Dynamic Equilibrium Models

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Road Map for Presentation

PART I (Rust): Background on solving DSGE models and stationary/non-stationary *recursive competitive equilibria* (RCE)

PART II (Rust): Analysis of a "Toy Model" of Heterogeneous Agent Equilibrium in the Auto Market

PART III (Schjerning): A Dynamic Model of Vehicle Ownership, Type Choice, and Usage

PART I

DSGE/RCE Models of the Auto Market

The challenge of modeling equilibrium in auto markets

1. There are many makes/models of cars, as well as different *ages* of vehicles
2. So finding an equilibrium is a *high dimensional* numerical problem
3. In addition, *dynamics are crucial:* there is an important decision of whether to keep your current car, or trade for a new one (or get rid of your car and not replace it with another)
4. With dynamics come *expectations:* how much can I sell my current car for, and how much will it cost to buy another one?
5. Macro shocks may also be important: do I want to buy a new car now if the economy is going into recession and I might lose my job?

A brief history of auto models

1. Manski (1982) and Berkovec (1985): recognized that separate prices need to be computed for new and used cars but used static choice models
2. Rust (1985): provided a dynamic framework for equilibrium prices and quantities, and showed that when *transactions costs are zero,* consumers trade every period for an optimal car, so the dynamic problem reduces to a static one
3. Konishi and Sandfort (2002): generalized Rust's analysis to allow for positive transaction costs and proved the existence of equilibrium, allowing also for multiple makes/models of cars
4. Schiraldi (2011): estimated a dynamic partial equilibrium model of consumer holdings/trading of cars on Italian data
5. Gavazza *et. al.* (2014): numerically calculated equilibrium with one car type and discrete ages/qualities of cars and analyzed the impact of the secondary market with varying levels of transaction costs

James A. Berkovec 1957-2009

1. "New Car Sales and Used Car Stocks: A Model of the Automobile Market" *RAND* 1985.
2. Estimated a discrete choice model of demand for cars using the *National Transportation Survey* – micro data on 1095 households in 1978
3. Using estimated demands for cars, he solved for equilibrium in the new and used car market, over 131 type/age classes, with 13 types and 10 ages (vintages) from 1969 to 1978 plus a residual (131st) category of all cars produced prior to 1969 (clunkers)

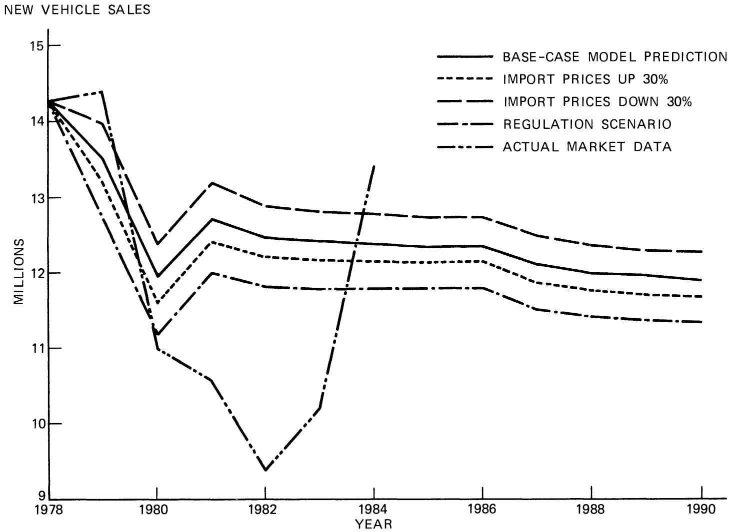
Berkovec's approach to calculating equilibrium

1. Let P be the 131×1 vector of prices of the 131 car type/age classes.
2. Let $E(P)$ be the 131×1 vector of excess demands for these 131 vehicles implied by the price vector P
3. An equilibrium in the auto market is a vector P^* satisfying $E(P^*) = 0$.
4. Berkovec defined $E(P) = D(P) - S(P) - Q - S(P)$ where Q is the vector of stocks of cars, and $S(P)$ is the number cars scrapped
5. Berkovec used a quasi-Newton method to find a P^* that satisfies $E(P^*) = 0$

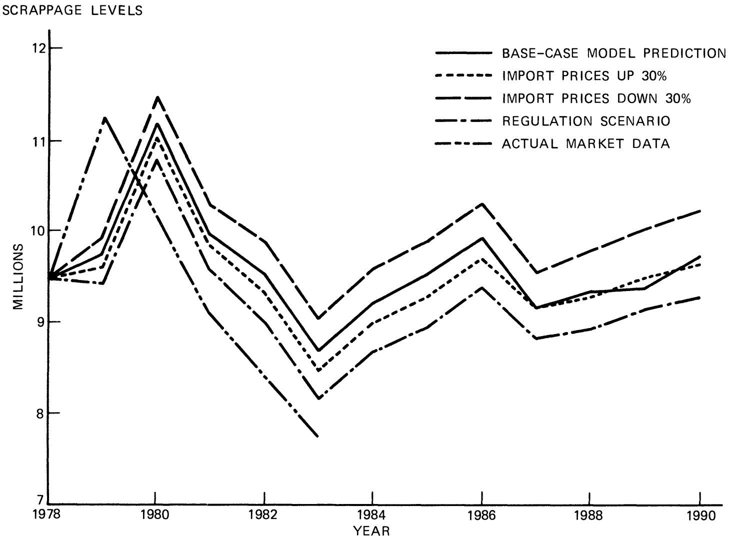
$$P' = P - \lambda[\nabla E(P)]^{-1} E(P) \tag{1}$$

where $\lambda \geq 0$ is a scalar *step size.*

New car sales predicted by Berkovec's model



Car scrappage predicted by Berkovec's model



A Dynamic Model of the Auto Market: Rust (1985)

1. Rust (1985) *Econometrica* "Stationary Equilibrium in a Market for Durable Goods" introduced a *durable asset pricing model* that includes particular durable goods such as cars that trade both on a *primary market* (new cars) and a *secondary market* (used cars)
2. "the essential benefit of a secondary market is to create dynamic trading opportunities similar to a securities market: the consumer can hold the current durable for any desired length of time and has the opportunity of trading it for a new asset or choosing from an array of used assets of different ages and physical conditions." (Rust, 1985, p. 783)

Equilibrium with a continuum of goods/consumers

1. Assume an infinitely elastic supply of new cars at price $\overline{P}$ and an infinite demand for cars by *scrappers* at price $\underline{P}$.
2. Suppose the condition car is summarized by its odometer x (no *lemon's problems*). Let $\phi(y|x)$ be the probability density of next period odometer given this period's value x. Example: $\phi(y|x) = \lambda \exp\{-\lambda(y-x)\}$ if $y \geq x$, 0 otherwise.
3. Consumers are *heterogeneous, infinitely lived,* must always own 1 car, and have *quasi-linear utility functions*

$$U(x, I, \tau) = a(\tau)I + q(x, \tau)$$

   where τ is the *type* of consumer, I is income, and x is the age/odometer of the car.
4. Let $m(x, \tau)$ be the maintenance costs incurred by consumer τ.

Equivalent cost-minimization formulation

1. Let $P(x)$ be an *equilibrium price function.* It should satisfy $P(0) = \overline{P}$ and $P(x) = \underline{P}$ where $x \geq \gamma$ where γ is the *scrapping threshold*. Let $T(x)$ be the *transactions cost* that a customer incurs when they sell a car of odometer x.
2. We can reframe the *car trading problem,* the consumer's dynamic utility maximization problem for owning an infinite sequence of autos, as a cost minimization problem using the "cost function" $G_\tau(x, d)$ given by

$$G_\tau(x, d) \quad = \quad \begin{cases} M(x, \tau) & \text{if} \quad d = x \\ M(z, \tau) + P(z) - P(x) + T(x) & \text{if} \quad d = z \end{cases}$$

where $M(x, \tau) = m(x, \tau) - q(x, \tau)/a(\tau)$ is the disutility of owning a car of odometer x expressed as a "generalized maintenance cost"

Bellman equation for the consumer

1. Let $V_\tau(x)$ be consumer τ's value function (optimal discounted cost of owning a car of odometer x). The *Bellman equation* is given by

$$V_\tau(x) = \min\Big[M(x,\tau) + \beta EV_\tau(x), \inf_{0\le z\le\gamma} M(z,\tau) + P(z) - P(x) + T(x) + \beta EV_\tau(z)\Big]$$

where

$$EV_\tau(x) = \int_x^\infty V_\tau(y)\phi(y|x)dy.$$

2. **Theorem** *Assume that transactions cost are zero, $T(x) = 0\ \forall x \ge 0$. Then it is optimal for each consumer τ to trade each period for their optimal car $z^*(\tau)$ given by*

$$z^*(\tau) = \underset{0\le z\le\gamma}{\operatorname{argmin}} \left[M(z,\tau) + P(z) + \beta EV_\tau(z)\right].$$

Implication of Zero Transactions Cost

1. **Lemma** If transactions costs are zero, $T(x) = 0$, then we have

$$V_\tau(x) = [M(x,\tau) + P(z^*(\tau)) - \beta EP(z^*(\tau))]/(1-\beta) - P(x)$$

where $z^*(\tau)$ is given by

$$z^*(\tau) = \underset{0 \le z \le \gamma}{argmin}\, [M(z,\tau) + P(z) - \beta EP(z)]\,. \tag{2}$$

2. What is no secondary market existed? Consumer's problem is now a *regenerative optimal stopping problem*

$$V_\tau(x) = \min\left[M(x,\tau) + \beta EV_\tau(x), M(0,\tau) + \overline{P} - \underline{P} + \beta EV_\tau(0)\right].$$

Optimal strategy is to keep the car x unless $x > \gamma(\tau)$ where $\gamma(\tau)$ is the solution to

$$\overline{P} - \underline{P} + V_\tau(0) = V_\tau(\gamma(\tau)).$$

Definition of Stationary Equilibrium

1. **Definition** The triple $\{P, F, \gamma\}$ is a *stationary equilibrium* if
2. $P(0) = \overline{P}$ and $P(x) = \underline{P}$ if $x \geq \gamma$
3. $F(x|\gamma)$ is a *stationary holdings distribution* the unique solution to

$$F(x|\gamma) = \int_0^\infty [1 - \Phi(\gamma|x') + \Phi(x|x')]\, F(dx'|\gamma)$$

4. Each consumer τ chooses an optimal car $z^*(\tau)$ given by equation (2) and $\forall x \in [0, \gamma]$ we have

$$F(x|\gamma) = \int_{\underline{\tau}}^{\overline{\tau}} I\left\{\tau | z^*(\tau) \leq x\right\} H(d\tau) \tag{3}$$

Intuitive Interpretation of Holdings Distribution $F(x|\gamma)$

1. Rust (1985) "Equilibrium Holding Distributions in Durable Asset Markets" *Transportation Research Part B: Methodological*
2. Using *renewal theory* Rust characterized $F(x|\gamma)$ as

$$F(x|\gamma) = \frac{\text{mean first passage time from 0 to } (x, \infty)}{\text{mean first passage time from 0 to } (\gamma, \infty)} \tag{4}$$

3. **Example:** if $\Phi(x'|x) = 1 - \exp\{-\lambda(x' - x)\} I\{x' \geq x\}$, then

$$\text{mean first passage time from 0 to } (x, \infty) = (1 + \lambda x) \tag{5}$$

so

$$F(x|\gamma) = \frac{1 + \lambda x}{1 + \lambda\gamma} \tag{6}$$

Stationary Equilibrium: Homogeneous Consumers

1. If all consumers have same τ, they must be indifferent about which car to buy

$$M(x,\tau) + P(x) - \beta EP(x) = \text{constant}$$

2. This implies that P is the solution to the following *functional equation*

$$P(x) = \max\left[\underline{P}, \overline{P} - \beta EP(0) - M(0,\tau) - M(x,\tau) + \beta EP(x)\right]$$

3. **Theorem** $P(x) = \overline{P} - [V_\tau(x) - V_\tau(0)]$.

Stationary Equilibrium: Homogeneous Consumers

1. **Theorem** If $\Phi(y|x) = \max[0, 1 - \exp\{-\lambda(y-x)\}]$, then

$$\begin{aligned} P(x) &= \max\Big[\underline{P}, \underline{P} + \\ &\qquad \frac{1}{1-\beta}\int_x^\gamma M'_\tau(y)\,[1-\beta\exp\{-\lambda(1-\beta)(y-x)\}]\,dy\Big] \\ F(x|\gamma) &= \frac{1+\lambda x}{1+\lambda\gamma} \quad x \in [0,\gamma] \end{aligned}$$

where γ is the unique solution to

$$\overline{P} - \underline{P} = \frac{1}{1-\beta}\int_0^\gamma M'(y,\tau)\,[1-\beta\exp\{-\lambda(1-\beta)(y-x)\}]\,dy$$

Stationary Equilibrium: Heterogeneous Consumers

1. **Theorem** If consumers are heterogeneous, a unique stationary equilibrium exists and is given by $F(x|\gamma)$ that is the unique solution to (3) and P is the solution to the the functional equation

$$P(x) = \max\left[\underline{P}, \overline{P} + \beta[EP(x) - EP(0)] - \int_0^x M'(y, H^{-1}(1 - F(y|\gamma)))dy\right]$$

where γ is the smallest solution to

$$\overline{P} - \beta EP(0) = (1-\beta)\underline{P} + \int_0^\gamma M'(y, H^{-1}(1 - F(y|\gamma)))dy$$

Declining Depreciation Result

1. The *economic depreciation* of a durable good is given by $P(x) - \beta EP(x)$.
2. In the homogeneous consumer case we have all consumers have type τ^* and
$$P(x) - \beta EP(x) = \overline{P} - \beta EP(0) - \int_0^x M'(y, \tau^*)dy.$$
3. In the heterogeneous consumer case we have all consumers have different types τ (with CDF H) and
$$P(x) - \beta EP(x) = \overline{P} - \beta EP(0) - \int_0^x M'(y, H^{-1}(1 - F(y|\gamma)))dy.$$
4. **Theorem** In either the homogeneous or heterogeneous consumer equilibrium, economic depreciation declines with x
$$\frac{d}{dx}P(x) - \beta EP(x) < 0.$$

Representative Agent Formulation

1. **Lemma** In equilibrium $z^*(\tau)$ is given by

$$z^*(\tau) = \begin{cases} F^{-1}(1 - H(\tau)|\gamma) & \text{for} \quad \tau \in [\underline{\tau}, \hat{\tau}) \\ 0 & \text{for} \quad \tau \in [\hat{\tau}, \overline{\tau}] \end{cases}$$

   where $\hat{\tau} = H^{-1}(1 - F(0|\gamma))$.

2. **Definition** The *modified utilitarian cost function* $M(x, \gamma)$ is given by

$$M(x, \gamma) = \int_0^x M'(y, H^{-1}(1 - F(y|\gamma)))dy$$

3. **Comment:** The modified utiitarian cost function puts no weight on consumers $\tau \in (\hat{\tau}, \overline{\tau}]$ who have the strongest preferences for new cars.

Shadow Prices to Social Planning Problem

1. **Definition** Let $V_\gamma(x)$ be the solution to the social planning problem with the modified utilitarian cost function

$$V_\gamma(x) = \min\left[M(x,\gamma) + \beta E V_\gamma(x), M(0,\gamma) + \overline{P} - \underline{P} + \beta E V_\gamma(0)\right].$$

where $\gamma = \Gamma(\gamma)$ is the smallest fixed point to the mapping $\Gamma : R_+ \to R_+$ given by the smallest solution to

$$\overline{P} - \underline{P} + V_\gamma(0) = V_\gamma(\Gamma(\gamma)).$$

2. **Theorem** *In the heterogeneous agent equilibrium, prices are given by*

$$P(x) = \overline{P} - [V_\gamma(x) - V_\gamma(0)]$$

where $\gamma = \Gamma(\gamma)$.

Closed-form solution in exponential case

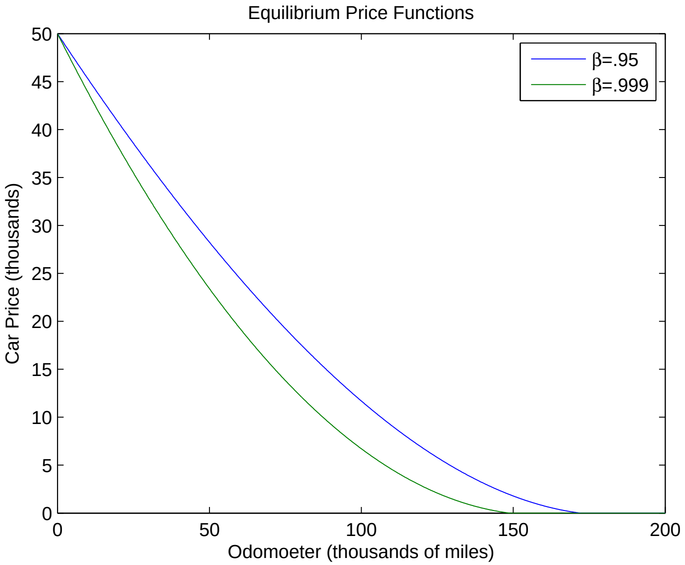
1. **Theorem** *If $\Phi(y|x) = \max[0, 1 - \exp\{\lambda(y-x)\}]$, then there is a unique stationary equilibrium given by*

$$
\begin{aligned}
F(x|\gamma) &= \frac{1+\lambda x}{1+\lambda\gamma} \\
P(x) &= \max\left[\underline{P}, \overline{P} + K(x,\gamma)/(1-\beta)\right] \\
K(x,\gamma) &= \int_x^\gamma M'(y,\gamma)[1-\beta\exp\{-\lambda(1-\beta)(y-x)\}]dy \\
M'(y,\gamma) &\equiv M'(y, H^{-1}(\gamma - y)/(1+\lambda\gamma)))
\end{aligned}
$$

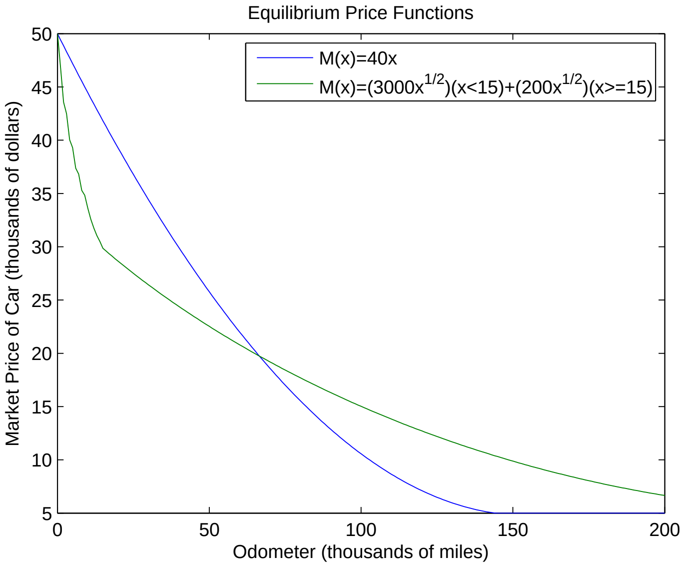
   where γ is the unique solution to $(1-\beta)(\overline{P} - \underline{P}) = K(0,\gamma)$.

2. **Comment** Notice the striking similarity of this solution and the solution to the homogeneous consumer case. The result underscores the observational equivalence of heterogeneous and homogeneous agent stationary equilibria.

Equilibrium Price Functions, effect of discount factor β



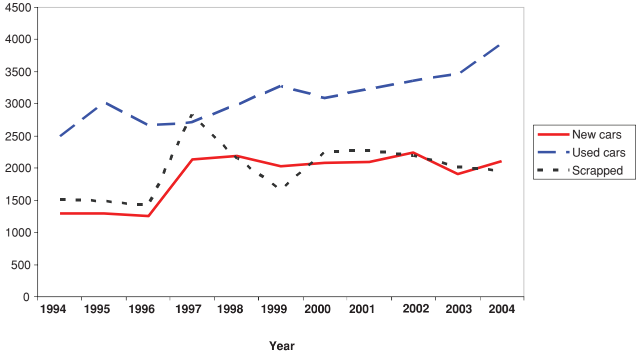
Equilibrium Price Functions, effect of maintenance costs M



Stationary Equilibria when car states are discrete

1. We can carry through a similar analysis if we treat the state of a car as a discrete “age” variable $a \in \{0, 1, 2, \dots\}$
2. Now, transitions are of the form $a \to a + 1$ with probability 1.
3. If cars are only scrapped at some age γ in the equilibrium in the automobile market when a secondary market exists, then Rust’s renewal result implies that $F(a|\gamma)$ is a discrete uniform distribution on the set $\{0, 1, \dots, \gamma\}$.
4. So $F(x|\gamma) = 1/\gamma$ and $1/\gamma$ of the stock of cars is scrapped each year, replaced by a corresponding fraction of new cars (flow equilibrium)

Empirical evidence for flow equilibrium: Schiraldi RAND 2011



Stationary Equilibria when car states are discrete

1. In the case of no transactions costs, each consumer has a preferred age of vehicle $a*$ that minimizes

$$a^* = \underset{a}{argmin} \left[m(a) + P(a) - \beta P(a+1) \right] \tag{7}$$

2. If there is an infinitely elastic supply of new cars at price $P(0) = \overline{P}$ and an infinitely elastic demand for scrapped cars at price $\underline{P}$, then $P(a) = \gamma$ for $a \geq \gamma$. The remaining prices $\{P(1), P(2), \ldots, P(\gamma - 1)\}$ are determined endogenously in equilibrium to set excess demand to zero at each age $a \in \{1, 2, \ldots, \gamma - 1\}$.
3. If consumers are homogeneous, then as before, we must have $m(a) + P(a) - \beta P(a+1) = K$ for some constant K for $a \in \{1, 2, \ldots, \gamma - 1\}$.

