5. If $\log_{10} 2 = a$, show that $\log_8 5 = \dfrac{1-a}{3a}$.

Solution
Changing the base to 10, we get

$$\log_8 5 = \frac{\log_{10} 5}{\log_{10} 8} = \frac{\log_{10}\left(\frac{10}{2}\right)}{\log_{10} 2^3} = \frac{\log_{10} 10 - \log_{10} 2}{3\log_{10} 2} = \frac{1-a}{3a}$$

6. Solve for x if $\log_3 x + \log_9 x^2 = 6$.

Solution
Changing the base to 3, we get

$$\log_3 x + \log_9 x^2 = 6 \quad \Rightarrow \quad \log_3 x + \frac{\log_3 x^2}{\log_3 9} = 6 \quad \Rightarrow \quad \log_3 x + \frac{2\log_3 x}{2\log_3 3} = 6$$

$$\Rightarrow \quad \log_3 x + \log_3 x = 6 \quad \Rightarrow \quad 2\log_3 x = 6 \quad \Rightarrow \quad \log_3 x = 3 \quad \Rightarrow \quad x = 3^3 = 27$$

Exercise:

1. (a) Solve for x if $\left(\dfrac{2}{3}\right)^x = \dfrac{1}{16}$. [ans: $x =$]
   (b) Solve for x given that $\log_{10} 4 + 2\log_{10} x = 2$. [ans: $x = 5$]
2. Find y in terms of x if $\log\left(\dfrac{x^2}{y}\right) = 5 - 2\log x$. [ans: $y = \dfrac{x^4}{10^5}$]
3. Solve for x in the following equations.
   (a) $4^x - 6(2^x) - 16 = 0$. [ans: $x = 3$]
   (b) $\log_2 x = \log_x 16$. [ans: $x = 4$]
   (c) $\log_2 8x^3 - \log_x 8x^3 = 8$. [ans: $x = 8$ or $x = 2^{-1/3}$]
   (d) $\log_3 x - 2\log_x 3 = 1$. [ans: $x = 9$ or $x = \dfrac{1}{3}$]
   (e) $\log_x 2 + \log_x 3 = 5$. [ans: $x = 6^{1/5}$]
4. Show that $\log_{16} xy = \dfrac{1}{2}\log_4 x + \dfrac{1}{2}\log_4 y$. Hence, solve the simultaneous equations:

$$\log_{16} xy = \sqrt{3} \qquad (i) \quad \text{and} \quad \frac{\log_4 x}{\log_4 y} = -8 \qquad (ii)$$

Lecture 3

1.4 Quadratic equations, functions and inequalities

1.4.1 Quadratic equations

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where a, b and c are constants, x is a variable and $a \neq 0$.

I: **Methods of solving quadratic equations**

(a) Factorization method
(b) Completing the square

(c) Quadratic formula
(d) Graphical method

(a) **Factorization method**

It involves expressing the quadratic expression as a product of two linear functions (polynomials of degree 1) and then solving for the unknown. To factorize a quadratic expression of the form $ax^2 + bx + c$, we determine two numbers whose sum is b and product is ac.

Example(s):

1. Solve the equation $2x^2 - 5x - 3 = 0$ by factorization method.

Solution
We need to find two number whose sum is -5 and product is -6. Thus, the numbers are -6 and 1. So, $2x^2 - 5x - 3 = 0$ becomes

$$2x^2 - 6x + x - 3 = 0 \quad \Rightarrow \quad 2x(x-3) + 1(x-3) = 0 \quad \Rightarrow \quad (2x+1)(x-3) = 0$$

$$\Rightarrow \quad (2x+1) = 0 \quad \text{or} \quad (x-3) = 0 \quad \Rightarrow \quad x = -\frac{1}{2} \quad \text{or} \quad x = 3$$

Hence, $(2x+1)$ and $(x-3)$ are called (linear) factors of the quadratic expression $2x^2 - 5x - 3$ while $x = -\frac{1}{2}$ and $x = 3$ are called roots/zeros/solutions of the quadratic equation $2x^2 - 5x - 3 = 0$.

Exercise:

Solve the following quadratic equations using factorization method

(i) $x^2 - 10x + 24 = 0$.
(ii) $x^2 + 6x + 9 = 0$.
(iii) $4x^2 + 10x - 6 = 0$.
(iv) $5x^2 - 17x + 6 = 0$.
(v) $25x^2 - 9 = 0$.

(b) **Completing the square**

Consider the expansion of $(x+a)^2$ for a real number a:

$$(x+a)^2 = x^2 + 2ax + a^2$$

For all such expansions, the constant term is the square of half coefficient of x. This forms the basis for the solution of quadratic equations by completing the square method. Thus if x^2+bx+c is a complete square, then $\boxed{c = \left(\frac{b}{2}\right)^2}$.

