Albert-Ludwigs-Universität Freiburg

Foundations of Deep Learning

Winter Semester 2018/2019

Exercise 01

Group members: Eduardo Alvarado, Manav Madan, Jatin Dhawan

October 23, 2018

Part I

Eigenvalue Decomposition

Given the matrix

$$A = \frac{1}{4}\begin{bmatrix} 7 & -\sqrt{3} \\ -\sqrt{3} & 5 \end{bmatrix}$$

(a) Compute the eigenvalue decomposition $A = Q\Lambda Q^T$ with $\Lambda = diag(\lambda_1, \lambda_2)$.

Solution:

Let Q be a square (NxN) matrix whose i^{th} column is the eigenvector v_i of A, and Λ the diagonal matrix whose diagonal elements are the corresponding eigenvalues, i.e., $\Lambda_{ii} = \lambda_i$. Then, the eigendecomposition of A states that:

$$AQ = Q\Lambda \rightarrow A = Q\Lambda Q^T$$

First, we calculate the eigenvalues of the matrix A, (λ_1 and λ_2). We know that an eigenvector v of the square matrix A must satisfy the following linear equation (also called **eigenvalue equation**) for its respective eigenvalue λ. This yields an equation for the eigenvalues:

$$Av = \lambda v \rightarrow det(A - \lambda I) = 0$$

By replacing the matrix A, and I as the identity matrix, we obtain that:

$$A - \lambda I = \begin{bmatrix} \frac{7}{4} & \frac{-\sqrt{3}}{4} \\ \frac{-\sqrt{3}}{4} & \frac{5}{4} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \frac{7}{4} - \lambda & \frac{-\sqrt{3}}{4} \\ \frac{-\sqrt{3}}{4} & \frac{5}{4} - \lambda \end{bmatrix}$$

$$det(A - \lambda I) = (\frac{7}{4} - \lambda)(\frac{5}{4} - \lambda) - (\frac{-\sqrt{3}}{4} \cdot \frac{-\sqrt{3}}{4}) = (\frac{7}{4} - \lambda)(\frac{5}{4} - \lambda) - \frac{3}{16} = \lambda^2 - 3\lambda + 2 = 0$$

$$\lambda = \frac{+3 \pm \sqrt{9 - 4 \cdot 1 \cdot 2}}{2} = \frac{+3 \pm \sqrt{1}}{2} \rightarrow \lambda_1 = 2; \lambda_2 = 1$$

Therefore, the matrix Λ is defined by:

$$\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Secondly, we calculate the eigenvectors of A. To do so, we define the eigenvalue equation for each eigenvalue λ_i, and solve it for its respective eigenvector v_i:

$$(A - \lambda_i I)v_i = 0$$

- For $\lambda_1 = 2$

$$\begin{bmatrix} \frac{7}{4} - \lambda_1 & \frac{-\sqrt{3}}{4} \\ \frac{-\sqrt{3}}{4} & \frac{5}{4} - \lambda_1 \end{bmatrix} = \begin{bmatrix} \frac{7}{4} - 2 & \frac{-\sqrt{3}}{4} \\ \frac{-\sqrt{3}}{4} & \frac{5}{4} - 2 \end{bmatrix} = \begin{bmatrix} \frac{-1}{4} & \frac{-\sqrt{3}}{4} \\ \frac{-\sqrt{3}}{4} & \frac{-3}{4} \end{bmatrix}$$

$$(A - \lambda_1 I)v_1 = 0 \rightarrow \begin{bmatrix} \frac{-1}{4} & \frac{-\sqrt{3}}{4} \\ \frac{-\sqrt{3}}{4} & \frac{-3}{4} \end{bmatrix} \cdot \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Normalize (unit-length) eigenvector } v_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 \\ -1/2 \end{bmatrix}$$

