System Theory Project - "Comfy Car"

Eduard Darbinyan
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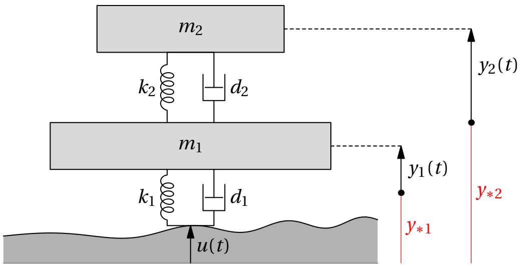
Contents

1 Introduction

In this report I will be investigating the behavior of a car and its systems: chair and body, in different scenarios, such as road bumps, slow and fast speeds and etc. The graphic below will show the inner workings of the cars system.



2 Calculation and simulations

2.1 Differential equations of the systems

To determine the second-order differential equations associated with each sub-system of the car (the chair and the body) we first need to differentiate the variables and forces acting on each of them.

The seat/chair system:

- Seat mass: m_2
- Upper spring: k_2
- Upper damper: d_2

Then, spring force k_2 depends on relative position of the seat y_2 and the car body y_1 and the damper force d_2 depends on relative velocity of the seat $\dot{y}_1$ and the car body $\dot{y}_1$:

$$k_2 = y_2 - y_1$$

$$d_2 = \dot{y}_2 - \dot{y}_1$$

By Newton's Second law $F = ma$:

$$m_2\ddot{y}_2 = -k_2(y_2 - y_1) - d_2(\dot{y}_2 - \dot{y}_1) \tag{1}$$

Spring and damper force are negative because they oppose displacement and velocity respectively. Rearranging to get our second-order differential equation for the seat:

$$m_2\ddot{y}_2 + d_2\dot{y}_2 + k_2y_2 = d_2\dot{y}_1 + k_2y_1 \tag{2}$$

Similarly for the body,

The Car Body System:

- Seat spring force: $k_2(y_2 - y_1)$
- Seat damper force: $d_2(\dot{y}_2 - \dot{y}_1)$
- Road spring force: $-k_1(y_1 - u)$
- Road damper force: $-d_1(\dot{y}_1 - \dot{u})$

Again, by Newton's second law:

$$m_1\ddot{y}_1 = -k_1(y_1 - u) - d_1(\dot{y}_1 - \dot{u} + k_2(y_2 - y_1) + d_2(\dot{y}_2 - \dot{y}_1) \tag{3}$$

The chair is modeled by the second-order differential equation:

$$m_2\ddot{y}_2 + d_2(\dot{y}_2 - \dot{y}_1) + k_2(y_2 - y_1) = 0. \tag{4}$$

Rearranging:

$$m_2\ddot{y}_2 + d_2\dot{y}_2 + k_2y_2 = d_2\dot{y}_1 + k_2y_1. \tag{5}$$

2.2 State space representation

Using resulting equations we can deduce the state space representation of the system. In a similar fashion we separate the chair and the body systems and later combine them into a single state space system.

Chair System (Σ_{chair})

From the equation differential equation of the chair $m_2\ddot{y}_2 + d_2\dot{y}_2 + k_2y_2 = d_2\dot{y}_1 + k_2y_1$ define the state variables:

$$x_3 = y_2, \quad x_4 = \dot{y}_2. \tag{6}$$

Then:

$$\dot{x}_3 = x_4, \tag{7}$$

$$\dot{x}_4 = -\frac{k_2}{m_2}x_3 - \frac{d_2}{m_2}x_4 + \frac{d_2}{m_2}\dot{y}_1 + \frac{k_2}{m_2}y_1. \tag{8}$$

The state-space representation becomes:

$$\begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k_2}{m_2} & -\frac{d_2}{m_2} \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k_2}{m_2} \end{bmatrix} y_1 + \begin{bmatrix} 0 \\ \frac{d_2}{m_2} \end{bmatrix} \dot{y}_1. \tag{9}$$

The output equation is:

$$y_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}. \tag{10}$$

Body System ($\Sigma_{\mathbf{body}}$)

The body is modeled by the second-order differential equation:

$$m_1\ddot{y}_1 + d_1(\dot{y}_1 - u) + k_1(y_1 - u) + d_2(\dot{y}_1 - \dot{y}_2) + k_2(y_1 - y_2) = 0. \tag{11}$$

