(a) $f(x) = 3x^3 + 2x^2 + x - 1$ by $g(x) = x - 1$

$$
\begin{array}{rl}
 & \quad 3x^2 + 5x + 6 \\ \cline{2-2}
x - 1) & \;3x^3 + 2x^2 \quad + x - 1 \\
 & -\,3x^3 + 3x^2 \\ \cline{2-2}
 & \qquad\quad 5x^2 \quad + x \\
 & \qquad -\,5x^2 + 5x \\ \cline{2-2}
 & \qquad\qquad\quad 6x - 1 \\
 & \qquad\qquad -\,6x + 6 \\ \cline{2-2}
 & \qquad\qquad\qquad\quad 5
\end{array}
$$

Thus, $Q(x) = 3x^2 + 5x + 6$ and $R(x) = 5$. Hence,

$$3x^3 + 2x^2 + x - 1 = (x - 1)(3x^2 + 5x + 6) + 5$$

(b) $f(x) = 3x^3 + 2x^2 + x - 1$ by $g(x) = x^2 - 2$

$$
\begin{array}{rl}
 & \qquad\qquad 3x + 2 \\ \cline{2-2}
x^2 - 2) & \;3x^3 + 2x^2 \quad + x - 1 \\
 & -\,3x^3 \qquad\quad + 6x \\ \cline{2-2}
 & \qquad 2x^2 + 7x - 1 \\
 & \quad -\,2x^2 \qquad + 4 \\ \cline{2-2}
 & \qquad\qquad 7x + 3
\end{array}
$$

Thus, $Q(x) = 3x + 2$ and $R(x) = 7x + 3$. Hence,

$$3x^3 + 2x^2 + x - 1 = (x^2 - 2)(3x + 2) + (7x + 3)$$

(c) $f(x) = 4x^3 - x + 2$ by $g(x) = 3x + 2$

$$
\begin{array}{rl}
 & \qquad \frac{4}{3}x^2 - \frac{8}{9}x + \frac{7}{27} \\ \cline{2-2}
3x + 2) & \;4x^3 \qquad\qquad - x \quad + 2 \\
 & -\,4x^3 - \frac{8}{3}x^2 \\ \cline{2-2}
 & \qquad -\,\frac{8}{3}x^2 \quad - x \\
 & \qquad\quad \frac{8}{3}x^2 + \frac{16}{9}x \\ \cline{2-2}
 & \qquad\qquad\qquad \frac{7}{9}x \quad + 2 \\
 & \qquad\qquad\quad -\,\frac{7}{9}x - \frac{14}{27} \\ \cline{2-2}
 & \qquad\qquad\qquad\qquad \frac{40}{27}
\end{array}
$$

Exercise:

2. Divide

   (a) $f(x) = x^3 - 2x + 4$ by $g(x) = x - 1$

   (b) $f(x) = 2x^3 - x^2 + 2$ by $g(x) = x - 3$

   (c) $f(x) = x^4 - 3x^3 + 5x$ by $g(x) = 2x + 1$

   (d) $f(x) = 9x^5 - 5x^2 + 2$ by $g(x) = 3x + 1$

   (e) $f(x) = 5x^3 - 2x^2 + 1$ by $g(x) = x - 3$

1.5.2 Remainder theorem

Let $f(x)$ be a polynomial and a be a real number. If $f(x)$ is divided by $(x - a)$, then the remainder is $R = f(a)$.

Proof. The polynomial $f(x)$ can be written as

$$f(x) = (x - a)Q(x) + R$$

Putting $x = a$ yields $R = f(a)$ □

Corollary: If a polynomial $f(x)$ is divided by $(px+q)$, where $p \neq 0$, then the remainder is $R = f\left(-\frac{q}{p}\right)$.

Proof. Suppose R is the remainder when $f(x)$ is divided by $(px+q)$. Then, $f(x) = (px+q)\cdot Q(x) + R$. Putting $x = -\dfrac{q}{p}$, we get

$$f\left(-\frac{q}{p}\right) = \left[p\left(-\frac{q}{p}\right) + q\right]\cdot Q\left(-\frac{q}{p}\right) + R = 0 + R \quad \Rightarrow \quad R = f\left(-\frac{q}{p}\right)$$

□

Example(s):

1. Find the remainder when $x^5 - 4x^3 + 2x + 3$ is divided by
   (a) $x - 1$
   (b) $x + 2$
   (c) $2x - 1$

   Solution
   Let $f(x) = x^5 - 4x^3 + 2x + 3$. Then,
   (a) The remainder when $f(x)$ is divided by $(x-1)$ is $R = f(1) = (1)^5 - 4(1)^3 + 2(1) + 3 = 2$.
   (b) The remainder when $f(x)$ is divided by $(x+2)$ is $f(-2) = (-2)^5 - 4(-2)^3 + 2(-2) + 3 = -1$.
   (c) The remainder when $f(x)$ is divided by $(2x-1)$ is $R = f(\frac{1}{2}) = (\frac{1}{2})^5 - 4(\frac{1}{2})^3 + 2(\frac{1}{2}) + 3 = \dfrac{113}{32}$.