Stationary Equilibria when car states are discrete

1. These indifference relations imply a system of $\gamma - 1$ linear equations in $\gamma - 1$ unknowns $\{P(1), \ldots, P(\gamma - 1)\}$ given by

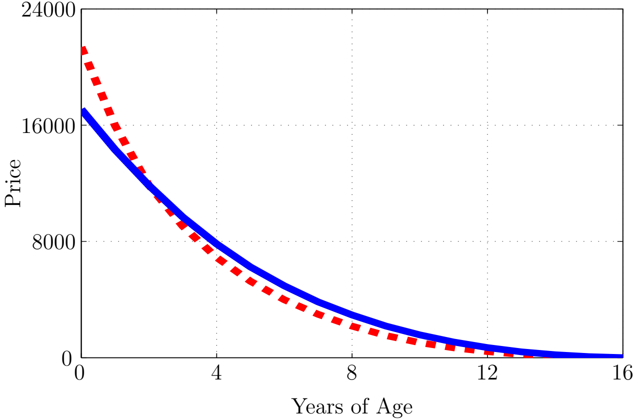
$$X \times P = Y \tag{8}$$

and

2. $P(a) = \overline{P} - [V(a) - V(0)]$ where
3. V is the value function to the optimal replacement problem

$$V(a) = \left[m(a) + \beta V(a+1), \overline{P} - \underline{P} + m(0) + \beta V(1)\right]. \tag{9}$$

Equilibrium Prices, U.S. vs France from Gavazza *et. al.*



Adding "Common" States (Macro shocks, fuel prices)

1. Hard! Why hard? Because the equilibrium prices will not only depend on the macro state and fuel price state variables, but *the entire holdings (age) distribution of vehicles from the previous period.* This holdings distribution represents the potential supply of different used car ages/odometer values.
2. But the distribution of car ages/odometer values is an *infinite dimensional state variable!*
3. Curse of dimensionality of DP implies that it is very very difficult to incorporate high but finite-dimensional state variables, much less infinite-dimensional state variables.

Krusell-Smith: use "moments" of the distribution

1. Per Krusell and Anthony Smith (1998) *Journal of Political Economy* "Income and Wealth Heterogeneity in the Macroeconomy"
2. "Most of dynamic general equilibrium macroeconomic theory relies heavily on the representative-agent abstraction: it is assumed that the economy behaves 'as if' it is inhabited by a single (type of) consumer."
3. "There are two circumstances, however, under which the representative-agent construct would be a reasonable modeling strategy. First, it is possible that the theoretical assumptions needed to justify the use of a representative consumer are roughly met in the data. . . . A second possibility is that the aggregate variables in theoretical models with a more realistic description of the microeconomic environment actually behave like those in the representative-agent models."

What is infinite-dimensional in Krusell-Smith?

1. **Wealth distribution** "We characterize stationary stochastic equilibria of this model numerically, and we then compare the aggregate properties of these equilibria with those implied by the corresponding representative agent model."
2. "An important component of our analysis involves dealing with the main computational difficulty of dynamic heterogeneous-agent models: in order to predict prices, consumers need to keep track of the evolution of the wealth distribution."
3. "One of the contributions of our work is to show how equilibria can be approximated numerically, despite the fact that the state of the economy at any point in time is an infinite-dimensional object (we assume a continuum of agents)."

The consumer's problem

1. Let Γ (continuous, infinite-dimensional) be the distribution of wealth (capital) and let z (discrete, uni-dimensional) denote a *technology shock.* (Γ, z) are *macro state variables.*
2. Let ν be a (discrete, uni-dimensional) *idiosyncratic (agent-specific) state variable* and let k be a (continuous, uni-dimensional) agent-specific wealth (capital stock ownership). Thus, (ν, k) evolve independently for different agents and agents do not observe other agents' (ν, k) values.
3. Bellman equation: an agent's value function is given by

$$\begin{aligned} V(k, \nu, \Gamma, z) &= \max_{c, k'} \left[u(c) + \beta E \left\{ V(k', \nu', \Gamma', z') | z, \nu \right\} \right]. \\ \Gamma' &= H(\Gamma, z, z') \\ c + k' &= r(\overline{k}, \overline{l}, z)k + w(\overline{k}, \overline{l}, z)l\nu + (1 - \delta)k \\ k' &\geq 0 \quad c \geq 0 \end{aligned}$$

The Krusell-Smith trick

1. Assume that we can summarize the distribution of capital Γ_t by its first moment: $\overline{k}_t = \int_0^\infty k d\Gamma_t(k)$ and this is a *sufficient statistic* for agents to predict future wages and interest rates.
2. Then we can reduce the agent's DP problem to a *finite-dimensional* state variable $(k, \nu, \overline{k}, z)$ and value function

$$
\begin{aligned}
V(k, \nu, \overline{k}, z) &= \max_{c, k'} \left[u(c) + \beta E \left\{ V(k', \nu', \overline{k}', z') | z, \nu \right\} \right]. \\
\log(\overline{k}') &= a(z) + b(z) \log(\overline{k}) \\
c + k' &= r(\overline{k}, \overline{l}, z)k + w(\overline{k}, \overline{l}, z) l \nu + (1 - \delta) k \\
k' &\geq 0 \quad c \geq 0
\end{aligned}
$$

Can $\overline{k}'$ be perfectly predicted using $\overline{k}$ as Krusell and Smith assume?

Almost!

1. Regression results from a numerical solution/simulation

   $$\log(\overline{k}') = 0.095 + 0.0962 \log(\overline{k}) \; z = 1 \; R^2 = 0.999998 \; \hat{\sigma} = 0.0028$$
   $$\log(\overline{k}') = 0.085 + 0.0965 \log(\overline{k}) \; z = 0 \; R^2 = 0.999998 \; \hat{\sigma} = 0.0036$$

   Why is today's mean capital stock such a good predictor of tomorrow's mean capital stock?

2. "the marginal propensities to consume are almost identical for agents with different employment states and levels of capital. As the agent's wealth increases, the slope increases toward one; a slope of one would amount to exact permanent income behavior, as discussed in Bewley (1977)."

3. "Most of the capital, however, is held by agents with essentially the same savings propensity. Very few agents — the very poorest ones — have a much lower propensity, and the capital that they hold is negligible."

"Approximate Aggregation"

1. "Why, then, are marginal savings propensities almost independent of wealth in the benchmark model? In the class of models that we consider here, the utility costs from accepting fluctuations in consumption are very small, even when these fluctuations are several times larger than for aggregate consumption."
2. "the access to one aggregate asset is sufficient for providing the agent with very good insurance in utility terms. . . . As a consequence, in the stationary state, most agents have enough capital that their savings behavior is guided mainly by intertemporal concerns rather than by insurance motives."

Krusell and Smith's Conclusions

1. "Our first and main finding is that a low-dimensional object — the total capital stock and the value of the aggregate productivity shock — seems to be sufficient for characterizing the stochastic behavior of all the macroeconomic aggregates, despite substantial heterogeneity in the population with respect to wealth as well as to some preference parameters. Hence, one need track only the evolution of the aggregate budget in order to analyze the dynamic behavior of the macroeconomic aggregates."
2. "A natural question is thus whether it is possible to describe preferences of an aggregate consumer such that when this agents utility is maximized subject to the aggregate budget the outcome is a set of time series that matches those calculated here. Our second finding is that this is sometimes, but not always, an easy task."

Krusell and Smith's Conclusions, continued

1. "In our benchmark economy and some extensions to it, the incomplete-markets (heterogeneous- agent) economy behaves almost identically to its complete- markets (representative-agent) counterpart, except for a difference in levels: in the stationary stochastic equilibrium, capital is slightly higher in the incomplete-markets version. In some extensions to the benchmark setup, however, especially those with preference heterogeneity, some second moments of the aggregate time series are substantially different."
2. "As we have shown in this paper, introducing preference heterogeneity into the standard model allows a closer match between model and data."

PART II

Analysis of a "Toy Model" of Heterogeneous Agent Equilibrium in the Auto Market

Homogeneous consumer economy

A stylized model for the automobile market

- Unit mass of homogeneous consumers that live forever.
- Cars vary only by age $a \in \{0, 1, \ldots, \overline{a}\}$
- Cars are scrapped at age $\overline{a}$
- Consumers can
  1. trade their current car for a car of age $d^* \in \{0, 1, \ldots, \overline{a} - 1\}$
  2. keep their current car, $(d = -1)$ (if $a < \overline{a}$)
  3. no out side option, consumers must own a car every period
- No transaction costs

Utility of owning a car

- Utility is a decreasing function $u(a)$ that reflects the net of
  1. decreasing utility of car services
  2. increasing cost of maintenance

The Bellman equation

For a consumer who owns a car of age $a \in \{1, \ldots, \bar{a} - 1\}$

$$V(a) = \max \left[u(a) + \beta V(a+1), \max_{d \in \{0,1,\ldots,\bar{a}-1\}} [u(d) + P(a) - P(d) + \beta V(d+1)] \right] \quad (10)$$

For a consumer who a car of age $a \geq \bar{a}$

$$V(a) = \max_{d \in \{0,\ldots,\bar{a}-1\}} [u(d) + \underline{P} - P(d) + \beta V(d+1)] \quad (11)$$

Trading costs

- Trading costs: $P(d^*) - P(a)$ when consumers sell their car of age a to buy a desired car of age d^*
- New cars: Infinitely elastic supply at fixed price $\overline{P}$
- Scrapped cars: Infinitely elastic demand at fixed price $\underline{P} < \overline{P}$.
- Secondary market: used cars are traded and consumers can buy or sell a car of age a at a price $P(a)$

The Bellman equation

With no transaction cost it is always optimal to trade for the preferred car age d^* This implies that

$$V(a) = u(d^*) + P(a) - P(d^*) + \beta V(d^* + 1) \tag{12}$$

and in particular,

$$V(d^*) = u(d^*) + \beta V(d^* + 1) \tag{13}$$

and

$$V(d^* + 1) = V(d^*) + P(d^* + 1) - P(d^*) \tag{14}$$

so these equations imply that

$$V(d^*) = \frac{u(d^*) - \beta[P(d^*) - P(d^* + 1)]}{1 - \beta}. \tag{15}$$

The Bellman equation

The Bellman equation for $a = d^*$

$$V(d^*) = \frac{u(d^*) - \beta[P(d^*) - P(d^*+1)]}{1-\beta}. \tag{16}$$

The Bellman equation for $a \neq d^*$

$$\begin{aligned} V(a) &= V(d^*) - [P(d^*) - P(a)] \\ &= \frac{u(d^*) - [P(d^*) - \beta P(d^*+1)]}{1-\beta} + P(a). \end{aligned} \tag{17}$$

If a consumer has a car of age $a \neq d^*$

- The consumer will have to sell this car in period $t = 0$ and buy the optimal age car d^*.
- The trading cost for this is $[P(d^*) - P(a)]$

Consumers muts be indifferent in equilibrium

From equation (17) we see that the discounted utility from a policy of trading for a car of age d in every period is given by

$$U(d) = \frac{u(d) - [P(d) - \beta P(d+1)]}{1-\beta} \tag{18}$$

If consumers are indifferent between all available ages of vehicles, then

$$u(d) - [P(d) - \beta P(d+1)] = K(1-\beta), \tag{19}$$

for $d \in \{0, 1, \ldots, \overline{a} - 1\}$.

These indifference restrictions imply a system of $\overline{a} - 1$ linear equations in the $\overline{a} - 1$ unknowns $P(1), \ldots, P(\overline{a} - 1)$.

Computing equilibrium prices

This system can be written in matrix form

$$X \times P = Y \tag{20}$$

where $P' = (P(1), \ldots, P(\overline{a}-1))$,

X is the $\overline{a}-1 \times \overline{a}-1$ matrix

$$X = \begin{bmatrix} -(1+\beta) & \beta & 0 & \cdots & 0 & 0 \\ 1 & -(1+\beta) & \beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & -(1+\beta) & \beta \\ 0 & 0 & \cdots & 0 & 1 & -(1+\beta) \end{bmatrix},$$

Y is a $\overline{a}-1 \times 1$ vector

$$Y = \begin{bmatrix} u(0) - u(1) - \overline{P} \\ u(1) - u(2) \\ \cdots \\ u(\overline{a}-3) - u(\overline{a}-2) \\ u(\overline{a}-2) - u(\overline{a}-1) - \beta \underline{P} \end{bmatrix}.$$

Heterogeneous consumer economy with transactions costs

Taste heterogeneity:

- Add an additive unobserved state variable ϵ to the value of each choice alternative
- Let ϵ be i.i.d. extreme value with (common) scale parameter $\sigma \geq 0$

Transaction cost

- We assume that in any secondary market transaction the seller of a car pays an exogenously fixed transaction cost $T \geq 0$.

The Bellman equation

Heterogeneous consumer economy with transactions costs

Bellman equation

If $a < \overline{a}$

$$V(a, \epsilon) = \max \left[v(-1, a) + \epsilon(-1), \max_{d \in \{0,1,\dots,\overline{a}-1\}} [v(d, a) + \epsilon(d)] \right],$$

If $a = \overline{a}$

$$V(\overline{a}, \epsilon) = \max_{d \in \{0,1,\dots,\overline{a}-1\}} [v(d, a) + \epsilon(d)].$$

The value of trading the current car of age a for a replacement car of age $d \in \{0, 1, \dots, \overline{a} - 1\}$

$$v(d, a) = u(d) + P(a) - P(d) - T + \beta EV(d + 1)$$

The value of keeping the currently held car of age a

$$v(-1, a) = u(a) + \beta EV(a + 1), \quad a \in \{1, \dots, \overline{a} - 1\}.$$

Expected value function

Heterogeneous consumer economy with transactions costs

EV is a fixed point to a contraction mapping, $EV = \Gamma(EV)$.

For $a = \overline{a}$

$$\begin{aligned} EV(\overline{a}) &= E\{V(\overline{a},\epsilon)\} = \int_\epsilon V(\overline{a},\epsilon) f(\epsilon) d\epsilon \\ &= \sigma \log\left(\sum_{d=0}^{\overline{a}-1} \exp\{v(d,\overline{a})/\sigma\}\right) \end{aligned} \tag{21}$$

For $a \in \{1, \ldots, \overline{a}-1\}$

$$\begin{aligned} EV(a) &= E\{V(a,\epsilon)\} = \int_\epsilon V(a,\epsilon) f(\epsilon) d\epsilon \\ &= \sigma \log\left(\exp\{v(-1,a)/\sigma\} + \sum_{d=0}^{\overline{a}-1} \exp\{v(d,a)/\sigma\}\right). \end{aligned} \tag{22}$$

Unique solution can be calculated by successive approximations (and Newton-Kantorovich algorithm).

The choice probabilities

Once we have found the solution $EV = (EV(1), \ldots, EV(\bar{a}))$, we can compute the multinomial choice probabilities

For $a < \bar{a} - 1$

$$
\begin{aligned}
Pr(-1|a) &= \frac{\exp\{v(-1,a)/\sigma\}}{\exp\{v(-1,a)/\sigma\} + \sum_{a'=0}^{\bar{a}-1} \exp\{v(a',a)/\sigma\}} \\
Pr(d|a) &= \frac{\exp\{v(d,a)/\sigma\}}{\exp\{v(-1,a)/\sigma\} + \sum_{a'=0}^{\bar{a}-1} \exp\{v(a',a)/\sigma\}},
\end{aligned}
\tag{23}
$$

For $a = \bar{a}$

$$
Pr(d|a) = \frac{\exp\{v(d,a)/\sigma\}}{\sum_{a'=0}^{\bar{a}-1} \exp\{v(a',a)/\sigma\}} \tag{24}
$$

where $d \in \{0, 1, \ldots, \bar{a} - 1\}$

We derived the solution taking prices $P(a)$ as given

- The value functions and choice probabilities were derived taking prices $P(a)$ as given.
- To emphasize this dependence on P we write the probabilities $Pr(d|a) = \Pi(d|a, P)$
- The solutions to the expected value functions above imply that $\Pi(d, a, P)$ are smooth implicit function of P

Equilibrium conditions

Now consider the equations for equilibrium in the market

$$\begin{aligned}
\sum_{a=1}^{\overline{a}} \Pi(1|a,P) &= 1-\Pi(-1|1,P) \\
\sum_{a=1}^{\overline{a}} \Pi(2|a,P) &= 1-\Pi(-1|2,P) \\
&\cdots \\
\sum_{a=1}^{\overline{a}} \Pi(\overline{a}-2|a,P) &= 1-\Pi(-1|\overline{a}-2,P) \\
\sum_{a=1}^{\overline{a}} \Pi(\overline{a}-1|a,P) &= 1-\Pi(-1|\overline{a}-1,P). \qquad (25)
\end{aligned}$$

That is

$$\textit{demand}(a,P) = \textit{supply}(a,P) \text{ for all } 0 < a < \overline{a}$$

Solving for the equilibrium prices

To solve for the equilibrium we search for prices that sets excess demand to zero in each of the $\overline{a} - 1$ second hand markets

$$E(P) = demand(P) - supply(P) = 0 \tag{26}$$

Since E is differentiable in P, we can solve it using Newton's method, iteratively as follows

$$P_{t+1} = P_t - [\nabla E(P_t)]^{-1} E(P_t) \tag{27}$$

where $\nabla E(P_t)$ is the $(\overline{a} - 1) \times (\overline{a} - 1)$ Jacobian matrix of $E(P)$.

Functional forms and parameters

Linear utility function $u(a) = c0 - c1 * a$

Parameters

- Price, new car: $\overline{P} = 10$
- Scrap price: $\underline{P} = 1$
- Discount factor $\beta = 0.95$
- Scale factor $\sigma = 1$
- Cost function parameters $c_0 = 0, c_1 = 0.01$
- Transaction costs $T = 1$
- Scarp date $T = 20$

Structural Estimation

Data: $(d_{i,t}, a_{i,t})$, $t = 1, ..., T_i$ and $i = 1, ..., n$

Likelihood function

$$\ell_i^f(\theta) = \sum_{t=2}^{T_i} log(\Pi(d_{i,t}|a_{i,t}, P, \theta))$$

where

- P must solve $E(P) = 0$
- $E()$ depends on Π
- Π depends on EV that solves $EV = \Gamma(EV)$
- All of this depends on parameters, θ
- A double nested fixed point problem?