Rearranging:

$$m_1\ddot{y}_1 + (d_1 + d_2)\dot{y}_1 + (k_1 + k_2)y_1 = d_1\dot{u} + k_1 u + d_2\dot{y}_2 + k_2 y_2. \tag{12}$$

Define the state variables:

$$x_1 = y_1, \quad x_2 = \dot{y}_1. \tag{13}$$

Then:

$$\dot{x}_1 = x_2, \tag{14}$$

$$\dot{x}_2 = -\frac{k_1 + k_2}{m_1}x_1 - \frac{d_1 + d_2}{m_1}x_2 + \frac{d_1}{m_1}\dot{u} + \frac{k_1}{m_1}u + \frac{d_2}{m_1}\dot{y}_2 + \frac{k_2}{m_1}y_2. \tag{15}$$

The state-space representation becomes:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k_1+k_2}{m_1} & -\frac{d_1+d_2}{m_1}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{k_1}{m_1}\end{bmatrix}u + \begin{bmatrix}0\\ \frac{k_2}{m_1}\end{bmatrix}y_2 + \begin{bmatrix}0\\ \frac{d_2}{m_1}\end{bmatrix}\dot{y}_2. \tag{16}$$

The output equation is:

$$y_1 = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}. \tag{17}$$

Combined System

The full state vector is:

$$x = \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}. \tag{18}$$

Substitute:

$$y_1 = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}, \quad y_2 = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_3\\ x_4\end{bmatrix}. \tag{19}$$

The combined state-space equations are:

$$\dot{x} = Ax + Bu, \tag{20}$$

$$y = Cx, \tag{21}$$

where:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{d_1+d_2}{m_1} & \frac{k_2}{m_1} & \frac{d_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{d_2}{m_2} & -\frac{k_2}{m_2} & -\frac{d_2}{m_2} \end{bmatrix}, \tag{22}$$

$$B = \begin{bmatrix} 0 \\ \frac{k_1}{m_1} \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{23}$$

2.3 Weak Chair Spring (French Car) Simulation

Simulation method

Given the parameters:

$$m_1 = 1400\,\mathrm{kg},\ d_1 = 1000\,\mathrm{kg/s},\ k_1 = 1000\,\mathrm{kg/s^2},$$

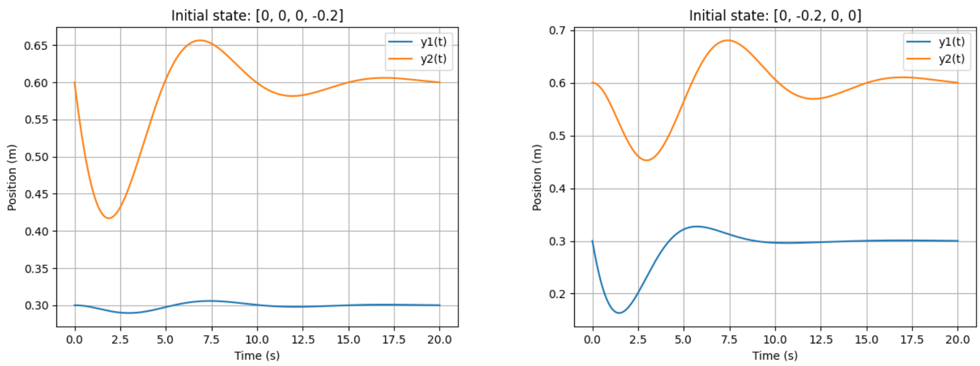
$$m_2 = 100\,\mathrm{kg},\ d_2 = 50\,\mathrm{kg/s},\ k_2 = 50\,\mathrm{kg/s^2},$$

the initial conditions for the state vector $x(t)$ are:

$$x(0) = (0, -0.2, 0, 0) \quad \text{(car body near the road)},$$

$$x(0) = (0, 0, 0, -0.2) \quad \text{(chair near the car body)}.$$

Euler's method is used to simulate the state $x(t)$ over the time interval $t \in [0, 20]$. (See the code for all the Appendix A)



(a) Initial conditions: $(0, 0, 0, -0.2)$ (b) Initial conditions: $(0, -0.2, 0, 0)$

Figure 1: Soft chair positions with different initial conditions

The positions $y_1(t) = y_1^* + y_1(t)$ and $y_2(t) = y_2^* + y_2(t)$ are plotted for both initial conditions. The differences in response due to initial conditions are evaluated, particularly in relation to the eigenvalues and time constants of the system's A-matrix.

