03 Exercise Notebook 3

March 29, 2023

1 Exercise 3

In this exercise, you will analyse a dataset obtained from the London transport system (TfL). The data is in a filled called `tfl_readership.csv` (comma-separated-values format). As in Exercise 2, we will load and view the data using `pandas`.

[1]:
```
# If you are running this on Google Colab, uncomment and run the following
↪lines; otherwise ignore this cell
# from google.colab import drive
# drive.mount('/content/drive')
```

[2]:
```
import math
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
```

[3]:
```
# Load data
df_tfl = pd.read_csv('tfl_ridership.csv')
# If running on Google Colab change path to '/content/drive/MyDrive/
↪IB-Data-Science/Exercises/tfl_ridership.csv'

df_tfl.head(3)
```

[3]:
```
      Year Period       Start         End Days  Bus cash (000s)  \
0  2000/01   P 01  01 Apr '00  29 Apr '00  29d              884
1  2000/01   P 02  30 Apr '00  27 May '00  28d              949
2  2000/01   P 03  28 May '00  24 Jun '00  28d              945

   Bus Oyster PAYG (000s)  Bus Contactless (000s)  \
0                       0                       0
1                       0                       0
2                       0                       0

   Bus One Day Bus Pass (000s)  Bus Day Travelcard (000s)  …  \
0                          210                        231  …
1                          214                        205  …
2                          209                        221  …
```

```
   Tube Contactless (000s)  Tube Day Travelcard (000s)  \
0                        0                         655
1                        0                         605
2                        0                         650

   Tube Season Travelcard (000s)  Tube Other incl free (000s)  \
0                           1066                          200
1                           1168                          217
2                           1154                          212

   Tube Total (000s)  TfL Rail (000s)  Overground (000s)  DLR (000s)  \
0               2509                0                  0          96
1               2598                0                  0          93
2               2623                0                  0          98

   Tram (000s)  Air Line (000s)
0         45.8              0.0
1         46.5              0.0
2         47.1              0.0

[3 rows x 26 columns]
```

Each row of our data frame represents the average daily ridership over a 28/29 day period for various types of transport and tickets (bus, tube etc.). We have used the `.head()` command to display the top 13 rows of the data frame (corresponding to one year). Focusing on the "Tube Total" column, notice the dip in ridership in row 9 (presumably due to Christmas/New Year's), and also the slight dip during the summer (rows 4,5).

[4]:
```
#df_tfl.sample(3)  #random sample of 3 rows
df_tfl.tail(3)  #last 3 rows
```

[4]:
```
        Year Period       Start         End Days  Bus cash (000s)  \
242  2018/19   P 09  11 Nov '18  08 Dec '18  28d                0
243  2018/19   P 10  09 Dec '18  05 Jan '19  28d                0
244  2018/19   P 11  06 Jan '19  02 Feb '19  28d                0

     Bus Oyster PAYG (000s)  Bus Contactless (000s)  \
242                    1110                    1089
243                    1001                     949
244                    1036                    1075

     Bus One Day Bus Pass (000s)  Bus Day Travelcard (000s)  …  \
242                            0                         41  …
243                            0                         38  …
244                            0                         30  …

     Tube Contactless (000s)  Tube Day Travelcard (000s)  \
242                     1399                         249
```

```
243                         1110                              242
244                         1310                              204

     Tube Season Travelcard (000s)  Tube Other incl free (000s)  \
242                          1017                          334
243                           632                          259
244                           924                          305

     Tube Total (000s)  TfL Rail (000s)  Overground (000s)  DLR (000s)  \
242               4221              996                557         355
243               3279              750                414         270
244               3809              929                517         333

     Tram (000s)  Air Line (000s)
242         84.1              2.6
243         66.3              3.2
244         79.3              2.3

[3 rows x 26 columns]
```

The dataframe contains $N = 245$ counting periods (of 28/29 days each) from 1 April 2000 to 2 Feb 2019. We now define a numpy array consisting of the values in the ' Tube Total (000s)' column:

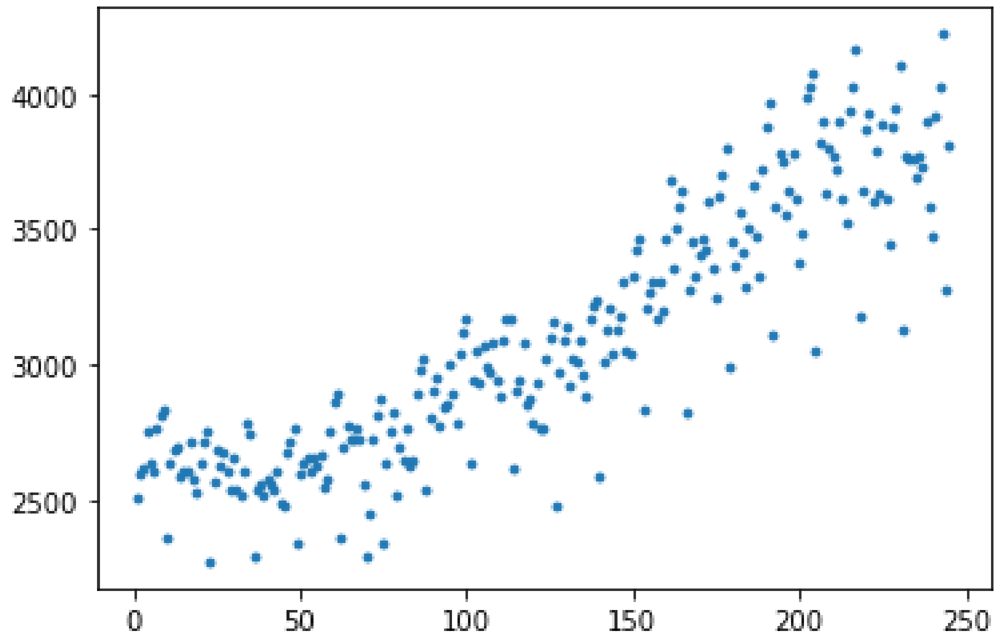
[5]:
```
yvals = np.array(df_tfl['Tube Total (000s)'])
N = np.size(yvals)
xvals = np.linspace(1,N,N) #an array containing the values 1,2....,N
```

We now have a time series consisting of points (x_i, y_i), for $i = 1, \ldots, N$, where y_i is the average daily tube rideship in counting period $x_i = i$.

1.1 2a) Plot the data in a scatterplot

[6]:
```
#Your code for scatterplot here

plt.scatter(xvals, yvals, s=8)
plt.rcParams['figure.figsize'] = [15, 8]
plt.rcParams['axes.titlesize'] = 20
plt.rcParams['axes.labelsize'] = 20
plt.rcParams['xtick.labelsize'] = 14
plt.rcParams['ytick.labelsize'] = 14
plt.show()
```



1.2 2b) Fit a linear model $f(x) = \beta_0 + \beta_1 x$ to the data

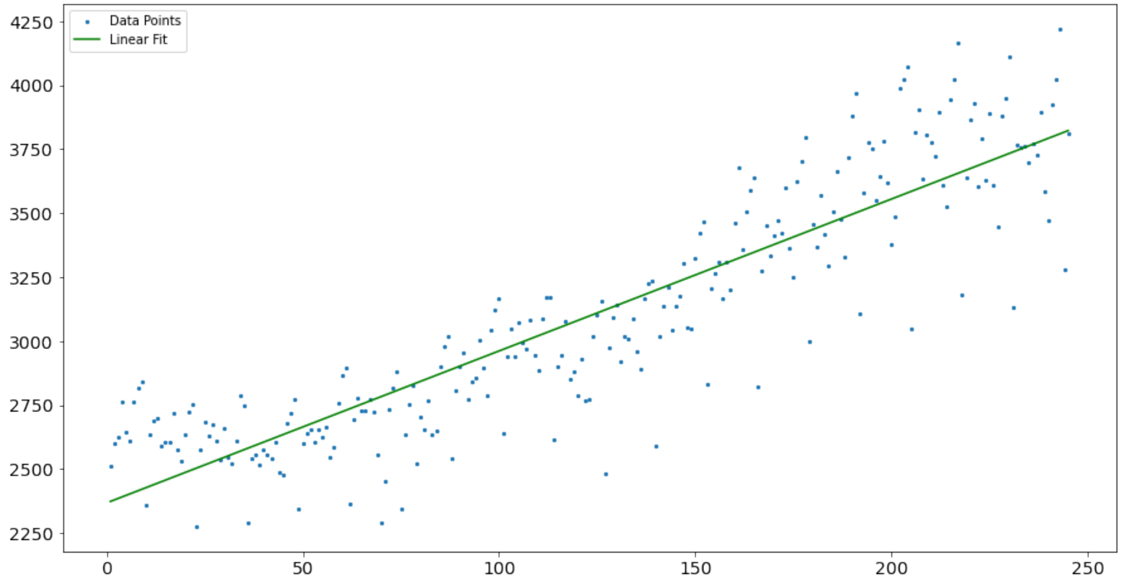
- Print the values of the regression coefficients β_0, β_1 determined using least-squares.
- Plot the fitted model and the scatterplot on the same plot.
- Compute and print the **MSE** and the R^2 coefficient for the fitted model.