→ Note: this method is appropriate when factorization is not possible.

Example(s):

1. Solve the equation $x^2 - 8x + 11 = 0$ by completing the square.

Solution
The given equation can be written as $x^2 - 8x = -11 \quad \Rightarrow \quad x^2 - 8x + c = -11 + c$, where $c = \left(\frac{-8}{2}\right)^2 = (-4)^2$. Thus, we have

$$x^2 - 8x + (-4)^2 = -11 + (-4)^2 \quad \Rightarrow \quad (x-4)^2 = 5 \quad \Rightarrow \quad (x-4) = \pm\sqrt{5}$$

$$\Rightarrow \quad x = 4 \pm \sqrt{5}. \quad \text{Either} \quad x = 4 + \sqrt{5} \quad \text{or} \quad x = 4 - \sqrt{5}$$

2. Solve the equation $5x^2 - 6x - 2 = 0$ by completing the square.

   Solution
   We first need to make the coefficient of x^2 be 1. Thus, the given equation can be written as $x^2 - \frac{6}{5}x = \frac{2}{5} \quad \Rightarrow \quad x^2 - \frac{6}{5}x + c = \frac{2}{5} + c$, where $c = \left(\frac{-6}{10}\right)^2 = \left(\frac{-3}{5}\right)^2$. Thus, we have

$$x^2 - \frac{6}{5}x + \left(\frac{-3}{5}\right)^2 = \frac{2}{5} + \left(\frac{-3}{5}\right)^2 \quad \Rightarrow \quad \left(x - \frac{3}{5}\right)^2 = \frac{19}{25} \quad \Rightarrow \quad \left(x - \frac{3}{5}\right) = \pm\sqrt{\frac{19}{25}} = \pm\frac{\sqrt{19}}{5}$$

$$\Rightarrow \quad x = \frac{3}{5} \pm \frac{\sqrt{19}}{5}. \quad \text{Either} \quad x = \frac{3 + \sqrt{19}}{5} \quad \text{or} \quad x = \frac{3 - \sqrt{19}}{5}$$

Exercise:

Solve by completing the square

(i) $2x^2 - 6x - 1 = 0$.
(ii) $x^2 + 7x - 3 = 0$.
(iii) $5x^2 + 12x + 6 = 0$.
(iv) $10 + 3x - 2x^2 = 0$.
(v) $2 - 2x - x^2 = 0$.
(vi) $-7 + 12x - 3x^2 = 0$.
(vii) $2x^2 + 6x + 13 = 0$.

(c) **Quadratic formula**

Solve the quadratic equation $ax^2 + bx + c = 0$ by completing the square method (where a, b and c are real numbers and $a \neq 0$).

Solution
We first need to make the coefficient of x^2 be 1. Thus, the given equation can be written as $x^2 + \frac{b}{a}x = -\frac{c}{a} \quad \Rightarrow \quad x^2 + \frac{b}{a}x + k = -\frac{c}{a} + k$, where $k = \left(\frac{b}{2a}\right)^2$. Thus, we have

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \quad \Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \quad \Rightarrow \quad \left(x + \frac{b}{2a}\right) = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\Rightarrow \quad x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \quad \Rightarrow \quad \boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}, \quad \text{which is the quadratic formula.}$$

The expression $\boxed{b^2 - 4ac}$ is known as the discriminant. The nature of the roots of a quadratic equation depend on the discriminant.

- If $b^2 - 4ac > 0$, then the roots of the quadratic equation $ax^2 + bx + c = 0$ are real are distinct.
- If $b^2 - 4ac = 0$, then the roots of the quadratic equation $ax^2 + bx + c = 0$ are real are equal.
- If $b^2 - 4ac < 0$, then the roots of the quadratic equation $ax^2 + bx + c = 0$ are complex conjugates.

Example(s):

1. Giving your answer correct to 2dp, solve $2x^2 - 6x - 3 = 0$ using the quadratic formula.

   Solution
   Here, $a = 2, b = -6$ and $c = -3$. Substituting these values into the quadratic formula, we get
   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm \sqrt{36 + 24}}{4} = \frac{6 \pm \sqrt{60}}{4} = \frac{6 \pm 2\sqrt{15}}{4} = \frac{3 \pm \sqrt{15}}{2}$$
   Either $x = \frac{3 + \sqrt{15}}{2} \approx 3.44$ or $x = \frac{3 - \sqrt{15}}{2} \approx -0.44$ (2dp).