2.4 Stiff Chair Spring Simulation

Following similar steps as Assignment 3, but with modified parameters:

$$m_1 = 1400\,\text{kg},\ d_1 = 1000\,\text{kg/s},\ k_1 = 1000\,\text{kg/s}^2,$$

$$m_2 = 100\,\text{kg},\ d_2 = 200\,\text{kg/s},\ k_2 = 100\,\text{kg/s}^2.$$

The initial states are:

$$x(0) = (0, -0.2, 0, 0) \quad \text{(car body near the road)},$$

$$x(0) = (0, 0, 0, -0.2) \quad \text{(chair near the car body)}.$$

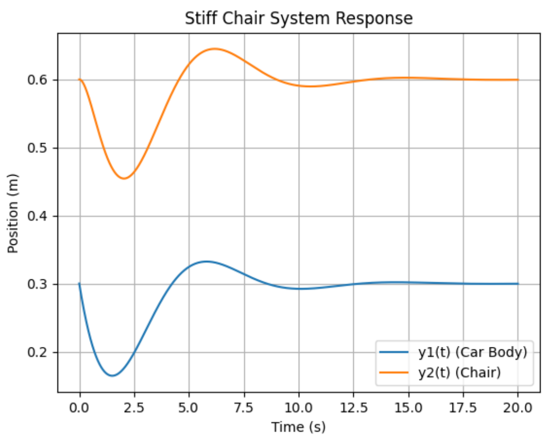


Figure 2: Stiff chair response

Euler's method is used for simulation over $t \in [0, 20]$. Plots of $y_1(t)$ and $y_2(t)$ are generated. The motion of the system is analyzed in terms of the eigenvalues of the updated A-matrix, focusing on the effects of increased stiffness and damping.

2.5 Parameter Variations

To analyze the effect of varying the damping coefficient d_1 on the displacement of the car body $y_1(t)$ when subjected to a step input representing a sudden road elevation.

The parameters for the system are:

$$m_1 = 1400\,\text{kg},\ k_1 = 1000\,\text{kg/s}^2,\ m_2 = 100\,\text{kg},\ k_2 = 100\,\text{kg/s}^2,\ d_2 = 200\,\text{kg/s}.$$

The damping coefficient of the car body d_1 is varied over the range:

$$d_1 \in \{250, 500, 750, \ldots, 2000\}\,\text{kg/s}.$$

The road surface elevation is modeled as a step function:

$$u(t) = \begin{cases} 0, & t < 1, \\ 1, & t \geq 1. \end{cases}$$

The state vector $x(t) = [y_1, \dot{y}_1, y_2, \dot{y}_2]$ is initialized as:

$$x(0) = [0, 0, 0, 0].$$

The equations of motion are solved using Euler's method over the time interval $t \in [0, 20]$ with a time step of $\Delta t = 0.01$. For each d_1, the displacement $y_1(t)$ is recorded and plotted.

The displacement of the car body $y_1(t)$ was plotted for various values of d_1. The following trends were observed:

- For small d_1 values (soft damper), the car body exhibited significant oscillations due to underdamping, with slower settling times.
- For large d_1 values (hard damper), the system showed reduced oscillations and quicker settling times, but at the cost of increased stiffness, which may reduce comfort.

The plot below illustrates the displacement $y_1(t)$ for the range of d_1 values considered.

2.6 Single Speed Bump

The goal is to analyze the response of the car body and seat as the car rides over a single sinusoidal speed bump. The speed bump is modeled as:

$$u(z) = 0.1 \sin\left(\frac{\pi}{5}(z - 5)\right), \quad \text{for } 5 \leq z \leq 9.5,$$

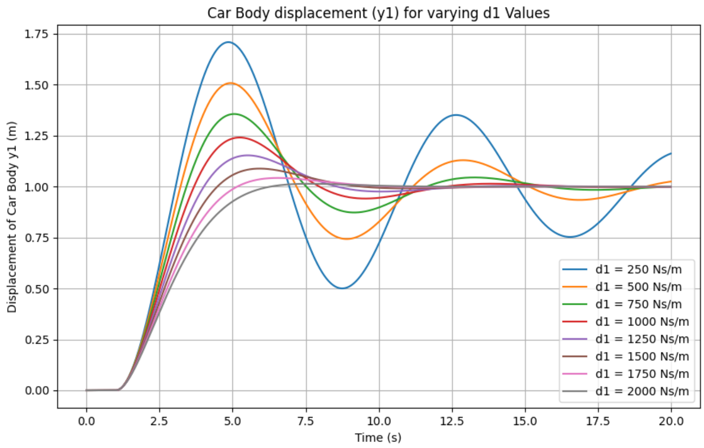


Figure 3: Car body displacement $y_1(t)$ for varying damping coefficients d_1.