All numerical outputs should be displayed to three decimal places.

[7]:
```
#Your code here
def polyreg(data_matrix: np.array, k: int)->np.array:
    '''
    The function returns the the coefficient vector beta, the fit X*beta, and␣
 ↪the vector of residuals y-X*beta
    '''
    N, _ = data_matrix.shape
    assert _ == 2
    t, y = [data_matrix[:,i] for i in range(2)]
    X = np.ones((N,k+1))
    for i in range(1,k+1):
        X[:,i]= t**i
    P = np.linalg.inv((X.T).dot(X))
    P = P.dot(X.T)
    beta = P.dot(y)
    fit = np.dot(X, beta)
    residual = y - fit
    return (beta,fit,residual)
```

```
xy_data= np.column_stack([xvals, yvals])
beta, fit, residual = polyreg(xy_data, 1)
MSE = residual.dot(residual) /N
R2 = 1- (MSE/ yvals.var())
print(f'Least squares coefficients for k=1 linear regression: y =␣
 ↪{round(beta[1],3)} x + {round(beta[0],3)}')
print(f'MSE1= {np.round(MSE,3)}, R2 = {np.round(R2,3)}')
plt.scatter(xvals, yvals, s=5, label='Data Points')
plt.plot(xvals, beta[0]+ beta[1]* xvals,'g',label='Linear Fit')
plt.legend()
plt.show()
```

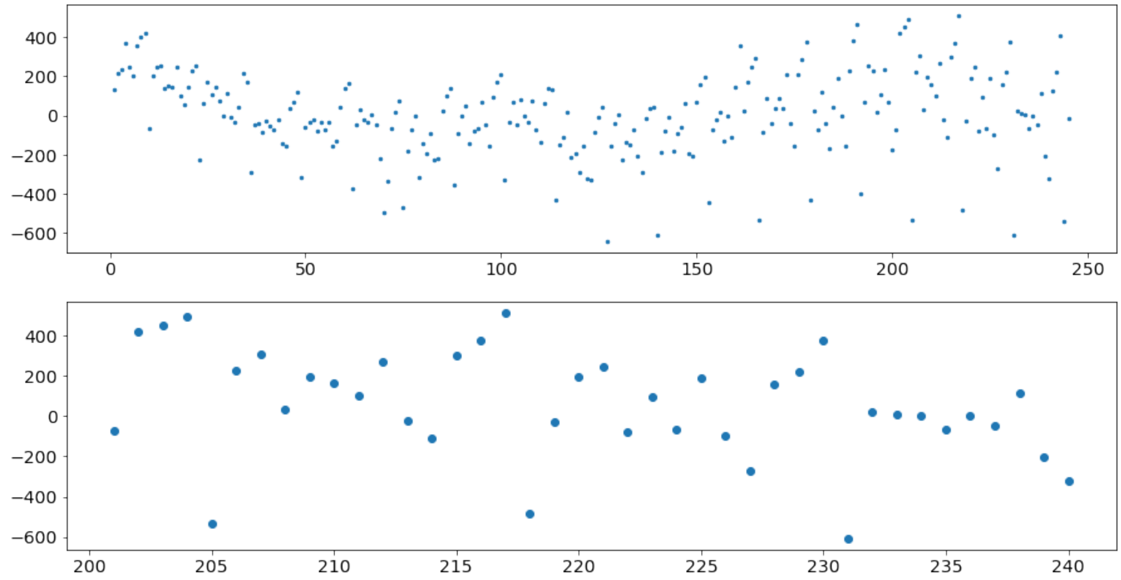
```
Least squares coefficients for k=1 linear regression: y = 5.939 x + 2367.382
MSE1= 45323.636, R2 = 0.796
```



