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COMPLEMENTO PARA
O PROFESSOR

FUNDAMENTOS DE

MATEMÁTICA 8 ELEMENTAR

LIMITES DERIVADAS
NOÇÕES DE INTEGRAL

ATUAL
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Apresentação

Este livro é o *Complemento para o Professor* do volume 8, Limites/Derivadas/Noções de integral, da coleção *Fundamentos de Matemática Elementar*.

Cada volume desta coleção tem um complemento para o professor, com o objetivo de apresentar a solução dos exercícios mais complicados do livro e sugerir sua passagem aos alunos.

É nossa intenção aperfeiçoar continuamente os *Complementos*. Estamos abertos às sugestões e críticas, que nos devem ser encaminhadas através da Editora.

Agradecemos à professora Irene Torrano Filisetti a colaboração na redação das soluções que são apresentadas neste *Complemento*.

Os Autores.

Sumário

Capítulo I — Funções

4. $\left.\begin{array}{l} f(x) = x - 1 \\ g(x) = 2x + 1 \end{array}\right\} \Rightarrow (g \circ f)(x) = g(f(x)) = 2(x - 1) + 1 = 2x - 1$

$x \in A = \{1, 2, 3\}$; temos:

$x = 1, (g \circ f)(x) = 2 \cdot 1 - 1 = 1$

$x = 2, (g \circ f)(x) = 2 \cdot 2 - 1 = 3$

$x = 3, (g \circ f)(x) = 2 \cdot 3 - 1 = 5$

Portanto: $\{(1, 1), (2, 3), (3, 5)\}$.

5. $f(x) = x^3$

$g(x) = x + 1$

$(g \circ f)(x) = g(f(x)) = x^3 + 1$

$(f \circ g)(x) = f(g(x)) = (x + 1)^3$

$(f \circ f)(x) = (x^3)^3 = x^9$

$(g \circ g)(x) = (x + 1) + 1 = x + 2$

6. $f(x) = x + 2$

$g(x) = x^2$

$h(x) = 2^x$

a) Obtemos, inicialmente, $g(f(x)) = (x + 2)^2$.

Então: $h[g(f(x))] = 2^{(x + 2)^2}$.

b) Obtemos, inicialmente, $g(h(x)) = (2^x)^2 = 2^{2x}$.

Então: $f[g(h(x))] = 2^{2x} + 2$.

7. a) $F(x) = |x^2 + 1|$

$x \xrightarrow[f]{} x^2 + 1 \xrightarrow[g]{} |x^2 + 1| \Rightarrow f(x) = x^2 + 1$ e $g(x) = |x|$

b) $F(x) = \text{sen}\,(x^2 + 4)$

$x \xrightarrow[f]{} x^2 + 4 \xrightarrow[g]{} \text{sen}\,(x^2 + 4) \Rightarrow f(x) = x^2 + 4$ e $g(x) = \text{sen}\,x$

c) $F(x) = \text{tg}\,x^3$

$x \xrightarrow[f]{} x^3 \xrightarrow[g]{} \text{tg}\,x^3 \Rightarrow f(x) = x^3$ e $g(x) = \text{tg}\,x$

d) $F(x) = \text{tg}^2\,x$

$x \xrightarrow[f]{} \text{tg}\,x \xrightarrow[g]{} \text{tg}^2\,x \Rightarrow f(x) = \text{tg}\,x$ e $g(x) = x^2$

e) $F(x) = 2^{\cos x}$

$x \xrightarrow[f]{} \cos x \xrightarrow[g]{} 2^{\cos x} \Rightarrow f(x) = \cos x$ e $g(x) = 2^x$

f) $F(x) = \text{sen } 3^x$

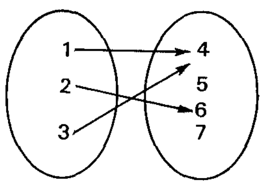
$x \xrightarrow[f]{} 3^x \xrightarrow[g]{} \text{sen } 3^x \Rightarrow f(x) = 3^x$ e $g(x) = \text{sen } x$

8. $F(x) = \cos 2^{x+3}$

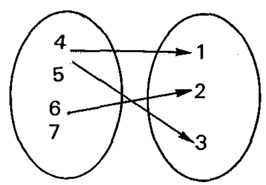
$x \xrightarrow[f]{} x + 3 \xrightarrow[g]{} 2^{x+3} \xrightarrow[h]{} \cos 2^{x+3} \Rightarrow f(x) = x + 3,\ g(x) = 2^x$ e $h(x) = \cos x$

9. a) $f^{-1}(x) = \{(a', a), (b', b), (c', c)\}$

b)



g é função



g^{-1} não é função

c) $h(x) = 1 - 5x \Leftrightarrow y = 1 - 5x$ é bijetora.

Trocando as variáveis *x* por *y* e isolando *y*, vem:

$x = 1 - 5y \Rightarrow y = \dfrac{1 - x}{5} \Rightarrow h^{-1}(x) = \dfrac{1 - x}{5}.$

d) $i(x) = x^3 - 2 \Leftrightarrow y = x^3 - 2$ é bijetora.

Trocando as variáveis *x* por *y* e isolando *y*, vem:

$x = y^3 - 2 \Rightarrow y = \sqrt[3]{x + 2} \Rightarrow i^{-1}(x) = \sqrt[3]{x + 2}.$

e) $j(x) : \mathbb{R}_- \to \mathbb{R}_+,\ j(x) = x^2 \Leftrightarrow y = x^2$ é bijetora.

Trocando as variáveis *x* por *y* e isolando *y*, vem:

$x = y^2 \Rightarrow y = \pm\sqrt{x}.$

Mas, como $j^{-1}(x) : \mathbb{R}_+ \to \mathbb{R}_-$, então $j^{-1}(x) = -\sqrt{x}.$

f) $p(x) = \dfrac{1}{x} \Leftrightarrow y = \dfrac{1}{x}$ é bijetora.

Trocando as variáveis *x* por *y*, *y* já fica isolado e, nesse caso, temos

$y = \dfrac{1}{x} \Rightarrow p(x) = p^{-1}(x).$

10. $f(x) = \begin{cases} x, & x \leqslant 1 \ \text{(I)} \\ \dfrac{x+1}{2}, & 1 < x \leqslant 3 \ \text{(II)} \\ x^2 - 7, & x > 3 \ \text{(III)} \end{cases}$

(I) $y = x,\ x \leqslant 1$

(II) $y = \dfrac{x+1}{2},\ 1 < x \leqslant 3 \Rightarrow \dfrac{1+1}{2} < \dfrac{x+1}{2} \leqslant \dfrac{3+1}{2} \Rightarrow 1 < y \leqslant 2$

Trocando as variáveis e isolando *y*, vem:

$x = \dfrac{y+1}{2} \Leftrightarrow y = 2x - 1,\ 1 < x \leqslant 2.$

(III) $y = x^2 - 7,\ x > 3 \Rightarrow x^2 > 9 \Rightarrow x^2 - 7 > 9 - 7 \Rightarrow y > 2$

Trocando as variáveis e isolando *y*, vem:

$x = y^2 - 7 \Leftrightarrow y = \sqrt{x + 7},\ x > 2.$

$f^{-1}(x) = \begin{cases} x, & \text{se } x \leqslant 1 \\ 2x - 1, & \text{se } 1 < x \leqslant 2 \\ \sqrt{x + 7}, & \text{se } x > 2 \end{cases}$

11. $f(x) = 2x - 3 \Rightarrow f^{-1}(x) = \dfrac{x + 3}{2}$

$g(x) = \sqrt[3]{x - 1} \Rightarrow g^{-1}(x) = x^3 + 1$

$(g^{-1} \circ f^{-1})(x) = g^{-1}(f^{-1}(x)) = \left(\dfrac{x + 3}{2}\right)^3 + 1 = \dfrac{x^3 + 9x^2 + 27x + 35}{8}$

12. $f : \mathbb{R}_+^* \to \mathbb{R},\ f(x) = \log\sqrt{x} \Leftrightarrow y = \log\sqrt{x}$

Trocando as variáveis e isolando *y*, vem:

$x = \log\sqrt{y} \Leftrightarrow 10^x = \sqrt{y} \Leftrightarrow 10^{2x} = y \Rightarrow f^{-1}(x) = 10^{2x}.$

13. $f : \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] \to [-1, 1],\ f(x) = \text{sen } \dfrac{x}{2} \Leftrightarrow y = \text{sen } \dfrac{x}{2}$

Trocando as variáveis e isolando *y*, vem:

$x = \text{sen } \dfrac{y}{2} \Leftrightarrow \dfrac{y}{2} = \text{arc sen } x \Leftrightarrow y = 2 \text{ arc sen } x$

$f^{-1}(x) = 2 \text{ arc sen } x$

Capítulo II — Limite

15. $|f(x) - 5| < 0{,}01 \Leftrightarrow |3x + 2 - 5| < 0{,}01 \Leftrightarrow$

$\Leftrightarrow |3x - 3| < 0{,}01 \Leftrightarrow 3 \cdot |x - 1| < 0{,}01 \Leftrightarrow |x - 1| < \dfrac{0{,}01}{3}$

Portanto: $0 < \delta \leqslant \dfrac{0{,}01}{3}.$

16. $|5 - 2x - 9| < 0{,}001 \Leftrightarrow |-2x - 4| < 0{,}001 \Leftrightarrow$

$\Leftrightarrow 2|-x - 2| < 0{,}001 \Leftrightarrow |-x - 2| < 0{,}0005$

Portanto: $0 < \delta \leqslant 0{,}0005.$

17. $f(x) = \dfrac{x^2 - 1}{x + 1} = \dfrac{(x + 1)(x - 1)}{x + 1} = x - 1$

$|x - 1 + 2| < 0{,}01 \Leftrightarrow |x + 1| < 0{,}01$

Portanto: $0 < \delta \leqslant 0{,}01$.

18. $f(x) = \dfrac{9x^2 - 4}{3x - 2} = \dfrac{(3x + 2)(3x - 2)}{3x - 2} = 3x + 2$

$|3x + 2 - 4| < 0{,}0001 \Leftrightarrow |3x - 2| < 0{,}0001 \Leftrightarrow$

$\Leftrightarrow 3 \cdot \left|x - \dfrac{2}{3}\right| < 0{,}0001 \Leftrightarrow \left|x - \dfrac{2}{3}\right| < \dfrac{0{,}0001}{3}$

Portanto: $0 < \delta \leqslant \dfrac{0{,}0001}{3}$.

20. a) $\lim\limits_{x \to 2} (4x - 1) = 7$

Devemos demonstrar que:

$\forall \epsilon > 0, \exists \delta > 0 \mid 0 < |x - 2| < \delta \Rightarrow |(4x - 1) - 7| < \epsilon$.

Temos:

$|(4x - 1) - 7| < \epsilon \Leftrightarrow |4x - 8| < \epsilon \Leftrightarrow 4|x - 2| < \epsilon \Leftrightarrow$

$\Leftrightarrow |x - 2| < \dfrac{\epsilon}{4}$.

Assim, sendo $\delta = \dfrac{\epsilon}{4}$, vem:

$\forall \epsilon > 0, \exists \delta = \dfrac{\epsilon}{4} \mid 0 < |x - 2| < \delta \Rightarrow |x - 2| < \dfrac{\epsilon}{4} \Rightarrow 4|x - 2| < \epsilon \Rightarrow$

$\Rightarrow |4x - 8| < \epsilon \Rightarrow |(4x - 1) - 7| < \epsilon$.

b) $\lim\limits_{x \to 3} (4 - 2x) = -2$

Devemos demonstrar que:

$\forall \epsilon > 0, \exists \delta > 0 \mid 0 < |x - 3| < \delta \Rightarrow |(4 - 2x) + 2| < \epsilon$.

Temos:

$|(4 - 2x) + 2| < \epsilon \Leftrightarrow |6 - 2x| < \epsilon \Leftrightarrow 2|3 - x| < \epsilon \Leftrightarrow$

$\Leftrightarrow 2|x - 3| < \epsilon \Leftrightarrow |x - 3| < \dfrac{\epsilon}{2}$.

Assim, sendo $\delta = \dfrac{\epsilon}{2}$, vem:

$\forall \epsilon > 0, \exists \delta = \dfrac{\epsilon}{2} \mid 0 < |x - 3| < \delta \Rightarrow |x - 3| < \dfrac{\epsilon}{2} \Rightarrow 2|x - 3| < \epsilon \Rightarrow$

$\Rightarrow 2|3 - x| < \epsilon \Rightarrow |6 - 2x| < \epsilon \Rightarrow |(4 - 2x) + 2| < \epsilon$.

c) $\lim\limits_{x \to -1} (3x - 2) = -5$

Devemos demonstrar que:

$\forall \epsilon > 0, \exists \delta > 0 \mid 0 < |x + 1| < \delta \Rightarrow |(3x - 2) + 5| < \epsilon$.