2. Find the positive value of k if the equation $x^2 + (2 + k)x + k^2 = 0$ has equal roots.

   Solution
   Here, $a = 1, b = (2 + k)$ and $c = k^2$. For equal roots, we require that the discriminant be zero. That is, $b^2 - 4ac = 0$. Substituting yields $(2 + k)^2 - 4k^2 = 0 \quad \Rightarrow \quad 3k^2 - 4k - 4 = 0$, which is a quadratic equation in the unknown k. Using the quadratic formula, we have
   $$k = \frac{4 \pm \sqrt{16 + 48}}{6} = \frac{4 \pm 8}{6} = \frac{2 \pm 4}{3}. \quad \text{Either} \quad k = 2 \quad \text{or} \quad k = -\frac{2}{3}$$
   Only $k = 2$ is positive.

(d) **Graphical method**

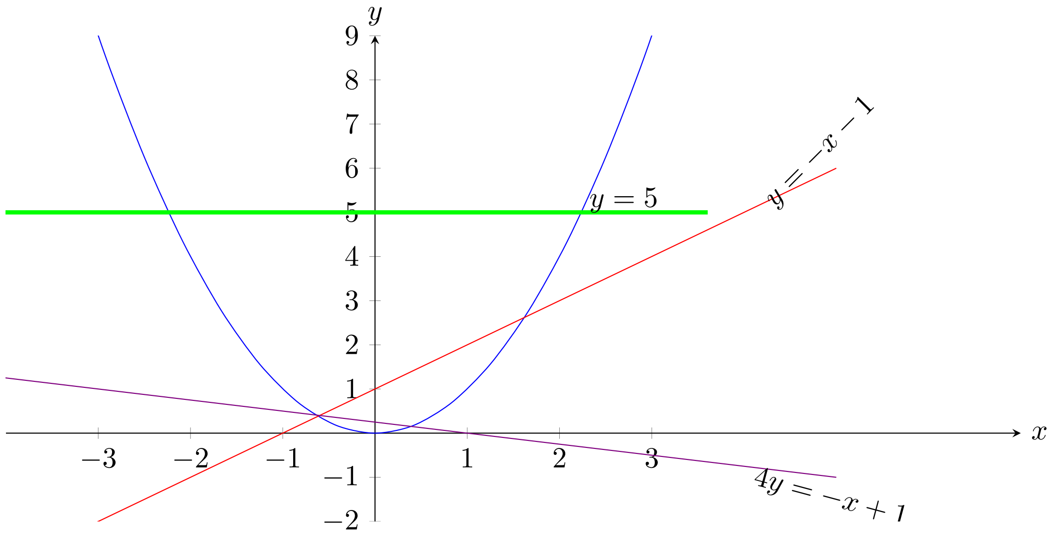
This method has low degree of accuracy since it involves some approximations.

Example(s):

1. Draw the graph of $y = x^2$ for $-3 \leq x \leq 3$. Hence, use the graph to:
   (a) find $\sqrt{5}$ correct to 2dp.
   (b) solve the equations
      (i) $x^2 - x - 1 = 0$.
      (ii) $4x^2 + x - 1 = 0$.

   Solution

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9



   (a) From the line $y = 5$ and the graph of $y = x^2 \quad \Rightarrow \quad x = \sqrt{y} = \pm\sqrt{5} \approx \pm 2.24$ (from the graph).

$$\begin{array}{l} y = x^2 \\ 0 = x^2 - x - 1 \\ \hline \end{array}$$

(b) (i) $y = x + 1$ (plot this line on the same axes then read the x coordinates at the intersection of the graphs of $y = x + 1$ and $y = x^2$, to obtain the required roots). These are: $x = -0.62$ and $x = 1.62$ (2dp).

$$\begin{array}{l} 4y = 4x^2 \\ 0 = 4x^2 + x - 1 \\ \hline \end{array}$$

(ii) $4y = -x + 1$ (plot this line on the same axes then read the x coordinates at the intersection of the graphs of $4y = -x + 1$ and $y = x^2$, to obtain the required roots). These are: $x = -0.64$ and $x = 0.39$ (2dp).

Exercise:

2. Draw the graph of $y = 2x^2 - 12x + 19$ for $1 \leq x \leq 5$. Hence, use the graph to find the roots of the equations.
   (a) $x^2 - 6x + 6 = 0$. [hint: $y = 7$, ans: $x \approx 1.3, x \approx 4.7$]
   (b) $4x^2 - 25x + 28 = 0$. [hint: $2y = x + 10$, ans: $x \approx 1.45, x \approx 4.80$]

II: **Roots of quadratic equations**

Consider the equation $ax^2 + bx + c = 0$. Then on rewriting

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad (a)$$

Now, suppose α and β are the roots of equation (a), then $x = \alpha$ or $x = \beta$. Thus, we have $(x - \alpha) = 0$ or $(x - \beta) = 0 \Rightarrow \quad (x - \alpha)(x - \beta) = 0$. Expanding yields