where z represents the horizontal position of the car. Assuming a constant car speed of $v = 3\,\mathrm{m/s}$, the bump elevation becomes a function of time, defined as

$$u(t) = u(vt), \quad \text{for } 5/v \leq t \leq 9.5/v,$$

and zero elsewhere.

The simulation system parameters are $m_1 = 1400\,\mathrm{kg}, k_1 = 1000\,\mathrm{N/m}, d_1 = 1000\,\mathrm{Ns/m}, m_2 = 100\,\mathrm{kg}, k_2 = 100\,\mathrm{N/m}$, and $d_2 = 200\,\mathrm{Ns/m}$. The state of the system, defined as $x(t) = [y_1, \dot{y}_1, y_2, \dot{y}_2]$, is initialized at $x(0) = [0, 0, 0, 0]$, where y_1 and y_2 are the positions of the car body and the seat, respectively. Using Euler's method, the equations of motion are solved numerically over the time interval $t \in [0, 20]$ with a time step of $\Delta t = 0.01$.

The simulation produces the elevation of the road $u(t)$, the displacements of the body of the car $y_1(t)$ and the seat $y_2(t)$, as well as the difference $y_2(t) - y_1(t)$. These are plotted to illustrate how the system responds to the speed bump. The maximum elevation difference between the seat and the car body is determined to be

$$\max(y_2(t) - y_1(t)) = 0.0134\,\mathrm{m}.$$

The results show the oscillatory behavior of both the car body and seat due to the bump input, with the magnitude and duration of the oscillations reflecting the system's damping and stiffness characteristics. The maximum elevation difference highlights the level of comfort experienced by the passenger. A plot of the results, including $u(t)$, $y_1(t)$, $y_2(t)$, and the difference $y_2(t) - y_1(t)$, is shown in Figure 4.

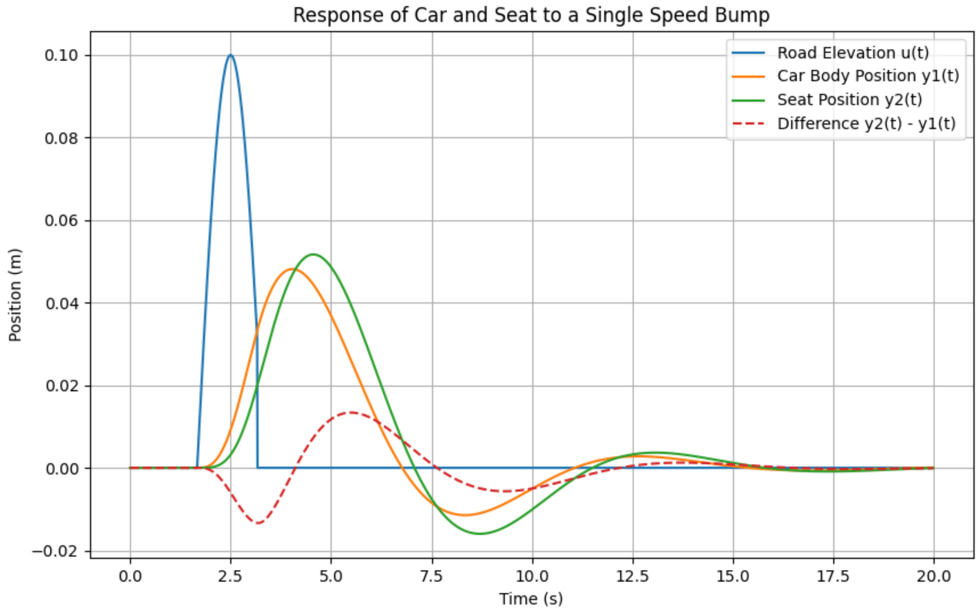


Figure 4: Response of the car body and seat to a single speed bump. The dashed line shows the difference between seat and car body positions.