A Double Nested Fixed Point Algorithm (N^2FXP)

N^2FXP solves the *unconstrained* optimization problem

$$\max_{\theta} L(\theta, P_\theta, EV_\theta)$$

Outer loop (Hill-climbing algorithm):

- Likelihood function $L(\theta, P_\theta, EV_\theta)$ is maximized w.r.t. θ
- Each evaluation of $L(\theta, P_\theta, EV_\theta)$ requires solution of $E_{(EV_\theta)}(P_\theta) = 0$
- Each evaluation of $E_{(EV_\theta)}$ requires solution of EV_θ

Middle loop (equilibrium):
The implicit function P_θ defined by $E_{(EV_\theta)}(P_\theta) = 0$ is solved by Newtons method

Inner loop (fixed point algorithm):
The implicit function EV_θ defined by $EV_\theta = \Gamma(EV_\theta)$ is solved by:

- Successive Approximations (SA) and Newton-Kantorovich (NK) Iterations

Alternatively treat P as parameters

NFXP solves the *unconstrained* optimization problem

$$\max_{\theta, P} L(\theta, P, EV_\theta)$$

Outer loop (Hill-climbing algorithm):

- Likelihood function $L(\theta, P, EV_\theta)$ is maximized w.r.t. θ and P
- Each evaluation of $L(\theta, P, EV_\theta)$ requires solution of EV_θ

Inner loop (fixed point algorithm):
The implicit function EV_θ defined by $EV_\theta = \Gamma(EV_\theta)$ is solved by:

- Successive Approximations (SA) and Newton-Kantorovich (NK) Iterations

Remark: Assumes that market is in equilibrium and that the sample reflect that

Sequential estimation, $NPL - NFXP$

If prices were known we could simply estimate θ using NFXP

Sequential $NPL - NFXP$ **algorithm**

Initialize let $k = 0$ and initialize with a guess on prices $P_k = P_0$

1. Set $k = k + 1$ and estimate θ_k using NFXP given prices P_{k-1}
2. Compute equilibrium prices, P_k using estimates $\hat{\theta}_k$ from Step 1
3. If $\theta_k - \theta_{k-1} < tol$ Go to step 1

Some remarks

- $NPL - NFXP$ produces a sequence of Pseudo ML estimators, $\hat{\theta}_k$, where each of them is obtained by NFXP given P_{k-1}.
- The likelihood function is a Pseudo likelihood unless the true equilibrium prices, P_θ is used.
- $\hat{\theta}_k$ converges to MLE obtained by N^2FXP - but much faster

PART III

A Dynamic Model of Vehicle Ownership, Type Choice, and Usage

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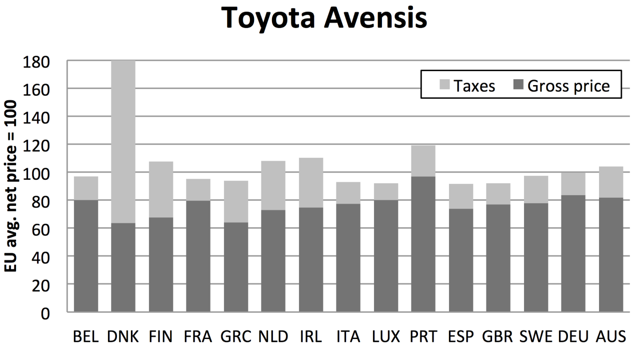
Fedor Iskhakov, Australian National University
Anders Munk-Nielsen, University of Copenhagen
John Rust, Georgetown University
Bertel Schjerning, University of Copenhagen

Policy Question and Data

Danish car tax: 180%!



Car taxes in Denmark

Annual revenue

- 30–50 billion DKK
- $\cong$ 2–3 pct. of GDP
- $\cong$ 4-7 pct. of total tax revenue

  Car taxes in DK, 1980-2012

- Most revenue originate from taxation of overship and registration of cars.
- Are transport externalities from car usage appropriately priced?
  - Underpriced congestion?
  - Incorrect taxation of gasoline?

Policy Question

IRUC reform

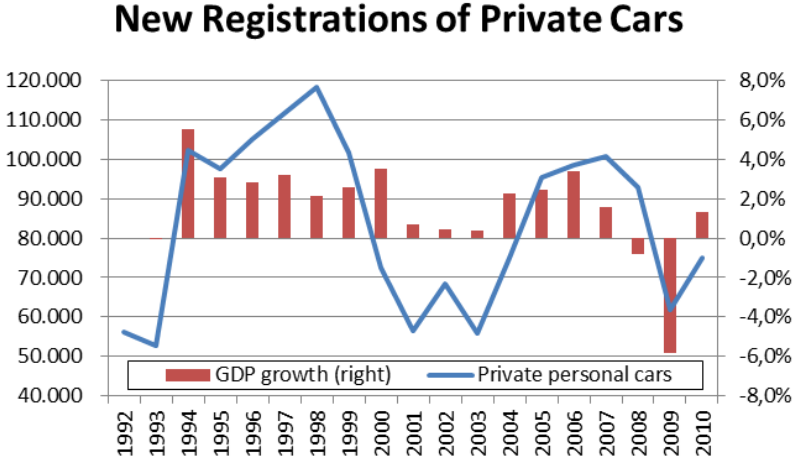
What would be the impact of a reform that a combination of

1. lower registration taxes, and
2. introducing road user charging

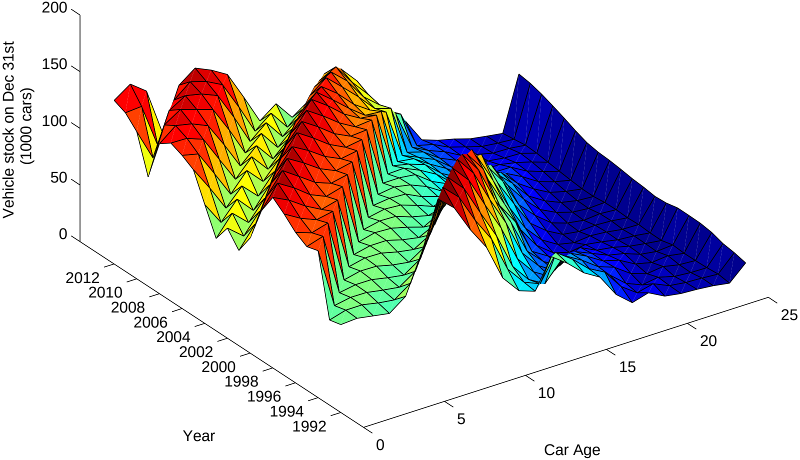
...on

1. car ownership:
    - new car purchases,
    - holding decisions,
    - scrappage,
    - value of the car stock,
2. driving, fuel demand and emissions,
3. redistribution and welfare.

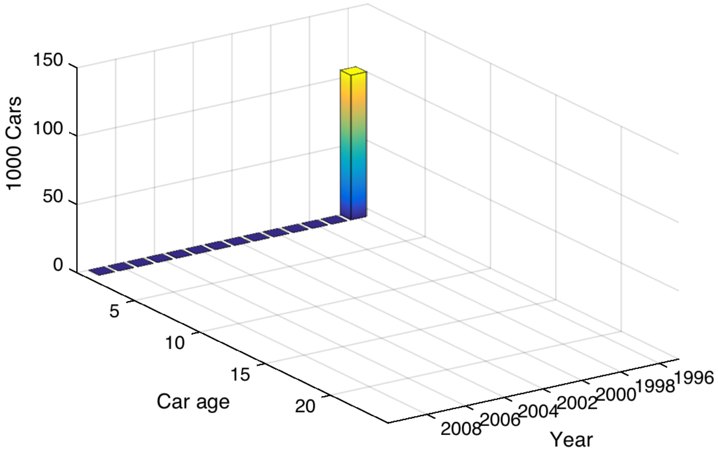
Car sales over the business cycle 1992-2010



Waves in the Age Distribution 1992-2012



The 1995 Cohort of Cars



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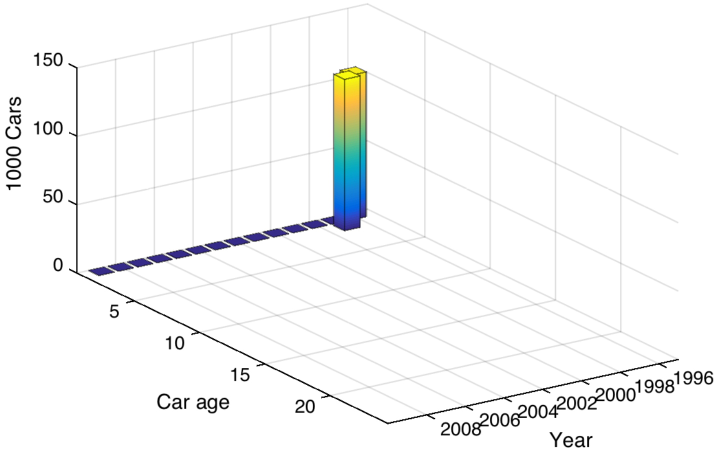

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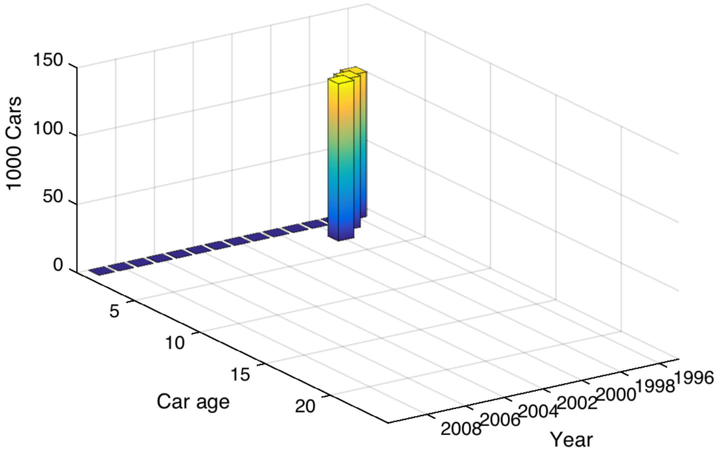

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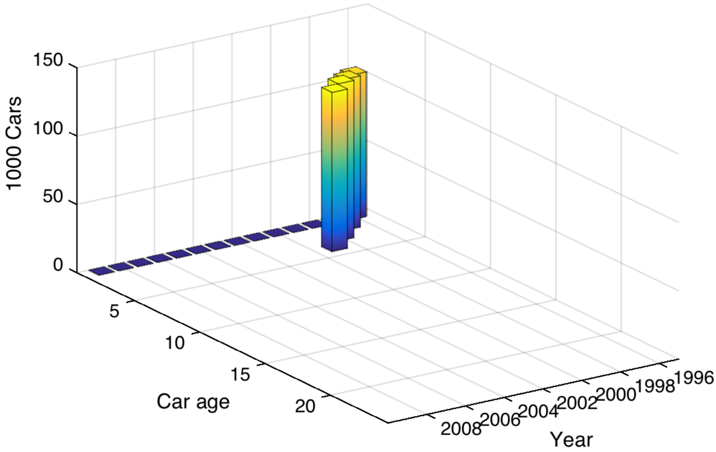

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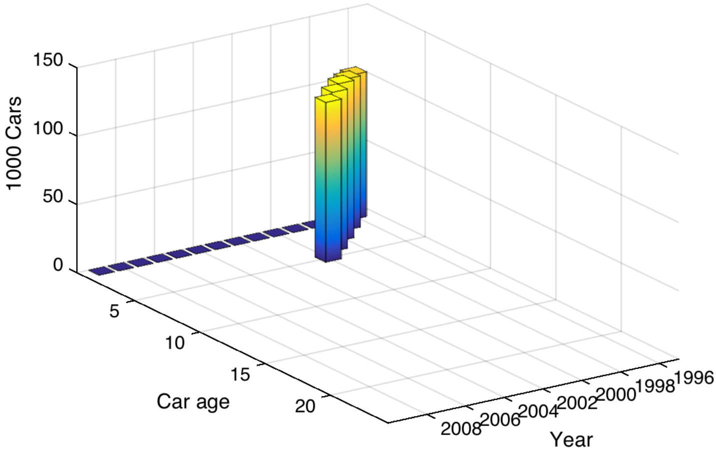

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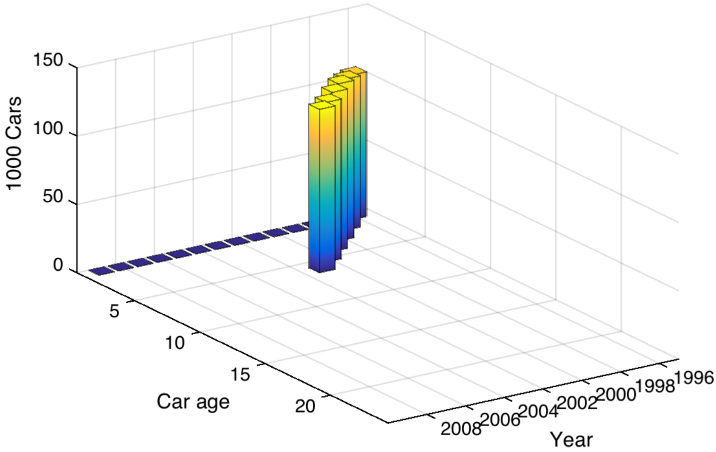

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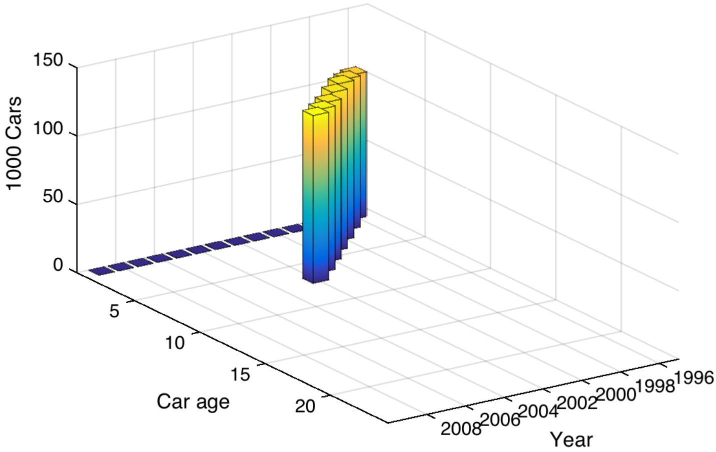

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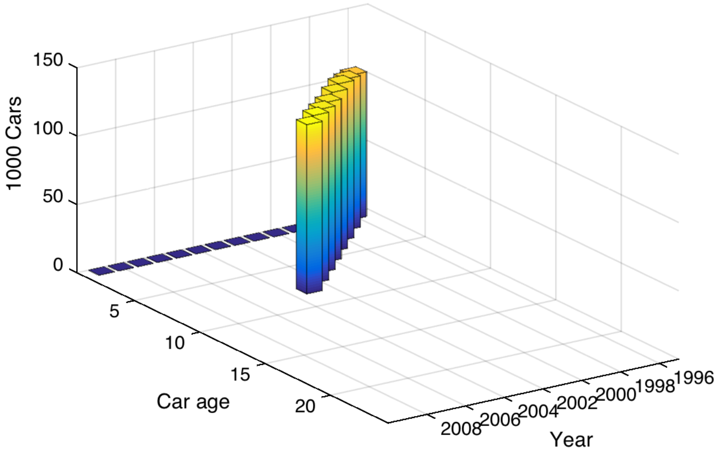

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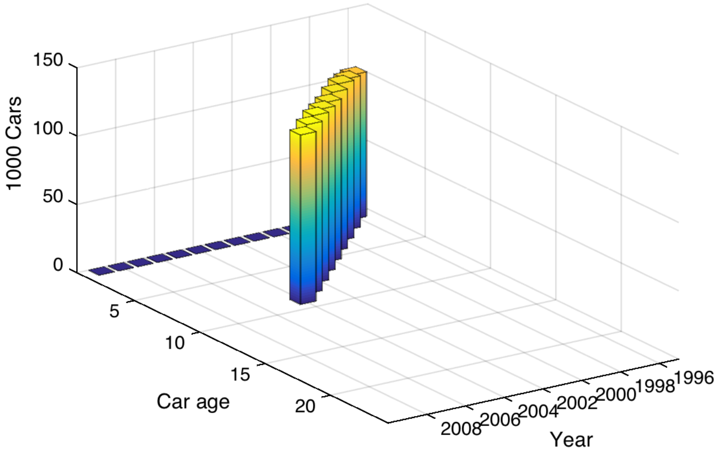

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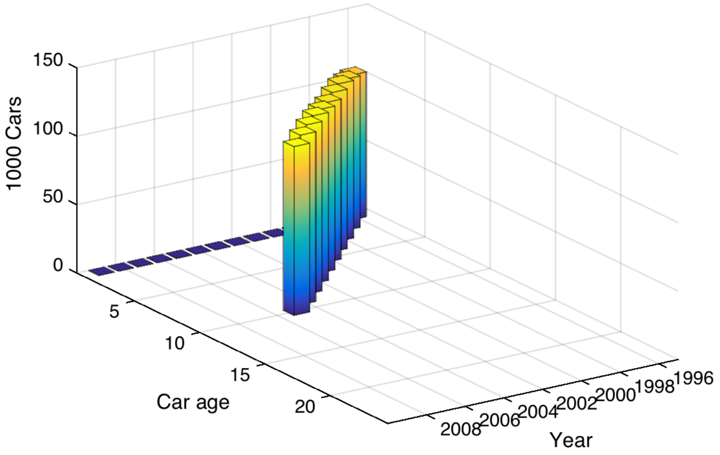

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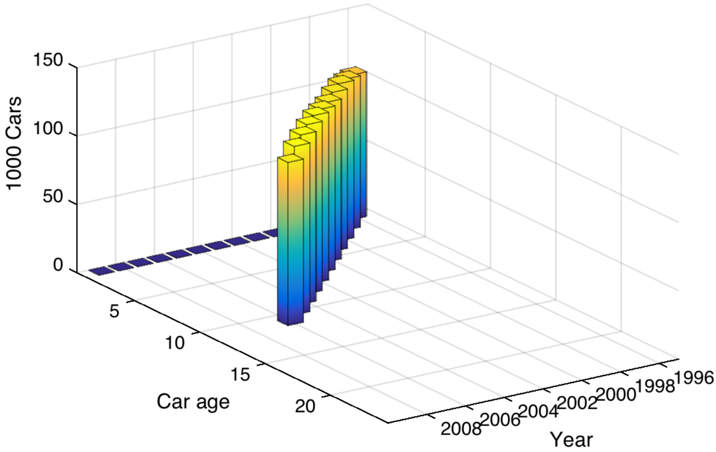

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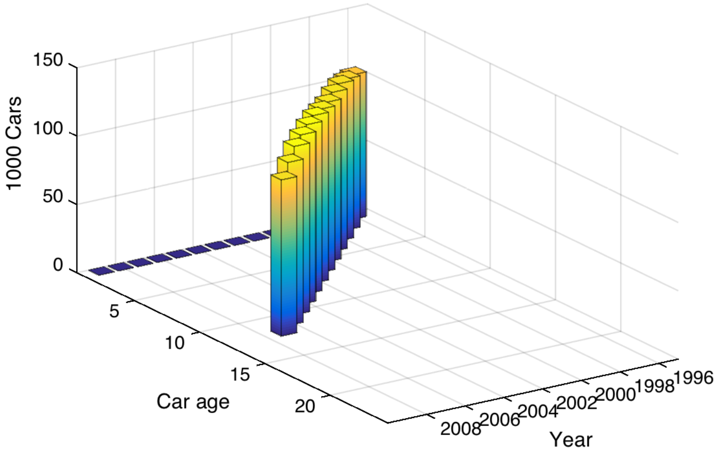

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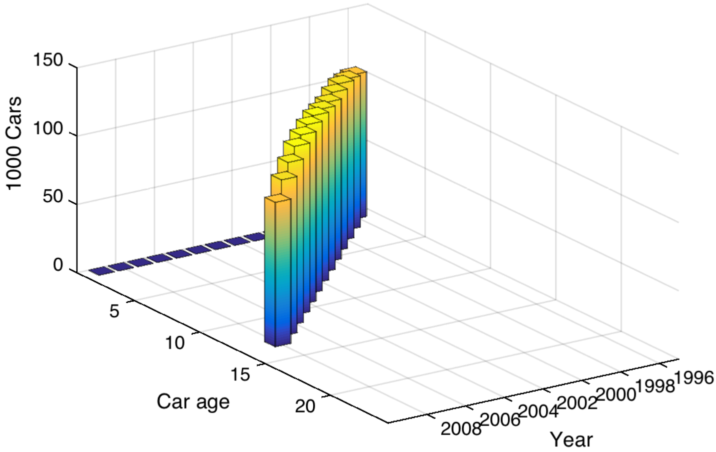

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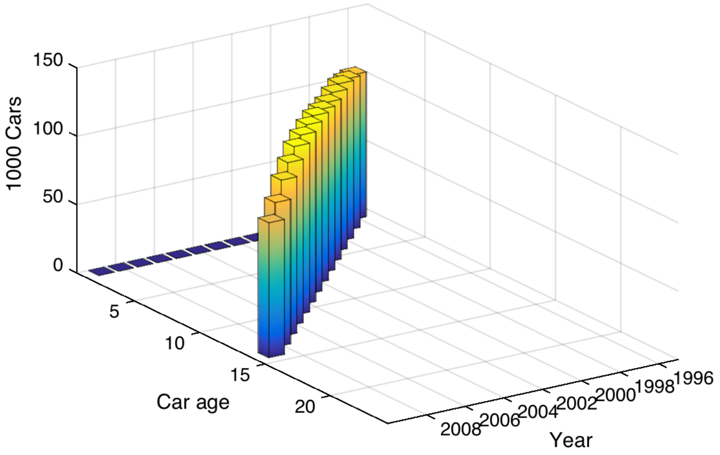
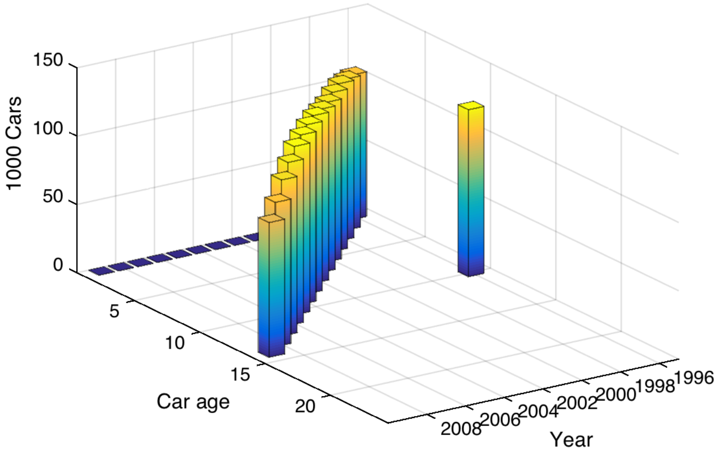


