1. **Jupyter notebooks**
   Most exercises will take place as jupyter notebooks: A document format to combine rich text with formulas, python code and plots.
   (a) Install jupyter with python 3 by following the instructions on `http://jupyter.org/install`. (0)
   (b) Learn how to start a notebook.. (0)
   (c) Work through the notebook `IntroductionToJupyterNotebooks.ipynb`. (0)
   (d) For course improvements, we would like your feedback about this question. At least tell us how much time you did invest, if you had major problems and if you think it's useful.

   Points for Question 1: 0

Note: The following questions are a repetition of linear algebra. Please also write down interim solutions. E.g. don't just use `numpy.linalg.norm(...)` to solve the question about vector norms.

2. **Eigenvalue Decomposition**
   Given the matrix
$$\mathbf{A} = \frac{1}{4}\begin{bmatrix} 7 & -\sqrt{3} \\ -\sqrt{3} & 5 \end{bmatrix}$$
   (a) Compute the eigenvalue decomposition $\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T$ with $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$. (2)
   (b) Show that the columns of $\mathbf{Q}$ are orthonormal, i.e. the columns are of unit length and orthogonal. (1)
   (c) Show that the matrix $\mathbf{Q}\mathbf{\Lambda}^{-1}\mathbf{Q}^T$ with $\mathbf{\Lambda}^{-1} = \mathrm{diag}(\lambda_1^{-1}, \lambda_2^{-1})$ is the inverse of $\mathbf{A}$ (1)
   (d) For course improvements, we would like your feedback about this question. At least tell us how much time you did invest, if you had major problems and if you think it's useful.

Solution:

(a) First we need to find the eigenvalues. They are defined as the roots of the characteristic polynomial
$$p(\lambda) = |\mathbf{A} - \lambda\mathbb{I}| = \left(\frac{7}{4} - \lambda\right)\left(\frac{5}{4} - \lambda\right) - \frac{3}{16} = \lambda^2 - 3\lambda - 2\,.$$
The two eigenvalues are then
$$\lambda_{1,2} = \frac{3}{2} \pm \sqrt{\frac{9}{4} - 2} = \frac{3 \pm 1}{2}\,.$$
Now we can find the corresponding eigenvectors:
$$\mathbf{A}\cdot\mathbf{x} = \lambda_1\mathbf{x} \quad \Longrightarrow \quad \begin{bmatrix} \frac{7}{4}x_1 - \frac{\sqrt{3}}{4}x_2 \\ -\frac{\sqrt{3}}{4}x_1 + \frac{5}{4}x_2 \end{bmatrix} = 2\cdot\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Longrightarrow \quad x_1 = -\sqrt{3}x_2$$
$$\mathbf{A}\cdot\mathbf{y} = \lambda_2\mathbf{y} \quad \Longrightarrow \quad \begin{bmatrix} \frac{7}{4}y_1 - \frac{\sqrt{3}}{4}y_2 \\ -\frac{\sqrt{3}}{4}y_1 + \frac{5}{4}y_2 \end{bmatrix} = 1\cdot\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \Longrightarrow \quad y_2 = \sqrt{3}y_1$$
Normalizing them to unit norm yields
$$\mathbf{x} = \frac{1}{2}\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}, \qquad \mathbf{y} = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \qquad \text{and} \qquad \mathbf{Q} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}.$$

To check our computation, we can now verify that

$$
\begin{aligned}
\mathbf{Q}\cdot\boldsymbol{\Lambda}\cdot\mathbf{Q}^T &= \frac{1}{2}\begin{bmatrix}\sqrt{3} & 1\\ -1 & \sqrt{3}\end{bmatrix}\cdot\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix}\sqrt{3} & -1\\ 1 & \sqrt{3}\end{bmatrix}\\
&= \frac{1}{4}\begin{bmatrix}\sqrt{3} & 1\\ -1 & \sqrt{3}\end{bmatrix}\cdot\begin{bmatrix}2\sqrt{3} & -2\\ 1 & \sqrt{3}\end{bmatrix}\\
&= \frac{1}{4}\begin{bmatrix}7 & -\sqrt{3}\\ -\sqrt{3} & 5\end{bmatrix}
\end{aligned}
$$

(b) We can show the orthonormality by showing that $\mathbf{Q}\cdot\mathbf{Q}^T=\mathbb{I}$:

$$
\begin{aligned}
\mathbf{Q}\cdot\mathbf{Q}^T &= \frac{1}{2}\begin{bmatrix}\sqrt{3} & 1\\ -1 & \sqrt{3}\end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix}\sqrt{3} & -1\\ 1 & \sqrt{3}\end{bmatrix}\\
&= \frac{1}{4}\begin{bmatrix}3+1 & 0\\ 0 & 1+3\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}
\end{aligned}
$$