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0 \qquad (b)$$

Comparing equations (a) and (b), we obtain

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

$\rightarrow$ Note: if we are required to write down an equation whose roots are known, then the required equation is given by

$$x^2 - (\text{sum of the roots})x + (\text{product of the roots}) = 0$$

Example(s):

1. Write down the sum and products of the roots of $3x^2 - 2x - 7 = 0$.

   Solution
   If the roots are α and β, then

$$\alpha + \beta = -\left(\frac{-2}{3}\right) = \frac{2}{3} \quad \text{and} \quad \alpha\beta = \frac{c}{a} = -\frac{7}{3}$$

2. Express each of the following in terms of $\alpha + \beta$ and $\alpha\beta$:
   (a) $\alpha^2 + \beta^2$.
   (b) $(\alpha - \beta)^2$.
   (c) $\alpha^3 + \beta^3$.

   Solution

(a) α^2 and β^2 occur in the expansion of $(\alpha+\beta)^2$. Thus,

$$(\alpha+\beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2 \quad \Rightarrow \quad \alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$$

(b)

$$\begin{aligned}(\alpha-\beta)^2 &= \alpha^2 - 2\alpha\beta + \beta^2 = \left(\alpha^2+\beta^2\right) - 2\alpha\beta \\ &= (\alpha+\beta)^2 - 2\alpha\beta - 2\alpha\beta \\ &= (\alpha+\beta)^2 - 4\alpha\beta\end{aligned}$$

(c) α^3 and β^3 occur in the expansion of $(\alpha+\beta)^3$. Thus,

$$\begin{aligned}(\alpha+\beta)^3 &= \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3 = \alpha^3+\beta^3+3\alpha\beta(\alpha+\beta) \\ \Rightarrow \quad \alpha^3+\beta^3 &= (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)\end{aligned}$$

3. The roots of the equation $3x^2+4x-5=0$ are α and β. Find the equation with integral coefficients whose roots are:
   (a) $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$.
   (b) α^2 and β^2.
   (c) $(\alpha+1)$ and $(\beta+1)$.
   (d) $\alpha^2\beta$ and $\alpha\beta^2$.
   (e) $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$.
   (f) $\dfrac{(1+\alpha)}{\beta}$ and $\dfrac{(1+\beta)}{\alpha}$.
   (g) $(\alpha-1)^2$ and $(\beta-1)^2$.

Solution

From the equation $3x^2+4x-5=0$, we have $(\alpha+\beta) = -\dfrac{b}{a} = -\dfrac{4}{3}$ and $\alpha\beta = \dfrac{c}{a} = -\dfrac{5}{3}$.

(a) The roots are $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$.

Sum of the roots: $\dfrac{1}{\alpha}+\dfrac{1}{\beta} = \dfrac{\alpha+\beta}{\alpha\beta} = \dfrac{-4/3}{-5/3} = \dfrac{4}{5}$

Product of the roots: $\dfrac{1}{\alpha}\cdot\dfrac{1}{\beta} = \dfrac{1}{\alpha\beta} = \dfrac{1}{-5/3} = -\dfrac{3}{5}$

Equation: $x^2 - (\text{sum of the roots})x + (\text{product of the roots}) = 0$

$$\Rightarrow \quad x^2 - \frac{4}{5}x - \frac{3}{5} = 0 \quad \Rightarrow \quad 5x^2 - 4x - 3 = 0$$

(b) The roots are α^2 and β^2.

Sum of the roots: $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = \left(-\dfrac{4}{3}\right)^2 - 2\left(-\dfrac{5}{3}\right) = \dfrac{16}{9}+\dfrac{10}{3} = \dfrac{46}{9}$

Product of the roots: $\alpha^2\cdot\beta^2 = (\alpha\beta)^2 = \left(-\dfrac{5}{3}\right)^2 = \dfrac{25}{9}$

Equation: $x^2 - (\text{sum of the roots})x + (\text{product of the roots}) = 0$

$$\Rightarrow \quad x^2 - \frac{46}{9}x + \frac{25}{9} = 0 \quad \Rightarrow \quad 9x^2 - 46x + 25 = 0$$

(c) The roots are $(\alpha+1)$ and $(\beta+1)$.

Sum of the roots: $(\alpha+1)+(\beta+1) = (\alpha+\beta)+2 = \left(-\dfrac{4}{3}\right)+2 = \dfrac{2}{3}$

Product of the roots: $(\alpha+1)(\beta+1) = (\alpha\beta)+(\alpha+\beta)+1 = \left(-\dfrac{5}{3}\right)+\left(-\dfrac{4}{3}\right)+1 = -2$

Equation: $x^2 - (\text{sum of the roots})x + (\text{product of the roots}) = 0$

$$\Rightarrow \quad x^2 - \frac{2}{3}x - 2 = 0 \quad \Rightarrow \quad 3x^2 - 2x - 6 = 0$$