The 1995 and 1986 Cohorts of Cars



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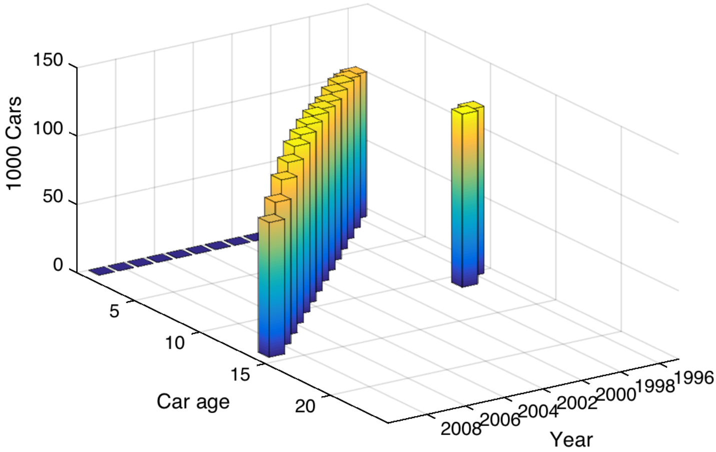

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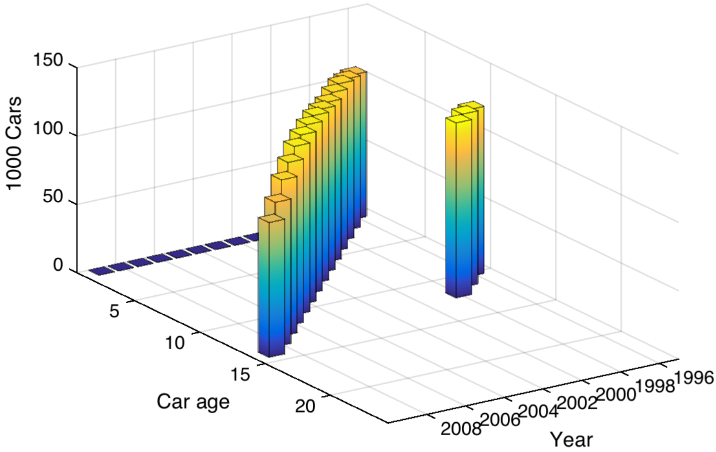

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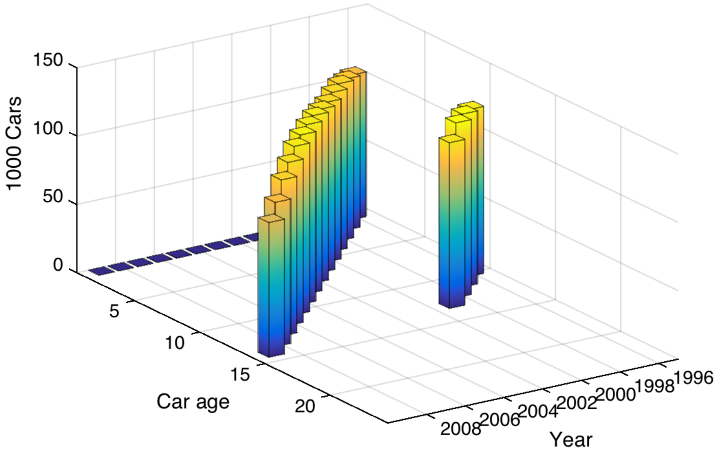

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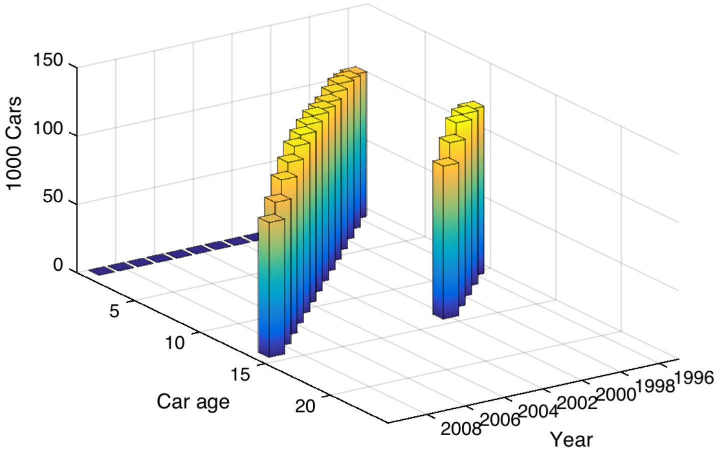

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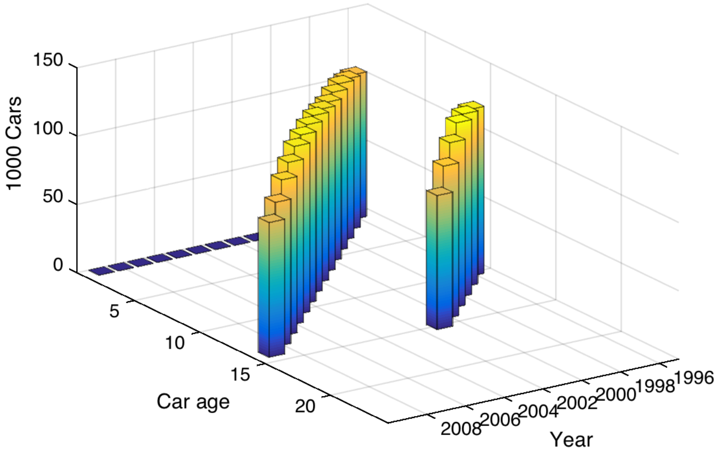

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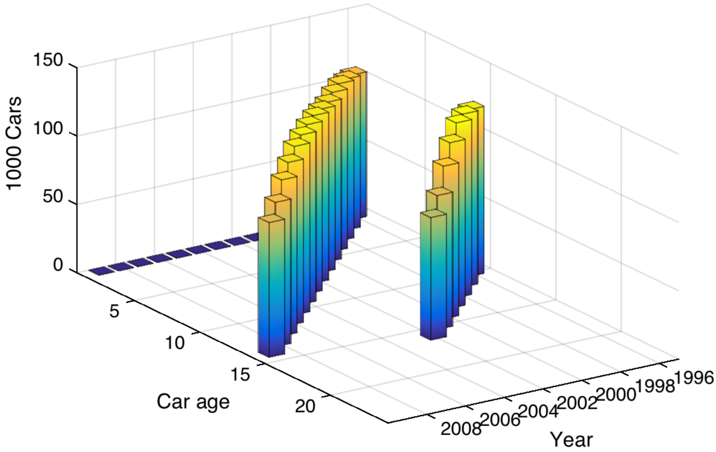

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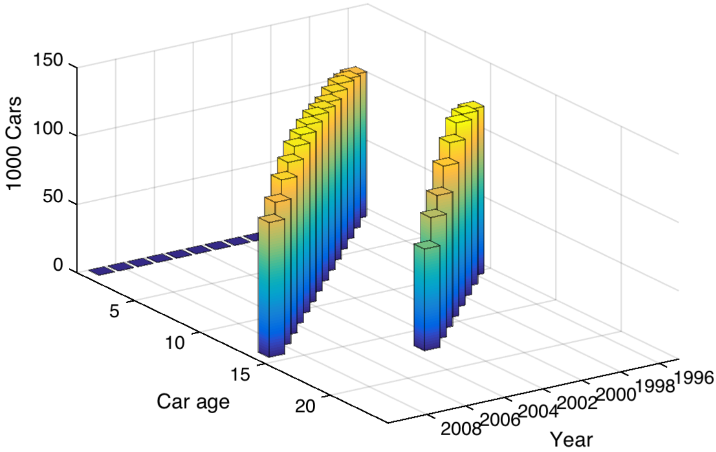

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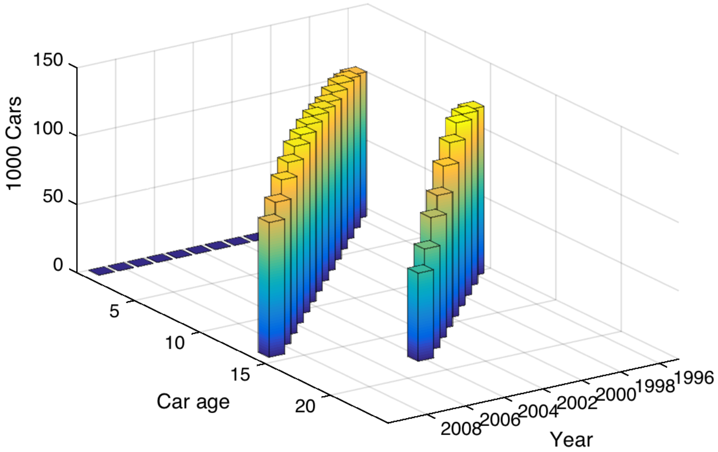

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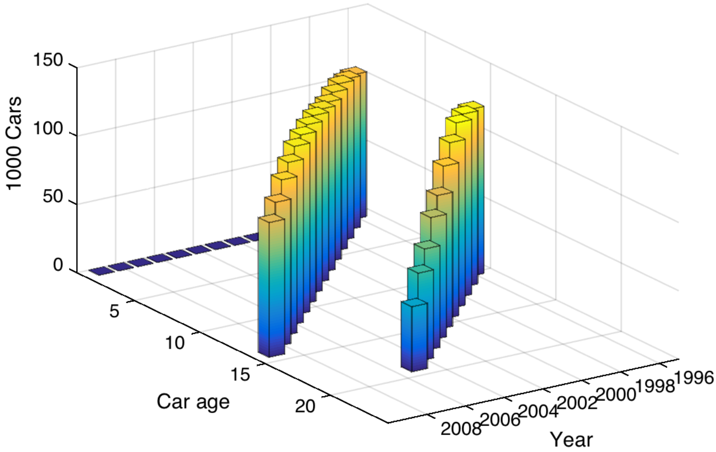

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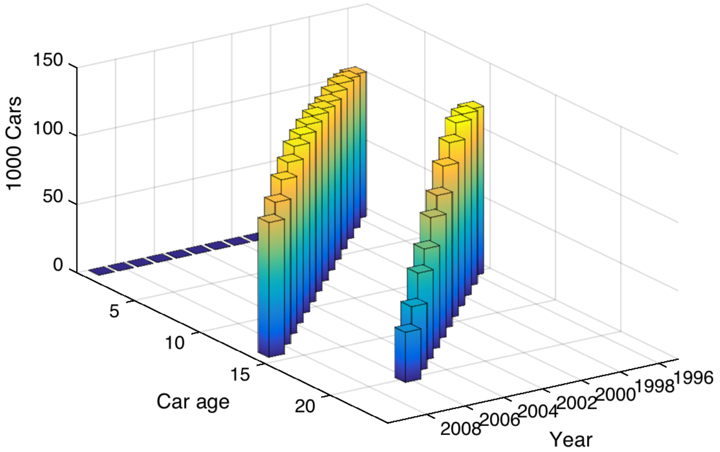

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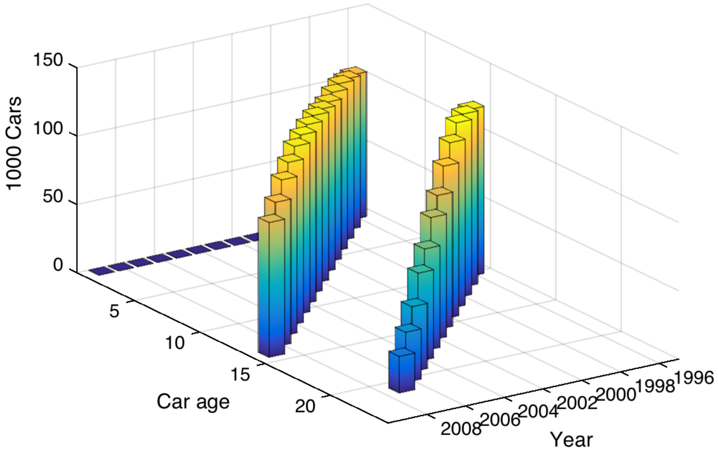

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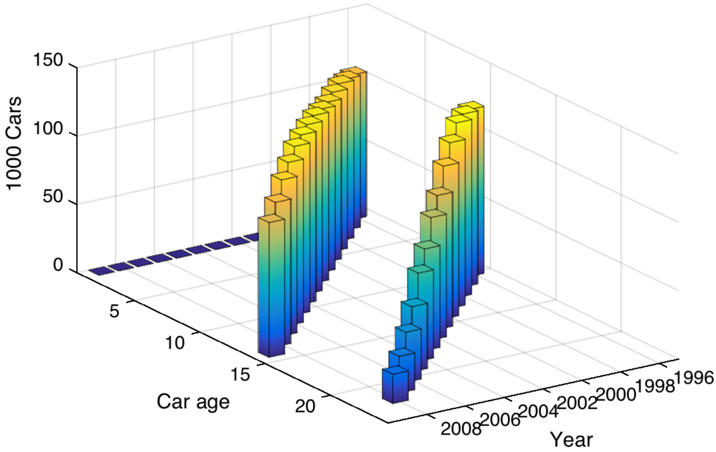

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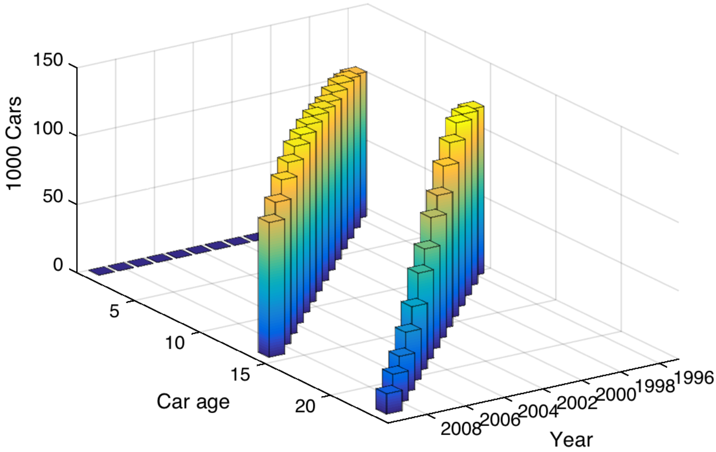
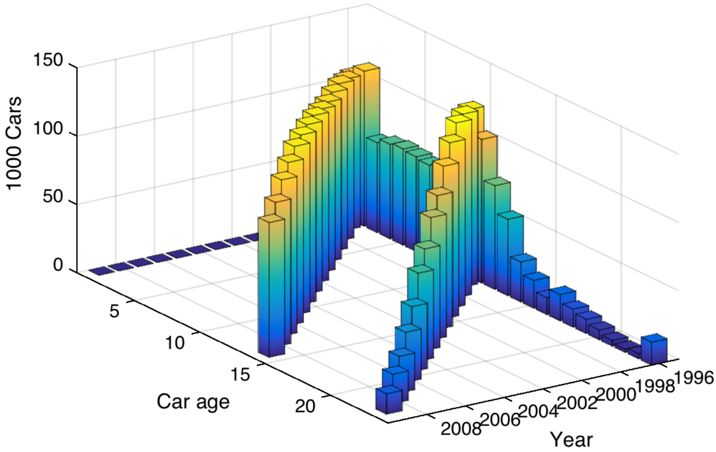


The Car Age Distribution Over Time



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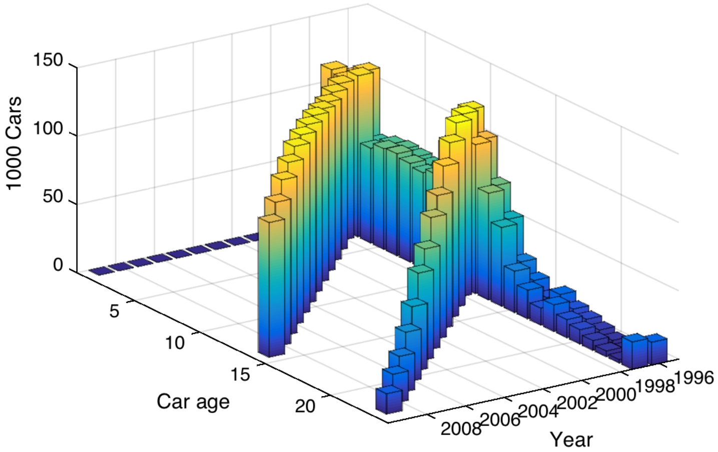

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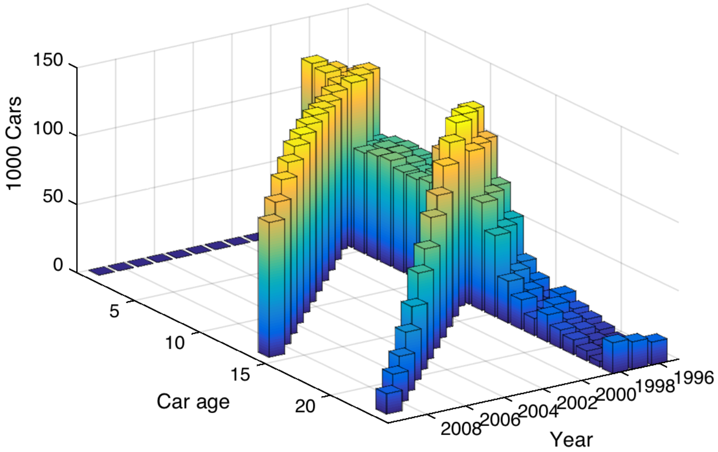

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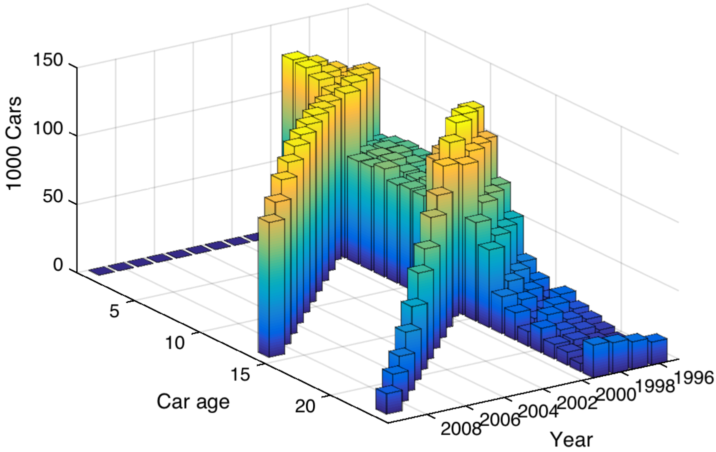

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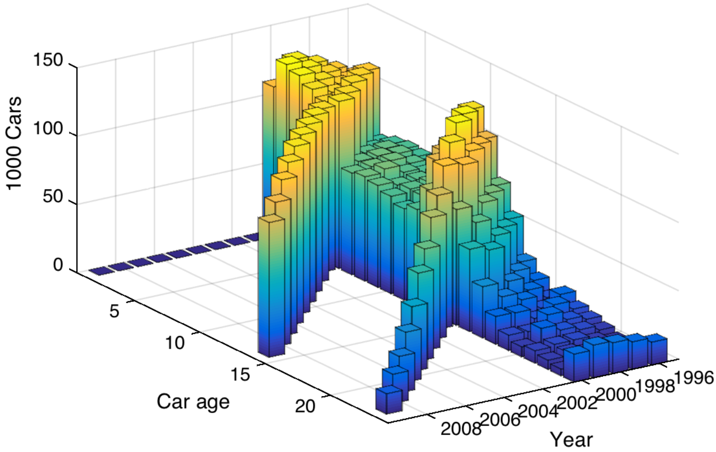