Temos:

$|(3x - 2) + 5| < \epsilon \Leftrightarrow |3x + 3| < \epsilon \Leftrightarrow 3|x + 1| < \epsilon \Leftrightarrow$

$\Leftrightarrow |x + 1| < \dfrac{\epsilon}{3}$.

Assim, sendo $\delta = \dfrac{\epsilon}{3}$, vem:

$\forall \epsilon > 0, \exists \delta = \dfrac{\epsilon}{3} \mid 0 < |x + 1| < \dfrac{\epsilon}{3} \Rightarrow 3|x + 1| < \epsilon \Rightarrow$

$\Rightarrow |3x + 3| < \epsilon \Rightarrow |(3x - 2) + 5| < \epsilon$.

22. a) $\lim\limits_{x \to 2} x^2 = 4$

Devemos provar que:

$\forall \epsilon > 0, \exists \delta > 0 \mid 0 < |x - 2| < \delta \Rightarrow |x^2 - 4| < \epsilon$.

Temos:

$|x^2 - 4| < \epsilon \Rightarrow -\epsilon < x^2 - 4 < \epsilon \Rightarrow 4 - \epsilon < x^2 < 4 + \epsilon$.

Suponhamos que o valor de δ que queremos encontrar seja menor ou igual a *2*, isto é:

$0 < |x - 2| < \delta \leqslant 2 \Rightarrow |x - 2| < 2 \Rightarrow -2 < x - 2 < 2 \Rightarrow 0 < x < 4$.

Sendo $\epsilon' > 0$ tal que $\begin{cases} \epsilon' = \epsilon, & \text{se } 0 < \epsilon < 4 \\ 0 < \epsilon' < 4, & \text{se } \epsilon \geqslant 4 \end{cases}$ temos:

$4 - \epsilon \leqslant 4 - \epsilon' < x^2 < 4 + \epsilon' \leqslant 4 + \epsilon \Rightarrow 0 < 4 - \epsilon' < x^2 < 4 + \epsilon' \Rightarrow$

$\Rightarrow \sqrt{4 - \epsilon'} < |x| < \sqrt{4 + \epsilon'} \Rightarrow \sqrt{4 - \epsilon'} < x < \sqrt{4 + \epsilon'} \Rightarrow$

$\Rightarrow \sqrt{4 - \epsilon'} - 2 < x - 2 < \sqrt{4 + \epsilon'} - 2 \Rightarrow \begin{cases} |x - 2| < \sqrt{4 + \epsilon'} - 2 \\ \text{e} \\ |x - 2| < 2 - \sqrt{4 - \epsilon'} \end{cases}$

Notando que:

$0 < \sqrt{4 + \epsilon'} - 2 < 2 - \sqrt{4 - \epsilon'} < 2$, temos:

para todo $\epsilon > 0$, $\exists \delta = \sqrt{4 + \epsilon'} - 2 > 0$, em que $\begin{cases} \epsilon' = \epsilon, \text{ se } 0 < \epsilon < 4 \\ 0 < \epsilon' < 4, \text{ se } \epsilon \geqslant 4 \end{cases}$

tal que $0 < |x - 2| < \delta \Rightarrow |x^2 - 4| < \epsilon$.

De fato:

$0 < |x - 2| < \delta \Rightarrow |x - 2| < \sqrt{4 + \epsilon'} - 2 \Rightarrow$

$\Rightarrow \sqrt{4 - \epsilon'} - 2 < 2 - \sqrt{4 + \epsilon'} < x - 2 < \sqrt{4 + \epsilon'} - 2 \Rightarrow$

$\Rightarrow \sqrt{4 - \epsilon'} - 2 < x - 2 < \sqrt{4 + \epsilon'} - 2 \Rightarrow$

$\Rightarrow \sqrt{4 - \epsilon'} < x < \sqrt{4 + \epsilon'} \Rightarrow 4 - \epsilon' < x^2 < 4 + \epsilon' \Rightarrow$

$\Rightarrow -\epsilon' < x^2 - 4 < \epsilon' \Rightarrow |x^2 - 4| < \epsilon' \leqslant \epsilon$.

b) $\lim\limits_{x \to -3} (x^2 + 1) = 10$

Devemos provar que:

$\forall \epsilon > 0, \exists \delta > 0 \mid 0 < |x + 3| < \delta \Rightarrow |(x^2 + 1) - 10| < \epsilon$.

Notemos que:

$|(x^2+1)-10| < \epsilon \Rightarrow |x^2-9| < \epsilon \Rightarrow -\epsilon < x^2-9 < \epsilon \Rightarrow$
$\Rightarrow 9-\epsilon < x^2 < 9+\epsilon.$

Suponhamos que o valor de δ que queremos encontrar seja menor ou igual a 3, isto é:

$0 < |x+3| < \delta < 3 \Rightarrow |x+3| < 3 \Rightarrow -3 < x+3 < 3 \Rightarrow -6 < x < 0.$

Sendo $\epsilon' > 0$ tal que $\epsilon' = \epsilon$ se $0 < \epsilon < 9$ ou $0 < \epsilon' < 9$ se $\epsilon \geqslant 9$, temos:

$9-\epsilon \leqslant 9-\epsilon' < x^2 < 9+\epsilon' \leqslant 9+\epsilon \Rightarrow 0 < 9-\epsilon' < x^2 < 9+\epsilon' \Rightarrow$
$\Rightarrow \sqrt{9-\epsilon'} < |x| < \sqrt{9+\epsilon'} \Rightarrow -\sqrt{9+\epsilon'} < x < -\sqrt{9-\epsilon'} \Rightarrow$

$$\Rightarrow 3-\sqrt{9+\epsilon'} < x+3 < 3-\sqrt{9-\epsilon'} \Rightarrow \begin{cases} |x+3| < 3-\sqrt{9-\epsilon'} \\ |x+3| < \sqrt{9+\epsilon'}-3 \end{cases}$$

Notando que

$0 < \sqrt{9+\epsilon'}-3 < 3-\sqrt{9-\epsilon'} < 3$, decorre que, para todo $\epsilon > 0$, existe $\delta = \sqrt{9+\epsilon'}-3$ em que $\epsilon' = \epsilon$ se $0 < \epsilon < 9$ ou $0 < \epsilon' < 9$ se $\epsilon \geqslant 9$, tal que:

$0 < |x+3| < \delta \Rightarrow |x+3| < \sqrt{9+\epsilon'}-3 \Rightarrow$
$\Rightarrow 3-\sqrt{9+\epsilon'} < x+3 < \sqrt{9+\epsilon'}-3 < 3-\sqrt{9-\epsilon'} \Rightarrow$
$\Rightarrow 3-\sqrt{9+\epsilon'} < x+3 < 3-\sqrt{9-\epsilon'} \Rightarrow -\sqrt{9+\epsilon'} < x < -\sqrt{9-\epsilon'} \Rightarrow$
$\Rightarrow 9-\epsilon' < x^2 < 9+\epsilon' \Rightarrow -\epsilon' < x^2+9 < \epsilon' \Rightarrow |x^2+9| < \epsilon' < \epsilon.$

c) $\lim\limits_{x\to 2}(1-x^2) = -3$

Devemos provar que:

$\forall \epsilon > 0, \exists \delta > 0 \mid 0 < |x-2| < \delta \Rightarrow |(1-x^2)+3| < \epsilon.$

Notemos que:

$|(1-x^2)+3| < \epsilon \Leftrightarrow |4-x^2| < \epsilon \Leftrightarrow |x^2-4| < \epsilon \Rightarrow$
$\Rightarrow -\epsilon < x^2-4 < \epsilon \Rightarrow 4-\epsilon < x^2 < 4+\epsilon.$

Sendo $\epsilon' > 0$, tal que se $0 < \epsilon < 4$, então $\epsilon' = \epsilon$ ou se $\epsilon \geqslant 4$ então $0 < \epsilon' < 4$, temos:

$4-\epsilon \leqslant 4-\epsilon' < x^2 < 4+\epsilon' \leqslant 4+\epsilon \Rightarrow 0 < 4-\epsilon' < x^2 < 4+\epsilon' \Rightarrow$
$\Rightarrow \sqrt{4-\epsilon'} < |x| < \sqrt{4+\epsilon'} \Rightarrow \sqrt{4-\epsilon'} < x < \sqrt{4+\epsilon'} \Rightarrow$

$$\Rightarrow \sqrt{4-\epsilon'}-2 < x-2 < \sqrt{4+\epsilon'}-2 \Rightarrow \begin{cases} |x-2| < \sqrt{4+\epsilon'}-2 \\ \text{e} \\ |x-2| < 2-\sqrt{4-\epsilon'} \end{cases}$$

Notando que $0 < \sqrt{4+\epsilon'}-2 < 2-\sqrt{4-\epsilon'}$, vem:
para todo $\epsilon > 0$, $\exists\delta = \sqrt{4+\epsilon'}-2$, em que $\epsilon' = \epsilon$ se $0 < \epsilon < 4$ ou $0 < \epsilon' < 4$ se $\epsilon \geqslant 4$, tal que $0 < |x-2| < \delta \Rightarrow |x^2-4| < \epsilon \Leftrightarrow$
$\Leftrightarrow |4-x^2| < \epsilon.$

De fato:

$0 < |x-2| < \delta \Rightarrow |x-2| < \sqrt{4+\epsilon'}-2 \Rightarrow$
$\Rightarrow \sqrt{4-\epsilon'}-2 < 2-\sqrt{4+\epsilon'} < x-2 < \sqrt{4+\epsilon'}-2 \Rightarrow$
$\Rightarrow \sqrt{4-\epsilon'}-2 < x-2 < \sqrt{4+\epsilon'}-2 \Rightarrow$
$\Rightarrow \sqrt{4-\epsilon'} < x < \sqrt{4+\epsilon'} \Rightarrow 4-\epsilon' < x^2 < 4+\epsilon' \Rightarrow$
$\Rightarrow -\epsilon' < x^2-4 < \epsilon' \Rightarrow |x^2-4| < \epsilon' \Leftrightarrow |4-x^2| < \epsilon' \leqslant \epsilon.$

24. $\lim\limits_{x\to 1}\dfrac{6}{x+2} = 2$

Devemos provar que:

$\forall \epsilon > 0, \exists \delta > 0 \mid 0 < |x-1| < \delta \Rightarrow \left|\dfrac{6}{x+2}-2\right| < \epsilon.$

Notemos que:

$\left|\dfrac{6}{x+2}-2\right| < \epsilon \Rightarrow -\epsilon < \dfrac{6}{x+2}-2 < \epsilon \Rightarrow 2-\epsilon < \dfrac{6}{x+2} < 2+\epsilon.$

Considerando $\epsilon' > 0$, tal que $\epsilon' = \epsilon$ se $0 < \epsilon < 2$ ou $0 < \epsilon' < 2$ se $\epsilon \geqslant 2$, temos:

$2-\epsilon \leqslant 2-\epsilon' < \dfrac{6}{x+2} < 2+\epsilon' \leqslant 2+\epsilon \Rightarrow$

$\Rightarrow 0 < 2-\epsilon' < \dfrac{6}{x+2} < 2+\epsilon' \Rightarrow \dfrac{1}{2-\epsilon'} > \dfrac{x+2}{6} > \dfrac{1}{2+\epsilon'} \Rightarrow$

$\Rightarrow \dfrac{6}{2-\epsilon'} > x+2 > \dfrac{6}{2+\epsilon'} \Rightarrow \dfrac{6}{2-\epsilon'}-3 > x-1 > \dfrac{6}{2+\epsilon'}-3 \Rightarrow$

$$\Rightarrow \frac{3\epsilon'}{2-\epsilon'} > x-1 > \frac{-3\epsilon'}{2+\epsilon'} \Rightarrow \begin{cases} |x-1| < \dfrac{3\epsilon'}{2-\epsilon'} \\ \text{e} \\ |x-1| < \dfrac{3\epsilon'}{2+\epsilon'} \end{cases}$$

Notando que $0 < \dfrac{3\epsilon'}{2+\epsilon'} < \dfrac{3\epsilon'}{2-\epsilon'}$ para todo $\epsilon > 0$, $\exists\delta = \dfrac{3\epsilon'}{2+\epsilon'} > 0$, em que $\epsilon' = \epsilon$, se $0 < \epsilon < 3$ ou $0 < \epsilon' < 3$, se $\epsilon \geqslant 3$, tal que

$0 < |x-1| < \delta \Rightarrow \left|\dfrac{6}{x+2}-2\right| < \epsilon.$

De fato:

$0 < |x-1| < \delta \Rightarrow |x-1| < \dfrac{3\epsilon'}{2+\epsilon'} \Rightarrow$

$\Rightarrow \dfrac{-3\epsilon'}{2+\epsilon'} < x-1 < \dfrac{3\epsilon'}{2+\epsilon'} < \dfrac{3\epsilon'}{2-\epsilon'} \Rightarrow$