1.3 2c) Plotting the residuals

- Plot the residuals on a scatterplot
- Also plot the residuals over a short duration and comment on whether you can discern any periodic components.

[8]:
```
# Your code here
plt.subplot(211)
plt.scatter(xvals, residual, s=6)
plt.subplot(212)
plt.scatter(xvals[200:240],residual[200:240])
plt.show()
```



Even though it is not a sinusoid, there is a repeating pattern whose period is about 13 time steps, which corresponds to about 364 days, so 1 year.

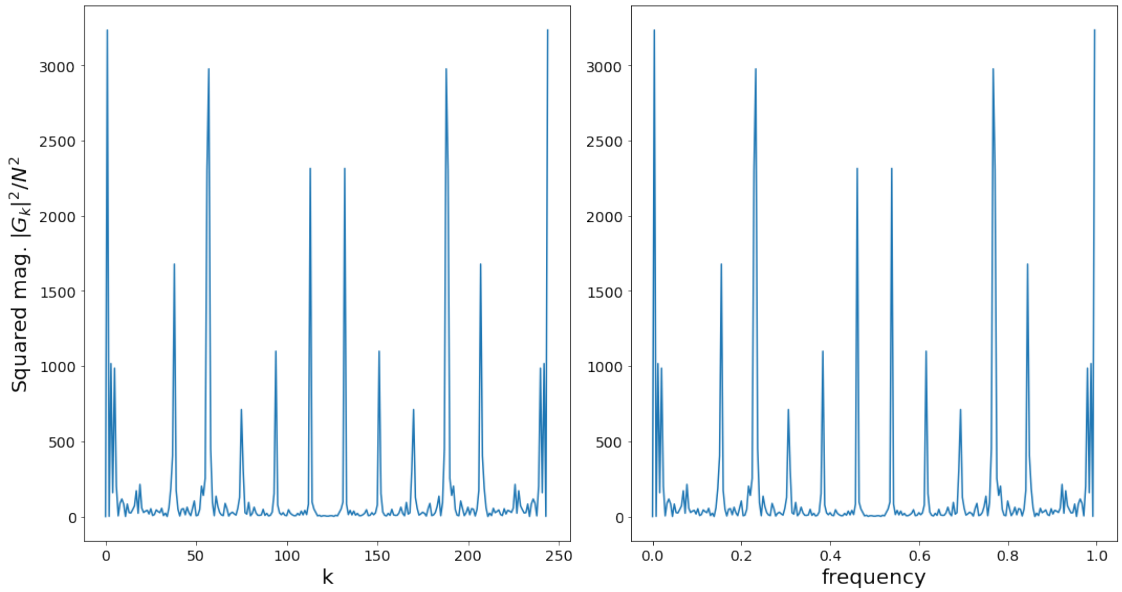
2d) Periodogram

- Compute and plot the peridogram of the residuals. (Recall that the periodogram is the squared-magnitude of the DFT coefficients.)
- Identify the indices/frequencies for which the periogram value exceeds **50%** of the maximum.

