdots$$

$$S_\infty = 9 + \frac{a}{1-r} = 9 + \frac{\frac{2}{3}(9)(2)}{\left\{1 - \frac{2}{3}\right\}} = 9 + \frac{12}{\left\{1 - \frac{2}{3}\right\}} = 9 + 36 = 45\text{m}$$

2. The fifth term of a convergent GP is the arithmetic mean of the preceding two terms. Find the common ration given that the common ration and the first term are non-zero. If the first term is 12, find the sum to infinity.

   Solution

   i) Let the series be $a + ar + ar^2 + \cdots$. Then, $u_5 = ar^4 = \frac{1}{2}\left(ar^2 + ar^3\right)$.

$$\Rightarrow \quad 2ar^4 = ar^2\,(1+r) \quad \Rightarrow \quad 2r^2 - r - 1 = 0 \quad \Rightarrow \quad r = 1 \text{ or } r = -\frac{1}{2}$$

   Since the series is convergent, then $|r| < 1$. Therefore, $r = -\frac{1}{2}$.

   ii) When $a = 12$, the sum to infinity is given by $S_\infty = \frac{a}{1-r} = \frac{12}{1 - \left(-\frac{1}{2}\right)} = 8$.

3. Express $0.\dot{4}\dot{5}$ as a fraction.

   Solution

$$0.\dot{4}\dot{5} = 0.454545\cdots = \frac{45}{100} + \frac{45}{10000} + \frac{45}{1000000} + \cdots$$

   This is a GP with $a = \frac{45}{100}$ and $r = \frac{1}{100}$. Therefore,

$$0.\dot{4}\dot{5} = S_\infty = \frac{45/100}{\left\{1 - \frac{1}{100}\right\}} = \frac{45}{100} \times \frac{100}{99} = \frac{45}{99} = \frac{5}{11}$$

Exercise:

1. Express the following recurring decimals as rational numbers

   (a) $0.0\dot{7}$ [ans: $= \frac{7}{90}$]

   (b) $1.00\dot{4}$ [ans: $= 1\frac{1}{225}$]

   (c) $2.9\dot{6}\dot{0}$ [ans: $= 2\frac{317}{330}$]

   (d) $0.2\dot{4}\dot{5}\dot{6}$ [ans: $= 2\frac{409}{1665}$]