$\Rightarrow \dfrac{-3\epsilon'}{2+\epsilon'} < x-1 < \dfrac{3\epsilon'}{2-\epsilon'} \Rightarrow \dfrac{6}{2+\epsilon'}-3 < x+2 < \dfrac{6}{2-\epsilon'}-3 \Rightarrow$

$\Rightarrow \dfrac{6}{2+\epsilon'} < x+2 < \dfrac{6}{2-\epsilon'} \Rightarrow \dfrac{1}{2+\epsilon'} < \dfrac{x+2}{6} < \dfrac{1}{2-\epsilon'} \Rightarrow$

$\Rightarrow 2-\epsilon' < \dfrac{6}{x+2} < 2+\epsilon' \Rightarrow -\epsilon' < \dfrac{6}{x+2}-2 < \epsilon' \Rightarrow$

$\Rightarrow \left|\dfrac{6}{x+2}-2\right| < \epsilon' \leqslant \epsilon.$

25. $\lim\limits_{x\to 2}\dfrac{x+2}{x-1} = 4$

Devemos provar que:

$\forall \epsilon > 0, \exists \delta > 0 \mid 0 < |x-2| < \delta \Rightarrow \left|\dfrac{x+2}{x-1}-4\right| < \epsilon.$

Notemos que:

$$\left|\frac{x+2}{x-1}-4\right|<\epsilon \Rightarrow -\epsilon<\frac{x+2}{x-1}-4<\epsilon \Rightarrow 4-\epsilon<\frac{x+2}{x-1}<4+\epsilon.$$

Considerando $\epsilon'>0$, tal que $\epsilon'=\epsilon$ se $0<\epsilon<4$ ou $0<\epsilon'<4$ se $\epsilon\geqslant 4$,

temos: $4-\epsilon \leqslant 4-\epsilon' < \dfrac{x+2}{x-1} < 4+\epsilon' \leqslant 4+\epsilon \Rightarrow$

$$\Rightarrow 0<4-\epsilon'<\frac{x+2}{x-1}<4+\epsilon'.$$

Considerando ainda que:

$\dfrac{x+2}{x-1}=\dfrac{x-1+3}{x-1}=\dfrac{x-1}{x-1}+\dfrac{3}{x-1}=1+\dfrac{3}{x-1}$, vem:

$$0<4-\epsilon'<\frac{3}{x-1}+1<4+\epsilon' \Rightarrow 3-\epsilon'<\frac{3}{x-1}<3+\epsilon'\Rightarrow$$

$$\Rightarrow \frac{1}{3-\epsilon'}>\frac{x-1}{3}>\frac{1}{3+\epsilon'}\Rightarrow \frac{3}{3-\epsilon'}>x-1>\frac{3}{3+\epsilon'}\Rightarrow$$

$$\Rightarrow \frac{3}{3-\epsilon'}-1>x-2>\frac{3}{3+\epsilon'}-1\Rightarrow \frac{\epsilon'}{3-\epsilon'}>x-2>\frac{-\epsilon'}{3+\epsilon'}\Rightarrow$$

$$\Rightarrow \begin{cases}|x-2|<\dfrac{\epsilon'}{3-\epsilon'}\\ \text{e} \\ |x-2|<\dfrac{\epsilon'}{3+\epsilon'}\end{cases}$$

Notando que $0<\dfrac{\epsilon'}{3+\epsilon'}<\dfrac{\epsilon'}{3-\epsilon'}$, temos que para todo $\epsilon>0$,

$\exists\delta=\dfrac{\epsilon'}{3+\epsilon'}>0$ em que $\epsilon'=\epsilon$, se $0<\epsilon<4$ ou $0<\epsilon'<4$, se $\epsilon\geqslant4$,

tal que: $0<|x-2|<\delta \Rightarrow \left|\dfrac{x+2}{x-1}-4\right|<\epsilon$.

De fato:

$$0<|x-2|<\delta\Rightarrow |x-2|<\frac{\epsilon'}{3+\epsilon'}\Rightarrow$$

$$\Rightarrow \frac{-\epsilon'}{3+\epsilon'}<x-2<\frac{\epsilon'}{3+\epsilon'}<\frac{\epsilon'}{3-\epsilon'}\Rightarrow$$

$$\Rightarrow \frac{-\epsilon'}{3+\epsilon'}<x-2<\frac{\epsilon'}{3-\epsilon'}\Rightarrow \frac{3}{3+\epsilon'}-1<x-2<\frac{3}{3-\epsilon'}-1\Rightarrow$$

$$\Rightarrow \frac{3}{3+\epsilon'}<x-1<\frac{3}{3-\epsilon'}\Rightarrow\frac{1}{3+\epsilon'}<\frac{x-1}{3}<\frac{1}{3-\epsilon'}\Rightarrow$$

$$3-\epsilon'<\frac{3}{x-1}<3+\epsilon'\Rightarrow 4-\epsilon'<\frac{3}{x-1}+1<4+\epsilon'\Rightarrow$$

$$4-\epsilon'<\frac{x+2}{x-1}<4+\epsilon'\Rightarrow -\epsilon'<\frac{x+2}{x-1}-4<\epsilon'\Rightarrow$$

$$\Rightarrow \left|\frac{x+2}{x-1}-4\right|<\epsilon'\leqslant\epsilon.$$

26. $\lim\limits_{x\to1} x^3 = 1$

Devemos provar que:

$\forall\epsilon>0, \exists\delta>0 \mid 0<|x-1|<\epsilon \Rightarrow |x^3-1|<\epsilon$.

Notemos que $|x^3-1|<\epsilon\Rightarrow -\epsilon<x^3-1<\epsilon\Rightarrow 1-\epsilon<x^3<1+\epsilon$.

Suponhamos que $\delta\leqslant 1$:

$0<|x-1|<\delta\leqslant1\Rightarrow |x-1|<1\Rightarrow -1<x-1<1\Rightarrow$

$\Rightarrow 0<x<2$ e, sendo $\epsilon'>0$ tal que se $0<\epsilon<1$, $\epsilon'=\epsilon$ ou se $\epsilon\geqslant1$, então $0<\epsilon'<1$, temos:

$1-\epsilon\leqslant 1-\epsilon'<x^3<1+\epsilon'\leqslant1+\epsilon\Rightarrow 0<1-\epsilon'<x^3<1+\epsilon'\Rightarrow$

$\Rightarrow \sqrt[3]{1-\epsilon'}<x<\sqrt[3]{1+\epsilon'}\Rightarrow$

$$\Rightarrow \sqrt[3]{1-\epsilon'}-1<x-1<\sqrt[3]{1+\epsilon'}-1\Rightarrow \begin{cases}|x-1|<\sqrt[3]{1+\epsilon'}-1\\ \text{e}\\ |x-1|<1-\sqrt[3]{1-\epsilon'}\end{cases}$$

Notando que $0<1-\sqrt[3]{1-\epsilon'}<\sqrt[3]{1+\epsilon'}-1<1$, temos que para todo $\epsilon>0$, $\exists\delta=1-\sqrt[3]{1-\epsilon'}$, em que $\epsilon'=\epsilon$, se $0<\epsilon<1$ ou $0<\epsilon'<1$ se $\epsilon\geqslant1$, tal que $0<|x-1|<\delta\Rightarrow|x^3-1|<\epsilon$.

De fato:

$0<|x-1|<\delta\Rightarrow|x-1|<1-\sqrt[3]{1-\epsilon'}\Rightarrow$

$\Rightarrow\sqrt[3]{1-\epsilon'}-1<x-1<1-\sqrt[3]{1-\epsilon'}<\sqrt[3]{1+\epsilon'}-1\Rightarrow$

$\Rightarrow \sqrt[3]{1-\epsilon'}-1<x-1<\sqrt[3]{1+\epsilon'}-1\Rightarrow$

$\Rightarrow\sqrt[3]{1-\epsilon'}<x<\sqrt[3]{1+\epsilon'}\Rightarrow 1-\epsilon'<x^3<1+\epsilon'\Rightarrow$

$\Rightarrow -\epsilon'<x^3-1<\epsilon'\Rightarrow |x^3-1|<\epsilon'\leqslant\epsilon.$

28. a) $\lim\limits_{x\to1}(4x^2-7x+5)\overset{(T)}{=}4\cdot1^2-7\cdot1+5=2$

b) $\lim\limits_{x\to-1}(x^3-2x^2-4x+3)\overset{(T)}{=}(-1)^3-2\cdot(-1)^2-4\cdot(-1)+3=4$

c) $\lim\limits_{x\to2}\dfrac{3x+2}{x^2-6x+5}\overset{(L_7)}{=}\dfrac{\lim\limits_{x\to2}(3x+2)}{\lim\limits_{x\to2}(x^2-6x+5)}\overset{(T)}{=}\dfrac{3\cdot2+2}{2^2-6\cdot2+5}=\dfrac{8}{-3}=\dfrac{-8}{3}$

d) $\lim\limits_{x\to-1}\dfrac{3x^2-5x+4}{2x+1}\overset{(L_7)}{=}\dfrac{\lim\limits_{x\to-1}(3x^2-5x+4)}{\lim\limits_{x\to-1}(2x+1)}\overset{(T)}{=}\dfrac{12}{-1}=-12$

e) $\lim\limits_{x\to-3}\dfrac{x^2+2x-3}{5-3x}\overset{(L_7)}{=}\dfrac{\lim\limits_{x\to-3}(x^2+2x-3)}{\lim\limits_{x\to-3}(5-3x)}=\dfrac{0}{14}=0$

f) $\lim\limits_{x\to2}\left(\dfrac{3x^2-2x-5}{-x^2+3x+4}\right)^3\overset{(L_6)}{=}\left[\lim\limits_{x\to2}\left(\dfrac{3x^2-2x-5}{-x^2+3x+4}\right)\right]^3\overset{(L_7)}{=}$

$$= \left[\frac{\lim\limits_{x\to 2}(3x^2 - 2x - 5)}{\lim\limits_{x\to 2}(-x^2 + 3x + 4)}\right]^3 \overset{(T)}{=} \left[\frac{3}{6}\right]^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

g) $$\lim_{x\to 4}\left(\frac{x^3 - 3x^2 - 2x - 5}{2x^2 - 9x + 2}\right)^2 \underset{(L_7)}{\overset{(L_6)}{=}} \left[\frac{\lim\limits_{x\to 4}(x^3 - 3x^2 - 2x - 5)}{\lim\limits_{x\to 4}(2x^2 - 9x + 2)}\right]^2 \overset{(T)}{=}$$

$$= \left[\frac{64 - 48 - 8 - 5}{32 - 36 + 2}\right]^2 = \left[\frac{3}{-2}\right]^2 = \left[-\frac{3}{2}\right]^2 = \frac{9}{4}$$

h) $$\lim_{x\to -1}\sqrt{\frac{2x^2 + 3x - 4}{5x - 4}} \overset{(L_8)}{=} \sqrt{\lim_{x\to -1}\left(\frac{2x^2 + 3x - 4}{5x - 4}\right)} \overset{(L_7)}{=}$$

$$= \sqrt{\frac{\lim\limits_{x\to -1}(2x^2 + 3x - 4)}{\lim\limits_{x\to -1}(5x - 4)}} \overset{(T)}{=} \sqrt{\frac{2 - 3 - 4}{-5 - 4}} = \sqrt{\frac{-5}{-9}} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$$

i) $$\lim_{x\to -2}\sqrt[3]{\frac{3x^3 - 5x^2 - x + 2}{4x + 3}} \overset{(L_8)}{=} \sqrt[3]{\lim_{x\to -2}\frac{3x^3 - 5x^2 - x + 2}{4x + 3}} \overset{(L_7)}{=}$$

$$= \sqrt[3]{\frac{\lim\limits_{x\to -2}(3x^3 - 5x^2 - x + 2)}{\lim\limits_{x\to -2}(4x + 3)}} \overset{(T)}{=} \sqrt[3]{\frac{-24 - 20 + 2 + 2}{-8 + 3}}$$

$$= \sqrt[3]{\frac{-40}{-5}} = \sqrt[3]{8} = 2$$

j) $$\lim_{x\to 2}\frac{\sqrt{2x^2 + 3x + 2}}{6 - 4x} \overset{(L_7)}{=} \frac{\lim\limits_{x\to 2}\sqrt{2x^2 + 3x + 2}}{\lim\limits_{x\to 2}(6 - 4x)} \overset{(L_8)}{=}$$

$$= \frac{\sqrt{\lim\limits_{x\to 2}(2x^2 + 3x + 2)}}{\lim\limits_{x\to 2}(6 - 4x)} \overset{(T)}{=} \frac{\sqrt{8 + 6 + 2}}{6 - 8} = \frac{\sqrt{16}}{-2} = -2$$