2.7 Two Bumps

What happens if there are two consecutive bumps on the road. Let's investigate the response of the car body and seat to two speed bumps, as well as the impact of the gaps between them on the system's behavior. Each bump is modeled as.

$$u(z) = 0.1 \sin\left(\frac{\pi}{5}(z-5)\right), \quad \text{for } 5 \le z \le 9.5,$$

where z is the horizontal position of the car. Assuming a constant speed of $v = 3\,\text{m/s}$, the elevation of the road for two bumps is determined based on the horizontal gap s between them. The first bump is at $5 \le z \le 9.5$, while the second is at $5 + s \le z \le 9.5 + s$. The resulting road elevation $u(t)$ as a function of time depends on $z(t) = vt$.

Consider two cases: a gap of $s = 1.5\,\text{m}$ between the bumps, and $s = 0$ (basically overleaping/consecutive bumps). The system parameters are $m_1 = 1400\,\text{kg}, k_1 = 1000\,\text{N/m}, d_1 = 1000\,\text{Ns/m}, m_2 = 100\,\text{kg}, k_2 = 100\,\text{N/m}$, and $d_2 = 200\,\text{Ns/m}$. The initial state is $x(0) = [0, 0, 0, 0]$ and we can use the Euler's method to simulate and solve the equations over $t \in [0, 20]$ with a time step of $\Delta t = 0.01$.

For both cases, the simulation produces the road elevation $u(t)$, the displacements of the car body $y_1(t)$ and seat $y_2(t)$, and the difference $y_2(t) - y_1(t)$. Then we can compare the resulting elevations by analyzing the local maximum differences of the car body and the seat, as well as the time at which the maxima occur. Plots of the road elevation, car body and seat displacements, and their difference for both scenarios are shown in Figures 5 and 6. Looking at the plots following can be observed:

- For $s = 1.5\,\mathrm{m}$, the second bump induces additional oscillations, with the response of the system depending on how the vibrations of the first bump interact with the second.
- For $s = 0$ (overlapping bumps), the input behaves as a single bump with a longer duration, leading to a more pronounced response and larger deviations.

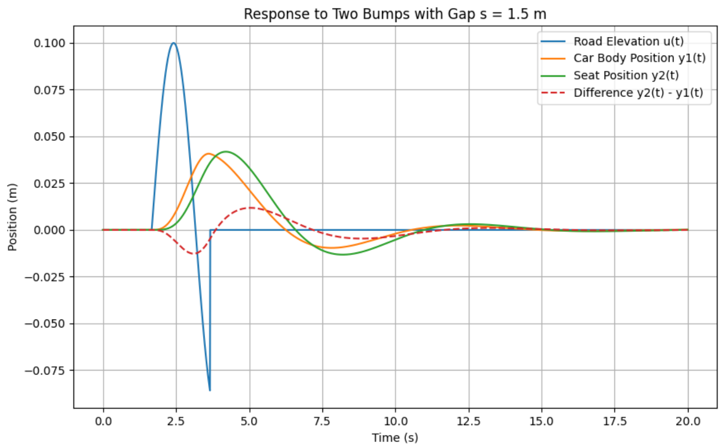


Figure 5: Response of the car body and seat for two bumps with $s = 1.5\,\mathrm{m}$. The dashed line shows the difference between seat and car body positions.

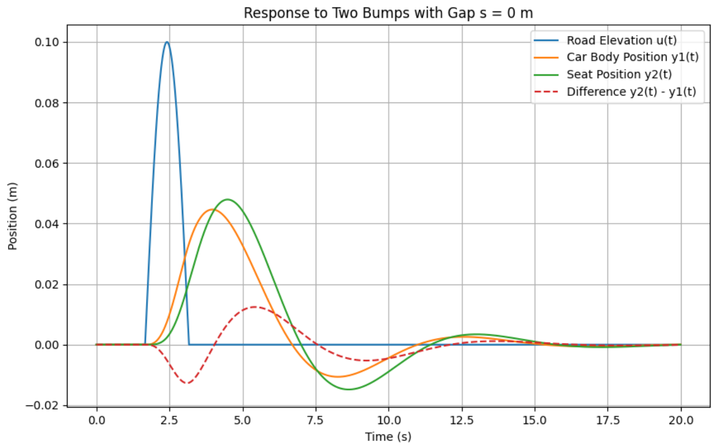


Figure 6: Response of the car body and seat for two overlapping bumps ($s = 0$).