[9]:
```
# Your code to compute and plot the histogram
from numpy.fft import fft
df_tfl['days_parsed'] = df_tfl.Days.apply(lambda x: int(x[:-1]))
T = df_tfl.days_parsed.mean() /365.5 ##Get the periodicity raported to 1 year␣
↪instead of period count
residual_freq = fft(residual, N) / N
pdgram = np.abs(residual_freq)**2
indices = np.linspace(0, N-1, N)
freq = indices/ N
angular_freq = 2* np.pi* freq

plt.title('Periodogram of the linear fit residuals')
plt.subplot(121)
plt.plot(indices, pdgram)
plt.xlabel('k')
plt.ylabel('Squared mag. $|G_k|^2/N^2$')
plt.subplot(122)
plt.plot(freq, pdgram)
plt.xlabel('frequency')
plt.tight_layout()
```

```
plt.show()
```



[10]:
```
# Your code to identify the indices for which the periodogram value exceeds 50%␣
↪of the maximum
high = pdgram.max()
top_ind  = indices[(pdgram >= 0.5 * high)]
top_freq =    freq[(pdgram >= 0.5 * high)]
assert top_ind.sum()%N==0
#Get rid of conjugate pairs
top_ind=top_ind[:top_ind.size//2]
top_freq=top_freq[:top_freq.size//2]
```

[11]:
```
#Get temporal frequencies
time_top_freq = 1/T * top_freq
print(f'The most significant harmonic components correspond to the following␣
↪freqencies in yr^-1 :\n{time_top_freq}')
```

```
The most significant harmonic components correspond to the following freqencies
in yr^-1 :
[0.05310956 2.01816332 2.97413543 3.02724499 6.00138041]
```

1.4 2e) To the residuals, fit a model of the form

$$\beta_{1s}\sin(\omega_1 x) + \beta_{1c}\cos(\omega_1 x) + \beta_{2s}\sin(\omega_2 x) + \beta_{2c}\cos(\omega_2 x) + \ldots + \beta_{Ks}\sin(\omega_K x) + \beta_{Kc}\cos(\omega_K x).$$

The frequencies $\omega_1, \ldots, \omega_K$ in the model are those corresponding to the indices identified in Part 2c. (Hint: Each of the sines and cosines will correspond to one column in your X-matrix.)

- Print the values of the regression coefficients obtained using least-squares.

- Compute and print the final **MSE** and R^2 coefficient. Comment on the improvement over the linear fit.

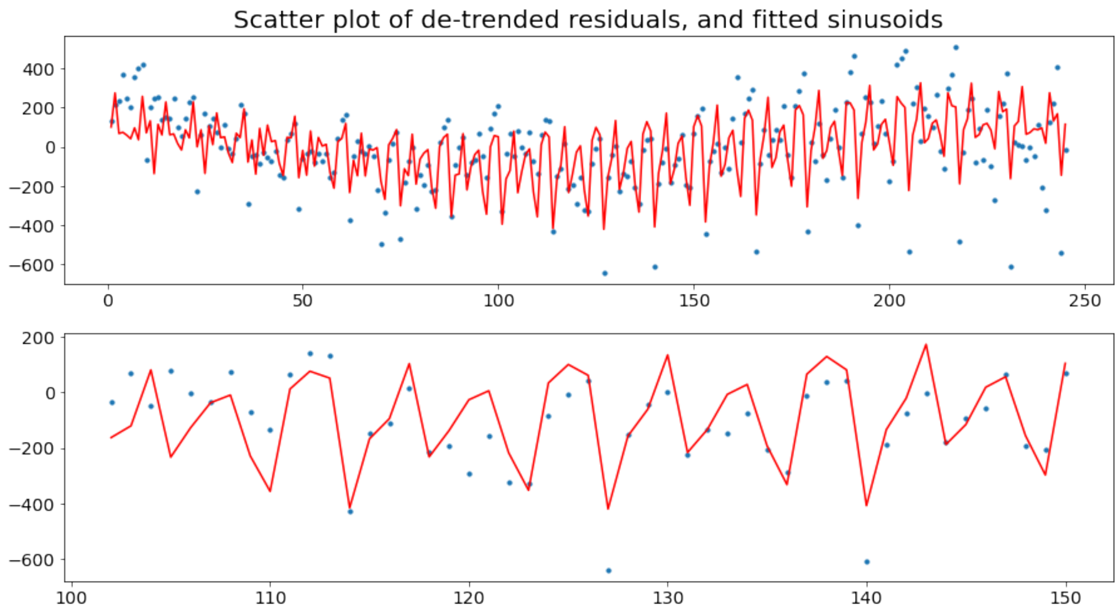
All numerical outputs should be displayed to three decimal places.