30. a) $$\lim_{x\to 1}\frac{x^2 - 1}{x - 1} = \lim_{x\to 1}\frac{(x + 1)(x - 1)}{x - 1} = \lim_{x\to 1}(x + 1) = 2$$

b) $$\lim_{x\to -2}\frac{4 - x^2}{2 + x} = \lim_{x\to -2}\frac{(2 - x)(2 + x)}{2 + x} = \lim_{x\to -2}(2 - x) = 4$$

c) $$\lim_{x\to \frac{3}{2}}\frac{4x^2 - 9}{2x - 3} = \lim_{x\to \frac{3}{2}}\frac{(2x - 3)(2x + 3)}{2x - 3} = \lim_{x\to \frac{3}{2}}(2x + 3) = 6$$

d) $$\lim_{x\to 3}\frac{x^2 - 4x + 3}{x^2 - x - 6} = \lim_{x\to 3}\frac{(x - 3)(x - 1)}{(x - 3)(x + 2)} =$$

$$= \lim_{x\to 3}\frac{x - 1}{x + 2} = \frac{\lim\limits_{x\to 3}(x - 1)}{\lim\limits_{x\to 3}(x + 2)} = \frac{2}{5}$$

e) $$\lim_{x\to \frac{1}{2}}\frac{2x^2 + 5x - 3}{2x^2 - 5x + 2} = \lim_{x\to \frac{1}{2}}\frac{2\left(x - \frac{1}{2}\right)(x + 3)}{2\left(x - \frac{1}{2}\right)(x - 2)} =$$

$$= \frac{\lim\limits_{x\to \frac{1}{2}}(x + 3)}{\lim\limits_{x\to \frac{1}{2}}(x - 2)} = \frac{\frac{1}{2} + 3}{\frac{1}{2} - 2} = \frac{-7}{3}$$

f) $$\lim_{x\to \frac{-3}{2}}\frac{6x^2 + 11x + 3}{2x^2 - 5x - 12} = \lim_{x\to \frac{-3}{2}}\frac{6\left(x + \frac{1}{3}\right)\left(x + \frac{3}{2}\right)}{2(x - 4)\left(x + \frac{3}{2}\right)} =$$

$$= \lim_{x\to \frac{-3}{2}}\frac{3\left(x + \frac{1}{3}\right)}{x - 4} = \frac{\lim\limits_{x\to \frac{-3}{2}}3\left(x + \frac{1}{3}\right)}{\lim\limits_{x\to \frac{-3}{2}}(x - 4)} = \frac{\frac{-7}{2}}{\frac{-11}{2}} = \frac{7}{11}$$

g) $$\lim_{x\to 1}\frac{x^3 - 1}{x^2 - 1} = \lim_{x\to 1}\frac{(x - 1)(x^2 + x + 1)}{(x - 1)(x + 1)} = \lim_{x\to 1}\frac{x^2 + x + 1}{x + 1} =$$

$$= \frac{\lim\limits_{x\to 1}(x^2 + x + 1)}{\lim\limits_{x\to 1}(x + 1)} = \frac{3}{2}$$

h) $$\lim_{x\to -2}\frac{8 + x^3}{4 - x^2} = \lim_{x\to -2}\frac{(2 + x)(x^2 - 2x + 4)}{(2 - x)(2 + x)} = \lim_{x\to -2}\frac{x^2 - 2x + 4}{2 - x} =$$

$$= \frac{\lim\limits_{x\to -2}(x^2 - 2x + 4)}{\lim\limits_{x\to -2}(2 - x)} = \frac{12}{4} = 3$$

i) $$\lim_{x\to 2}\frac{x^4 - 16}{8 - x^3} = \lim_{x\to 2}\frac{(x - 2)(x + 2)(x^2 + 4)}{(x - 2)(-x^2 - 2x - 4)} =$$

$$= \frac{\lim\limits_{x\to 2}(x + 2)\cdot\lim\limits_{x\to 2}(x^2 + 4)}{\lim\limits_{x\to 2}(-x^2 - 2x - 4)} = \frac{4\cdot 8}{-12} = \frac{-8}{3}$$

32. $$\lim_{x\to 2}\frac{2x^2 - 3x - 2}{x - 2} = \lim_{x\to 2}\frac{2(x - 2)\left(x + \frac{1}{2}\right)}{x - 2} = \lim_{x\to 2}2\left(x + \frac{1}{2}\right) = 5$$

33. $\lim\limits_{x\to -3} \dfrac{2x^2 + 9x + 9}{x + 3} = \lim\limits_{x\to -3} \dfrac{2(x + 3)\left(x + \frac{3}{2}\right)}{(x + 3)} = \lim\limits_{x\to -3} 2\left(x + \dfrac{3}{2}\right) =$

$= 2 \cdot \left(-3 + \dfrac{3}{2}\right) = 2\left(\dfrac{-3}{2}\right) = -3$

35. a) $\lim\limits_{x\to -1} \dfrac{x^3 + 3x^2 - x - 3}{x^3 - x^2 + 2} = \lim\limits_{x\to -1} \dfrac{(x + 1)(x^2 + 2x - 3)}{(x + 1)(x^2 - 2x + 2)} =$

$= \lim\limits_{x\to -1} \dfrac{x^2 + 2x - 3}{x^2 - 2x + 2} = \dfrac{-4}{5}$

b) $\lim\limits_{x\to 3} \dfrac{x^3 - 6x - 9}{x^3 - 8x - 3} = \lim\limits_{x\to 3} \dfrac{(x - 3)(x^2 + 3x + 3)}{(x - 3)(x^2 + 3x + 1)} =$

$= \lim\limits_{x\to 3} \dfrac{x^2 + 3x + 3}{x^2 + 3x + 1} = \dfrac{21}{19}$

c) $\lim\limits_{x\to 1} \dfrac{x^3 - 3x^2 + 6x - 4}{x^3 - 4x^2 + 8x - 5} = \lim\limits_{x\to 1} \dfrac{(x - 1)(x^2 - 2x + 4)}{(x - 1)(x^2 - 3x + 5)} =$

$= \lim\limits_{x\to 1} \dfrac{x^2 - 2x + 4}{x^2 - 3x + 5} = 1$

d) $\lim\limits_{x\to 2} \dfrac{x^4 - 10x + 4}{x^3 - 2x^2} = \lim\limits_{x\to 2} \dfrac{(x - 2)(x^3 + 2x^2 + 4x - 2)}{x^2(x - 2)} =$

$= \lim\limits_{x\to 2} \dfrac{x^3 + 2x^2 + 4x - 2}{x^2} = \dfrac{11}{2}$

37. a) $\lim\limits_{x\to 1} \dfrac{x^3 - 3x + 2}{x^4 - 4x + 3} = \lim\limits_{x\to 1} \dfrac{(x - 1)(x^2 + x - 2)}{(x - 1)(x^3 + x^2 + x - 3)} = \lim\limits_{x\to 1} \dfrac{x^2 + x - 2}{x^3 + x^2 + x - 3} =$

$= \lim\limits_{x\to 1} \dfrac{(x - 1)(x + 2)}{(x - 1)(x^2 + 2x + 3)} = \lim\limits_{x\to 1} \dfrac{x + 2}{x^2 + 2x + 3} = \dfrac{1}{2}$

b) $\lim\limits_{x\to -2} \dfrac{x^4 + 4x^3 + x^2 - 12x - 12}{2x^3 + 7x^2 + 4x - 4} = \lim\limits_{x\to -2} \dfrac{(x + 2)(x^3 + 2x^2 - 3x - 6)}{(x + 2)(2x^2 + 3x - 2)} =$

$= \lim\limits_{x\to -2} \dfrac{x^3 + 2x^2 - 3x - 6}{2x^2 + 3x - 2} = \lim\limits_{x\to -2} \dfrac{(x + 2)(x^2 - 3)}{(x + 2)(2x - 1)} =$

$= \lim\limits_{x\to -2} \dfrac{x^2 - 3}{2x - 1} = \dfrac{-1}{5}$

c) $\lim\limits_{x\to -1} \dfrac{x^4 - x^3 - x^2 + 5x + 4}{x^3 + 4x^2 + 5x + 2} = \lim\limits_{x\to -1} \dfrac{(x + 1)(x^3 - 2x^2 + x + 4)}{(x + 1)(x^2 + 3x + 2)} =$

$= \lim\limits_{x\to -1} \dfrac{x^3 - 2x^2 + x + 4}{x^2 + 3x + 2} = \lim\limits_{x\to -1} \dfrac{(x + 1)(x^2 - 3x + 4)}{(x + 1)(x + 2)} =$

$= \lim\limits_{x\to -1} \dfrac{x^2 - 3x + 4}{x + 2} = 8$

d) $\lim\limits_{x\to -2} \dfrac{x^4 + 2x^3 - 5x^2 - 12x - 4}{2x^4 + 7x^3 + 2x^2 - 12x - 8} = \lim\limits_{x\to -2} \dfrac{(x + 2)(x^3 - 5x - 2)}{(x + 2)(2x^3 + 3x^2 - 4x - 4)} =$

$= \lim\limits_{x\to -2} \dfrac{x^3 - 5x - 2}{2x^3 + 3x^2 - 4x - 4} = \lim\limits_{x\to -2} \dfrac{(x + 2)(x^2 - 2x - 1)}{(x + 2)(2x^2 - x - 2)} =$

$= \lim\limits_{x\to -2} \dfrac{x^2 - 2x - 1}{2x^2 - x - 2} = \dfrac{7}{8}$

38. a) $\lim\limits_{x\to a} \dfrac{x^2 - a^2}{x - a} = \lim\limits_{x\to a} \dfrac{(x - a)(x + a)}{x - a} = \lim\limits_{x\to a} (x + a) = 2a$

b) $\lim\limits_{x\to -a} \dfrac{a^2 - x^2}{a^3 + x^3} = \lim\limits_{x\to -a} \dfrac{(a - x)(a + x)}{(a + x)(x^2 - ax + a^2)} =$

$= \lim\limits_{x\to -a} \dfrac{a - x}{x^2 - ax + a^2} = \dfrac{2a}{3a^2} = \dfrac{2}{3a}$

c) $\lim\limits_{x\to 1} \dfrac{x^n - 1}{x - 1} = \lim\limits_{x\to 1} \dfrac{(x - 1)(x^{n-1} + x^{n-2} + x^{n-3} + \ldots + 1)}{(x - 1)} =$

$= \lim\limits_{x\to 1} (x^{n-1} + x^{n-2} + x^{n-3} + \ldots + 1) = \underbrace{1 + 1 + 1 + \ldots + 1}_{n} = n$

d) $\lim\limits_{x\to 1} \dfrac{x^m - 1}{x^n - 1} = \lim\limits_{x\to 1} \dfrac{(x - 1)(x^{m-1} + x^{m-2} + \ldots + 1)}{(x - 1)(x^{n-1} + x^{n-2} + \ldots + 1)} =$

$= \lim\limits_{x\to 1} \dfrac{x^{m-1} + x^{m-2} + \ldots + 1}{x^{n-1} + x^{n-2} + \ldots + 1} = \dfrac{m}{n}$

e) $\lim\limits_{x\to a} \dfrac{x^n - a^n}{x - a} = \lim\limits_{x\to a} \dfrac{(x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-1})}{x - a} =$

$= \lim\limits_{x\to a} (x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-1}) =$

$= a^{n-1} + a \cdot a^{n-2} + a^2 \cdot a^{n-3} + \ldots + a^{n-1} =$

$= \underbrace{a^{n-1} + a^{n-1} + a^{n-1} + \ldots + a^{n-1}}_{n} = n \cdot a^{n-1}$

f) $\lim\limits_{x\to a} \dfrac{x^m - a^m}{x^n - a^n} = \lim\limits_{x\to a} \dfrac{(x - a)(x^{m-1} + ax^{m-2} + a^2x^{m-3} + \ldots + a^{m-1})}{(x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-1})} =$

$= \lim\limits_{x\to a} \dfrac{x^{m-1} + ax^{m-2} + a^2x^{m-3} + \ldots + a^{m-1}}{x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-1}} =$

$= \dfrac{a^{m-1} + a \cdot a^{m-2} + a^2 \cdot a^{m-3} + \ldots + a^{m-1}}{a^{n-1} + a \cdot a^{n-2} + a^2 \cdot a^{n-3} + \ldots + a^{n-1}} = \dfrac{m \cdot a^{m-1}}{n \cdot a^{n-1}} =$