(c) Let us first compute the proposed inverse:

$$
\begin{aligned}
\mathbf{Q}\cdot\boldsymbol{\Lambda}^{-1}\cdot\mathbf{Q}^T &= \frac{1}{2}\begin{bmatrix}\sqrt{3} & 1\\ -1 & \sqrt{3}\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{2} & 0\\ 0 & 1\end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix}\sqrt{3} & -1\\ 1 & \sqrt{3}\end{bmatrix}\\
&= \frac{1}{4}\begin{bmatrix}\frac{\sqrt{3}}{2} & 1\\ -\frac{1}{2} & \sqrt{3}\end{bmatrix}\cdot\begin{bmatrix}\sqrt{3} & -1\\ 1 & \sqrt{3}\end{bmatrix}\\
&= \frac{1}{4}\begin{bmatrix}\frac{5}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{7}{2}\end{bmatrix}\\
&= \frac{1}{8}\begin{bmatrix}5 & \sqrt{3}\\ \sqrt{3} & 7\end{bmatrix}.
\end{aligned}
$$

Now we can verify that this is indeed the inverse:

$$
\begin{aligned}
\mathbf{A}\cdot\left(\mathbf{Q}\cdot\boldsymbol{\Lambda}^{-1}\cdot\mathbf{Q}^T\right) &= \frac{1}{4}\begin{bmatrix}7 & -\sqrt{3}\\ -\sqrt{3} & 5\end{bmatrix}\cdot\frac{1}{8}\begin{bmatrix}5 & \sqrt{3}\\ \sqrt{3} & 7\end{bmatrix}\\
&= \frac{1}{32}\begin{bmatrix}35-3 & 7\sqrt{3}-7\sqrt{3}\\ -5\sqrt{3}+5\sqrt{3} & -3+35\end{bmatrix}\\
&= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.
\end{aligned}
$$

Points for Question 2: 4

3. **Matrix inversion**

Given the matrix

$$\mathbf{A} = \begin{bmatrix}3 & 4\\ 6 & 13\end{bmatrix}$$

(a) Compute the LU decomposition of it, i.e. $\mathbf{A}=\mathbf{L}\cdot\mathbf{U}$, where $\mathbf{L}$ is a lowertriangular matrix and $\mathbf{U}$ is an uppertriangular matrix. Use this decomposition to solve $\mathbf{A}\cdot\mathbf{x}=\mathbf{b}$ with $\mathbf{b}=(1,2)^T$ via forward and backward substitution. (2)

(b) Use Gauss elimination to explicitly calculate the inverse of $\mathbf{A}$ and show that it yields the same solution $\mathbf{x}=\mathbf{A}^{-1}\cdot\mathbf{b}$ from part (a). (2)

(c) For course improvements, we would like your feedback about this question. At least tell us how much time you did invest, if you had major problems and if you think it's useful.

Solution:

(a) First, we have to find the lowertriangular matrix L and the upper triangular matrix U by solving the following system of equations:

$$\begin{bmatrix}3 & 4\\6 & 13\end{bmatrix} = \begin{bmatrix}l_{11} & 0\\l_{21} & l_{22}\end{bmatrix} \cdot \begin{bmatrix}u_{11} & u_{12}\\0 & u_{22}\end{bmatrix} = \begin{bmatrix}l_{11}u_{11} & l_{11}u_{12}\\l_{21}u_{11} & l_{21}u_{12}+l_{22}u_{22}\end{bmatrix}$$

We choose among the infinitely many solutions of the system the one with $l_{11} = l_{22} = 1$ (a common choice):

$$A = \begin{bmatrix}1 & 0\\2 & 1\end{bmatrix} \cdot \begin{bmatrix}3 & 4\\0 & 5\end{bmatrix}$$

Once the $LU-$decomposition is found, the original system $b = Ax$ can be rewritten also as $b = LUx$. The forward and backward substitution is a two step procedure that solves the original problem $b = Ax$ by

1. solving for y from $b = Ly$ (*forward substitution*)
2. solving for x from $y = Ux$ (*backward substitution*)

In particular, in our case

1. $\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1 & 0\\2 & 1\end{bmatrix} \begin{bmatrix}y_{11}\\y_{21}\end{bmatrix} \rightarrow \begin{bmatrix}y_{11}\\y_{21}\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$ (*forward substitution*)
2. $\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}3 & 4\\0 & 5\end{bmatrix} \begin{bmatrix}x_{11}\\x_{21}\end{bmatrix} \rightarrow \begin{bmatrix}x_{11}\\x_{21}\end{bmatrix} = \frac{1}{3} \cdot \begin{bmatrix}1\\0\end{bmatrix}$ (*backward substitution*)