[12]:
```
def harmonic_reg(t: np.array, y: np.array, frequencies: np.array)->np.array:
    '''
    The function returns the the coefficient vector beta, the fit X*beta, and␣
↪the vector of residuals y-X*beta
    '''
    assert t.shape==y.shape
    N = t.size
    X = np.ones((N, 2* len(frequencies)))
    for i in range(len(frequencies)):
        X[:,2*i  ] = np.sin(2 *np.pi *frequencies[i] *t)
        X[:,2*i+1] = np.cos(2 *np.pi *frequencies[i] *t)
    P = np.linalg.inv((X.T).dot(X))
    P = P.dot(X.T)
    beta = P.dot(y)
    fit = np.dot(X, beta)
    residual = y - fit
    return (beta,fit,residual)

beta_sc, fit_sc, residual_sc = harmonic_reg(xvals, residual, top_freq)
MSE_sc = residual_sc.dot(residual_sc) / residual_sc.size
R2_sc = 1 - MSE_sc / residual.var()
print(f'Least squares coefficients for linear regression:{np.round(beta_sc,3)}')
print(f'MSE_sc= {np.round(MSE_sc,3)}, R2 = {np.round(R2_sc,3)}')

plt.subplot(211)
plt.scatter(xvals, residual, s=10)
plt.plot(xvals, fit_sc, 'r')
plt.title('Scatter plot of de-trended residuals, and fitted sinusoids')
# Zoom in to a few values
plt.subplot(212)
plt.scatter(xvals[101:150], residual[101:150], s=10)
plt.plot(xvals[101:150], fit_sc[101:150], 'r')
plt.show()
```

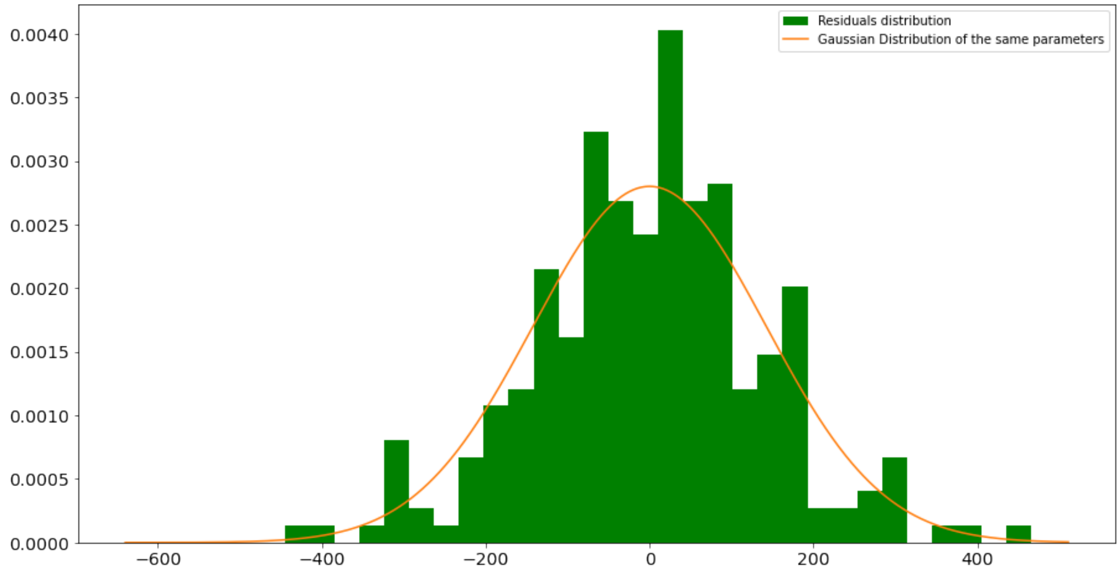
```
Least squares coefficients for linear regression:[-51.253 101.556  61.628
-54.006 -15.581 -94.797  81.659  72.381  32.472
  90.589]
MSE_sc= 20297.501, R2 = 0.552
```



The MSE is halved by adding the seasonality correction, and a small R suggests that we can improve the precision by including more terms.