$= \dfrac{m}{n} \cdot a^{(m-1)-(n-1)} = \dfrac{m}{n} \cdot a^{m-n}$

40. a) $$\lim_{x\to1}\frac{\sqrt{x}-1}{x-1}=\lim_{x\to1}\frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(x-1)(\sqrt{x}+1)}=$$
$$=\lim_{x\to1}\frac{x-1}{(x-1)(\sqrt{x}+1)}=\lim_{x\to1}\frac{1}{\sqrt{x}+1}=\frac{1}{2}$$

b) $$\lim_{x\to0}\frac{1-\sqrt{1-x}}{x}=\lim_{x\to0}\frac{(1-\sqrt{1-x})(1+\sqrt{1-x})}{x(1+\sqrt{1-x})}=$$
$$=\lim_{x\to0}\frac{1-(1-x)}{x(1+\sqrt{1-x})}=\lim_{x\to0}\frac{x}{x(1+\sqrt{1-x})}=$$
$$=\lim_{x\to0}\frac{1}{1+\sqrt{1-x}}=\frac{1}{2}$$

c) $$\lim_{x\to1}\frac{\sqrt{x+3}-2}{x-1}=\lim_{x\to1}\frac{(\sqrt{x+3}-2)(\sqrt{x+3}+2)}{(x-1)(\sqrt{x+3}+2)}=$$
$$=\lim_{x\to1}\frac{1}{\sqrt{x+3}+2}=\frac{1}{4}$$

d) $$\lim_{x\to0}\frac{\sqrt{1-2x-x^2}-1}{x}=\lim_{x\to0}\frac{(\sqrt{1-2x-x^2}-1)(\sqrt{1-2x-x^2}+1)}{x(\sqrt{1-2x-x^2}+1)}=$$
$$=\lim_{x\to0}\frac{-x(2+x)}{x(\sqrt{1-2x-x^2}+1)}=\lim_{x\to0}\frac{-x-2}{\sqrt{1-2x-x^2}+1}=-1$$

e) $$\lim_{x\to0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}=\lim_{x\to0}\frac{(\sqrt{1+x}-\sqrt{1-x})(\sqrt{1+x}+\sqrt{1-x})}{x(\sqrt{1+x}+\sqrt{1-x})}=$$
$$=\lim_{x\to0}\frac{2}{\sqrt{1+x}+\sqrt{1-x}}=1$$

f) $$\lim_{x\to1}\frac{\sqrt{2x}-\sqrt{x+1}}{x-1}=\lim_{x\to1}\frac{(\sqrt{2x}-\sqrt{x+1})(\sqrt{2x}+\sqrt{x+1})}{(x-1)(\sqrt{2x}+\sqrt{x+1})}=$$
$$=\lim_{x\to1}\frac{1}{\sqrt{2x}+\sqrt{x+1}}=\frac{1}{2\sqrt{2}}=\frac{\sqrt{2}}{4}$$

41. a) $$\lim_{x\to1}\frac{3-\sqrt{10-x}}{x^2-1}=\lim_{x\to1}\frac{(3-\sqrt{10-x})(3+\sqrt{10-x})}{(x^2-1)(3+\sqrt{10-x})}=$$
$$=\lim_{x\to1}\frac{x-1}{(x-1)(x+1)(3+\sqrt{10-x})}=\lim_{x\to1}\frac{1}{(x+1)(3+\sqrt{10-x})}=\frac{1}{12}$$

b) $$\lim_{x\to3}\frac{2-\sqrt{x+1}}{x^2-9}=\lim_{x\to3}\frac{(2-\sqrt{x+1})(2+\sqrt{x+1})}{(x^2-9)(2+\sqrt{x+1})}=$$
$$=\lim_{x\to3}\frac{-(x-3)}{(x+3)(x-3)(2+\sqrt{x+1})}=\lim_{x\to3}\frac{-1}{(x+3)(2+\sqrt{x+1})}=\frac{-1}{24}$$

c) $$\lim_{x\to1}\frac{\sqrt{x+3}-2}{x^2-3x+2}=\lim_{x\to1}\frac{(\sqrt{x+3}-2)(\sqrt{x+3}+2)}{(x^2-3x+2)(\sqrt{x+3}+2)}=$$

$$=\lim_{x\to1}\frac{x-1}{(x-1)(x-2)(\sqrt{x+3}+2)}=\lim_{x\to1}\frac{1}{(x-2)(\sqrt{x+3}+2)}=\frac{-1}{4}$$

d) $$\lim_{x\to2}\frac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}=\lim_{x\to2}\frac{(x^2-4)(\sqrt{x+2}+\sqrt{3x-2})}{(\sqrt{x+2}-\sqrt{3x-2})(\sqrt{x+2}+\sqrt{3x-2})}=$$
$$=\lim_{x\to2}\frac{(x-2)(x+2)(\sqrt{x+2}+\sqrt{3x-2})}{-2(x-2)}=$$
$$=\lim_{x\to2}\frac{(x+2)(\sqrt{x+2}+\sqrt{3x-2})}{-2}=-8$$

e) $$\lim_{x\to1}\frac{\sqrt{x^2-3x+3}-\sqrt{x^2+3x-3}}{x^2-3x+2}=$$
$$=\lim_{x\to1}\frac{(\sqrt{x^2-3x+3}-\sqrt{x^2+3x-3})(\sqrt{x^2-3x+3}+\sqrt{x^2+3x-3})}{(x^2-3x+2)(\sqrt{x^2-3x+3}+\sqrt{x^2+3x-3})}=$$
$$=\lim_{x\to1}\frac{-6(x-1)}{(x-1)(x-2)(\sqrt{x^2-3x+3}+\sqrt{x^2+3x-3})}=$$
$$=\lim_{x\to1}\frac{-6}{(x-2)(\sqrt{x^2-3x+3}+\sqrt{x^2+3x-3})}=\frac{-6}{-2}=3$$

43. a) $$\lim_{x\to4}\frac{\sqrt{2x+1}-3}{\sqrt{x-2}-\sqrt{2}}=\lim_{x\to4}\frac{(\sqrt{2x+1}-3)(\sqrt{2x+1}+3)(\sqrt{x-2}+\sqrt{2})}{(\sqrt{x-2}-\sqrt{2})(\sqrt{2x+1}+3)(\sqrt{x-2}+\sqrt{2})}=$$
$$=\lim_{x\to4}\frac{2(\sqrt{x-2}+\sqrt{2})}{\sqrt{2x+1}+3}=\frac{2(2\sqrt{2})}{3+3}=\frac{2\sqrt{2}}{3}$$

b) $$\lim_{x\to6}\frac{4-\sqrt{10+x}}{2-\sqrt{10-x}}=\lim_{x\to6}\frac{(4-\sqrt{10+x})(4+\sqrt{10+x})(2+\sqrt{10-x})}{(2-\sqrt{10-x})(4+\sqrt{10+x})(2+\sqrt{10-x})}=$$
$$=\lim_{x\to6}\frac{(6-x)(2+\sqrt{10-x})}{(-6+x)(4+\sqrt{10+x})}=\lim_{x\to6}\frac{2+\sqrt{10-x}}{-(4+\sqrt{10+x})}=\frac{2+2}{-(4+4)}=\frac{-1}{2}$$

c) $$\lim_{x\to0}\frac{\sqrt{3x+4}-\sqrt{x+4}}{\sqrt{x+1}-1}=$$
$$=\lim_{x\to0}\frac{(\sqrt{3x+4}-\sqrt{x+4})(\sqrt{3x+4}+\sqrt{x+4})(\sqrt{x+1}+1)}{(\sqrt{x+1}-1)(\sqrt{3x+4}+\sqrt{x+4})(\sqrt{x+1}+1)}=$$
$$=\lim_{x\to0}\frac{2(\sqrt{x+1}+1)}{\sqrt{3x+4}+\sqrt{x+4}}=\frac{2\cdot2}{2+2}=1$$

d) $$\lim_{x\to2}\frac{\sqrt{x^2+x-2}-\sqrt{x^2-x+2}}{\sqrt{x+2}-2}=$$
$$=\lim_{x\to2}\frac{(\sqrt{x^2+x-2}-\sqrt{x^2-x+2})(\sqrt{x^2+x-2}+\sqrt{x^2-x+2})(\sqrt{x+2}+2)}{(\sqrt{x+2}-2)(\sqrt{x^2+x-2}+\sqrt{x^2-x+2})(\sqrt{x+2}+2)}=$$
$$=\lim_{x\to2}\frac{2(\sqrt{x+2}+2)}{\sqrt{x^2+x-2}+\sqrt{x^2-x+2}}=\frac{2(2+2)}{2+2}=2$$

44. a) $$\lim_{x\to 2}\frac{\sqrt{2x^2-3x+2}-2}{\sqrt{3x^2-5x-1}-1}=$$

$$=\lim_{x\to 2}\frac{(\sqrt{2x^2-3x+2}-2)(\sqrt{2x^2-3x+2}+2)(\sqrt{3x^2-5x-1}+1)}{(\sqrt{3x^2-5x-1}-1)(\sqrt{2x^2-3x+2}+2)(\sqrt{3x^2-5x-1}+1)}=$$

$$=\lim_{x\to 2}\frac{(2x^2-3x-2)(\sqrt{3x^2-5x-1}+1)}{(3x^2-5x-2)(\sqrt{2x^2-3x+2}+2)}=$$

$$=\lim_{x\to 2}\frac{2(x-2)\left(x+\frac{1}{2}\right)(\sqrt{3x^2-5x-1}+1)}{3(x-2)\left(x+\frac{1}{3}\right)(\sqrt{2x^2-3x+2}+2)}=$$

$$=\frac{2\left(2+\frac{1}{2}\right)(\sqrt{12-10-1}+1)}{3\left(2+\frac{1}{3}\right)(\sqrt{8-6+2}+2)}=\frac{5\,(1+1)}{7(2+2)}=\frac{5}{14}$$

b) $$\lim_{x\to -1}\frac{\sqrt{3x^2+4x+2}-1}{\sqrt{x^2+3x+6}-2}=$$

$$=\lim_{x\to -1}\frac{(\sqrt{3x^2+4x+2}-1)(\sqrt{3x^2+4x+2}+1)(\sqrt{x^2+3x+6}+2)}{(\sqrt{x^2+3x+6}-2)(\sqrt{3x^2+4x+2}+1)(\sqrt{x^2+3x+6}+2)}=$$

$$=\lim_{x\to -1}\frac{(3x^2+4x+1)(\sqrt{x^2+3x+6}+2)}{(x^2+3x+2)(\sqrt{3x^2+4x+2}+1)}=$$

$$=\lim_{x\to -1}\frac{3\left(x+\frac{1}{3}\right)(x+1)(\sqrt{x^2+3x+6}+2)}{(x+1)(x+2)(\sqrt{3x^2+4x+2}+1)}=$$

$$=\frac{3\left(-1+\frac{1}{3}\right)(\sqrt{1-3+6}+2)}{(-1+2)(\sqrt{3-4+2}+1)}=-4$$

46. a) $$\lim_{x\to 0}\frac{\sqrt[3]{x+1}-1}{x}=\lim_{x\to 0}\frac{(\sqrt[3]{x+1}-1)(\sqrt[3]{(x+1)^2}+\sqrt[3]{x+1}+1)}{x(\sqrt[3]{(x+1)^2}+\sqrt[3]{x+1}+1)}=$$

$$=\lim_{x\to 0}\frac{(x+1)-1}{x(\sqrt[3]{(x+1)^2}+\sqrt[3]{x+1}+1)}=\lim_{x\to 0}\frac{1}{\sqrt[3]{(x+1)^2}+\sqrt[3]{x+1}+1}=$$

$$=\frac{1}{3}$$

b) $$\lim_{x\to -1}\frac{x+1}{\sqrt[3]{2x+3}-1}=\lim_{x\to -1}\frac{(x+1)(\sqrt[3]{(2x+3)^2}+\sqrt[3]{2x+3}+1)}{(\sqrt[3]{2x+3}-1)(\sqrt[3]{(2x+3)^2}+\sqrt[3]{2x+3}+1)}=$$

$$=\lim_{x\to -1}\frac{\sqrt[3]{(2x+3)^2}+\sqrt[3]{2x+3}+1}{2}=\frac{3}{2}$$

c) $$\lim_{x\to 0}\frac{\sqrt[3]{8-2x+x^2}-2}{x-x^2}=$$

$$=\lim_{x\to 0}\frac{(\sqrt[3]{8-2x+x^2}-2)(\sqrt[3]{(8-2x+x^2)^2}+2\sqrt[3]{8-2x+x^2}+4)}{x(1-x)(\sqrt[3]{(8-2x+x^2)^2}+2\sqrt[3]{8-2x+x^2}+4)}=$$

$$=\lim_{x\to 0}\frac{-(2-x)}{(1-x)(\sqrt[3]{(8-2x+x^2)^2}+2\sqrt[3]{8-2x+x^2}+4)}=$$

$$=\frac{-2}{1(\sqrt[3]{64}+2\sqrt[3]{8}+4)}=\frac{-1}{6}$$

47. a) $$\lim_{x\to 0}\frac{1-\sqrt[3]{1-x}}{1+\sqrt[3]{3x-1}}=$$

$$=\lim_{x\to 0}\frac{(1-\sqrt[3]{1-x})(1+\sqrt[3]{1-x}+\sqrt[3]{(1-x)^2})(1-\sqrt[3]{3x-1}+\sqrt[3]{(3x-1)^2})}{(1+\sqrt[3]{3x-1})(1-\sqrt[3]{3x-1}+\sqrt[3]{(3x-1)^2})(1+\sqrt[3]{1-x}+\sqrt[3]{(1-x)^2})}=$$