- For $\lambda_2 = 1$

$$\begin{bmatrix} \frac{7}{4} - \lambda_2 & \frac{-\sqrt{3}}{4} \\ \frac{-\sqrt{3}}{4} & \frac{5}{4} - \lambda_2 \end{bmatrix} = \begin{bmatrix} \frac{7}{4} - 1 & \frac{-\sqrt{3}}{4} \\ \frac{-\sqrt{3}}{4} & \frac{5}{4} - 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{-\sqrt{3}}{4} \\ \frac{-\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}$$

$$(A - \lambda_2 I)v_2 = 0 \rightarrow \begin{bmatrix} \frac{3}{4} & \frac{-\sqrt{3}}{4} \\ \frac{-\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Normalize (unit-length) eigenvector } v_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$$

Therefore, the matrix Q is:

$$Q = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$$

and we can verify that:

$$A = Q\Lambda Q^T = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} 7/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 5/4 \end{bmatrix}$$

(b) Show that the columns of Q are orthonormal, i.e. the columns are of unit length and orthogonal.

Solution:

To proof that the eigenvectors have unit lenght:

$$v_1^T v_1 = 1 \rightarrow \begin{bmatrix} \sqrt{3}/2 & -1/2 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{3}/2 \\ -1/2 \end{bmatrix} = (\sqrt{3}/2)^2 + (-1/2)^2 = 1$$
$$v_2^T v_2 = 1 \rightarrow \begin{bmatrix} 1/2 & \sqrt{3}/2 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} = (1/2)^2 + (\sqrt{3}/2)^2 = 1$$

To proof that they are orthogonal, the dot product between them must be zero:

$$v_2 \cdot v_2 = 0 \rightarrow= (\frac{\sqrt{3}}{2} \cdot \frac{1}{2}) + (\frac{-1}{2} \cdot \frac{\sqrt{3}}{2}) = 0$$

(c) Show that the matrix $Q\Lambda^{-1}Q^T$ with $\Lambda^{-1} = diag(\lambda_1^{-1}, \lambda_2^{-1})$ is the inverse of A.

Solution:

First, we calculate the inverse of Λ by using the general formula for a 2x2 matrix:

$$M^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{det(M)} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\Lambda^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \frac{1}{det(\Lambda)} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, we replace in the eigenvalue decomposition equation:

$$A^{-1} = Q\Lambda^{-1}Q^T = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$
$$= \begin{bmatrix} \sqrt{3}/4 & 1/2 \\ -1/4 & \sqrt{3}/2 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} 5/8 & \sqrt{3}/8 \\ \sqrt{3}/8 & 7/8 \end{bmatrix}$$

If we calculate A^{-1} separately using the general formula, we reach the same solution:

$$A^{-1} = \frac{1}{det(A)} \cdot \begin{bmatrix} 5/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 7/4 \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} 5/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 7/4 \end{bmatrix} = \begin{bmatrix} 5/8 & \sqrt{3}/8 \\ \sqrt{3}/8 & 7/8 \end{bmatrix}$$

(d) For course improvements, we would like your feedback about this question. At least tell us how much time you did invest, if you had major problems and if you think it's useful.

Time invested: Around 45 min, specially to remember the knowledge and do the mathematical operations by hand. It was an useful question in order to recover some previous concepts. In my opinion, I would have empathized more on the meaning of eigenvectors and eigenvalues with respect to real data.

Part II

Matrix inversion

Given the matrix

$$A = \begin{bmatrix} 3 & 4 \\ 6 & 13 \end{bmatrix}$$

(a) Compute the LU decomposition of it, i.e. $A = L \cdot U$, where L is a lowertriangular matrix and U is an uppertriangular matrix. Use this decomposition to solve $A \cdot x = b$ with $b = (1, 2)^T$ via forward and backward substitution.