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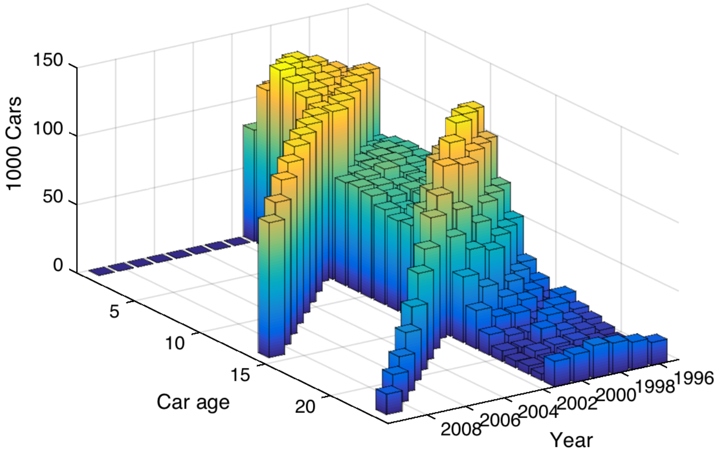

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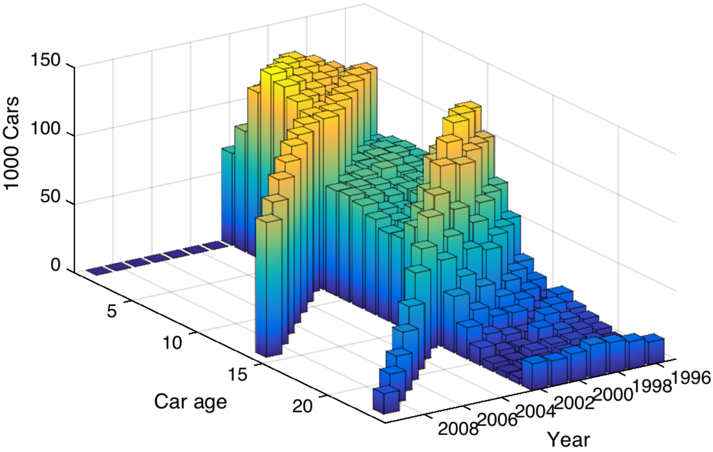

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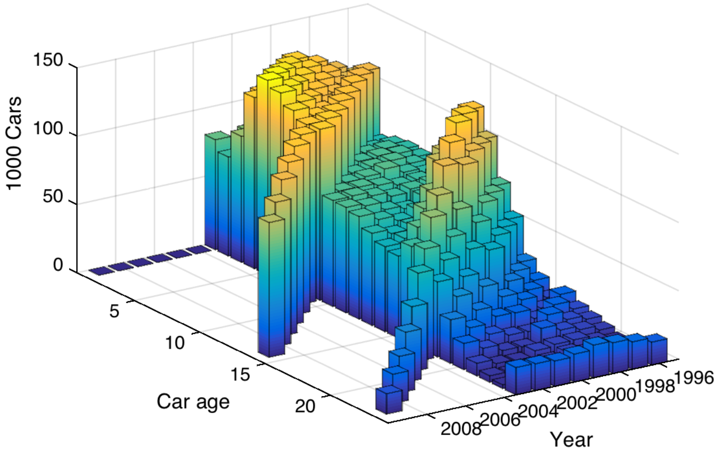

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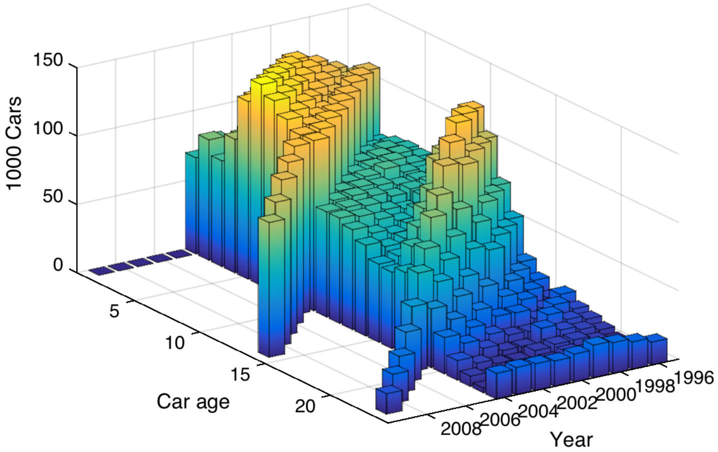

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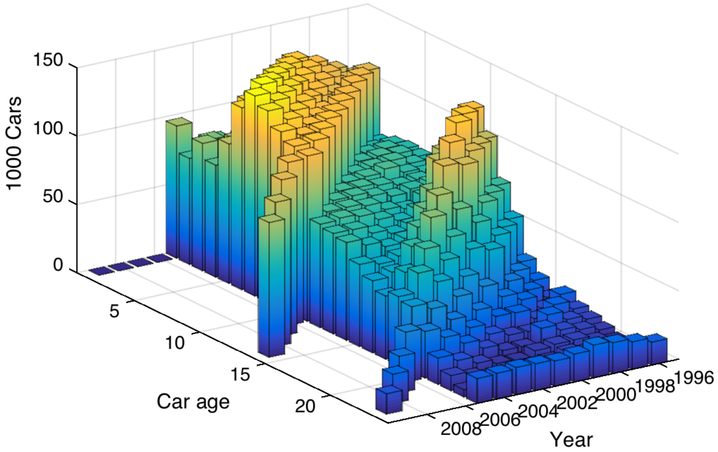

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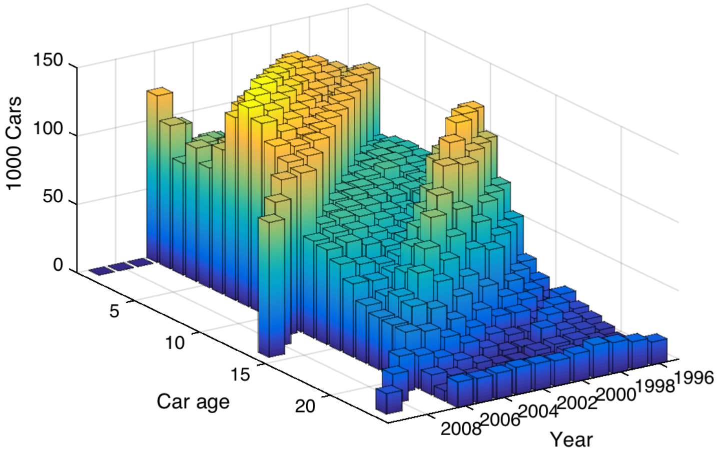

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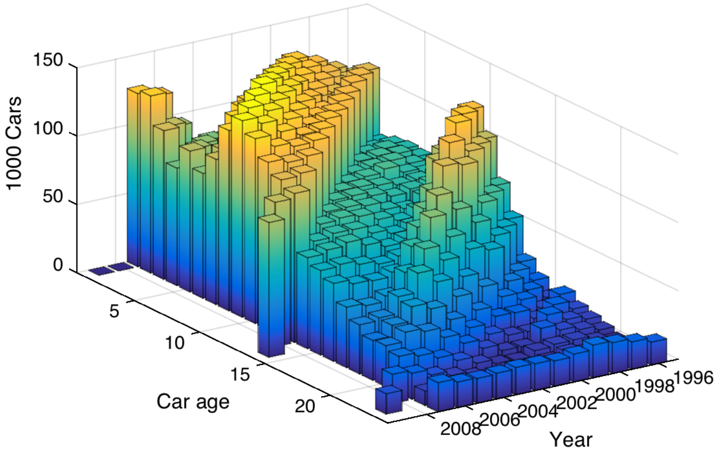

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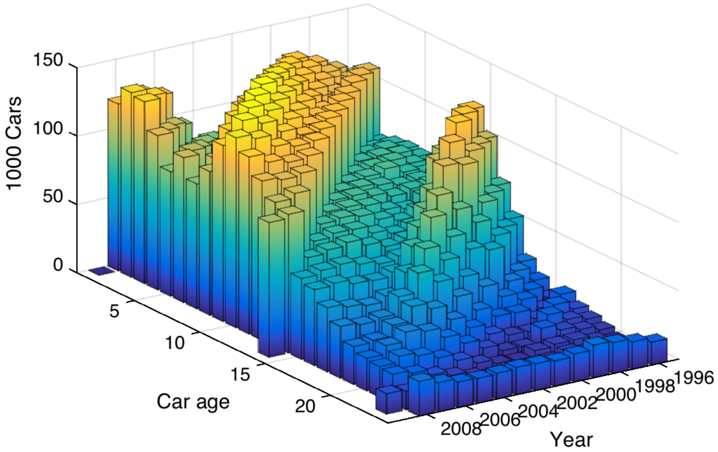

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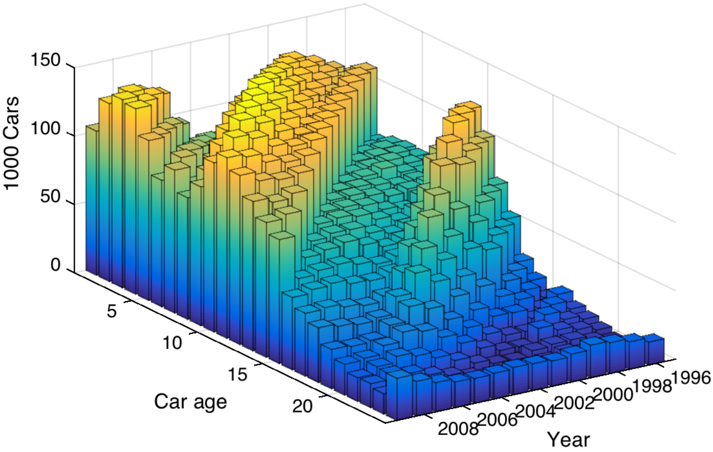
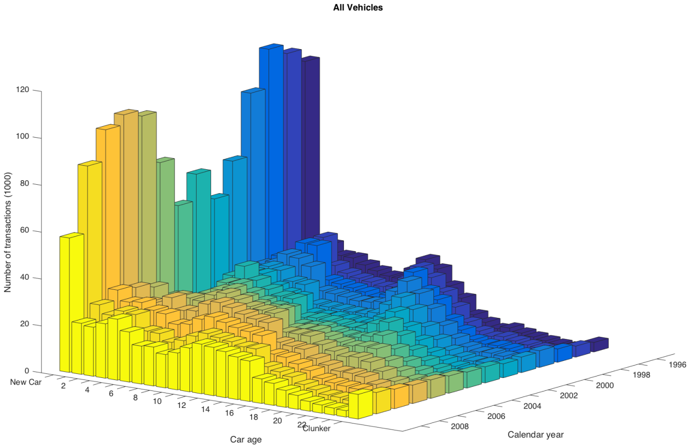
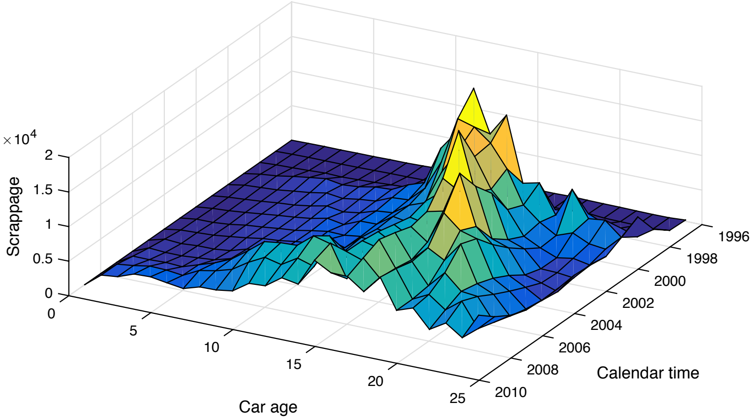


Waves in car purchases 1995-2009



Waves in scrappage 1995-2010



Our paper

Goal: analyze the effects of an IRUC reform on

- Age of the car fleet,
- Value of the car fleet,
- Demand for fuel.

We develop a model of ownership type/age choice and usage

- Allows us to simulate the counterfactual equilibrium,
- Study effects of an important policy reform that is actually being considered.

Focal points:

- Business cycle variation in new/used car purchases,
- Equilibrium price mechanism in the used-car market,
- Scrappage, replacement timing, new vs. used-car tradeoffs.

Data Sources

Purpose	Source	Time	Availability
Ownership	Central motor register	Day	1990–2011
Usage	Ministry of Transportation tests (occur approx. at car age 4, 6, 8, ...)	Day	1997–2007*
Car characteristics	Vehicle type approval documents	n.a.	1997–2011**
New car prices	Danish Automobile Dealer Association	Year	1997–2009
Demographics	Danish registers.	Year	1980–2012
Fuel prices	Danish Oil Industry Association	Day	1980–2014

* Driving periods for used cars observed until [2009; 2011]-periods.

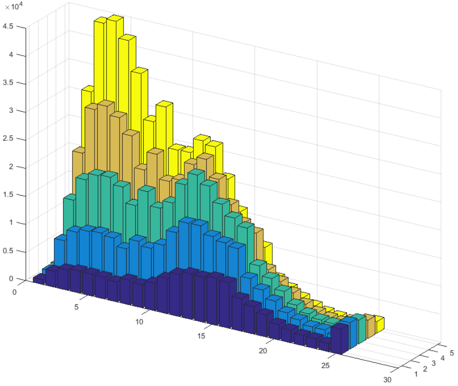
** Early on, car characteristics are almost only available for new cars.

Summary Statistics

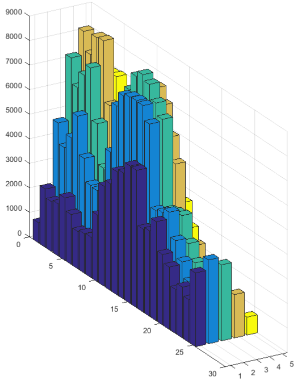
Variable	N	Mean	sd
Age of H.	22,041,601	38.93	11.66
Real income (2005 kr)	22,041,601	403,820.70	403,550.81
Urban resident	22,041,601	0.32	0.47
Work distance of H.	22,041,601	20.81	86.96
Unemployment for H.	16,242,835	0.08	0.28
Dummy for couple	22,041,601	0.45	0.50
Num of kids	22,041,601	0.61	0.97
Car age in years	7,085,310	7.27	4.88
Fuel price (period)	6,362,373	8.76	0.63
Fuel price (annual)	22,041,601	8.37	1.32
Dummy for diesel car	22,041,601	0.02	0.14
Total weight of car	7,085,310	1,576.66	217.83
Fuel efficiency (km/l)	3,547,818	14.21	2.49
VKT (km traveled/day)	7,085,310	46.52	24.22
Years to test	7,085,310	4.03	3.65

Notes: "H." refers to the head of the household.
All Danish kroner (kr) in 2005 kroner.

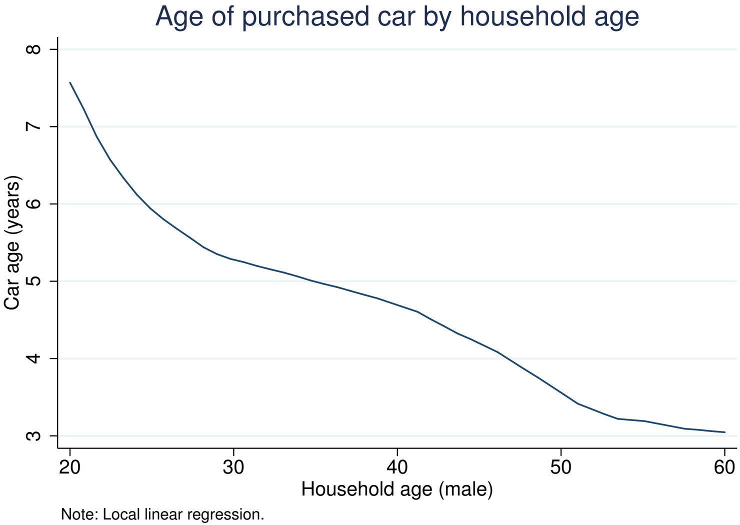
Holdings by income quintiles, Heavy Vehicle



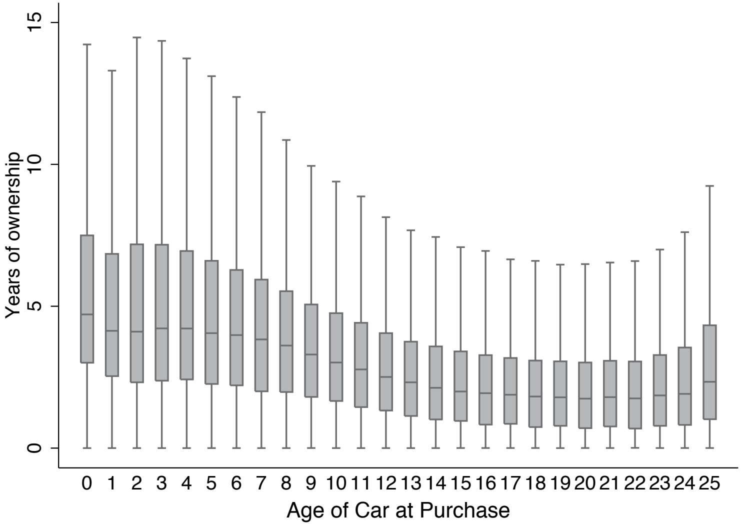
Holdings by income quintiles, Light Vehicle



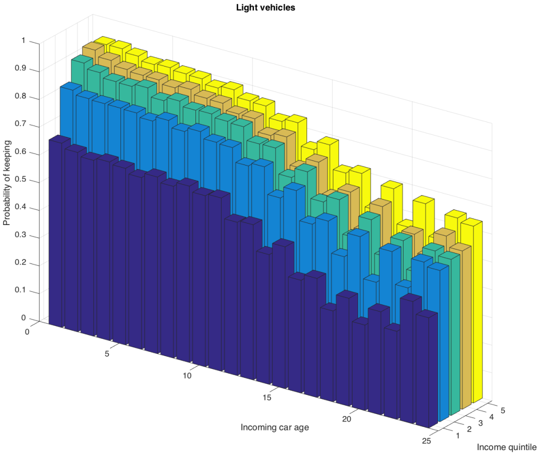
Heterogeneity - Household and Car Age (male)



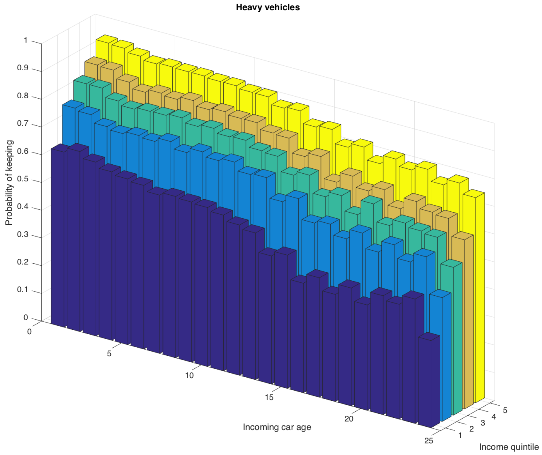
Years of ownership by car age at purchase



Keep Decisions – by income and age of car



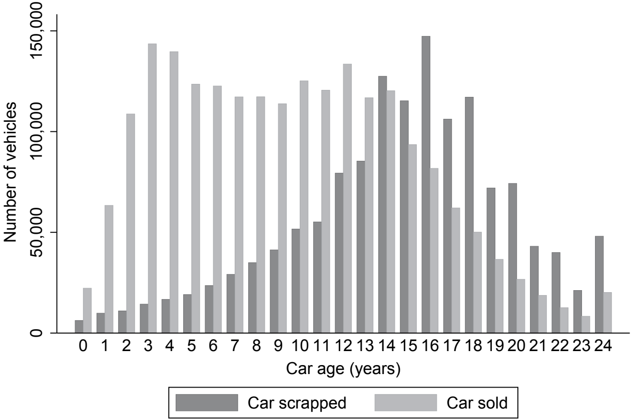
Keep Decisions – by income and age of car



Delayed Scrappage Decisions: Bertel' 1986 Volvo

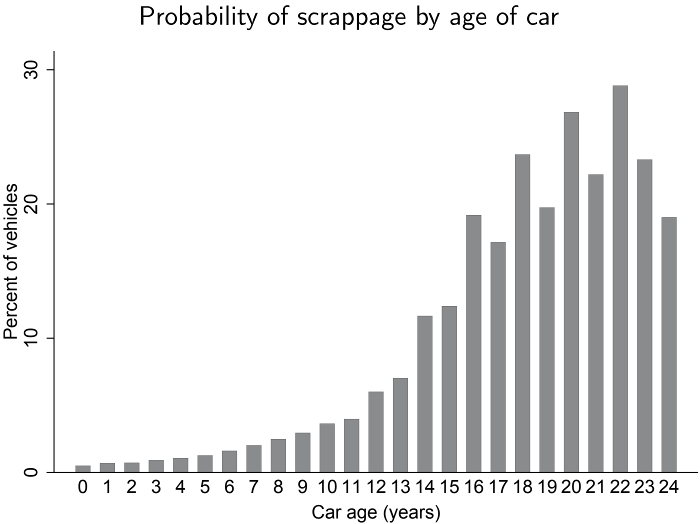
Scrappage Decisions – by car age

Definition (Scrap): The last ownership spell ends for a given car.