$$=\lim_{x\to 0}\frac{1-\sqrt[3]{3x-1}+\sqrt[3]{(3x-1)^2}}{3(1+\sqrt[3]{1-x}+\sqrt[3]{(1-x)^2})}=\frac{1-(-1)+1}{3(1+1+1)}=\frac{1}{3}$$

b) $$\lim_{x\to -2}\frac{\sqrt[3]{2-3x}-2}{1+\sqrt[3]{2x+3}}=$$

$$=\lim_{x\to -2}\frac{(\sqrt[3]{2-3x}-2)(\sqrt[3]{(2-3x)^2}+2\sqrt[3]{2-3x}+4)(1-\sqrt[3]{2x+3}+\sqrt[3]{(2x+3)^2})}{(1+\sqrt[3]{2x+3})(1-\sqrt[3]{2x+3}+\sqrt[3]{(2x+3)^2})(\sqrt[3]{(2-3x)^2}+2\sqrt[3]{2-3x}+4)}$$

$$=\lim_{x\to -2}\frac{-3(1-\sqrt[3]{2x+3}+\sqrt[3]{(2x+3)^2})}{2(\sqrt[3]{(2-3x)^2}+2\sqrt[3]{2-3x}+4)}=\frac{-3(1+1+1)}{2(4+4+4)}=\frac{-3}{8}$$

c) $$\lim_{x\to 2}\frac{\sqrt[3]{3x^2-7x+1}+1}{\sqrt[3]{2x^2-5x+3}-1}=$$

$$=\lim_{x\to 2}\frac{(\sqrt[3]{3x^2-7x+1}+1)(\sqrt[3]{(3x^2-7x+1)^2}-\sqrt[3]{3x^2-7x+1}+1)}{(\sqrt[3]{2x^2-5x+3}-1)(\sqrt[3]{(2x^2-5x+3)^2}+\sqrt[3]{2x^2-5x+3}+1)}\cdot$$

$$\cdot\frac{(\sqrt[3]{(2x-5x+3)^2}+\sqrt[3]{2x^2-5x+3}+1)}{(\sqrt[3]{(3x^2-7x+1)^2}-\sqrt[3]{3x^2-7x+1}+1)}=$$

$$=\lim_{x\to 2}\frac{3\left(x-\frac{1}{3}\right)(\sqrt[3]{(2x^2-5x+3)^2}+\sqrt[3]{2x^2-5x+3}+1)}{2\left(x-\frac{1}{2}\right)(\sqrt[3]{(3x^2-7x+1)^2}-\sqrt[3]{3x^2-7x+1}+1)}=$$

$$=\frac{5(1+1+1)}{3(1+1+1)}=\frac{5}{3}$$

48. a) $\lim_{x\to 1} \dfrac{\sqrt{5x+4}-3}{\sqrt[3]{x-2}+1} =$

$$= \lim_{x\to 1} \frac{(\sqrt{5x+4}-3)(\sqrt{5x+4}+3)(\sqrt[3]{(x-2)^2}-\sqrt[3]{x-2}+1)}{(\sqrt[3]{x-2}+1)(\sqrt[3]{(x-2)^2}-\sqrt[3]{x-2}+1)(\sqrt{5x+4}+3)} =$$

$$= \lim_{x\to 1} \frac{5(\sqrt[3]{(x-2)^2}-\sqrt[3]{x-2}+1)}{\sqrt{5x+4}+3} = \frac{5(1+1+1)}{3+3} = \frac{5}{2}$$

b) $\lim_{x\to 2} \dfrac{\sqrt[3]{5x-2}-2}{\sqrt{x-1}-1} =$

$$= \lim_{x\to 2} \frac{(\sqrt[3]{5x-2}-2)(\sqrt[3]{(5x-2)^2}+2\sqrt[3]{5x-2}+4)(\sqrt{x-1}+1)}{(\sqrt{x-1}-1)(\sqrt{x-1}+1)(\sqrt[3]{(5x-2)^2}+2\sqrt[3]{5x-2}+4)} =$$

$$= \lim_{x\to 2} \frac{5(\sqrt{x-1}+1)}{\sqrt[3]{(5x-2)^2}+2\sqrt[3]{5x-2}+4} = \frac{5\cdot 2}{4+4+4} = \frac{5}{6}$$

c) $\lim_{x\to 1} \dfrac{\sqrt{3x^3-5x+6}-2}{\sqrt[3]{x^2-3x+1}+1} =$

$$= \lim_{x\to 1} \frac{(\sqrt{3x^3-5x+6}-2)(\sqrt{3x^3-5x+6}+2)}{(\sqrt[3]{x^2-3x+1}+1)(\sqrt[3]{(x^2-3x+1)^2}-\sqrt[3]{x^2-3x+1}+1)} \cdot$$

$$\cdot \frac{(\sqrt[3]{(x^2-3x+1)^2}-\sqrt[3]{x^2-3x+1}+1)}{(\sqrt{3x^3-5x+6}+2)} =$$

$$= \lim_{x\to 1} \frac{(3x^2+3x-2)(\sqrt[3]{(x^2-3x+1)^2}-\sqrt[3]{x^2-3x+1}+1)}{(x-2)(\sqrt{3x^3-5x+6}+2)} =$$

$$= \frac{4(1+1+1)}{(-1)(2+2)} = -3$$

50. a) $\lim_{x\to -1} \dfrac{\sqrt[3]{x}+1}{x+1}$

Fazendo $\sqrt[3]{x} = y \Rightarrow x = y^3$, vem:

$$\lim_{y\to -1} \frac{y+1}{y^3+1} = \lim_{y\to -1} \frac{y+1}{(y+1)(y^2-y+1)} = \lim_{y\to -1} \frac{1}{y^2-y+1} = \frac{1}{3}.$$

b) $\lim_{x\to 1} \dfrac{\sqrt{x}-1}{\sqrt[3]{x}-1}$

Fazendo $\sqrt[6]{x} = y$, vem: $\sqrt{x} = (\sqrt[6]{x})^3 = y^3$ e $\sqrt[3]{x} = (\sqrt[6]{x})^2 = y^2$.

$$\lim_{y\to 1} \frac{y^3-1}{y^2-1} = \lim_{y\to 1} \frac{(y-1)(y^2+y+1)}{(y-1)(y+1)} = \lim_{y\to 1} \frac{y^2+y+1}{y+1} = \frac{3}{2}$$

c) $\lim_{x\to 1} \dfrac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}$

Fazendo $\sqrt[12]{x} = y$, vem: $\sqrt[3]{x} = (\sqrt[12]{x})^4 = y^4$ e $\sqrt[4]{x} = (\sqrt[12]{x})^3 = y^3$.

$$\lim_{y\to 1} \frac{y^4-1}{y^3-1} = \lim_{y\to 1} \frac{(y^2+1)(y+1)(y-1)}{(y-1)(y^2+y+1)} = \lim_{y\to 1} \frac{(y^2+1)(y+1)}{y^2+y+1} = \frac{4}{3}$$

d) $\lim_{x\to 0} \dfrac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1}$

Fazendo $\sqrt[6]{1+x} = y$, vem: $\sqrt{1+x} = (\sqrt[6]{1+x})^3 = y^3$ e $\sqrt[3]{1+x} = (\sqrt[6]{1+x})^2 = y^2$.

$$\lim_{y\to 1} \frac{y^3-1}{y^2-1} = \lim_{y\to 1} \frac{(y-1)(y^2+y+1)}{(y-1)(y+1)} = \lim_{y\to 1} \frac{y^2+y+1}{y+1} = \frac{3}{2}$$

51. a) $\lim_{x\to a} \dfrac{x\sqrt{x}-a\sqrt{a}}{\sqrt{x}-\sqrt{a}}$

Fazendo $\sqrt{x} = y$, vem $x = y^2$ e $\sqrt{a} = z$, vem $a = z^2$.

Além disso, $\lim_{x\to a} \sqrt{x} = \sqrt{a} = z = \lim_{y\to z} y$.

$$\lim_{y\to z} \frac{y^2\cdot y - z^2\cdot z}{y-z} = \lim_{y\to z} \frac{y^3-z^3}{y-z} = \lim_{y\to z} \frac{(y-z)(y^2+yz+z^2)}{y-z} =$$

$$= \lim_{y\to z} (y^2+yz+z^2) = 3z^2 = 3a.$$

b) $\lim_{x\to 1} \dfrac{\sqrt[n]{x}-1}{x-1}$

Fazendo $\sqrt[n]{x} = y$, vem $x = y^n$ e $\lim_{x\to 1} \sqrt[n]{x} = \lim_{y\to 1} y$.

$$\lim_{y\to 1} \frac{y-1}{y^n-1} = \lim_{y\to 1} \frac{y-1}{(y-1)(y^{n-1}+y^{n-2}+\ldots+y+1)} =$$

$$= \lim_{y\to 1} \frac{1}{y^{n-1}+y^{n-2}+\ldots+y+1} = \frac{1}{\underbrace{1+1+\ldots+1+1}_{n}} = \frac{1}{n}$$

c) $\lim_{x\to 1} \dfrac{\sqrt[m]{x}-1}{\sqrt[n]{x}-1}$

Fazendo $\sqrt[mn]{x} = y \Rightarrow \begin{cases} \sqrt[m]{x} = (\sqrt[mn]{x})^n = y^n \\ \text{e} \\ \sqrt[n]{x} = (\sqrt[mn]{x})^m = y^m \end{cases}$

Além disso, $\lim_{x\to 1} \sqrt[mn]{x} = 1 = \lim_{x\to 1} y$.

$$\lim_{y\to 1} \frac{y^n-1}{y^m-1} = \lim_{y\to 1} \frac{(y-1)(y^{n-1}+y^{n-2}+\ldots+y+1)}{(y-1)(y^{m-1}+y^{m-2}+\ldots+y+1)} =$$

$$= \lim_{y\to 1} \frac{y^{n-1}+y^{n-2}+\ldots+y+1}{y^{m-1}+y^{m-2}+\ldots+y+1} = \frac{\overbrace{1+1+\ldots+1+1}^{n}}{\underbrace{1+1+\ldots+1+1}_{m}} = \frac{n}{m}$$

d) $\lim_{x\to a} \dfrac{\sqrt[n]{x} - \sqrt[n]{a}}{x - a}$

$\left.\begin{array}{l}\sqrt[n]{x} = y \Rightarrow x = y^n \\ \sqrt[n]{a} = z \Rightarrow a = z^n\end{array}\right\}$ e $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a} = z = \lim_{y\to z} y$

$$\lim_{y\to z} \frac{y - z}{y^n - z^n} = \lim_{y\to z} \frac{y - z}{(y - z)(y^{n-1} + y^{n-2}\cdot z + y^{n-3}\cdot z^2 + \dots + yz^{n-2} + z^{n-1})} =$$

$$= \lim_{y\to z} \frac{1}{y^{n-1} + y^{n-2}\cdot z + y^{n-3}\cdot z^2 + \dots + y\cdot z^{n-2} + z^{n-1}} =$$

$$= \frac{1}{\underbrace{z^{n-1} + z^{n-1} + z^{n-1} + \dots + z^{n-1}}_{n}} = \frac{1}{n\cdot z^{n-1}} = \frac{1}{n(\sqrt[n]{a})^{n-1}} =$$

$$= \frac{\sqrt[n]{a}}{n(\sqrt[n]{a})^{n-1}(\sqrt[n]{a})} = \frac{\sqrt[n]{a}}{na}$$

e) $\lim_{x\to a} \dfrac{\sqrt[m]{x} - \sqrt[m]{a}}{\sqrt[n]{x} - \sqrt[n]{a}}$

Fazendo:

$\sqrt[mn]{x} = y \Rightarrow \begin{cases}\sqrt[m]{x} = (\sqrt[mn]{x})^n = y^n \\ \sqrt[n]{x} = (\sqrt[mn]{x})^m = y^m\end{cases}$

$\sqrt[mn]{a} = z \Rightarrow \begin{cases}\sqrt[m]{a} = (\sqrt[mn]{a})^n = z^n \\ \sqrt[n]{a} = (\sqrt[mn]{a})^m = z^m\end{cases}$

Além disso:

$\lim_{x\to a} \sqrt[mn]{x} = \sqrt[mn]{a} = z = \lim_{y\to z} y$

Assim, temos:

$$\lim_{y\to z} \frac{y^n - z^n}{y^m - z^m} =$$

$$= \lim_{y\to z} \frac{(y - z)(y^{n-1} + y^{n-2}\cdot z + y^{n-3}\cdot z^2 + \dots + yz^{n-2} + z^{n-1})}{(y - z)(y^{m-1} + y^{m-2}\cdot z + y^{m-3}\cdot z^2 + \dots + yz^{m-2} + z^{m-1})} =$$

$$= \lim_{y\to z} \frac{y^{n-1} + y^{n-2}\cdot z + y^{n-3}\cdot z^2 + \dots + yz^{n-2} + z^{n-1}}{y^{m-1} + y^{m-2}\cdot z + y^{m-3}\cdot z^2 + \dots + yz^{m-2} + z^{m-1}} =$$

$$= \frac{z^{n-1} + z^{n-2}\cdot z + z^{n-3}\cdot z^2 + \dots + z^{n-1}}{z^{m-1} + z^{m-2}\cdot z + z^{m-3}\cdot z^2 + \dots + z^{n-1}} = \frac{n\cdot z^{n-1}}{m\cdot z^{m-1}} =$$

$$= \frac{n(\sqrt[mn]{a})^{n-1}}{m\cdot(\sqrt[mn]{a})^{m-1}} = \frac{n}{m}\cdot(\sqrt[mn]{a})^{n-m} = \frac{n}{m}\sqrt[mn]{a^{n-m}}$$

52. a) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (3x - 2) = 1$

b) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (4x + 1) = 5$

c) Não existe $\lim_{x\to 1} f(x)$ porque $\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$.