(b) The Gauss elimination is a method to solve general linear systems $AX = B$ by appyling elementary row operations (interchange two rows, add a multiple of one row to another or multiply a row by a nonzero number) on the augmented matrix $[A|B]$. In the particular case of matrix inversion, the system to be solved becomes $AX = I$.

$$\left[\begin{array}{cc|cc}3 & 4 & 1 & 0\\6 & 13 & 0 & 1\end{array}\right] \xrightarrow{\mathrm{R_2-2R_1 \to R_2}} \left[\begin{array}{cc|cc}3 & 4 & 1 & 0\\0 & 5 & -2 & 1\end{array}\right] \xrightarrow{\frac{1}{5}\mathrm{R_2 \to R_2}} \left[\begin{array}{cc|cc}3 & 4 & 1 & 0\\0 & 1 & -\frac{2}{5} & \frac{1}{5}\end{array}\right] \xrightarrow{\frac{1}{3}\mathrm{R_1 \to R_1}}$$

$$\left[\begin{array}{cc|cc}1 & \frac{4}{3} & \frac{1}{3} & 0\\0 & 1 & -\frac{2}{5} & \frac{1}{5}\end{array}\right] \xrightarrow{\mathrm{R_1}-\frac{4}{3}R_2 \to \mathrm{R_1}} \left[\begin{array}{cc|cc}1 & 0 & \frac{13}{15} & -\frac{4}{15}\\0 & 1 & -\frac{2}{5} & \frac{1}{5}\end{array}\right]$$

Now that we found the inverse

$$\mathbf{A}^{-1} = \frac{1}{15} \cdot \begin{bmatrix}13 & -4\\-6 & 3\end{bmatrix},$$

we can verify the result from (a):

$$\mathbf{A}^{-1} \cdot \mathbf{b} = \frac{1}{15} \cdot \begin{bmatrix}13 & -4\\-6 & 3\end{bmatrix} \cdot \begin{bmatrix}1\\2\end{bmatrix} = \frac{1}{15} \begin{bmatrix}5\\0\end{bmatrix} = \frac{1}{3} \cdot \begin{bmatrix}1\\0\end{bmatrix}$$

Points for Question 3: 4

4. **Vector Norms**
The length of a vector is not a single number but can be defined in different ways. These vector norms share common properties but also have different characteristics.

(a) Compute the $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_8$ and $\|\cdot\|_\infty$ norms of the following vectors: (2)

$$\mathbf{x}_1 = \begin{bmatrix} 24 \\ 3 \\ 2 \\ 31 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 27 \\ 20 \\ 26 \\ 21 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 30 \\ 21 \\ 27 \\ 5 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_4 = \begin{bmatrix} 26 \\ 28 \\ 25 \\ 14 \end{bmatrix}.$$

(b) Draw the set of points with $\|\mathbf{x}\|_i = 1$ for $i \in \{1, 2, 8, \infty\}$ and $\mathbf{x} \in \mathbb{R}^2$ (2)

(c) For course improvements, we would like your feedback about this question. At least tell us how much time you did invest, if you had major problems and if you think it's useful.

Solution:

(a) First, recall that the p-norm of a vector is defined as

$$\|\mathbf{x}\|_p = \left(\sum_i |x_i|^p \right)^{1/p}.$$

This leads us to the following norms

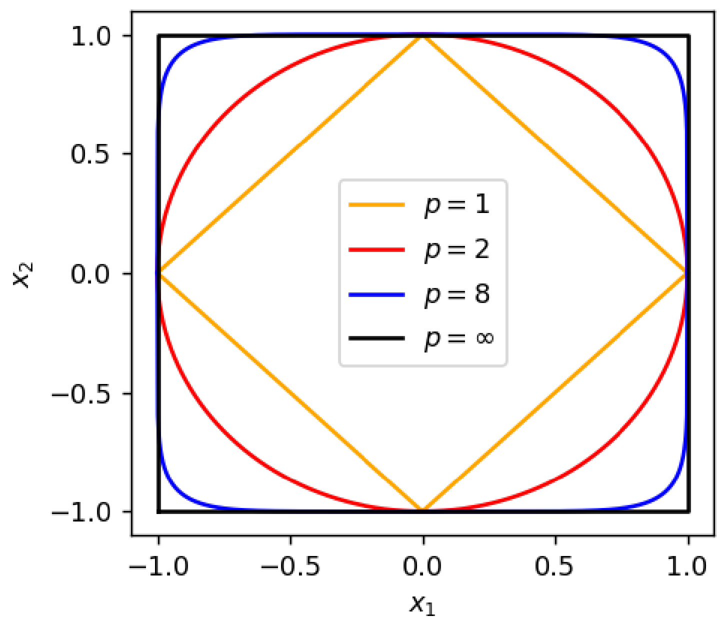
	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$
$\|\cdot\|_1$	60	94	83	93
$\|\cdot\|_2$	39.37	47.39	45.77	47.76
$\|\cdot\|_8$	31.47	29.38	31.53	30.46
$\|\cdot\|_\infty$	31	27	30	28

Note how a different vector is the largest in each norm.