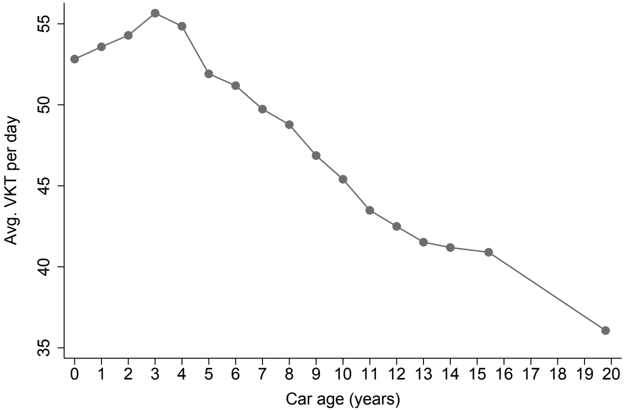
Note: each bar shows the percentage of cars and vans that were scrapped or sold to another household at each age.

Scrappage Decisions – by car age



Note: each bar shows the percentage of cars and vans that were scrapped within each car age.

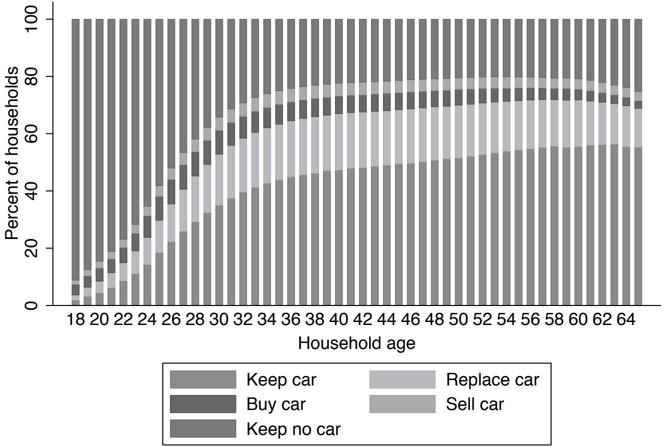
Vehicle Km Travelled (VKT) by car age



Controls: none.
Selection: VKT within 1% and 99% percentiles, year in [1996;2009] and household age in [18;65].

By income By HH age

Life-cycle patterns

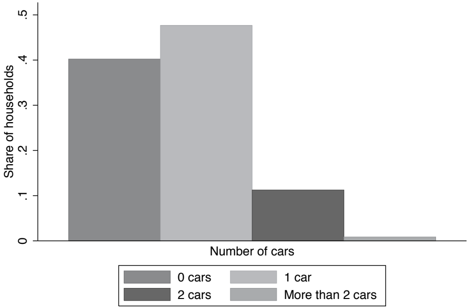


Note: each bar shows the share in each car state within each household age group.

By income

Number of cars

Simplification: Treat multi-car households as independent.



Note: each bar shows the share of households in the sample that owns the particular number of cars in a given year.

By age

Summing up

- Older households and richer tend to buy newer and own more cars high quality cars,
- Older cars are
  - Held shorter durations,
  - Driven less intensively.
- Cars have many owners (typically 5 for a 15 y/o car).
- Profound waves in the age distribution and scrappage frequency over time.

Choice model + market equilibrium

Choice Model

Finite-horizon discrete time lifecycle model of

1. Holding and trading decisions
2. Scrappage decision
3. Choice of the amount of driving (usage decision)

State variables:

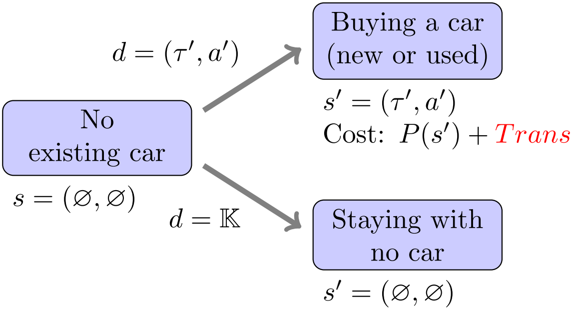
- Household "car state" given by $s = (\tau, a)$
- Household income $y \in [\underline{y}, \overline{y}] \subset \mathbb{R}_+$
- Fuel price $p \in [\underline{p}, \overline{p}] \subset \mathbb{R}_+$
- Macro state of the economy $m \in \{0, 1\}$

Cars

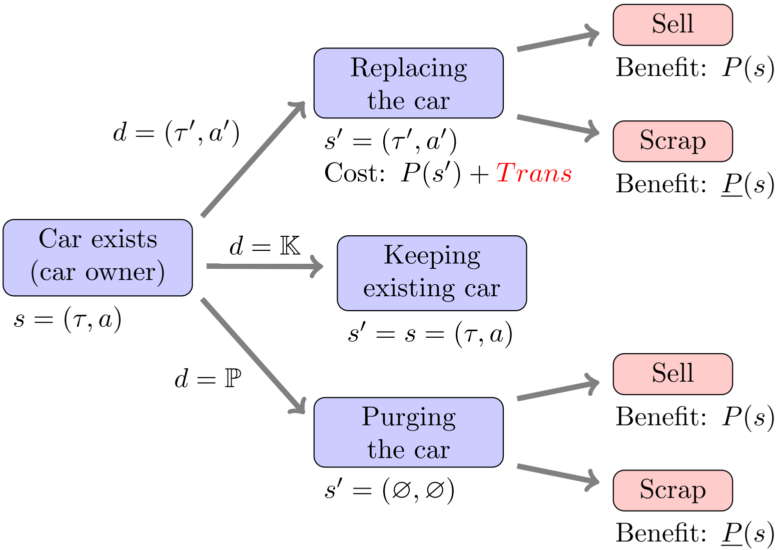
- Type of car τ, $\tau \in \{1, \ldots \overline{\tau}\}$
  make-model, class (compact, luxury, SUV, etc.) or other
- Car age a:
  $a = 0$ new car, $a = \overline{a}$ oldest car in the model ("clunker")
- A car is defined by $s = (\tau, a)$
  - Abstract from technological progress $\Leftrightarrow$ Constant choice set
  - Assume that conditional on a, the effect of odometer reading on the consumer characteristics and the price of the car enters through EV shocks
- Assume there can be at most one car per household

Currently we have at most 2 types of cars (light/cheap and heavy/expensive) and $\overline{a} = 24$

Decisions for households with no car



Decisions of car owners



Timing of decisions

1. State variables are realized
   - Existing car $s_t = (\tau, a)$
   - Income y_t, AR(1)
   - Fuel price p_t, AR(1)
   - Macro state $m_t \in \{0, 1\}$, Markov process
2. Decision to keep | replace | purge $\rightarrow s_t' = (\tau', a')$ is car to drive

$$d_t \in \Big\{\underbrace{(1, ..., \overline{\tau})}_{\text{car type}} \times \underbrace{(0, 1, ..., \overline{a})}_{\text{car age}} \times \underbrace{\{-1, 1\}}_{d_s}, \mathbb{P} \times \underbrace{\{-1, 1\}}_{d_s}, \mathbb{K}\Big\}$$

3. How much to drive $\mathrm{vkt}_t \in \mathbb{R}_+$
4. Utility of car ownership and driving is realized

Next period existing car is $s_{t+1} = (\tau', a' + 1)$

Utility specification

$$u_t(s_t, d_t, d_t^s, \mathrm{vkt}_t, y_t, p_t, m_t) =$$
$$\theta(y_t, m_t)\underbrace{\left[y_t - p_{km}(\tau', p_t)\cdot \mathrm{vkt}_t - TC(s_t, s_t', d_t^s, p_t, m_t)\right]}_{\text{consumption}}$$
$$+\underbrace{\gamma(a', m_t)\mathrm{vkt}_t + \phi \mathrm{vkt}_t^2 - q(a') + \delta_\tau + \delta_n \mathbb{1}\{a'=0\}}_{\text{utility from car services}} + \psi(d_t)$$

$\theta(y_t, m_t)$	marginal utility of money
$p_{km}(\tau', p_t)$	per km cost of fuel and maintenance
$TC(\dots)$	trade cost (price difference + transaction cost)
$\phi, \gamma(a', m_t)$	parameters of the utility of driving
$q(a')$	annual maintenance cost (increasing in age)
δ_τ, δ_n	type specific dummies and the effect of new car
$\psi(d_t)$	psychological + search cost when not keeping

Bellman equation

$$V_t(s_t, y_t, p_t, m_t) = \max_{d_t, d_t^s, \mathrm{vkt}_t} \Big\{ u_t(s_t, d_t, d_t^s, \mathrm{vkt}_t, y_t, p_t, m_t) + \lambda\varepsilon(d_t, d_t^s) + \\ + \beta \mathbb{E}_t \Big[V_{t+1}(s_{t+1}, y_{t+1}, p_{t+1}, m_{t+1}) \Big] \Big\}$$

- $\varepsilon(d_t, d_t^s)$ are EV i.i.d. $\Rightarrow E\big(max\{\dots\}\big)$ is Logsum
- CCPs are logit of discrete choice specific value functions

$$v_t(s_t, y_t, p_t, m_t, d_t, d_t^s, \mathrm{vkt}_t) = u_t(s_t, d_t, d_t^s, \mathrm{vkt}_t, y_t, p_t, m_t) + \\ + \beta \mathbb{E}_t \Big[\max_{d_{t+1}, d_{t+1}^s, \mathrm{vkt}_{t+1}} v_{t+1}(s_{t+1}, y_{t+1}, p_{t+1}, m_{t+1}, d_{t+1}, d_{t+1}^s, \mathrm{vkt}_{t+1}) \Big]$$

- $v_t(s_t, y_t, p_t, m_t, d_t, d_t^s, \mathrm{vkt}_t)$ is very high-dimensional object

A: Separate static continuous choice

Next period value function $V_{t+1}(s_{t+1}, y_{t+1}, p_{t+1}, m_{t+1})$ is independent of vkt_t, therefore the Bellman equation simplifies to

$$V_t(\tau_t, a_t, y_t, p_t, m_t) = \max_{d_t, d_t^s}\Big\{\max_{\mathrm{vkt}_t}\left[u_t(s_t, s_t', d_t^s, \mathrm{vkt}_t, y_t, p_t, m_t)\right] + \lambda\varepsilon(d_t, d_t^s) + \beta\mathbb{E}_t\Big[V_{t+1}(s_{t+1}, y_{t+1}, p_{t+1}, m_{t+1})\Big]\Big\}$$

$$\frac{\partial u_t(\cdot)}{\partial \mathrm{vkt}_t} = \gamma(a', m_t) - \theta(y_t, m_t)p_{km}(\tau', p_t) + 2\phi\mathrm{vkt}_t = 0 \quad \Rightarrow$$

$$\mathrm{vkt}_t^* = \frac{\theta(y_t, m_t)p_{km}(\tau', p_t) - \gamma(a', m_t)}{2\phi}$$

Pure discrete choice model conditional on optimal driving.

B: Separate static scrappage choice

Next period value function $V_{t+1}(s_{t+1}, y_{t+1}, p_{t+1}, m_{t+1})$ is independent of d_t^s, therefore the Bellman equation further simplifies to

$$V_t(\tau_t, a_t, y_t, p_t, m_t) = \max_{d_t} \Big\{ \max \Big\{ \underbrace{u_t(s_t, s_t', -1, \mathrm{vkt}_t^*, y_t, p_t, m_t)}_{\text{scrap}}, \underbrace{u_t(s_t, s_t', +1, \mathrm{vkt}_t^*, y_t, p_t, m_t)}_{\text{sell}} \Big\} + \lambda\varepsilon(d_t) + \beta\mathbb{E}_t\Big[V_{t+1}(s_{t+1}, y_{t+1}, p_{t+1}, m_{t+1})\Big]\Big\}$$

After adding extreme value shocks, the alternatives of the scrappage decision are characterized by the expected maximum utility (logsum) of the highlighted utilities of selling and scrapping

λ_s – scale parameter for the EV shocks on the scrappage decision

C: Reformulate in terms of expected value function

Next period value function $V_{t+1}(s_{t+1}, y_{t+1}, p_{t+1}, m_{t+1})$ is only dependent of the car to drive s'_t and not on the existing car s_t, therefore solving the Bellman equation in terms of expected value function has much lower dimensionality

$$EV_t(s_{t+1}, y_t, p_t, m_t) = \mathbb{E}_t\Big[V_{t+1}(s_{t+1}, y_{t+1}, p_{t+1}, m_{t+1})\Big] =$$

$$= \sum_{m_{t+1}} \int_{y_{t+1}} \int_{p_{t+1}} \lambda \log \left(\sum_{d_{t+1}} \exp \frac{v_{t+1}(s_{t+1}, y_{t+1}, p_{t+1}, m_{t+1}, d_{t+1})}{\lambda} \right)$$
$$g_s(y_{t+1}|y_t, p_{t+1}, m_{t+1}) h(p_{t+1}, m_{t+1}|p_t, m_t) \mathrm{d}y_{t+1} \mathrm{d}p_{t+1} \mathrm{d}m_{t+1}$$

Assume the above dependence structure between y_t, p_t and m_t.

Choice specific value functions

Decision to keep existing car or no car

$$v_s(\tau, a, y_t, p_t, m_t, d_t = \mathbb{K}) =$$

$$u_t(\underbrace{\tau, a}_{s_t}, \underbrace{\tau, a}_{s_t'}, \mathrm{vkt}_t^*, y_t, p_t, m_t) + \beta EV_t(\underbrace{\tau, a+1}_{s_{t+1}}, y_t, p_t, m_t)$$

Decision to purge existing car

$$v_s(\tau, a, y_t, p_t, m_t, d_t = \mathbb{P}) =$$

$$u_t^{ls}(\underbrace{\tau, a}_{s_t}, \underbrace{\emptyset, \emptyset}_{s_t'}, \pm 1, \mathrm{vkt}_t^*, y_t, p_t, m_t) + \beta EV_t(\underbrace{\emptyset, \emptyset}_{s_{t+1}}, y_t, p_t, m_t)$$

Decision to buy or replace the car

$$v_s(\tau, a, y_t, p_t, m_t, d_t = (\tau', a')) =$$

$$u_t^{ls}(\underbrace{\tau, a}_{s_t}, \underbrace{\tau', a'}_{s_t'}, \pm 1, \mathrm{vkt}_t^*, y_t, p_t, m_t) + \beta EV_t(\underbrace{\tau', a'+1}_{s_{t+1}}, y_t, p_t, m_t)$$

Bellman equation in expected values

$$EV_t(s', y, p, m) = \sum_{m'} \int_{y'} \int_{p'} \Bigg[\lambda \log \sum_{d'} \exp \frac{u(s', s''(d'), y', p', m') + \beta EV_{t+1}(s''(d'), y', p', m')}{\lambda} \Bigg] g_s(y'|y_t, p', m') h(p', m'|p_t, m_t) \mathrm{d}y' \mathrm{d}p' \mathrm{d}m'$$

Once $EV_t(s', y, p, m)$ are computed, the choice probabilities are calculated with standard logit formula

$$P_t(d) = \frac{\exp\left(\frac{1}{\lambda} u(s, d, y, p, m) + \frac{\beta}{\lambda} EV_{t+1}(s'(d), y, p, m)\right)}{\sum_{d'} \exp\left(\frac{1}{\lambda} u(s, d', y, p, m) + \frac{\beta}{\lambda} EV_{t+1}(s'(d'), y, p, m)\right)}$$

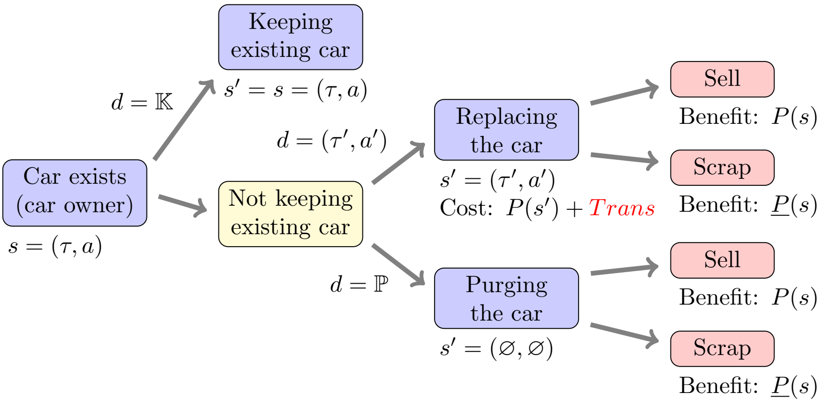
Computational details

1. Interpolation over (y, p)
   - Chebyshev polynomials to approximate $EV_t(s', y, p, m)$
   - Expected value function is smooth, so this works very well
   - Only the Chebyshev coefficients are stored in memory to represent the whole function
2. Integrals over errors in AR(1) processes
   - Need to compute integrals over distributions of (y, p)
   - Use two-dimensional Gaussian quadrature
3. Implementation: Matlab + C

Alternative specifications and additional model details

- Accidents happen with a fixed probability α_τ which is independent of car age, result in making "keep" decision $d = \mathbb{K}$ infeasible, and incur a penalty to be paid
- Clunkers ($a = \bar{a}$) may be allowed or not allowed to be traded in the secondary market
- Nested choice structure to allow for "Not to keep" decisions to have correlated EV error terms

Alternative nested choice specification



Choice model + market equilibrium

Excess demand and market equilibrium

1. For each value of structural parameters θ
2. Solution of the model is given by $EV_t(s', y, p, m, \theta)$
3. Choice probabilities $P_t(d, s, y, p, m, \theta)$
4. Simulate choices for large number of agents N
5. For each (τ, a, p, m) define excess demand as

$$\mathrm{ED}\Big(\tau, a, p, m, P(\tau, a, p, m, \theta), \theta\Big) =$$

$$\sum_{i=1}^{N}\Big(\mathbb{1}\{\text{buy}\,(a, \tau, P(\tau, a, p, m, \theta))\} - \mathbb{1}\{\text{sell}\,(a, \tau, P(\tau, a, p, m, \theta))\}\Big)$$

6. Equilibrium prices minimize ED
$$P(\tau, a, p, m, \theta) = \arg\min \Big\{\mathrm{ED}\big(\tau, a, p, m, \theta, P(\tau, a, p, m, \theta)\big)\Big\}$$

Supply and Demand

Buyers:
Agents **purchasing** a car $d = (\tau, a) = s'$

Sellers:
Owners of $s = (\tau, a)$ car who **replace** $d = (\tau', a')$ or **purge** $d = \mathbb{P}$, but do not **scrap** $d_s = +1$.

No contribution
Agents choosing to keep $d = \mathbb{K}$

Problem:
Simulating buyers and sellers $\rightarrow$ piecewise flat objective function

Expected excess demand

1. Compute choice probabilities of large number of individuals
2. For each (τ, a, p, m) define:

$$\text{EED}(\tau, a, p, m, P(\tau, a, p, m, \theta), \theta) =$$

$$\sum_{i=1}^{N} \Big(P(\text{buy}|a, \tau, P(\tau, a, p, m), \theta) - P(\text{sell}|a, \tau, P(\tau, a, p, m), \theta) \Big)$$

3. Equilibrium prices minimize EED
$P(\tau, a, p, m, \theta) = \arg\min \Big\{ \text{EED}\big(\tau, a, P(\tau, a, p, m, \theta), \theta\big) \Big\}$
4. Smoothness through the choice probabilities.
5. This objective function is never flat
(although may still have discontinuities due to finite set of decision makers)

Pinning down the equilibrium

New cars: Infinitely elastic supply, world market price $\overline{P}(\tau, p, m)$

- Imported from large market
- small open economy

Clunkers: Infinitely elastic demand at the scrap value $\underline{P}(\tau, p, m)$

- Scrap subsidy 1,500 DKK since 1 Jul 2000

$\Rightarrow$ For all τ, θ and price functions $P(\tau, a, p, m, \theta)$

- $EED(\tau, 0, P(\tau, a, p, m, \theta), \theta) = 0$
- $EED(\tau, \overline{a}, P(\tau, a, p, m, \theta), \theta) = 0$

How to parameterize prices?

