TÉCNICO LISBOA

Deadline: Please deliver your resolution (preferably hand-written, except computational part) at the theory class, October 31st

B. Malaca, M. Pardal, J. Vieira

Física e Tecnologia dos Plasmas

MEFT 2022/23 – **Homework problem**

Instructions: Read the assignment carefully. Answer **every** question **clearly** in the conditions specified. Carefully justify your answers and present in detail all the calculations made. **Team work to be completed in groups. Please identify resolution with name and student numbers of all group members.** All points have the same score. You are free to use (**without presenting any derivation**) all known dispersion relations and other results that appear in the class slides/textbooks.

Cherenkov radiation occurs when a charged particle travels faster than the phase velocity of electromagnetic waves in a given medium. In this problem, we will explore Cherenkov radiation emission in a plasma. **Consider that Cherenkov radiation is emitted in the particle direction of motion.**

(a) Cherenkov emission is forbidden in un-magnetised plasma. Justify.

(b) Consider a plasma with fixed ions in an uniform, static magnetic field. Which types of high frequency electromagnetic waves can be excited in this plasma?

(c) Assume now that the plasma is perturbed by a point-like charged particle traveling along the longitudinal x direction with velocity v_0. The magnetic field points along the y direction, and the amplitude is constant and equal to B_0. Under certain conditions, this particle can emit Cherenkov radiation. What class of waves in magnetised plasma does such Cherenkov emission belong too? Plot the corresponding dispersion relation $(\omega, c^2k^2/\omega^2)$ and indicate which branch of the dispersion relation can correspond to Cherenkov radiation. Justify.

(d) The radiated intensity by the charged particle, per unit frequency and per solid angle, far from the charged particle is given by:

$$\frac{d^2 I}{\mathrm{d}\omega \mathrm{d}\Omega} = \frac{e^2\omega^2}{16\pi^3\epsilon_0 c}\left|\int_0^{t_e} \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\exp\left[i\omega\left(t-\mathbf{n}\cdot r(t)/v(\omega)\right)\right]\mathrm{d}t\right|^2, \tag{1}$$

where $\mathbf{n}$ is the unit vector that points from the origin to the observer. Assume $\mathbf{n} = (n_x, n_y, n_z) = [\cos(\theta), \sin(\theta), 0]$, where θ is the angle of emission. In addition, $t = 0$ is the initial emission time, $t = t_e$ is the final emission time. Furthermore, Ω is the solid angle, $\boldsymbol{\beta} = (1/c)\mathrm{d}r(t)/\mathrm{d}t$, $r(t)$ is the particle trajectory, ω is the radiation frequency, and $v(\omega)$ is the phase velocity of the radiation. Consider that the phase velocity of e.m. waves is constant and given by $v(\omega) = v_n$. Show that the radiated intensity at the Cherenkov angle, given by $\cos(\theta) = v_n/v_0$, is given by:

$$\left(\frac{d^2 I}{\mathrm{d}\omega \mathrm{d}\Omega}\right)_{\text{Cherenkov}} = \frac{e^2\omega^2 \sin^2(\theta) v_0^2 t_e^2}{16\pi^3 \epsilon_0 c^3}, \tag{2}$$

i.e. that the radiated intensity grows quadratically with the propagation distance (or time). The scaling of the intensity with emission time cannot be higher than t_e^2 for a particle moving with constant velocity. Hence, the radiated intensity scales with $\simeq t_e^\alpha$, with $\alpha < 2$, at any other angle. *Note: It is this favourable scaling of the radiated intensity with emission time that leads to a strong and visible bluish tone colours when fast particles cross the water pools of nuclear reactors.*

(e) Consider a strongly magnetised plasma (i.e. assume that the upper hybrid frequency $\omega_{\mathrm{uh}} \gg \omega$) and emission along the x-axis. Show that the Cherenkov radiation frequency is given by:

$$\frac{\omega^2}{\omega_p^2} = \frac{1}{1 - \frac{\omega_{\mathrm{h}}^2}{\omega_p^2}\frac{1}{\gamma_0^2 - 1}}, \tag{3}$$

where $\gamma_0^2 = 1/(1 - v_0^2/c^2)$ and ω_h is the upper hybrid frequency. In addition, take the ultra-relativistic limit $\gamma_0^2 - 1 \gg \omega_{\mathrm{h}}^2/\omega_p^2$.