53. a) $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (3 - 2x) = 5$

b) $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (4 - x) = 5$

c) $\lim_{x\to -1} f(x) = 5$

54. a) $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (2x - 5) = 1$

b) $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (4 - 5x) = -11$

c) Não existe $\lim_{x\to 3} f(x)$ porque $\lim_{x\to 3^+} f(x) \neq \lim_{x\to 3^-} f(x)$.

55. a) $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x - 1) = 1$

b) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (1 - x^2) = -3$

c) Não existe $\lim_{x\to 2} f(x)$ porque $\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$.

56. a) $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (8 - 2x) = 2$

b) $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2 - 3x + 2) = 2$

c) $\lim_{x\to 3} f(x) = 2$

57. a) $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (-x^2 + 6x - 7) = 1$

b) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2x^2 - 3x - 1) = 1$

c) $\lim_{x\to 2} f(x) = 1$

59. $f(x) = \dfrac{|x + 1|}{x + 1}$

$|x + 1| = \begin{cases} x + 1, \text{ se } x \geqslant -1 \\ -x - 1, \text{ se } x < -1\end{cases}$

a) $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \dfrac{x + 1}{x + 1} = \lim_{x\to -1^+} 1 = 1$

b) $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \dfrac{-(x + 1)}{x + 1} = \lim_{x\to -1^-} (-1) = -1$

c) $\nexists \lim_{x\to -1} f(x)$ porque $\lim_{x\to -1^+} f(x) \neq \lim_{x\to -1^-} f(x)$

60. $f(x) = \dfrac{|3x - 2|}{2 - 3x}$

$$|3x - 2| = \begin{cases} 3x - 2, & \text{se } x \geqslant \frac{2}{3} \\ -(3x - 2), & \text{se } x < \frac{2}{3} \end{cases}$$

a) $\lim\limits_{x \to \frac{2}{3}^+} f(x) = \lim\limits_{x \to \frac{2}{3}^+} \dfrac{3x - 2}{2 - 3x} = \lim\limits_{x \to \frac{2}{3}^+} \dfrac{3x - 2}{-(3x - 2)} = \lim\limits_{x \to \frac{2}{3}^+} (-1) = -1$

b) $\lim\limits_{x \to \frac{2}{3}^-} f(x) = \lim\limits_{x \to \frac{2}{3}^-} \dfrac{-(3x - 2)}{2 - 3x} = \lim\limits_{x \to \frac{2}{3}^-} \dfrac{2 - 3x}{2 - 3x} = \lim\limits_{x \to \frac{2}{3}^-} 1 = 1$

c) $\nexists \lim\limits_{x \to \frac{2}{3}} f(x)$

61. $f(x) = \dfrac{x^2 - 5x + 4}{|x - 1|}$

$$|x - 1| = \begin{cases} x - 1, & \text{se } x \geqslant 1 \\ -(x - 1), & \text{se } x < 1 \end{cases}$$

$x^2 - 5x + 4 = (x - 1)(x - 4)$

a) $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \dfrac{(x - 1)(x - 4)}{(x - 1)} = \lim\limits_{x \to 1^+} (x - 4) = -3$

b) $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} \dfrac{(x - 1)(x - 4)}{-(x - 1)} = \lim\limits_{x \to 1^-} -(x - 4) = 3$

c) $\nexists \lim\limits_{x \to 1} f(x)$

62. $f(x) = \dfrac{|3x^2 - 5x - 2|}{x - 2}$

$$|3x^2 - 5x - 2| = \begin{cases} 3x^2 - 5x - 2, & \text{se } x \leqslant -\frac{1}{3} \text{ ou } x \geqslant 2 \\ -(3x^2 - 5x - 2), & \text{se } -\frac{1}{3} < x < 2 \end{cases}$$

Além disso, $3x^2 - 5x - 2 = 3\left(x + \frac{1}{3}\right)(x - 2)$.

a) $\lim\limits_{x \to 2^+} \dfrac{3\left(x + \frac{1}{3}\right)(x - 2)}{x - 2} = \lim\limits_{x \to 2^+} 3\left(x + \dfrac{1}{3}\right) = 7$

b) $\lim\limits_{x \to 2^-} \dfrac{-3\left(x + \frac{1}{3}\right)(x - 2)}{x - 2} = \lim\limits_{x \to 2^-} \left[-3\left(x + \dfrac{1}{3}\right)\right] = -7$

c) $\nexists \lim\limits_{x \to 2} f(x)$

63. $f(x) = \dfrac{x^3 - 6x^2 + 11x - 6}{|x - 2|}$

$x^3 - 6x^2 + 11x - 6 = (x - 1)(x - 2)(x - 3)$

$$|x - 2| = \begin{cases} x - 2, & \text{se } x \geqslant 2 \\ -(x - 2), & \text{se } x < 2 \end{cases}$$

a) $\lim\limits_{x \to 2^+} \dfrac{(x - 1)(x - 2)(x - 3)}{x - 2} = \lim\limits_{x \to 2^+} (x - 1)(x - 3) = -1$

b) $\lim\limits_{x \to 2^-} \dfrac{(x - 1)(x - 2)(x - 3)}{-(x - 2)} = \lim\limits_{x \to 2^-} [-(x - 1)(x - 3)] = 1$

c) $\nexists \lim\limits_{x \to 2} f(x)$

64. $f(x) = \dfrac{x^3 - 4x^2 + x + 6}{|2x^2 - 9x + 10|}$

$$|2x^2 - 9x + 10| = \begin{cases} (2x^2 - 9x + 10) = 2(x - 2)\left(x - \frac{5}{2}\right), & \text{se } x \leqslant 2 \text{ ou } x \geqslant \frac{5}{2} \\ -(2x^2 - 9x + 10) = -2(x - 2)\left(x - \frac{5}{2}\right), & \text{se } 2 < x < \frac{5}{2} \end{cases}$$

$x^3 - 4x^2 + x + 6 = (x + 1)(x - 2)(x - 3)$

a) $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} \dfrac{(x + 1)(x - 2)(x - 3)}{-2(x - 2)\left(x - \frac{5}{2}\right)} = \lim\limits_{x \to 2^+} \dfrac{(x + 1)(x - 3)}{-2\left(x - \frac{5}{2}\right)} = -3$

b) $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} \dfrac{(x + 1)(x - 2)(x - 3)}{2(x - 2)\left(x - \frac{5}{2}\right)} = \lim\limits_{x \to 2^-} \dfrac{(x + 1)(x - 3)}{2\left(x - \frac{5}{2}\right)} = 3$

c) $\nexists \lim\limits_{x \to 2} f(x)$

65. a) $\lim\limits_{x \to 1^+} [x] = 1$

b) $\lim\limits_{x \to 1^-} [x] = 0$

c) $\nexists \lim\limits_{x \to 1} [x]$

d) $\lim\limits_{x \to 1^+} (x - [x]) = 1 - 1 = 0$

e) $\lim\limits_{x \to 1^-} (x - [x]) = 1 - 0 = 1$

f) $\nexists \lim\limits_{x \to 1} (x - [x])$

g) $\lim\limits_{x \to 1^+} (x + [x]) = 1 + 1 = 2$

h) $\lim\limits_{x \to 1^-} (x + [x]) = 1 + 0 = 1$

i) $\nexists \lim\limits_{x \to 1} (x + [x])$

66. $f(x) = \begin{cases} 3x - 2 & \text{se } x > -1 \\ 3 & \text{se } x = -1 \\ 5 - ax & \text{se } x < -1 \end{cases}$

$\lim\limits_{x \to -1} f(x)$ existe se $\lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^-} f(x)$

$$\left.\begin{array}{l} \lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^+} (3x - 2) = -5 \\ \lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^-} (5 - ax) = 5 + a \end{array}\right\} \Rightarrow 5 + a = -5 \Rightarrow a = -10$$

67. $f(x) = \begin{cases} 4x + 3 & \text{se } x \leqslant -2 \\ 3x + a & \text{se } x > -2 \end{cases}$

$\lim\limits_{x \to -2} f(x)$ existe se $\lim\limits_{x \to -2^+} f(x) = \lim\limits_{x \to -2^-} f(x)$

$$\left.\begin{array}{l} \lim\limits_{x \to -2^+} f(x) = \lim\limits_{x \to -2^+} (3x + a) = -6 + a \\ \lim\limits_{x \to -2^-} f(x) = \lim\limits_{x \to -2^-} (4x + 3) = -5 \end{array}\right\} \Rightarrow -6 + a = -5 \Rightarrow a = 1$$

68. $f(x) = \begin{cases} \dfrac{3x^2 - 5x - 2}{x - 2} & \text{se } x < 2 \\ 3 - ax - x^2 & \text{se } x \geqslant 2 \end{cases}$

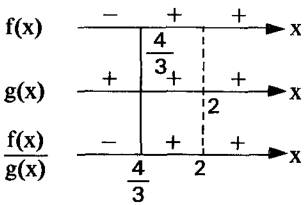
$\lim\limits_{x \to 2} f(x)$ existe se $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^-} f(x)$

$$\left.\begin{array}{l} \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (3 - ax - x^2) = 3 - 2a - 4 = -2a - 1 \\ \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} \dfrac{3(x - 2)\left(x + \frac{1}{3}\right)}{x - 2} = \lim\limits_{x \to 2^-} 3\left(x + \dfrac{1}{3}\right) = 7 \end{array}\right\} \Rightarrow$$

$$\Rightarrow -2a - 1 = 7 \Rightarrow a = -4$$

Capítulo III – O infinito

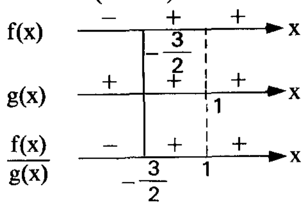
70. a) $\dfrac{f(x)}{g(x)} = \dfrac{3x - 4}{(x - 2)^2}$



Notemos que $\dfrac{f(x)}{g(x)} > 0$ quando x está próximo de 2.

Então, $\lim\limits_{x \to 2} \dfrac{3x - 4}{(x - 2)^2} = +\infty$.

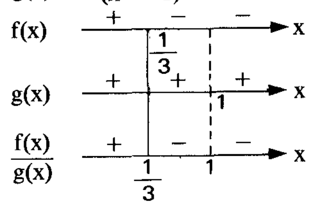
b) $\dfrac{f(x)}{g(x)} = \dfrac{2x + 3}{(x - 1)^2}$



$\dfrac{f(x)}{g(x)} > 0$ quando x está próximo de 1.

Então, $\lim\limits_{x \to 1} \dfrac{2x + 3}{(x - 1)^2} = +\infty$.

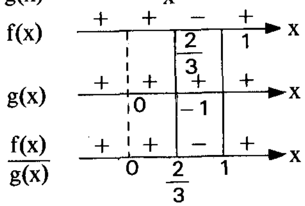
c) $\dfrac{f(x)}{g(x)} = \dfrac{1 - 3x}{(x - 1)^2}$



$\dfrac{f(x)}{g(x)} < 0$ quando x está próximo de 1.

Então, $\lim\limits_{x \to 1} \dfrac{1 - 3x}{(x - 1)^2} = -\infty$.

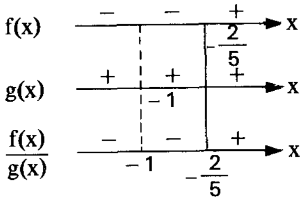
d) $\dfrac{f(x)}{g(x)} = \dfrac{3x^2 - 5x + 2}{x^2}$



$\dfrac{f(x)}{g(x)} > 0$ quando x está próximo de 0.

Então, $\lim\limits_{x \to 0} \dfrac{3x^2 - 5x + 2}{x^2} = +\infty$.