Solution:

- For U

$$A = \begin{bmatrix} 3 & 4 \\ 6 & 13 \end{bmatrix} \longrightarrow_{R_2 - 2R_1} \longrightarrow \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} = U$$

where k = 2

- For L

$$L = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

We calculate x via **forward and backward substitution**. To do so, we need first to perform forward substitution:

$$Ax = b \rightarrow LUx = b \rightarrow LY = b$$

where $Y = Ux$

If $LY = b$, then:

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \longrightarrow \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

With the values of Y, we perform backward substitution and calculate the values of x:

$$Ux = Y \rightarrow \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow$$
$$x = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 0 \end{bmatrix}$$

(b) Use Gauss elimination to explicitly calculate the inverse of A and show that it yields the same solution $x = A^{-1} \cdot b$ from part (a).

Solution:

To calculate the inverse of A explicitly by Gauss elimination, we proceed in the following way:

$$[A|I] = [I|A^{-1}]$$

$$[\begin{bmatrix}3 & 4\\6 & 13\end{bmatrix} \mid \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}] \rightarrow_{R_1/3}\rightarrow [\begin{bmatrix}1 & 4/3\\6 & 13\end{bmatrix} \mid \begin{bmatrix}1/3 & 0\\0 & 1\end{bmatrix}]$$

$$[\begin{bmatrix}1 & 4/3\\6 & 13\end{bmatrix} \mid \begin{bmatrix}1/3 & 0\\0 & 1\end{bmatrix}] \rightarrow_{R_2-6R_1}\rightarrow [\begin{bmatrix}1 & 4/3\\0 & 5\end{bmatrix} \mid \begin{bmatrix}1/3 & 0\\-2 & 1\end{bmatrix}]$$

$$[\begin{bmatrix}1 & 4/3\\0 & 5\end{bmatrix} \mid \begin{bmatrix}1/3 & 0\\-2 & 1\end{bmatrix}] \rightarrow_{R_2/5}\rightarrow [\begin{bmatrix}1 & 4/3\\0 & 1\end{bmatrix} \mid \begin{bmatrix}1/3 & 0\\-2/5 & 1/5\end{bmatrix}]$$

$$[\begin{bmatrix}1 & 4/3\\0 & 1\end{bmatrix} \mid \begin{bmatrix}1/3 & 0\\-2/5 & 1/5\end{bmatrix}] \rightarrow_{R_1-(4/3)R_2}\rightarrow [\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \mid \begin{bmatrix}13/15 & -4/15\\-2/5 & 1/5\end{bmatrix}]$$

If we calculate the inverse of A separately, we reach the same result:

$$A^{-1} = \frac{1}{det(A)}\begin{bmatrix}13 & -4\\-6 & 3\end{bmatrix} = \frac{1}{15}\begin{bmatrix}13 & -4\\-6 & 3\end{bmatrix} = \begin{bmatrix}13/15 & -4/15\\-2/5 & 1/5\end{bmatrix}]$$

If we use this matrix and b to calculate x, it gives the same result than before:

$$x = A^{-1}b = \begin{bmatrix}13/15 & -4/15\\-2/5 & 1/5\end{bmatrix} \cdot \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1/3\\0\end{bmatrix}$$

(c) For course improvements, we would like your feedback about this question. At least tell us how much time you did invest, if you had major problems and if you think it's useful.

Time invested: Around 30 min. Nothing special to remark.

Part III

Vector Norms

The length of a vector is not a single number but can be defined in different ways. These vector norms share common properties but also have different characteristics.

(a) Compute the $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_8$ and $\|\cdot\|_\infty$ norms of the following vectors:

$$x_1 = \begin{bmatrix} 24 \\ 3 \\ 2 \\ 31 \end{bmatrix}, x_2 = \begin{bmatrix} 27 \\ 20 \\ 26 \\ 21 \end{bmatrix}, x_3 = \begin{bmatrix} 30 \\ 21 \\ 27 \\ 5 \end{bmatrix} \text{ and } x_4 = \begin{bmatrix} 26 \\ 28 \\ 25 \\ 14 \end{bmatrix}$$

Solution:

The general formula for any p-order norm is:

$$\|x\|_p = (\sum_i |x_i|^p)^{\frac{1}{p}}$$

- For x_1:

$$\begin{gathered} \|x_1\|_1 = 24 + 3 + 2 + 31 = 60 \\ \|x_1\|_2 = \sqrt{24^2 + 3^2 + 2^2 + 31^2} = 39,37 \\ \|x_1\|_8 = (24^8 + 3^8 + 2^8 + 31^8)^{\frac{1}{8}} = 31,47 \\ \|x_1\|_\infty = max(|x_1|, ..., |x_n|) = 31 \end{gathered}$$

- For x_2:

$$\begin{gathered} \|x_2\|_1 = 27 + 20 + 26 + 21 = 94 \\ \|x_2\|_2 = \sqrt{27^2 + 20^2 + 26^2 + 21^2} = 47,39 \\ \|x_2\|_8 = (24^8 + 3^8 + 2^8 + 31^8)^{\frac{1}{8}} = 29,37 \\ \|x_2\|_\infty = max(|x_1|, ..., |x_n|) = 27 \end{gathered}$$

- For x_3:

$$\begin{gathered} \|x_3\|_1 = 30 + 21 + 27 + 5 = 83 \\ \|x_3\|_2 = \sqrt{30^2 + 21^2 + 27^2 + 5^2} = 45,77 \\ \|x_3\|_8 = (30^8 + 21^8 + 27^8 + 5^8)^{\frac{1}{8}} = 31,52 \\ \|x_3\|_\infty = max(|x_1|, ..., |x_n|) = 30 \end{gathered}$$

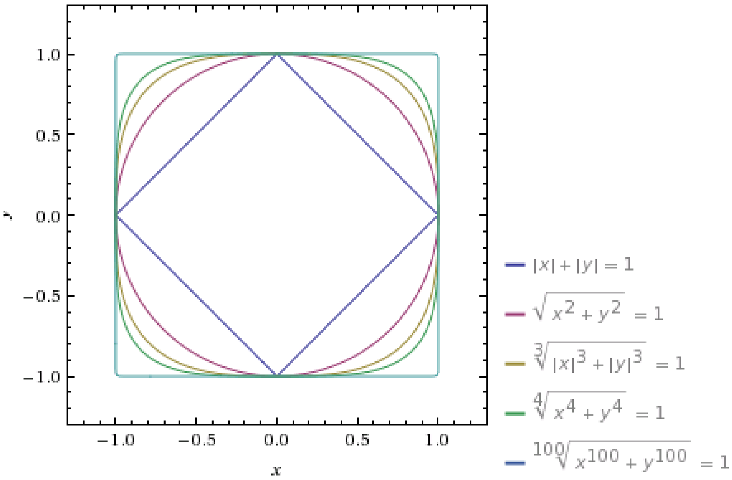
- For x_4:

$$\begin{gathered} \|x_4\|_1 = 26 + 28 + 25 + 14 = 93 \\ \|x_4\|_2 = \sqrt{26^2 + 28^2 + 25^2 + 14^2} = 47,75 \\ \|x_4\|_8 = (26^8 + 28^8 + 25^8 + 14^8)^{\frac{1}{8}} = 30,45 \\ \|x_4\|_\infty = max(|x_1|, ..., |x_n|) = 28 \end{gathered}$$

(b) Draw the set of points with $\|x\|_i = 1$ for $i \in \{1, 2, 8, \infty\}$ and $x \in \mathbb{R}^2$

Solution:

In the following plot, we can observe different norms for many values of x. The norm of order 1 ($L1$ or Lasso) has totally straight sides, and and as we increase the order, the sides start to curve. In the case of the $2^{nd}-norm$ ($L2$ or quadratic norm), we obtain a perfect circle.



For the $8^{th} - norm$, we approach the shape of a square, but with rounded corners. In the case of $\infty - norm$, the shape is a perfect square.

(c) For course improvements, we would like your feedback about this question. At least tell us how much time you did invest, if you had major problems and if you think it's useful.

Time invested: 20 min. I found the question about plotting the norms interesting to refer later on to different regularization techniques (L2 or L1 for example).