(f) This mechanism can be observed in gaseous plasmas, where the transition between the plasma and vacuum is smooth. In this case, radiation may cross an evanescent layer before reaching vacuum. Consider that the radiated frequency is equal to the plasma frequency in the region of maximum plasma density, ω_{p0}. Show that there is a cut-off layer as long as $\omega_{p0}^2 - \omega_{p0}|\omega_{ce}| < \omega_p(x)^2 < \omega_{p0}^2 - |\omega_{ce}|^2$, where $\omega_p(x)$ is the local plasma frequency at a given position x. Is there a condition for which no cut-off occurs? If yes, what is it?

(g) Assume that the cutoff layer does exist. This layer will attenuate the radiation. Taking this effect into consideration is important to compare theoretical predictions with experiments. In order to capture the attenuation effect, assume that the plasma frequency, during the plasma-vacuum transition, is given by $\omega_p^2 = \omega_{p0}^2(L - x)/L$, where L is the plasma layer length. Assume that the radiated frequency is $\omega = \omega_{p0}$. The attenuation factor for the transmitted electric field through the evanescent layer is $\Gamma = e^{-\int k_i \mathrm{d}x}$, where k_i is the imaginary part of the e.m. wavenumber, and the integral is took over the evanescent layer. Show that $\Gamma = e^{-\alpha L/(c\omega_{p0}^2)}$, where:

$$\alpha = \int_{\omega_{ce}^2 - \omega_{p0}\omega_{ce}}^{0} \sqrt{\frac{(\Theta - \omega_{ce}^2 - \omega_{p0}\omega_{ce})(\Theta - \omega_{ce}^2 + \omega_{p0}\omega_{ce})}{\Theta}}\mathrm{d}\Theta. \tag{4}$$

By plotting the integral above as a function of ω_{ce}/ω_p (numerically) show that the maximum attenuation is when $\omega_{ce}/\omega_p \simeq 0.65$. *Hint: Use a good variable transformation in your integration and mind the integration limits!*

(h) Demonstrate the existence of Cherenkov emission only in a magnetised plasma using ZPIC. Consider both a sharp interface and smooth plasma-vacuum transition. The following parameters can be used as a starting point for the exploration: use an ultra-relativistic ($\gamma_0 = 10^5$), ultra-short electron bunch (length $0.2c/\omega_p$) to mimic the effect of a point charge, with peak density $n_b/n_0 = 0.1$. Use the 1D version of the code. Use a simulation box with a length $100c/\omega_p$. Use $\omega_{ce}/\omega_p = 0.1$ (i.e. $B_z = 0.1$). Use plasma ramps with length $10 - 20c/\omega_p$. Produce figures of $E_x(x)$ and $E_y(x)$ across the plasma-vacuum boundary. What can you say about your answer to question a)?

Note: Use the notebooks under the folder `notebooks_basic` to design a new simulation that addresses the question above. These notebooks are the following:

- `density.ipynb`: Use this notebook to understand how to create charged particle beams with custom density profiles, and use it to initialise an ultrashort, spatially localised electron beam with a peak density lower than the background plasma. Use the `ufl` parameter to select initial momentum (for an ultra-relativistic electron bunch, $\gamma_0 \simeq \texttt{ufl} \gg 1$).
- `External Fields.ipynb`: Use this notebook to add an external magnetic field.
- `Direct.ipynb`: Use this notebook to plot E_x and E_y.