[13]:
```
from scipy.stats import norm

n, bins, patches = plt.hist(residual_sc, bins=30, density=True,␣
 ↪facecolor='green',label='Residuals distribution')
x_pdf = np.linspace(residual.min(), residual.max(),1000)
y_pdf = norm.pdf(x_pdf,loc = residual_sc.mean(), scale = residual_sc.std() )
plt.plot(x_pdf,y_pdf,label='Gaussian Distribution of the same parameters')
plt.legend()
plt.show()
```



1.5 2f) The combined fit

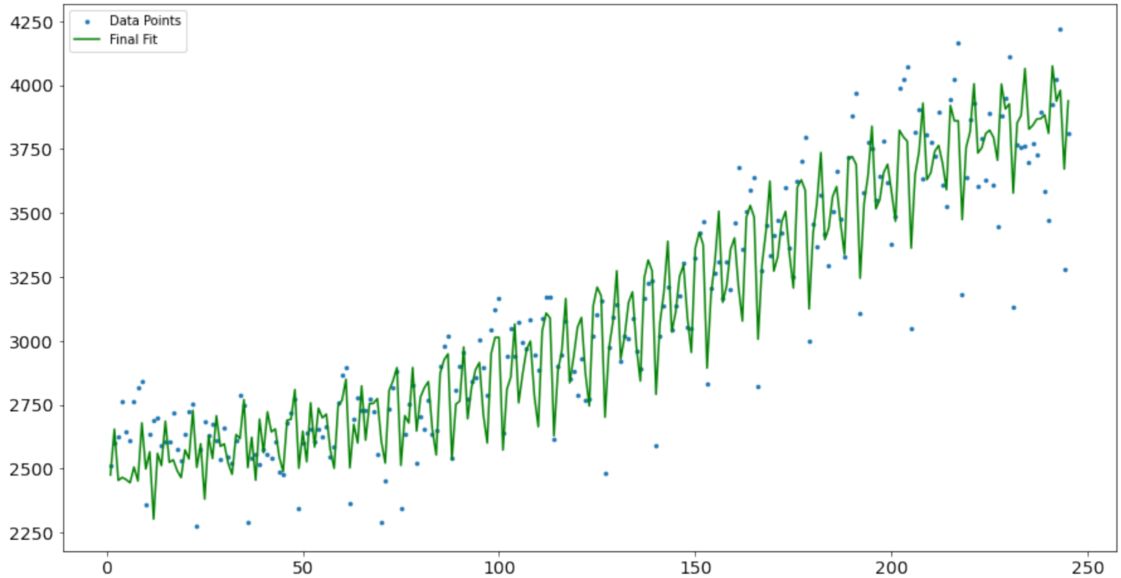
- Plot the combined fit together with a scatterplot of the data
- Compute and print the final **MSE** and R^2 coefficient. Comment on the improvement over the linear fit.

The combined fit, which corresponds to the full model

$$f(x) = \beta_0 + \beta_1 x + \beta_{s1}\sin(\omega_1 x) + \beta_{c1}\cos(\omega_1 x) + ... + \beta_{sk}\sin(\omega_k x) + \beta_{ck}\cos(\omega_k x),$$

can be obtained by adding the fits in parts 2b) and 2e).

[14]:
```
y_pred = fit + fit_sc
plt.scatter(xvals, yvals, s=7, label='Data Points')
plt.plot(xvals, y_pred,'g',label='Final Fit')
plt.legend()
plt.show()
```



[15]:
```
residual_total= yvals - y_pred
mean_total = residual_total.mean()
print(f'The mean of the new residuals is {round(mean_total,3)}')
MSE_final = residual_total.dot(residual_total) / N
R2_final  = 1- MSE_final/ yvals.var()
print('Mean squared error for the combined fit = ', np.round(MSE_final,3))
print('R^2 coefficient of combined fit = ', np.round(R2_final, 3))
```

```
The mean of the new residuals is 0.0
Mean squared error for the combined fit =  20297.501
R^2 coefficient of combined fit =  0.908
```

R^2 got improved from 0.8 to 0.9, while the MSE halved, so fitting sinusoidal components proved to model really well the seasonality of the data.