(b) For drawing the set of points with a given Norm in $\mathbb{R}^2$, we can use definition above to find an explicit expression for x_2 as a function of x_1

$$x_2 = \pm \left(1 - |x_1|^p\right)^{1/p}.$$

For the $p = \infty$ case, the set consists of all points with $\max(|x_1|, |x_2|) = 1$ which is the square of length 2 centered at the origin. The following plot shows the different sets:



Points for Question 4: 4

5. **Special Orthorgonal Matrices**

A very important class of matrices are the special orthogonal group $SO(n)$. These matrices are characterized by a unit determinant and correspond to rotations around the origin.

(a) Given the matrix (2)

$$\mathbf{A} = \frac{1}{2}\begin{bmatrix} 7 & \sqrt{3} \\ \sqrt{3} & 5 \end{bmatrix}$$

compute $\det(\mathbf{A})$, $\mathrm{Tr}(\mathbf{A})$ and its eigenvalues.

(b) The matrix (1)

$$\mathbf{Q}(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

represents a rotation around an axis by the angle α. Compute $\mathbf{A}' = \mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^T$, $\det(\mathbf{A}')$, $\mathrm{Tr}(\mathbf{A}')$ and its eigenvalues for $\alpha = {}^{\pi}/_{12}$.

(c) What would $\mathbf{A}'$ be for $\alpha = {}^{\pi}/_{3}$? (1)

(d) For course improvements, we would like your feedback about this question. At least tell us how much time you did invest, if you had major problems and if you think it's useful.

Solution:

(a) Let us start with the trace and the determinant:

$$\mathrm{Tr}(\mathbf{A}) = \frac{7+5}{2} = 6 \qquad \text{and} \qquad \det(\mathbf{A}) = \frac{1}{4}(35-3) = 8\,.$$

To find the two eigenvalues, we could find the roots of the characteristic polynomial, but for a 2×2 matrix, we can also use the following equations

$$\mathrm{Tr}(\mathbf{A}) = \lambda_1 + \lambda_2 \qquad \text{and} \qquad \det(\mathbf{A}) = \lambda_1 \cdot \lambda_2\,.$$

Solving these, leads to $\lambda_1 = 4$ and $\lambda_2 = 2$.

(b) For the case of $\alpha = \frac{\pi}{12}$:

$$\mathbf{Q}(\alpha) = \frac{1}{4}\begin{bmatrix} (\sqrt{2}+\sqrt{6}) & (\sqrt{2}-\sqrt{6}) \\ (\sqrt{6}-\sqrt{2}) & (\sqrt{2}+\sqrt{6}) \end{bmatrix}$$

By matrix multiplication, we get:

$$A' = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

With

$$\det(\mathbf{Q}) = \cos^2\alpha + \sin^2\alpha = 1\,,$$

and $\det(AB) = \det(A)\det(B)$, we obtain that:

$$\det(\mathbf{A}') = \det(\mathbf{Q}^T)\det(\mathbf{A})\det(\mathbf{Q}) = 8\,.$$

The eigenvalues are the roots of the characteristic polynomial:

$$\lambda^2 - 6\lambda + 8 \overset{!}{=} 0 \longrightarrow \lambda_1 = 4\,, \lambda_2 = 2$$

Finally, by applying the definition of trace of a matrix, we obtain:

$$\mathrm{Tr}(A) = 3 + 3 = 6$$

(c) For the case $\alpha = \frac{\pi}{3}$:

$$\mathbf{Q}(\alpha) = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

By matrix multiplication, we get:

$$A' = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

We directly see from the diagonal matrix A' that all quantities are the same as in (a).

Points for Question 5: 4

You can achieve a total of **16 points** for this exercise. Additionally you can achieve **1 bonus point** for answering the feedback questions.

Please send a **single solution** as a **group of three** via ILIAS until **29.10.2018 12 pm**. You can hand in a jupyter notebook (**recommended**), a pdf or a scan (please write legibly).

Note: Jupyter notebooks will be executed **from top to bottom**. To **avoid point deduction** check your notebook by the following steps: 1. Use the python 3 kernel (`Kernel > Change kernel > Python 3`), 2. Run the full notebook (`Kernel > Restart & Run All`), 3. Save (`File > Save and Checkpoint`).