1.6 Permutations and combinations

1.6.1 Permutations

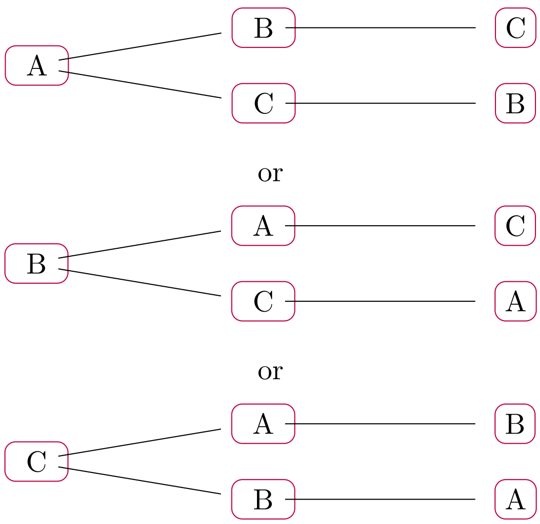
A permutation is an arrangement of a given number of objects in a particular order. Now, if k independent choices are to be made where there are r_1 possibilities for the first choice, r_2 possibilities for the second choice, and so on, the the total number of choices is $r_1 \times r_2 \times \cdots \times r_k$.

Example(s):

1. In how many ways can the letters A, B and C be arranged in three consecutive positions.

   Solution

1st position	2nd position	3rd position



   The 1st position can be filled in 3 different ways (we can have either A or B or C). Once the 1st position is filled, 2 letters remain and any of them can be used to fill the 2nd position. So, for each of the 3 ways of filling the 1st position we have 2 ways of filling the 2nd position. After filling the 1st and 2nd positions, the 3rd position can be filled in any 1 way as we are left with only 1 letter. Thus, the total number of distinct arrangements of the three letters is $3 \times 2 \times 1 = 6$ different ways. Each of these 6 arrangements is called a permutation of the letters A,B and C. The six permutations are ABC, ACB, BAC, BCA, CAB, and CBA.

Definition 1.6.1 (The factorial notation)**.** *The number of ways of arranging n distinct objects is $n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$ which is denoted by $n!$ and read as "n factorial". For example,*

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120, \quad 4! = 4 \times 3 \times 2 \times 1 = 24, \quad 3! = 3 \times 2 \times 1 = 6, \quad 2! = 2 \times 1 = 2$$

By convention:

$$1! = 1 \quad \text{is the number of ways of arranging one object.}$$

$$0! = 1 \quad \text{is the number of ways of arranging zero object.}$$

Thus,

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 \cdot 0!$$

→ Note: most calculators have a key for permutations.

Example(s):

1. Evaluate

   (a) $\dfrac{9!}{2!7!}$

(b) $\frac{6!}{(3!)^2}$

(c) $\frac{6!2!}{8!}$

Solution

(a) $\frac{9!}{2!7!} = \frac{9 \times 8 \times \cancel{7!}}{2! \times \cancel{7!}} = \frac{9 \times 8}{2 \times 1} = 36$

(b) $\frac{6!}{(3!)^2} = \frac{6 \times 5 \times 4 \times \cancel{3!}}{3! \times \cancel{3!}} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$

(c) $\frac{6!2!}{8!} = \frac{\cancel{6!} \times 2!}{8 \times 7 \times \cancel{6!}} = \frac{2 \times 1}{8 \times 7} = \frac{1}{28}$

2. Write $40 \times 39 \times 38 \times 37$ in factorial notation.

Solution

$$40 \times 39 \times 38 \times 37 = \frac{40 \times 39 \times 38 \times 37 \times 36 \times 35 \times \cdots \times 3 \times 2 \times 1}{36 \times 35 \times \cdots \times 3 \times 2 \times 1} = \frac{40!}{36!}$$

3. How many arrangements are there for 2 objects chosen from 4 distinct objects?

Solution

There are 4 ways of filling the 1st position. Once the 1st position is filled, three objects remain. Thus, there are 3 ways of filling the 2nd position. Therefore, the number of arrangements of two objects chosen from four different objects is $4 \times 3 = 12$. But

$$4 \times 3 = \frac{4!}{2!} = \frac{4!}{(4-2)!}$$

Similarly, $5 \times 4 = \frac{5!}{3!} = \frac{5!}{(5-2)!} = 20$ is the number of arrangements of two objects from five distinct objects. Also, $5 \times 4 \times 3 = \frac{5!}{2!} = \frac{5!}{(5-3)!} = 60$ is the number of arrangements of three objects from five distinct objects.