2.8 Slow and fast speed over a bump

Here the goal is to observe the response of the car's suspension system to the bump at slow and fast speeds. Simulating the cars response to a speed bump at different speeds: $v_1 = 3$m/s and $v_2 = 10$ m/s.

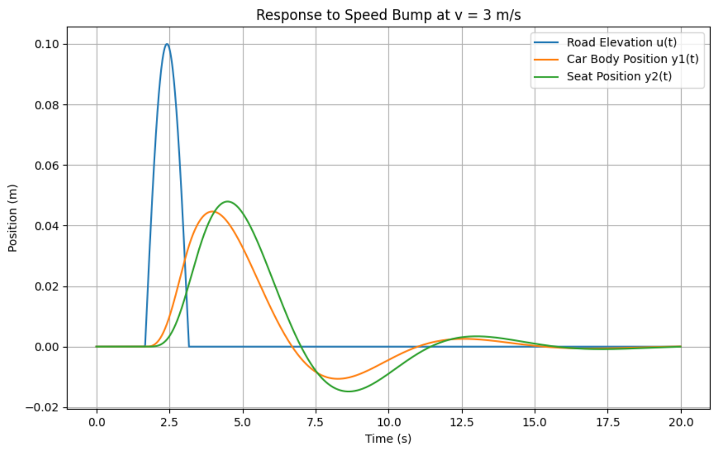


Figure 7: Response of the cars suspension to the bump at $v = 3$m/s.

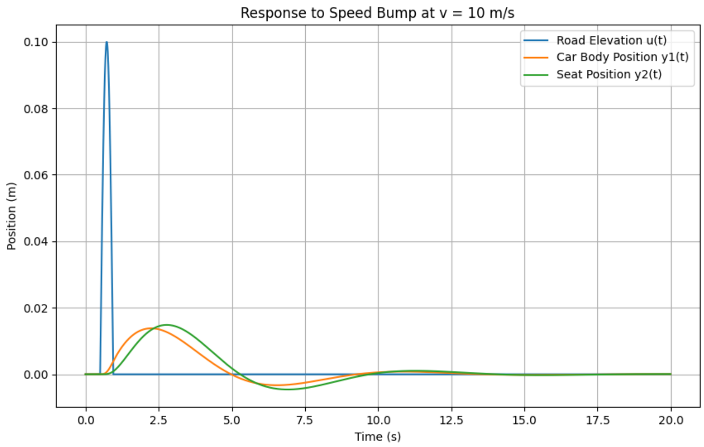


Figure 8: Response of the cars suspension to the bump at $v = 10$m/s.

2.9 Differentiability of cars speed and position functions

Firstly to determine whether both speed and position functions are differentiable we need to plot them together and observe their behavior. As seen in Figure 9 both functions are smooth curves, therefore we can conclude that they are continuous and, thus, in this case differentiable.

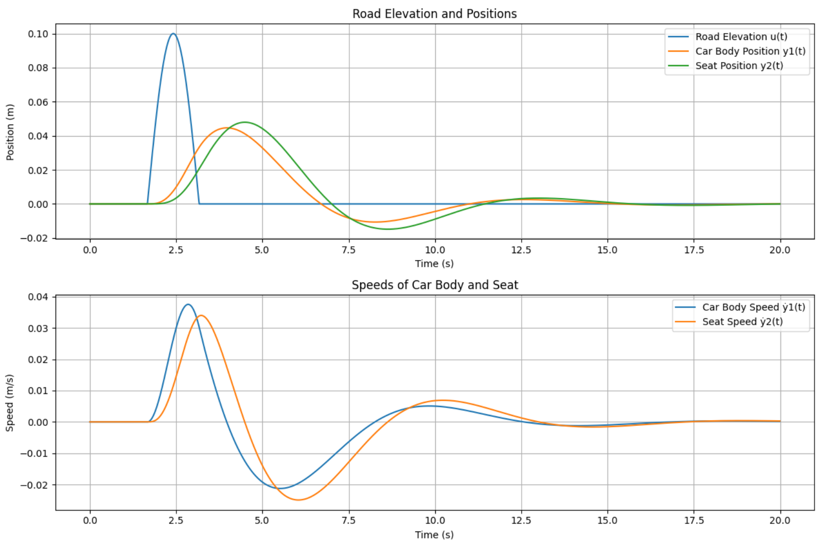


Figure 9: Car's speed and its systems position plots

2.10 Active Suspension Design

To adapt the suspension design to make it active we are introducing an additional control force $w(t)$ between the two masses m_1 and m_2. This control force acts in opposite directions on the two masses, effectively applying a push-pull force to control the motion of the system. With the addition of this control input $w(t)$, the state-space representation of the system becomes:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}w(t) \ \mathbf{y}(t) \qquad\qquad = \mathbf{C}\mathbf{x}(t)$$