Exercise:

1. Find the remainder when
   (a) $f(x) = x^3 - 2x^2 + 5x + 8$ is divided by $x - 2$. [ans: $R = f(2) =$]
   (b) $f(x) = x^3 + 3x^2 + 3x + 1$ is divided by $x + 2$. [ans: $R = f(-2) =$]
   (c) $f(x) = 4x^3 + 6x^2 + 3x + 2$ is divided by
      (i) $2x + 3$ [ans: $R = f(-\frac{3}{2}) =$]
      (ii) $-2x + 3$ [ans: $R = f(\frac{3}{2}) =$]
      (iii) $-2x - 3$ [ans: $R = f(-\frac{3}{2}) =$]
      (iv) x [ans: $R = f(0) = 2$]

1.5.3 Factor theorem

Let $f(x)$ be a polynomial and a be a real number. If $f(a) = 0$, then $(x-a)$ is a factor of $f(x)$. That is, $(x-a)$ is a factor of $f(x)$ if and only if $f(x)$ leaves remainder zero when it is divided by $(x-a)$.

Example(s):

1. Factorize $x^4 - 3x^3 + 4x^2 - 8$ completely.

   Solution
   We use trial and error method together with the factor theorem. Let $f(x) = x^4 - 3x^3 + 4x^2 - 8$. The factors of 8 are $\pm 1, \pm 2, \pm 4$ and ± 8. Now, by trial and error:

$$\text{Put } x = 1: \quad f(1) = -6 \neq 0 \quad \Rightarrow \quad (x-1) \quad \text{is not a factor of } f(x)$$

$$\text{Put } x = 2: \quad f(2) = (2)^4 - 3(2)^3 + 4(2)^2 - 8 = 0 \quad \Rightarrow \quad (x-2) \quad \text{is a factor of } f(x)$$

By long division, we have

$$\begin{array}{r l}
 & x^3 \quad - x^2 + 2x + 4 \\ \cline{2-2}
x-2) & x^4 - 3x^3 + 4x^2 \qquad - 8 \\
 & -x^4 + 2x^3 \\ \cline{2-2}
 & \quad -x^3 + 4x^2 \\
 & \quad x^3 - 2x^2 \\ \cline{2-2}
 & \qquad 2x^2 \\
 & \qquad -2x^2 + 4x \\ \cline{2-2}
 & \qquad\qquad 4x - 8 \\
 & \qquad\qquad -4x + 8 \\ \cline{2-2}
 & \qquad\qquad\qquad 0
\end{array}$$

$$\Rightarrow \quad x^4 - 3x^3 + 4x^2 - 8 = (x-2)(x^3 - x^2 + 2x + 4)$$

Next, we factorize $f_1(x) = x^3 - x^2 + 2x + 4$. The factors of 4 are $\pm 1, \pm 2$ and ± 4. Now, by trial and error:

$$\text{Put } x = 2: \quad f_1(2) = 12 \neq 0 \quad \Rightarrow \quad (x-2) \quad \text{is not a factor of } f_1(x)$$

$$\text{Put } x = -1: \quad f_1(-1) = (-1)^3 - (-1)^2 + 2(-1) + 4 = 0 \quad \Rightarrow \quad (x+1) \quad \text{is a factor of } f_1(x)$$

By long division, we have

$$\begin{array}{r l}
 & x^2 - 2x + 4 \\ \cline{2-2}
x+1) & x^3 \quad - x^2 + 2x + 4 \\
 & -x^3 \quad - x^2 \\ \cline{2-2}
 & \quad -2x^2 + 2x \\
 & \quad 2x^2 + 2x \\ \cline{2-2}
 & \qquad 4x + 4 \\
 & \qquad -4x - 4 \\ \cline{2-2}
 & \qquad\qquad 0
\end{array}$$

$$\Rightarrow \quad x^3 - x^2 + 2x + 4 = (x+1)(x^2 - 2x + 4)$$

Since $(x^2 - 2x + 4)$ has no linear factors, therefore,

$$x^4 - 3x^3 + 4x^2 - 8 = (x-2)(x+1)(x^2 - 2x + 4)$$

2. Solve the cubic equation $x^3 - 7x - 6 = 0$.

Solution

Let $f(x) = x^3 - 7x - 6$. The factors of 6 are $\pm 1, \pm 2, \pm 3$ and ± 6. Clearly,

$$\text{Put } x = -1: \quad f(-1) = (-1)^3 - 7(-1) - 6 = 0 \quad \Rightarrow \quad (x+1) \quad \text{is a factor of } f(x)$$

$$\text{Put } x = -2: \quad f(-2) = (-2)^3 - 7(-2) - 6 = 0 \quad \Rightarrow \quad (x+2) \quad \text{is a factor of } f(x)$$

$$\text{Put } x = 3: \quad f(3) = (3)^3 - 7(3) - 6 = 0 \quad \Rightarrow \quad (x-3) \quad \text{is a factor of } f(x)$$

Therefore,

$$x^3 - 7x - 6 = (x+2)(x+1)(x-3) = 0. \quad \text{The roots are } x = -1, x = -2, x = 3$$

3. A polynomial $f(x)$ has remainder 9 when divided by $(x-3)$ and remainder -5 when divided by $(2x+1)$. Find the remainder when $f(x)$ is divided by $(x-3)(2x+1)$.