Definition 1.6.2. *The number of permutations/arrangements of r objects chosen from n distinct objects is given by*

$$^nP_r = \frac{n!}{(n-r)!}$$

and is read as "n permutation r".

Example(s):

1. (a) Suppose that 7 people enter a swim meet. Assuming that there are no ties, in how many ways could the gold, silver, and bronze medals be awarded?

Solution

Gold	Silver	Bronze
7	6	5

$$^7P_3 = \frac{7!}{(7-3)!} = \frac{7 \times 6 \times 5 \times \cancel{4!}}{\cancel{4!}} = 7 \times 6 \times 5 = 210 \text{ ways.}$$

(b) In a lottery, a total of 1000 tickets were sold. Determine the number of ways of obtaining winners of the 1st, 2nd and 3rd prizes if three tickets are drawn one after the other.

Solution

1st	2nd	3rd
1000	999	998

$$^{1000}P_3 = \frac{1000!}{(1000-3)!} = \frac{1000 \times 999 \times 998 \times \cancel{997!}}{\cancel{997!}} = 1000 \times 999 \times 998 = 997002000 \text{ ways.}$$

2. Five letter words are formed from the letters of the word INCOMPUTABLE. How many of these words have no repeated letters?

Solution

$$^{12}P_5 = \frac{12!}{(12-5)!} = \frac{12 \times 11 \times 10 \times 9 \times 8 \times \cancel{7!}}{\cancel{7!}} = 12 \times 11 \times 10 \times 9 \times 8 = 95040 \text{ ways.}$$

3. (a) Find how many different arrangements of 12 letters can be obtained from the letters of the word SPEEDOMETERS

Solution

If the 12 letters are all different, there could be 12! different arrangements. In any of these arrangements, the two S's can be arranged in $2! = 2$ ways without altering the appearance of the letters in the given arrangement. Similarly, the four E's can be arranged in $4! = 24$ ways without altering the appearance of the letters in an arrangement. This in turn implies that in any arrangement, there can be $2! \times 4! = 48$ arrangements without altering the appearance of the letters of the given arrangement. Since the two S's and four E's are alike, these are

$$\frac{12!}{2!4!} = 9979200 \text{ arrangements}$$

In general, if we have n objects of which r_1 of the first kind are alike, r_2 of the second kind are alike,$\cdots$, r_k of the kth kind are alike, then the total number of distinct permutations is

$$\frac{n!}{r_1! \times r_2! \times \cdots \times r_k!}$$

(b) Determine the number of permutations of the letters of the words

(i) ASSIGNMENT
(ii) ASSASSINATION
(iii) MISSISSIPPI
(iv) FUNDAMENTALISM
(v) INTERCOMMUNICATION

(c) In how many ways can 5 blue beads, 4 green beads, 2 red beads and 1 white bead be arranged in a row if beads of the same color are indistinguishable. [ans: 83,160 ways]

4. How many different 7-digit telephone numbers are possible if the first digit cannot be zero and no digit may repeat?

Solution

Our number system consists of 10 digits, which are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, \text{ and } 9\}$. Since the first digit cannot be a zero, then there are 9 choices for the first digit, 9 choices for the second digit since a zero can be used and no digits may repeat, 8 choices for the third digit, 7 choices for the fourth digit, and so on. Thus, there are $\boxed{9 \mid 9 \mid 8 \mid 7 \mid 6 \mid 5 \mid 4} = 9\times9\times8\times7\times6\times5\times4 = 544{,}320$ possible telephone numbers.

5. How many even numbers, greater that 50000, can be formed using the digits 0,3,4,5,6,7
   (a) without replacement
   (b) if repetitions are allowed

Solution

The number can have either 5 or 6 digits and cannot begin with zero.

(a) If there is no repetition, the problem is split up into four cases:

i) Numbers with 5 digits, the ten thousand's (T Th) digit being even > 5

T Th	Th	H	T	O
1	4	3	2	2

$= 1 \times 4 \times 3 \times 2 \times 2 = 48$ possibilities

ii) Numbers with 5 digits, the T Th digit being odd ≥ 5

T Th	Th	H	T	O
2	4	3	2	3

$= 2 \times 4 \times 3 \times 2 \times 3 = 144$ possibilities

iii) Numbers with 6 digits, the H Th digit being even.