Price parameterizations

1. Constant depreciation rate:
   - Good starting point: suggestive rates given as data.
   - But unrealistic first-year drop.
2. Piecewise-constant rates: pick $1 < K < \bar{a} - 2$ knots,
   - Estimate the rates within each interval.
3. Fully nonparametric: estimate $P(\tau, a)$ for all τ and $a = 1, ..., \bar{a} - 1$.
   - System of $\bar{a} - 1$ equations (car types) with $\bar{a} - 1$ unknowns (prices).
   - Efficient gradient-based numerical solvers can be used.

We found: going directly for 2 works fine

Structural Estimation

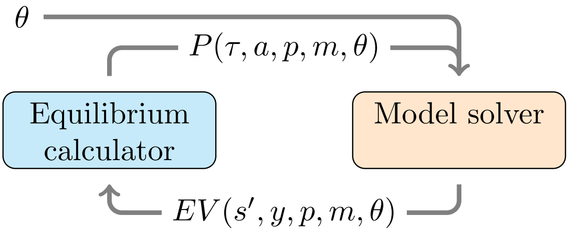
Log likelihood

- Likelihood contribution of household i in period t

$$\ell_{it}(\theta) = \log\Big[(1-\alpha_\tau)\cdot P_{it}(d_{it}|s_{it}, y_{it}, p_t, m_t, \theta) + \alpha_\tau \cdot P_{it}^{\mathcal{A}}(d_{it}|s_{it}, y_{it}, p_t, m_t, \theta)\Big] + \log\phi\left(\frac{vkt_{it} - vkt_{it}^*(s_{it}'(d_{it}), y_{it}, p_t, m_t, \theta)}{\sigma_{vkt}}\right)$$

- Data on discrete choices d_{it}, driving vkt_{it}, existing cars s_{it}, income y_{it}, fuel prices p_t and macro state m_t
- Voluntary and involuntary scrappage decisions are indistinguishable in the data ⇒Mixture in the likelihood, $\mathcal{A}$ denotes the "accident" event
- Transition probabilities for income y_t, fuel price p_t and macro state m_t are estimation on Stage 1 outside

Estimation circle



- Equilibrium prices P must solve $EED(P) = 0$
- $EED(P)$ depends on choice probabilities P_i
- P_i depends on EV which solve the Bellman equation
- EV depends on P that represents future prices
- Everything depends on structural parameter θ

A: Double Nested Fixed Point Algorithm (N^2FXP)

N^2FXP solves the *unconstrained* optimization problem

$$\max_{\theta} L\Big(\theta, EV_\theta(P_\theta)\Big)$$

Outer loop (Hill-climbing algorithm):

- Likelihood function is maximized w.r.t. θ
- Each evaluation of likelihood requires solution of $EED_{(EV_\theta)}(P_\theta) = 0$
- Each evaluation of $EED_{(EV_\theta)}$ requires solution of EV_θ

Middle loop (equilibrium):
Implicit function P_θ defined by $EED_{(EV_\theta)}(P_\theta) = 0$ is found by Newtons method

Inner loop (model solution):
Expected values $EV_\theta(P_\theta)$ and corresponding choice probabilities are found by backward induction

B: NFXP with P as parameters

NFXP solves the *unconstrained* optimization problem

$$\max_{\theta, P} L\Big(\theta, EV_\theta(P)\Big)$$

Outer loop (Hill-climbing algorithm):

- Likelihood function is maximized w.r.t. θ and P
- Each evaluation of likelihood requires solution of $EV_\theta(P)$

Inner loop (model solution):
The implicit function $EV_\theta(P)$ and corresponding choice probabilities are found by backward induction

- Assume that markets are in equilibrium and observed data reflects that
- If prices are observed, they can be used in place of P

C: Sequential estimation NPL-NFXP

Sequential NPL-NFXP algorithm

1. Initialize prices with $P = P_0$
2. Given current estimate P, obtain $\hat{\theta}$ with NFXP
3. Calculate and update equilibrium prices P using $\hat{\theta}$
4. Return to step 2 unless convergence in $\hat{\theta}$ is achived

- $NPL - NFXP$ produces a sequence of Pseudo ML estimators $\hat{\theta}$, each of them is obtained by NFXP using current estimate of equilibrium prices
- Sequence of $\hat{\theta}$ converges to MLE obtained by N^2FXP
- Computationally much faster than N^2FXP

D: Separate estimation of vkt_t

- Optimal driving is given by

$$\mathrm{vkt}_t^* = \big(\theta(y_t, m_t) p_{km}(\tau', p_t) - \gamma(a', m_t)\big)/2\phi$$

- Parametrization

$$\begin{aligned} \gamma(a, m) &= \gamma_0 + \gamma_1 a + \gamma_2 a^2 + \gamma_3 m + \eta + u^{vkt1} \\ \theta(y, m) &= \theta_0 + \theta_1 y + \theta_2 y^2 + \theta_3 m \end{aligned}$$

- Fixing $\phi = -1/2$, estimate with OLS

$$\begin{aligned} \mathrm{vkt}_t^* = {} & \kappa_0 + \kappa_2 a + \kappa_2 a^2 + \kappa_3 m \\ & + \kappa_4 p^{km} + \kappa_5 p^{km} a + \kappa_6 p^{km} a^2 + \kappa_7 p^{km} m + \tilde{\eta} + \tilde{u}^{vkt1} \end{aligned}$$

- Let ϕ be estimated with other structural parameters, and set

$$\begin{aligned} (\gamma_0, \gamma_1, \gamma_2, \gamma_3) &= 2\phi(\kappa_0, \kappa_2, \kappa_2, \kappa_3) \\ (\theta_0, \theta_1, \theta_2, \theta_3) &= -2\phi(\kappa_4, \kappa_5, \kappa_6, \kappa_7) \end{aligned}$$

Cross-equational restrictions in ϕ, $\gamma(a, m)$ and $\theta(y, m)$

These restrictions are testable

- Estimate κ in *vkt* equation and estimate discrete choice model with unrestricted coefficients for $\gamma(a', m_t)$, $\theta(y_t, m_t)$ and ϕ
- Test if these coefficients are different from parameters estimated in first step.
- If test is rejected, this suggest that marginal utility on spending a dollar on gasoline is different from spending it on durables such as cars (e.g. due to liquidity constraints, commitment).
- We are still in the process of estimating the unrestricted model.

Results

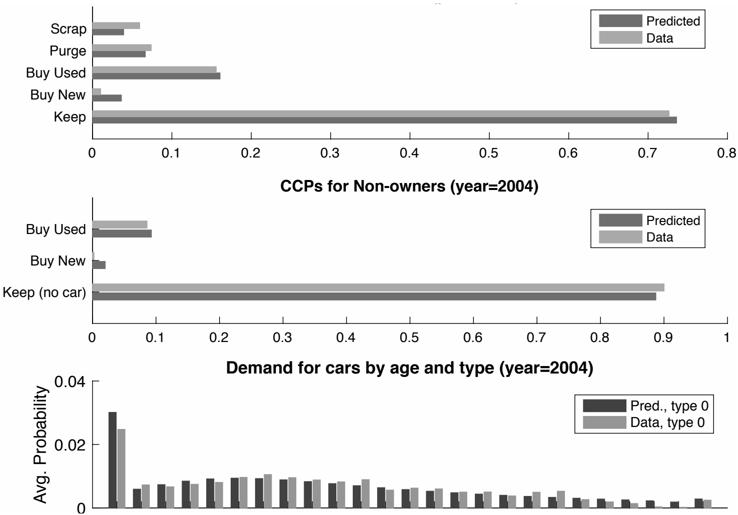
Structural Parameter Estimates

Fixed parameters

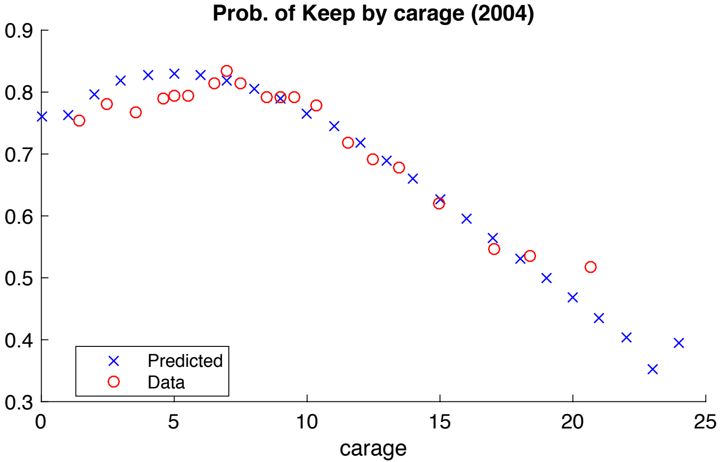
	Variable	**Value**
	Min. Hh. age	20
	Max. Hh. age	85
	# of car ages	25
	# of car types	1
	Clunkers in choiceset	1
β	Discount factor	0.95
ρ	Inc. AR(1) term	0.86818
σ_y	Inc. s.d.	0
ρ_p	Fuel price AR(1) term	1
σ_y	Fuel price s.d.	0
Pr(0|0)	Macro transition	0.75
Pr(1|1)	Macro transition	0.8
	Accident prob.	0.0004
λ	Logit error var.	1

	Variable	**Estimate**	**Std.err.**
	Monetary Utility		
θ_0	Intercept	0.037476	
θ_1	Inc.	-1.2043e-05***	2.54e-08
θ_2	Inc. sq.	1.171e-08***	0
θ_3	Macro	-0.00084546***	9.604e-06
	Car Utility		
$q(a)$	Car age, linear	0.12737***	0.0002645
$q(a)$	Car age, squared	-0.0010925***	1.157e-05
δ_1	Car type dummy	1.1433***	0.004135
	Transaction costs		
	Fixed cost	138.8***	0.06326
	Proportional cost	0	
	Scrappage		
λ^{scrap}	Scrappage error var.	1.2154***	0.002351

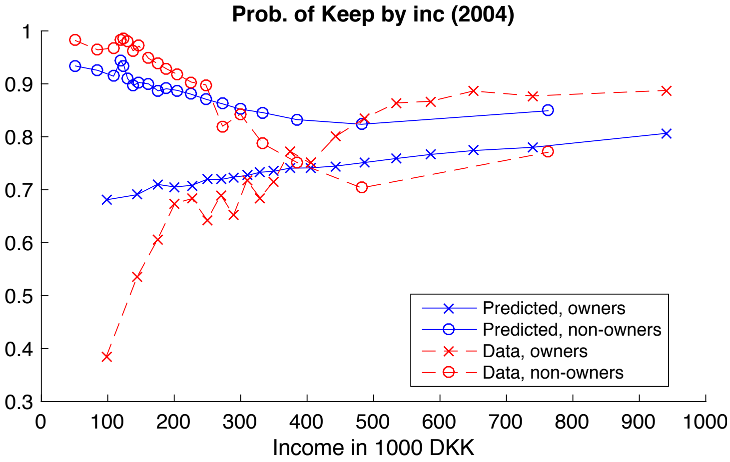
Model fit by state variables



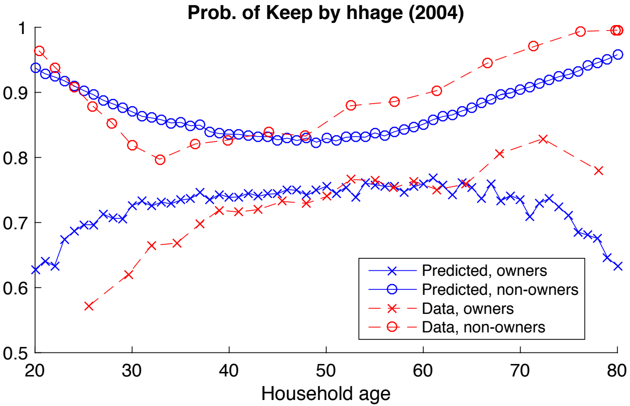
Fit by car age: Keep probability



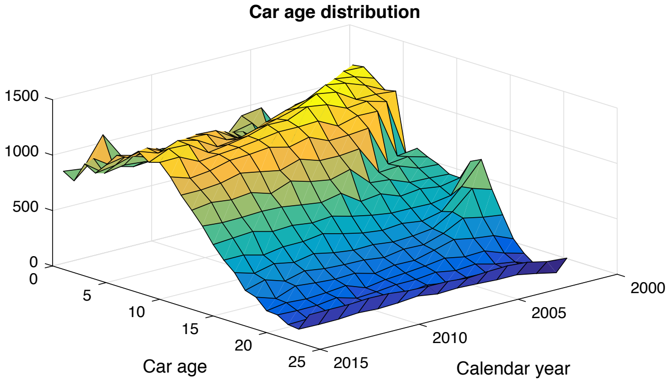
Fit by household income: Keep probability



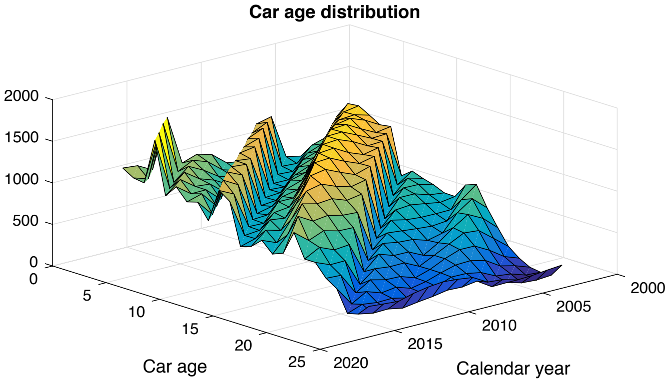
Fit by household age: Keep probability



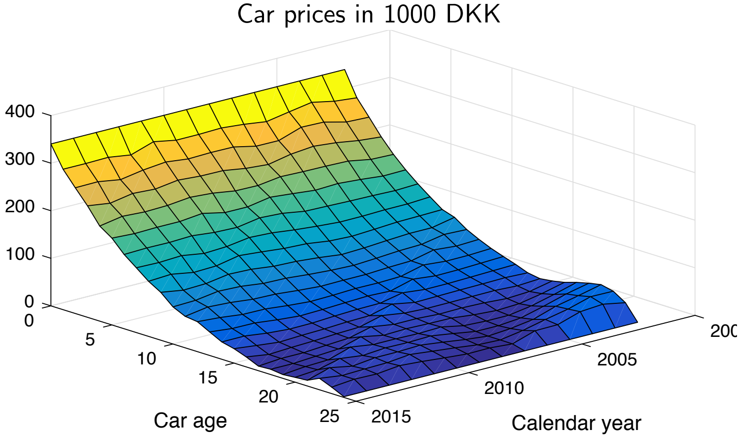
Non-equilibrium simulations: no clear waves



Equilibrium simulations: clear waves

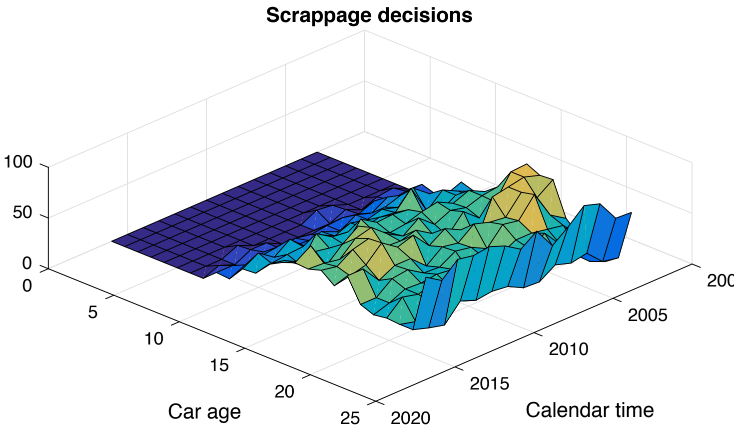


Equilibrium prices

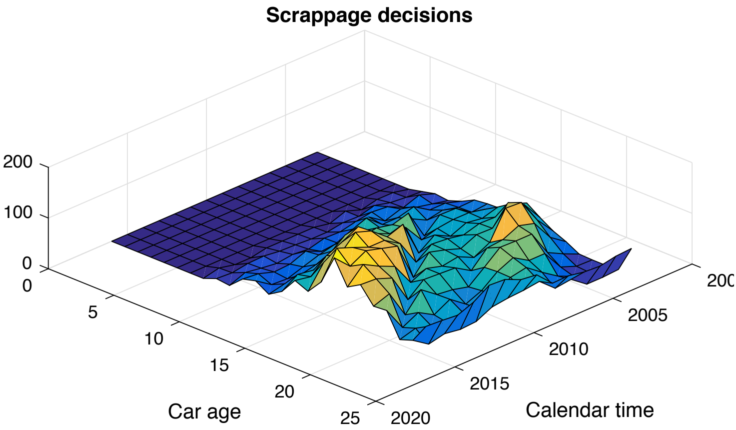


400,000 DKK ≅ 60,000 USD

Scrappage: non-equilibrium simulation



Scrappage: equilibrium simulation

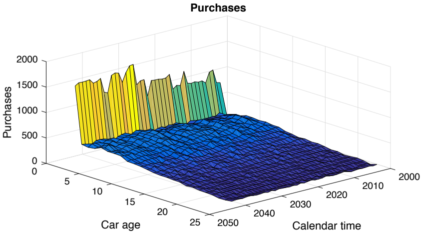


Counterfactual

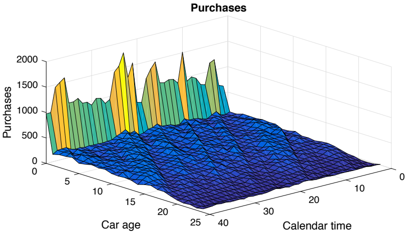
- **Goal:** Study interactions between car tax policies and the non-stationary equilibrium.
- **Eventually:** Lower registration tax and increase fuel (or driving) taxes.
- **Today:** Lower new car prices by 10%.
  - Non-equilibrium: Lower new car prices and assume depreciation rates are unchanged.
  - Equilibrium: Change new-car prices and solve for the rest.