Solution

Since the divisor $(x-3)(2x+1)$ is of degree 2, the remainder R should be of degree 1 or 0. Let $R = ax + b$ ($a \neq 0$ if R is of degree 1 and $a = 0$ if R is of degree 0). Thus, the polynomial can be written as:

$$f(x) = (x-3)(2x+1) \cdot Q(x) + (ax + b)$$

- The remainder when $f(x)$ is divided by $(x-3)$ is $f(3) \equiv 3a + b = 9 \cdots (i)$.
- The remainder when $f(x)$ is divided by $(2x+1)$ is $f(-\frac{1}{2}) \equiv -\dfrac{1}{2}a + b = -5 \cdots (ii)$.

Solving equations (i) and (ii) simultaneously yields $a = 4$ and $b = -3$. Therefore, the remainder when $f(x)$ is divided by $(x-3)(2x+1)$ is $R = 4x - 3$.

4. When a polynomial $f(x)$ is divided by $(x-1)$, the remainder is 3. When $f(x)$ is divided by $(x+1)$, the remainder is 5. When $f(x)$ is divided by $(x-2)$, the remainder is 20. Find the remainder when $f(x)$ is divided by $(x^2-1)(x-2)$.

Solution

Since the divisor $(x^2-1)(x-2) = (x-1)(x+1)(x-2)$ is of degree 3, the remainder R should be of degree 2 or 1 or 0. Let $R = ax^2 + bx + c$. Thus, the polynomial can be written as:

$$f(x) = (x-1)(x+1)(x-2) \cdot Q(x) + (ax^2 + bx + c)$$

- The remainder when $f(x)$ is divided by $(x-1)$ is $f(1) \equiv a + b + c = 3 \cdots (i)$.
- The remainder when $f(x)$ is divided by $(x+1)$ is $f(-1) \equiv a - b + c = 5 \cdots (ii)$.
- The remainder when $f(x)$ is divided by $(x-2)$ is $f(2) \equiv 4a + 2b + c = 20 \cdots (iii)$.

Solving equations (i), (ii) and (iii) simultaneously yields $a = 6, b = -1$ and $c = -2$. Therefore, the remainder when $f(x)$ is divided by $(x^2-1)(x-2)$ is $R = 6x^2 - x - 2$.

Exercise:

1. Solve the equations
   (a) $x^4 + 5x^3 + 5x^2 - 5x - 6 = 0$
   (b) $2x^3 + 11x^2 + 17x + 6 = 0$
   (c) $3x^3 + x^2 - 5x + 2 = 0$
2. When the polynomial $x^5 + 4x^2 + ax + b$ is divided by (x^2-1), the remainder is $2x+3$. Find the values of a and b. [ans: $a = 1, b = -1$]
3. When the polynomial $x^3 + 3x^2 + ax + b$ is divided by (x^2-4), the remainder is $x+16$. Find the values of a and b. [ans: $a = -3, b = 4$]
4. A cubic polynomial $ax^3 + bx + 6$ is divisible by $(x+2)$. It leaves a remainder of -3 when divided by $(x-1)$. Determine the values of a and b and hence factorize the polynomial completely. [ans: $a = 4, b = -13$]
5. The expression $ax^2 + bx + c$ is divisible by $(x-1)$. It has remainder 2 when divided by $(x+1)$, and has remainder 8 when divided by $(x-2)$. Find the values of a, b and c. [ans: $a = 3, b = -1, c = -2$]
6. $(x-1)$ and $(x+1)$ are factors of the expression $x^3 + ax^2 + bx + c$. It leaves a remainder of 12 when divided by $(x-2)$. Find the values of a, b and c. [ans: $a = 2, b = -1, c = -2$]
7. [***Assignment 2:***] A polynomial $Ax^3 + Bx^2 + Cx + D$ leaves a remainder of $x^2 - 3x + 2$ when divided by $(x+2)^3$, and a remainder of $ax + b$ when divided by $(x+2)^2$. Determine the values of a and b. [ans: $a = -7, b = -2$]
8. The expression $ax^4 + bx^3 + 3x^2 - 2x + 3$ has a remainder of $x+1$ when divided by $(x^2 - 3x + 2)$. Find the values of a and b. [ans: $a = 1, b = -3$]
9. What is the value of a if the polynomials $2x^2 - x - 6$, $3x^2 - 8x + 4$ and $ax^3 - 10x - 4$ have a common factor. [ans: $a = 3$ or $a = -250$]

LECTURE 5