H Th	T Th	Th	H	T	O
2	4	3	2	1	2

$= 2 \times 4 \times 3 \times 2 \times 1 \times 2 = 96$ possibilities

iv) Numbers with 6 digits, the H Th digit being odd.

H Th	T Th	Th	H	T	O
3	4	3	2	1	3

$= 3 \times 4 \times 3 \times 2 \times 1 \times 3 = 216$ possibilities

Therefore, there are $48 + 144 + 96 + 216 = 504$ possible numbers.

(b) If repetition is allowed, we have two cases to consider:

i) Numbers with 5 digits, T Th digit ≥ 5

T Th	Th	H	T	O
3	6	6	6	3

$= 3 \times 6 \times 6 \times 6 \times 3 = 1944$ possibilities

ii) Numbers with 6 digits

H Th	T Th	Th	H	T	O
5	6	6	6	6	3

$= 5 \times 6 \times 6 \times 6 \times 6 \times 3 = 19440$ possibilities

Therefore, there are $1944 + 19440 = 21384$ possible numbers.

6. A man dines at the same hotel for three consecutive days and the menu each day is one of any 4 types of goat dish, or 2 types of chicken dish or 1 type of vegetarian dish. In how many ways can he arrange his lunches over the three days if he doesn't have a goat dish two days running nor repeat any dish.

Solution

We have four cases to consider

i) Goat dish on the first day:

Day 1 (Goat dish)	Day 2 (Other dish)	Day 3 (Any remaining dish)
4	3	5

$= 4 \times 3 \times 5 = 60$ ways

ii) Goat dish on the second day:

Day 1 (Other dish)	Day 2 (Goat)	Day 3 (Any of the two remaining other)
3	4	2

$= 3 \times 4 \times 2 = 24$ ways

iii) Goat dish on the third day:

Day 1 (Other dish)	Day 2 (Any of the two remaining other)	Day 3 (Goat)
3	2	4

$= 3 \times 2 \times 4 = 24$ ways

iv) No goat dish:

Day 1 (Other dish)	Day 2 (Any of the two remaining other dishes)	Day 3 (Single remaining other dish)
3	2	1

$= 3 \times 2 \times 1 = 6$ ways

Therefore, there are $60 + 24 + 24 + 6 = 114$ ways.

Exercise:

6. How many odd numbers greater than 500,000 can be made from the digits 2,3,4,5,6,7.
   (a) without repetition. [ans: 168 ways]
   (b) if repetition is allowed. [ans: 11,664 ways]
7. How many odd numbers greater than 70,000 can be formed using the digits 0,1,4,7,8,9.
   (a) without repetition. [ans: 456 ways]
   (b) if repetitions are allowed. [ans: 21,384 ways]
8. In how many ways can 8 people be scattered at a round table? [ans: $7! = 5040$]
9. How many arrangements of the letters of the word BEGIN are there which start with a vowel, without repetition? [ans: 48 ways]
10. In how many ways can 5 boys and 4 girls be arranged on a bench if boys and girls alternate? [ans: $5! \times 4! = 2880$ ways]
11. How many arrangements of the letters of the word REMAND are possible if
    (a) they begin with RE? [ans: $1! \times 4! = 24$ ways]
    (b) they don't begin with RE? [ans: $6! - 24 = 696$ ways]
12. How many numbers between 10 and 300 can be made from the digits 1,2,3 if
    (a) each digit may be used only once. [ans: 10 numbers]
    (b) each digit may be used more than once. [ans: 27 numbers]
12. Five letters from the word DRILLING are to be arranged in a row. Find the number of ways in which this can be done when the first letter is I and the last letter is L.
    (a) if no letter may be repeated. [ans: 24 ways]
    (b) if each letter may occur as many times as it does in DRILLING. [ans: 120 ways]
13. In how many ways can two mathematics books and 4 physics books be arranged in a shelf if
    (a) the mathematics books must be placed next to each other. [ans: $2! \times 5! = 240$ ways]
    (b) the mathematics books must NOT be placed next to each other. [ans: $6! - 240 = 480$ ways]

1.6.2 Combinations

In permutations, the order in which objects are chosen in important. However, in some cases, the order of selection is irrelevant. When the selection of objects is made randomly with no regard being paid to the order, it is referred to as a *combination*. Thus, ABC, ACB, BAC are different permutations but they are the same combination of letters.

Example(s):

1. How many different committees of 3 people can be chosen to work on a special project from a group of 5 people? A,B,C,D and E?