New car sales

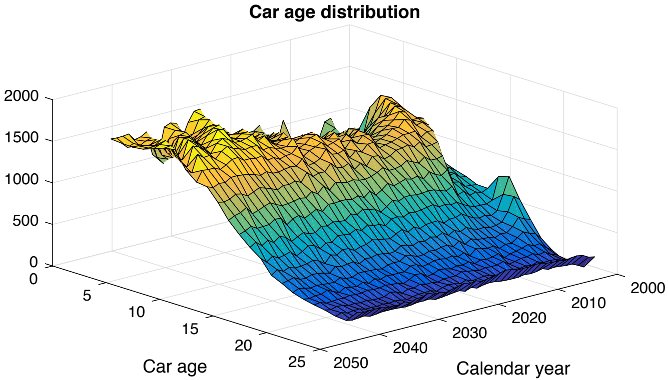
Non-equilibrium



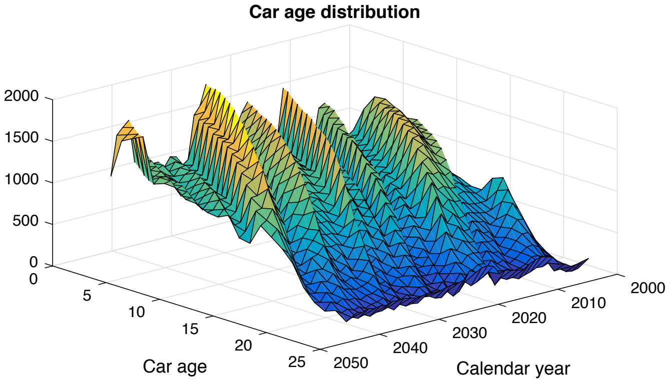
Equilibrium



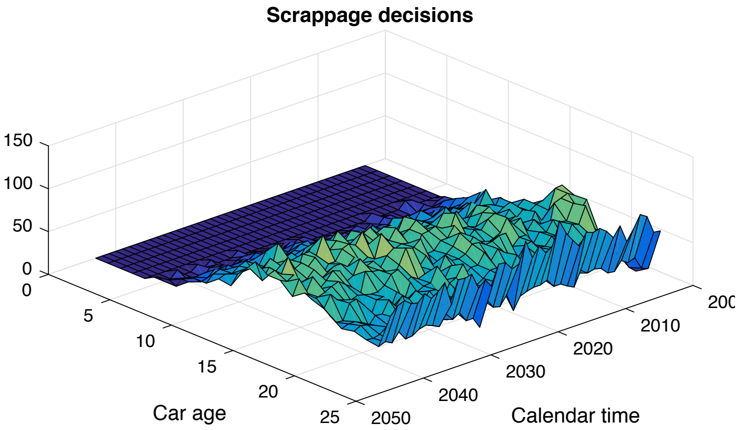
Simulated car stock: non-equilibrium



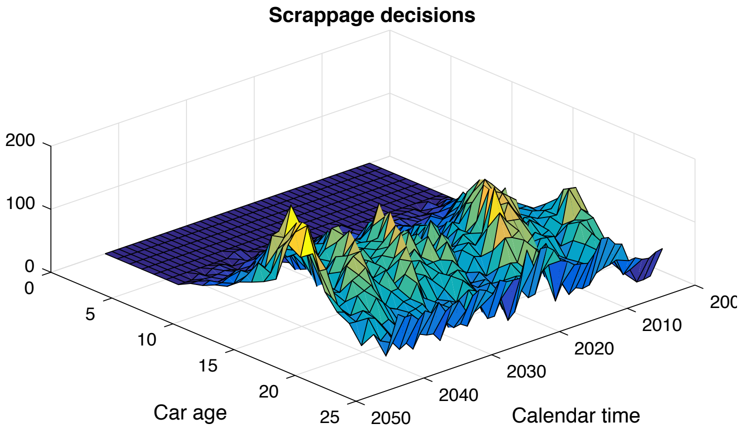
Simulated car stock: equilibrium



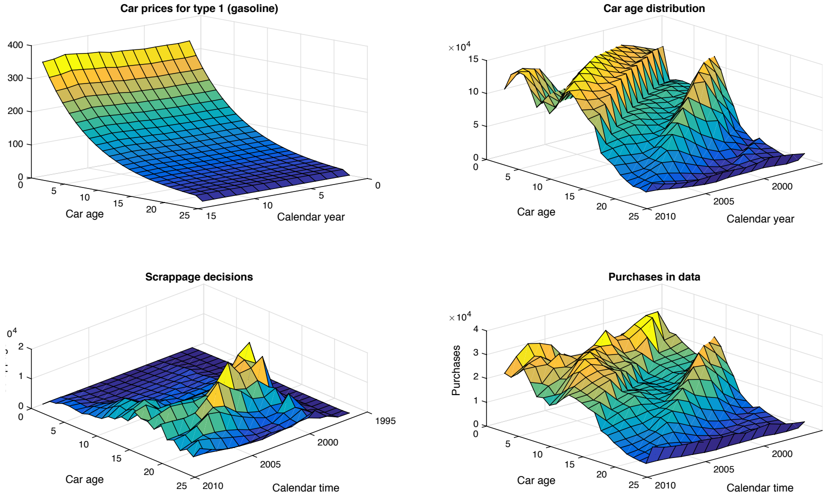
Simulated scrappage: non-equilibrium



Simulated scrappage: equilibrium



Data


Car prices for type 1 (gasoline)
400
300
200
100
0
Car age
Calendar year
Car age distribution
×10^4
15
10
5
0
Car age
Calendar year
2000
2005
2010
Scrappage decisions
2
1
0
Car age
Calendar time
1995
2000
2005
2010
Purchases in data
×10^4
4
3
2
1
0
Purchases
Car age
Calendar time
2000
2005
2010

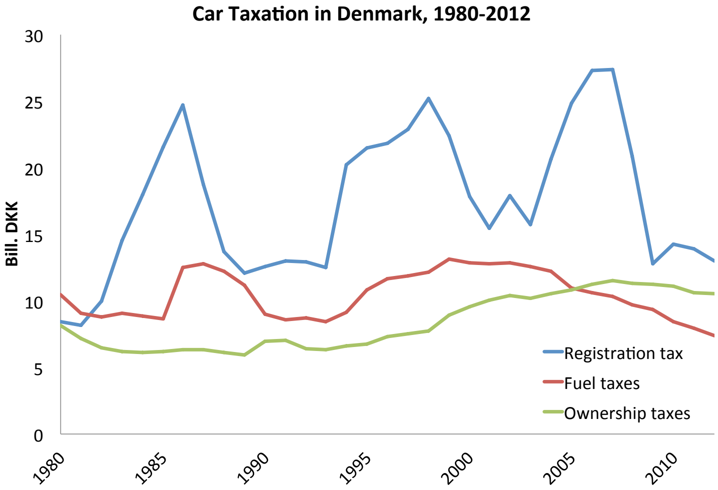
Conclusion

- **Simulated waves:** Preliminary estimates can reproduce waves in simulations, but it is essential to ensure that prices are in equilibrium.

Thank you

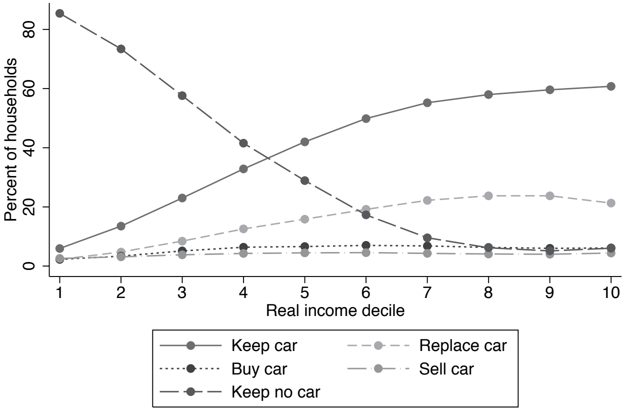
Thank you for all your attention

Car taxes 1980-2012



Go Back

Heterogeneity — Discrete Choice by Income



Note: each curve shows the percentage of households within the given income decile that is in the particular car state in a given year.

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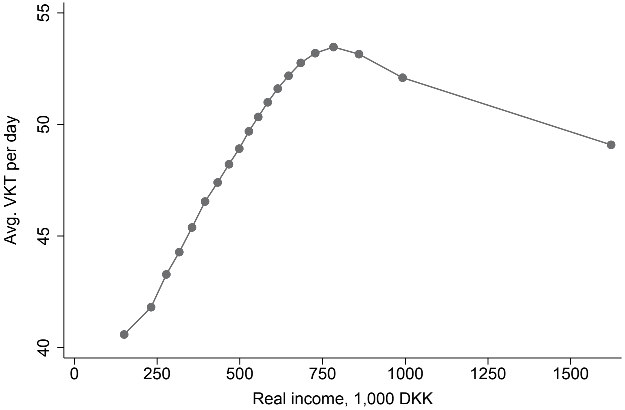
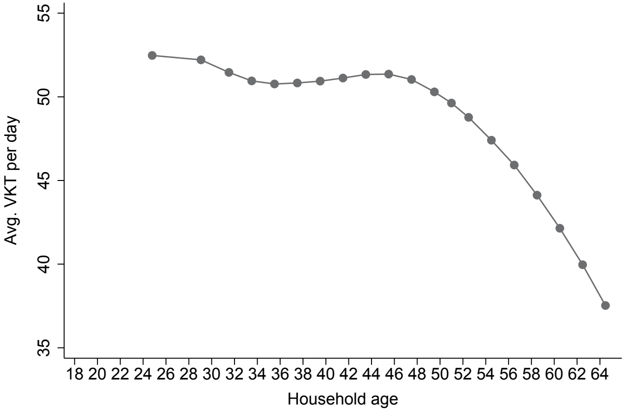
Vehicle Km Travelled (VKT) by household income



Controls: none.
Selection: VKT within 1% and 99% percentiles, year in [1996;2009] and household age in [18;65].

Back

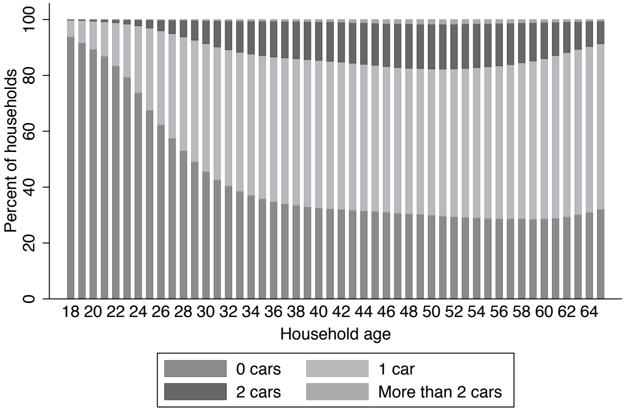
Vehicle Km Travelled (VKT) by household age



Controls: none.
Selection: VKT within 1% and 99% percentiles, year in [1996;2009] and household age in [18;65].

Back

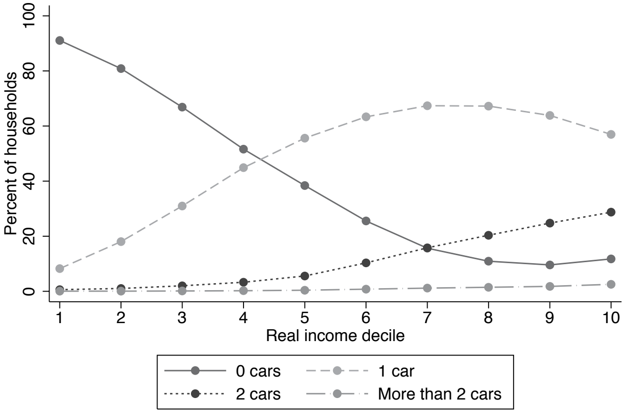
Vehicles ownership over the life cycle



Note: each bar shows the percentage of households within the given age category that own the particular number of cars in a particular year.

Back

Heterogeneity - Num. Cars Owned by Income



Note: each curve shows the percentage of households within the given income decile that own the particular number of cars in a given year.

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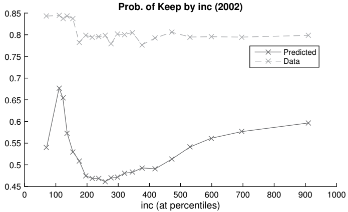
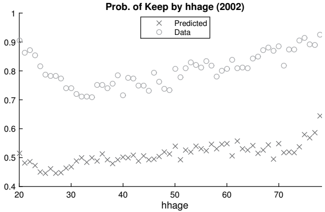
Lower transaction costs

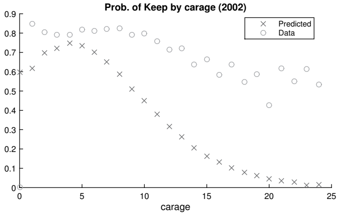
Motivation:	Estimated transaction costs are unrealistically high.
Solution:	Fix at lower value.
Fixed part:	20,000 DKK (about 5%–10% of new car price),
Proportional:	20% of the car's price.
Who?	Levied both on the seller and buyer.
Result:	Vast underprediction of keep probability.

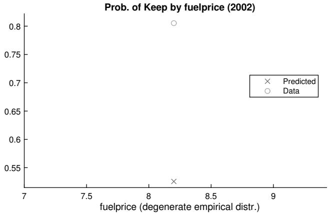
Back

Lower transaction costs: model fit









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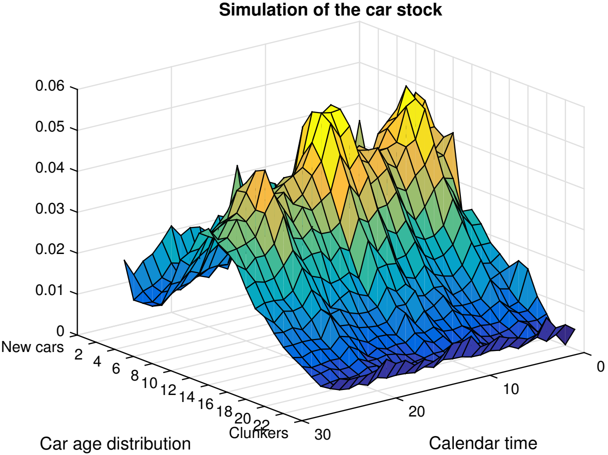
Lower transaction costs: simulation forward in time



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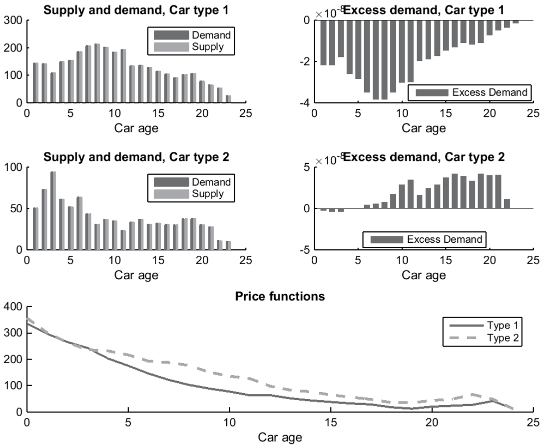
Searching for equilibrium: nonparametric prices



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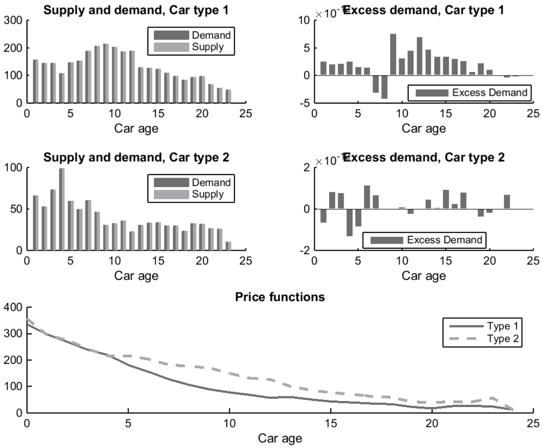
Searching for equilibrium: nonparametric prices



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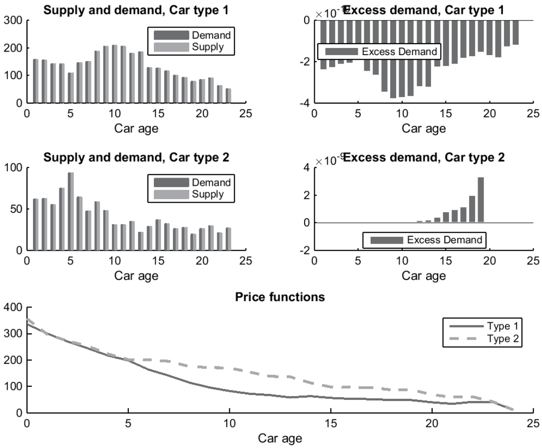
Searching for equilibrium: nonparametric prices



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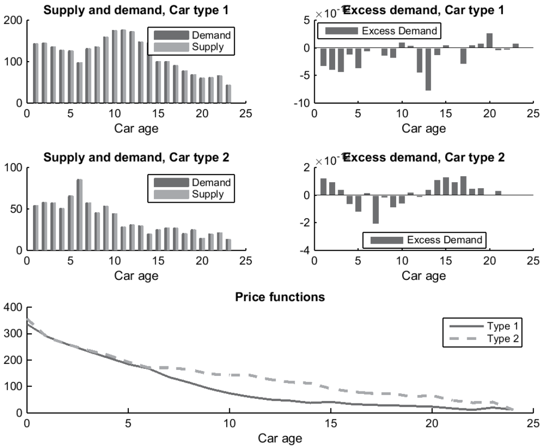
Searching for equilibrium: nonparametric prices



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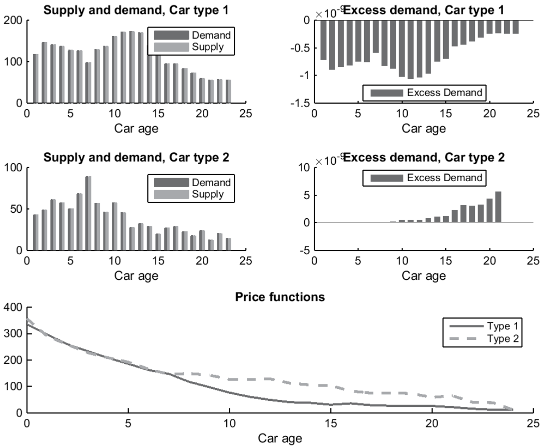
Searching for equilibrium: nonparametric prices



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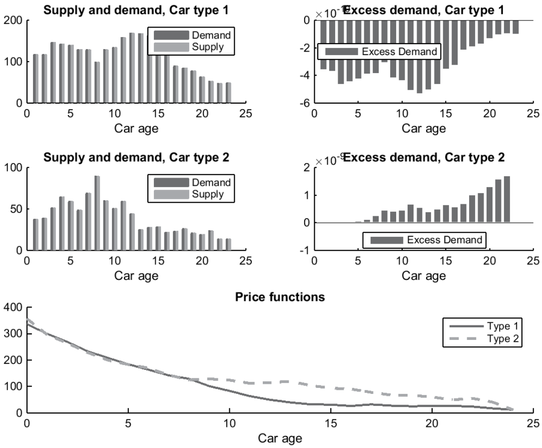
Searching for equilibrium: nonparametric prices



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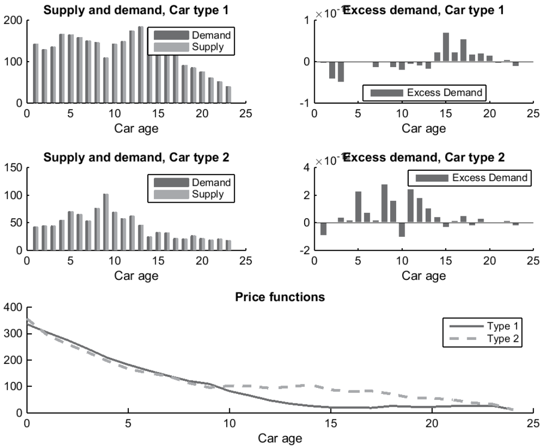
Searching for equilibrium: nonparametric prices



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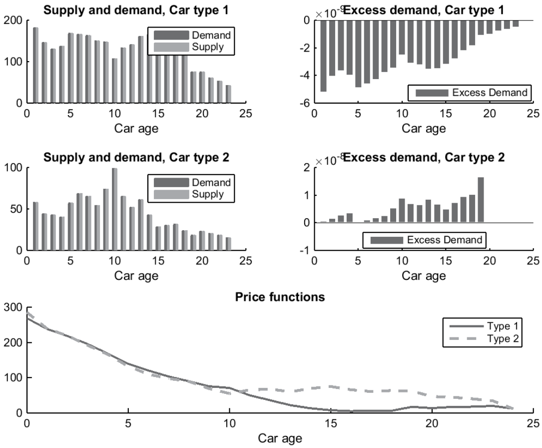
Searching for equilibrium: nonparametric prices



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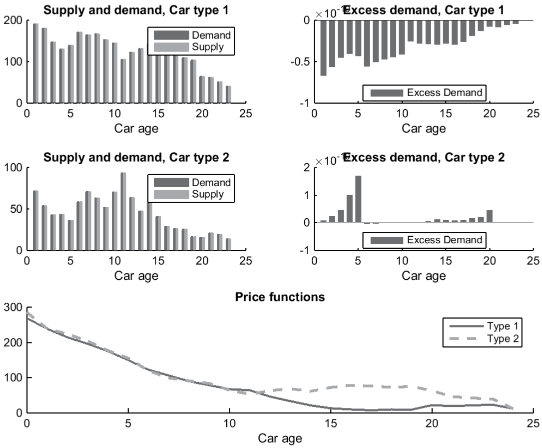
Searching for equilibrium: nonparametric prices



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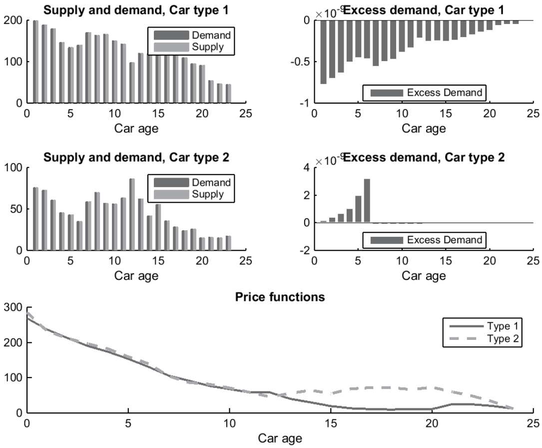
Searching for equilibrium: nonparametric prices



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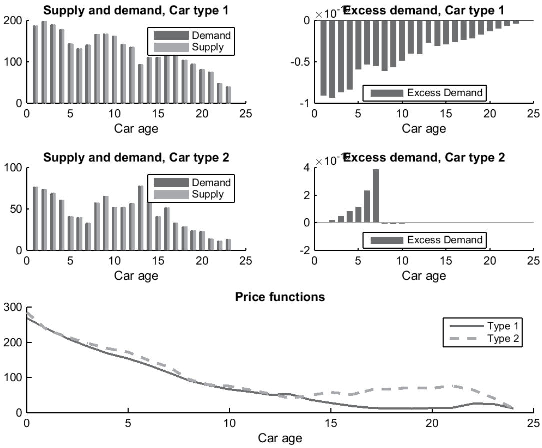
Searching for equilibrium: nonparametric prices



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