where the state vector is now defined as $\mathbf{x} = [y_1, y_2, \dot{y}_1, \dot{y}_2]'$, containing the positions and velocities of the car body and seat. The input $u(t)$ representing the road elevation has been removed, and the control force $w(t)$ has been added as the new input. We are assuming the "French car" parameter values as in Assignment 3:

$$m_1 = 1400, \quad d_1 = 1000, \quad k_1 = 1000 \ m_2 \quad = 100, \quad d_2 = 50, \quad k_2 = 50$$

Now we need to show controllability of this with the new control input $w(t)$ system and to demonstrate that we need to check the rank of the controllability matrix $\mathbf{W}_c$, defined as:

$$\mathbf{W}_c = [\mathbf{B}; \mathbf{AB}; \mathbf{A}^2\mathbf{B}; \mathbf{A}^3\mathbf{B}]$$

If the rank of $\mathbf{W}_c$ is equal to the dimension of the state vector (4 in this case), then the system is controllable. Using a python script to check the matrix' rank we can verify that the system is indeed controllable as the rank matches the dimension of the state vector (see "active suspension.py") But before applying the active control, we should first simulate the uncontrolled state $\mathbf{x}(t)$ for $t \in [0, 20]$ with the initial condition $\mathbf{x}(0) = (0, -0.2, 0, 0)$ and $w(t) = 0$. The uncontrolled response should be similar to what was observed in Assignment 3, with the body and seat positions exhibiting oscillatory behavior due to the system dynamics. We will apply a state feedback control law of the form $w(t) = -\mathbf{F}\mathbf{x}(t)$, where $\mathbf{F}$ is the state feedback gain matrix. We will use the values $\mathbf{F} = [22; 62; 35; 108]$, which were designed using LQ control theory. See the simulation of the resulting closed-loop system for $t \in [0, 20]$ with the same initial condition $\mathbf{x}(0) = (0, -0.2, 0, 0)$ in Figure 10.

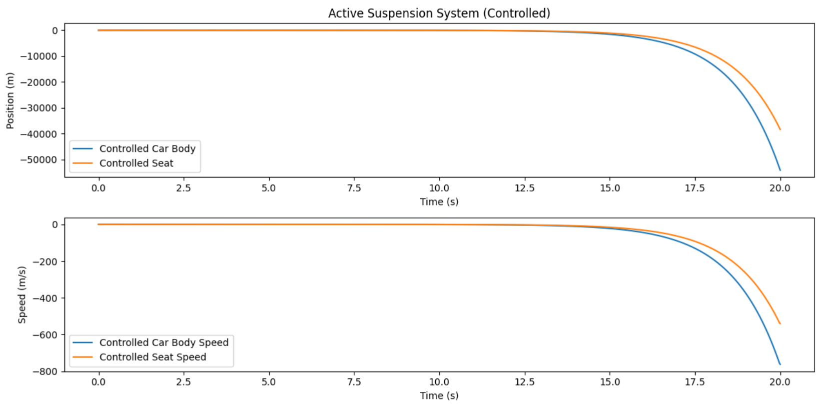


Figure 10: Car's speed and its systems position plots

3 Appendix A: Python code

For readability purposes each assignment has it's own python file. All the source files can be found here GItHub Repository. Files are named according to the assignment they were used in.

4 Appendix B: Variables

Symbol	Description	Unit
t	time	s
m_1	mass of car body	kg
m_2	mass of seat	kg
k_1	spring constant between road and car body	N/m
k_2	spring constant between car body and seat	N/m
d_1	damping coefficient between road and car body	Ns/m
d_2	damping coefficient between car body and seat	Ns/m
$u(t)$	road height function	m
$y_1(t)$	vertical position of car body	m
$y_2(t)$	vertical position of seat	m
s	horizontal distance between bumps	m
z	horizontal position of the car	m
v	horizontal speed of the car	m/s

Table 1: Variables used