Part IV

Special Orthorgonal Matrices

A very important class of matrices are the special orthogonal group *SO(n)*. These matrices are charac- terized by a unit determinant and correspond to rotations around the origin.

(a) Given the matrix

$$A = \frac{1}{2}\begin{bmatrix} 7 & \sqrt{3} \\ \sqrt{3} & 5 \end{bmatrix}$$

compute $det(A), Tr(A)$ and its eigenvalues.

Solution:

We calculate the determinant, trace and eigenvalues of A:

$$det(A) = (\frac{35}{4} - \frac{3}{4}) = 8$$
$$Tr(A) = (\frac{7}{2} + \frac{5}{2}) = 6$$

$$A - \lambda I = \begin{bmatrix} \frac{7}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{5}{2} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \frac{7}{2} - \lambda & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{5}{2} - \lambda \end{bmatrix}$$

$$det(A - \lambda I) = (\frac{7}{2} - \lambda)(\frac{5}{2} - \lambda) - (\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}) = (\frac{7}{2} - \lambda)(\frac{5}{2} - \lambda) - \frac{3}{4} = \lambda^2 - 6\lambda + 8 = 0$$

$$\lambda = \frac{+6 \pm \sqrt{36 - 4 \cdot 1 \cdot 8}}{2} = \frac{+6 \pm \sqrt{4}}{2} \rightarrow \lambda_1 = 4; \lambda_2 = 2$$

(b) The matrix

$$Q(\alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

represents a rotation around an axis by the angle α.
Compute $A' = Q \cdot A \cdot Q^T$, $det(A')$, $Tr(A')$ and its eigenvalues for $\alpha = \pi/12$

Solution:

$$A' = QAQ^T = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \cdot \begin{bmatrix} 7/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 5/2 \end{bmatrix} \cdot \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} (7/2)\cos(\alpha) - (\sqrt{3}/2)\sin(\alpha) & (\sqrt{3}/2)\cos(\alpha) - (5/2)\sin(\alpha) \\ (7/2)\sin(\alpha) + (\sqrt{3}/2)\cos(\alpha) & (\sqrt{3}/2)\sin(\alpha) + (5/2)\cos(\alpha) \end{bmatrix} \cdot \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} 3.156 & 0.188 \\ 1.736 & 2.63 \end{bmatrix} \cdot \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} (3.156)\cos(\alpha) - (0.188)\sin(\alpha) & (3.156)\sin(\alpha) + (0.188)\cos(\alpha) \\ (1.736)\cos(\alpha) - (2.63)\sin(\alpha) & (1.736)\sin(\alpha) + (2.63)\cos(\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$det(A') = (9 - 1) = 8$$

$$Tr(A') = (3 + 3) = 6$$

$$A - \lambda I = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$$

$$det(A - \lambda I) = (3-\lambda)(3-\lambda) - (1) = \lambda^2 - 6\lambda + 8 = 0$$

$$\lambda = \frac{+6 \pm \sqrt{36 - 4 \cdot 1 \cdot 8}}{2} = \frac{+6 \pm \sqrt{4}}{2} \rightarrow \lambda_1 = 4; \lambda_2 = 2$$

We can notice that neither, determinant, trace nor eigenvalues have changed after the rotation.

(c) What would A' be for $\alpha = \pi/3$

The cos and sin values for α are:

$$cos(\frac{\pi}{3}) = \frac{1}{2},\ sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$$

If we replace the new α in the same previous matrix multiplication, we obtain:

$$A' = \begin{bmatrix} (7/4) - (3/4) & (\sqrt{3}/4) - (5\sqrt{3}/4) \\ (7\sqrt{3}/4) + (\sqrt{3}/4) & (3/4) + (5/4) \end{bmatrix} \cdot \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\sqrt{3} \\ 2\sqrt{3} & 2 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

(d) For course improvements, we would like your feedback about this question. At least tell us how much time you did invest, if you had major problems and if you think it's useful.

Time invested: Around 45-60 min, specially for the mathematical operations of the question (b).