   Solution

   There are 10 possible combinations of the committees of 3 people chosen from a group of 5 people: ABC, ABD, ABE, ACD, AEC, ADE, BCD, BCE, BDE, and CDE. Thus, there are 10 ways of selecting the project group.

   → Note: On the other hand, there are ${}^5P_3 = 60$ permutations of 3 objects from 5 distinct objects.

In general, the number of combinations/selections of r objects from n available objects can be found as follows:

- there are $\dfrac{n!}{(n-r)!}$ permutations of r objects chosen from n distinct objects.
- but each combination of r objects can be arranged in $r!$ ways, therefore,

$$\text{the number of permutations } = r! \times (\text{the number of combinations})$$

$$\text{i.e., } \frac{n!}{(n-r)!} = r! \times (\text{the number of combinations})$$

- so the total number of combinations/selections is $\dfrac{n!}{(n-r)!r!}$ written as

$${}^nC_r \text{ or } \binom{n}{r} = \frac{n!}{(n-r)!r!}$$

→ Note: most calculators have a key for combinations.

Example(s):

1. Five policemen are to be selected for duty from a force of 20. In how many ways can this be done if

   (a) there is no restriction?

   (b) it has already been decided that a certain person must be on duty?

   Solution

   (a)

   $${}^{20}C_5 = \frac{20!}{(20-5)!5!} = \frac{20!}{15!5!} = 15,504$$

   (b) The problem has been reduced to selecting 4 more policemen from the remaining 19 policemen. Thus, we have ${}^{19}C_4 = \dfrac{19!}{(19-4)!4!} = \dfrac{19!}{15!4!} = 3876$. There are 3,876 ways of selecting the policemen for duty.

2. A mixed hokey team containing 5 men and 6 women is to be chosen from 7 men and 9 women. In how many ways can this be done?

   Solution

   Five men can be chosen from 7 men in ${}^7C_5$ ways and 6 women can be chosen from 9 women in ${}^9C_6$ ways. For each of the ${}^7C_5$ ways of selecting men, there are ${}^9C_6$ ways of selecting women. Therefore, the total number of selecting the team is

   $${}^7C_5 \times {}^9C_6 = \frac{7!}{(7-5)!5!} \times \frac{9!}{(9-6)!6!} = \frac{7!}{2!5!} \times \frac{9!}{3!6!} = 21 \times 84 = 1764 \text{ ways}$$

3. A person wants to invite 8 friends but there is only room for 4 of them. In how many ways can the four to be invited be chosen if two of the eight are twins and must not be separated?

   Solution

   There are two cases to consider

   i) The twins are invited: thus, we have to choose 2 out of the remaining 6 in ${}^6C_2 = 15$ ways
   ii) The twins are left out: thus, we have to choose 4 out of the remaining 6 in ${}^6C_4 = 15$ ways

   Therefore, the total number of ways is $15 + 15 = 30$.

4. Nine people are going to travel in two taxi. The larger has 5 seats and the smaller has 4 seats. In how many ways can the party be split?

   Solution

   Once the group of 5 has been selected, then the remaining 4 people will automatically comprise the other group. Thus, we have to select 5 from 9 in ${}^9C_5 = 126$ ways.

5. Five books are to be selected from 20 books of which 8 are paperback and 12 are hardback. How many selections are possible if at least one paperback book has to be included?

   Solution

   - If there is no restriction, we have ${}^{20}C_5 = 15,504$ ways of selecting the 5 books
   - If only hardback books are selected, we have ${}^{12}C_5 = 792$ ways of selecting the 5 books

   Therefore, the number of ways of selecting at least one paperback book is $15,504 - 792 = 14,712$.

Exercise:

1. A committee of six is to be formed from nine women and three men. In how many ways can the members be chosen so as to include at least one man? [ans: 840 ways]
2. A committee of ten is to be formed from nine men and six women. In how many ways can it be formed if at least four women are to be in the committee? [ans: 2142 ways]
3. In how many ways can a class of 20 children be split into two groups of 8 members and 12 members, respectively if there are two twins in the class who must not be separated. [ans: ${}^{18}C_6 +^{18} C_8 = 62,322$ ways]
4. Nine players are available to play for a table tennis team of 4 players. In how many ways can the team be selected if 2 of the players are brothers and must either both be included or both be excluded and if 2 other players have recently quarreled and should not both play in a team? [ans: 45 ways]

LECTURE 6

1.7 Binomial expansion

A binomial is the sum or difference of two terms. For example,

$$\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4
\end{aligned}$$