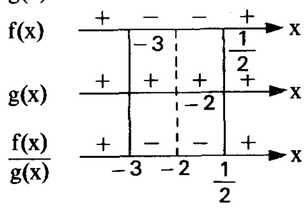
e) $\dfrac{f(x)}{g(x)} = \dfrac{5x + 2}{|x + 1|}$



$\dfrac{f(x)}{g(x)} < 0$ quando x está próximo de -1.

Então, $\lim\limits_{x \to -1} \dfrac{5x + 2}{|x + 1|} = -\infty$.

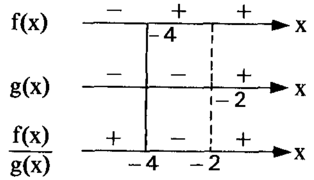
f) $\dfrac{f(x)}{g(x)} = \dfrac{2x^2 + 5x - 3}{|x + 2|}$



$\dfrac{f(x)}{g(x)} < 0$ quando x está próximo de -2.

Então, $\lim\limits_{x \to -2} \dfrac{2x^2 + 5x - 3}{|x + 2|} = -\infty$.

72. a) $\dfrac{f(x)}{g(x)} = \dfrac{x+4}{x+2}$



À esquerda de -2, $\dfrac{f(x)}{g(x)} < 0$.

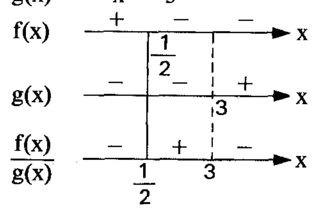
Então, $\lim\limits_{x \to -2^-} \dfrac{x+4}{x+2} = -\infty$.

b) $\dfrac{f(x)}{g(x)} = \dfrac{x+4}{x+2}$

Pelo esquema anterior, verifica-se que à direita de -2, $\dfrac{f(x)}{g(x)} > 0$.

Então, $\lim\limits_{x \to -2^+} \dfrac{x+4}{x+2} = +\infty$.

c) $\dfrac{f(x)}{g(x)} = \dfrac{1-2x}{x-3}$



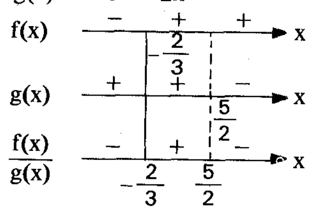
À esquerda de 3, $\dfrac{f(x)}{g(x)} > 0$.

Então, $\lim\limits_{x \to 3^-} \dfrac{1-2x}{x-3} = +\infty$.

d) Pelo esquema do item anterior, à direita de 3, $\dfrac{f(x)}{g(x)} < 0$.

Então, $\lim\limits_{x \to 3^+} \dfrac{1-2x}{x-3} = -\infty$.

e) $\dfrac{f(x)}{g(x)} = \dfrac{3x+2}{5-2x}$



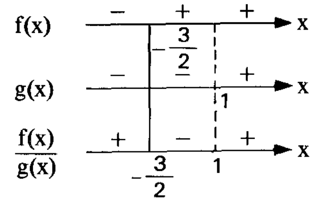
À esquerda de $\dfrac{5}{2}$, $\dfrac{f(x)}{g(x)} > 0$.

Então, $\lim\limits_{x \to \frac{5}{2}^-} \dfrac{3x+2}{5-2x} = +\infty$.

f) Pelo esquema do item anterior, à direita de $\dfrac{5}{2}$, $\dfrac{f(x)}{g(x)} < 0$.

Então, $\lim\limits_{x \to \frac{5}{2}^+} \dfrac{3x+2}{5-2x} = -\infty$.

g) $\dfrac{f(x)}{g(x)} = \dfrac{2x+3}{(x-1)^3}$



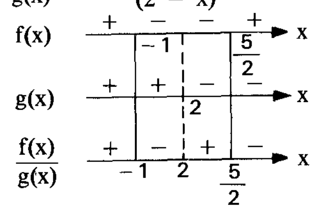
À esquerda de 1, $\dfrac{f(x)}{g(x)} < 0$.

Então, $\lim\limits_{x \to 1^-} \dfrac{2x+3}{(x-1)^3} = -\infty$.

h) Pelo esquema do item anterior, à direita de 1, $\dfrac{f(x)}{g(x)} > 0$.

Então, $\lim\limits_{x \to 1^+} \dfrac{2x+3}{(x-1)^3} = +\infty$.

i) $\dfrac{f(x)}{g(x)} = \dfrac{2x^2-3x-5}{(2-x)^3}$



À esquerda de 2, $\dfrac{f(x)}{g(x)} < 0$.

Então, $\lim\limits_{x \to 2^-} \dfrac{2x^2-3x-5}{(2-x)^3} = -\infty$.

j) Pelo esquema do item anterior, à direita de 2, $\dfrac{f(x)}{g(x)} > 0$.

Então, $\lim\limits_{x \to 2^+} \dfrac{2x^2-3x-5}{(2-x)^3} = +\infty$.

73. Provemos que:

$\forall M > 0, \exists \delta > 0 \mid 0 < |x - 0| < \delta \Rightarrow f(x) > M$.

Como $\lim\limits_{x \to 0} x^2 = 0$, temos:

$\forall \epsilon > 0, \exists \delta > 0 \mid 0 < |x - 0| < \delta \Rightarrow x^2 < \epsilon \Rightarrow \dfrac{1}{x^2} > \dfrac{1}{\epsilon}$.

Chamando $\dfrac{1}{\epsilon} = M$, temos:

$\forall M > 0, \exists \delta > 0 \mid 0 < |x - 0| < \delta \Rightarrow f(x) > M$.

74. a) $\lim\limits_{x \to 0^-} \dfrac{1}{x^3} = -\infty$

Para $x < 0$, $f(x) = \dfrac{1}{x^3} < 0$.

Provemos que:

$\forall M > 0, \exists \delta > 0 \mid 0 < |x - 0| < \delta \Rightarrow |f(x)| > M$.

Como $\lim\limits_{x \to 0} x^3 = 0$, temos:

$\forall \epsilon > 0, \exists \delta > 0 \mid 0 < |x - 0| < \delta \Rightarrow |x^3| < \epsilon \Rightarrow \left|\frac{1}{x^3}\right| > \frac{1}{\epsilon}$.

Chamando $\frac{1}{\epsilon} = M$, temos:

$\forall M > 0, \exists\, \delta > 0 \mid 0 < |x - 0| < \delta \Rightarrow |f(x)| > M$.

b) Analogamente, notando que para $x > 0$, $f(x) = \frac{1}{x^3} > 0$.

76. a) $\lim\limits_{x \to +\infty} (2x + 3) = \lim\limits_{x \to +\infty} (2x) = +\infty$

b) $\lim\limits_{x \to -\infty} (4 - 5x) = \lim\limits_{x \to -\infty} (-5x) = +\infty$

c) $\lim\limits_{x \to +\infty} (5x^2 - 4x + 3) = \lim\limits_{x \to +\infty} (5x^2) = +\infty$

d) $\lim\limits_{x \to +\infty} (4 - x^2) = \lim\limits_{x \to +\infty} (-x^2) = -\infty$

e) $\lim\limits_{x \to -\infty} (3x^3 - 4) = \lim\limits_{x \to -\infty} (3x^3) = -\infty$

f) $\lim\limits_{x \to -\infty} (8 - x^3) = \lim\limits_{x \to -\infty} (-x^3) = +\infty$

77. a) $\lim\limits_{x \to +\infty} (x^n - 1) = \lim\limits_{x \to +\infty} (x^n) = +\infty$

$(n \in \mathbb{N}^*)$

b) $\lim\limits_{x \to -\infty} (1 - x^n) = \lim\limits_{x \to -\infty} (-x^n) = \begin{cases} +\infty, \text{ se } n \text{ é ímpar} \\ -\infty, \text{ se } n \text{ é par} \end{cases}$

$(n \in \mathbb{N}^*)$

c) $\lim\limits_{x \to +\infty} (c \cdot x) = \begin{cases} +\infty, \text{ se } c > 0 \\ -\infty, \text{ se } c < 0 \end{cases}$

$(c \in \mathbb{R}^*)$

d) $\lim\limits_{x \to -\infty} \left(\frac{x}{c}\right) = \begin{cases} +\infty, \text{ se } c < 0 \\ -\infty, \text{ se } c > 0 \end{cases}$

78. a) $\lim\limits_{x \to +\infty} \sqrt{x^2 - 2x + 2} = \lim\limits_{x \to +\infty} \sqrt{x^2} = \lim\limits_{x \to +\infty} |x| = +\infty$

b) $\lim\limits_{x \to -\infty} \sqrt{x^2 - 3x + 5} = \lim\limits_{x \to -\infty} \sqrt{x^2} = \lim\limits_{x \to -\infty} |x| = +\infty$

80. a) $\lim\limits_{x \to +\infty} \frac{3 - 2x}{5x + 1} = \lim\limits_{x \to +\infty} \frac{-2x}{5x} = \frac{-2}{5}$

b) $\lim\limits_{x \to -\infty} \frac{4x - 3}{3x + 2} = \lim\limits_{x \to -\infty} \frac{4x}{3x} = \frac{4}{3}$

c) $\lim\limits_{x \to +\infty} \frac{x^2 - 4}{x + 1} = \lim\limits_{x \to +\infty} \frac{x^2}{x} = \lim\limits_{x \to +\infty} x = +\infty$

d) $\lim\limits_{x \to -\infty} \frac{x^3 - 1}{x^2 + 1} = \lim\limits_{x \to -\infty} \frac{x^3}{x^2} = \lim\limits_{x \to -\infty} x = -\infty$

e) $\lim\limits_{x \to +\infty} \frac{x^2 - 3x + 4}{3x^3 + 5x^2 - 6x + 2} = \lim\limits_{x \to +\infty} \frac{x^2}{3x^3} = \lim\limits_{x \to +\infty} \frac{1}{3x} = 0$

f) $\lim\limits_{x \to -\infty} \frac{x^2 + 4}{8x^3 - 1} = \lim\limits_{x \to -\infty} \frac{x^2}{8x^3} = \lim\limits_{x \to -\infty} \frac{1}{8x} = 0$

g) $\lim\limits_{x \to -\infty} \frac{x^2 + x + 1}{(x + 1)^3 - x^3} = \lim\limits_{x \to -\infty} \frac{x^2 + x + 1}{3x^2 + 3x + 1} =$

$= \lim\limits_{x \to -\infty} \frac{x^2}{3x^2} = \lim\limits_{x \to -\infty} \frac{1}{3} = \frac{1}{3}$

h) $\lim\limits_{x \to +\infty} \frac{(2x - 3)^3}{x(x + 1)(x + 2)} = \lim\limits_{x \to +\infty} \frac{8x^3}{x^3} = \lim\limits_{x \to +\infty} 8 = 8$

(Neste caso não é necessário efetuar todos os cálculos. Basta obter os termos de maior expoente.)

i) $\lim\limits_{x \to -\infty} \frac{(3x + 2)^3}{2x(3x + 1)(4x - 1)} = \lim\limits_{x \to -\infty} \frac{27x^3}{24x^3} = \lim\limits_{x \to -\infty} \frac{27}{24} = \frac{9}{8}$

j) $\lim\limits_{x \to +\infty} \frac{(2x - 3)^3 \cdot (3x - 2)^2}{x^5} = \lim\limits_{x \to +\infty} \frac{72x^5}{x^5} = \lim\limits_{x \to +\infty} 72 = 72$

k) $\lim\limits_{x \to -\infty} \frac{(x + 2)^4 - (x - 1)^4}{(2x + 3)^3} = \lim\limits_{x \to -\infty} \frac{[(x + 2)^2 + (x - 1)^2][(x + 2)^2 - (x - 1)^2]}{(2x + 3)^3} =$

$= \lim\limits_{x \to -\infty} \frac{(2x^2 + 2x + 5)(6x + 3)}{(2x + 3)^3} = \lim\limits_{x \to -\infty} \frac{12x^3}{8x^3} = \lim\limits_{x \to -\infty} \frac{12}{8} = \frac{3}{2}$

82. a) $\lim\limits_{x \to +\infty} \frac{\sqrt{x^2 + x + 1}}{x + 1} = \lim\limits_{x \to +\infty} \frac{\sqrt{x^2\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)}}{x\left(1 + \frac{1}{x}\right)} =$

$= \lim\limits_{x \to +\infty} \frac{|x| \cdot \sqrt{1 + \frac{1}{x} + \frac{1}{x^2}}}{x\left(1 + \frac{1}{x}\right)} = \lim\limits_{x \to +\infty} \frac{\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}}}{1 + \frac{1}{x}} = 1$

b) $\lim\limits_{x \to -\infty} \frac{\sqrt{x^2 + x + 1}}{x + 1} = \lim\limits_{x \to -\infty} \frac{\sqrt{x^2\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)}}{x\left(1 + \frac{1}{x}\right)} =$

$$= \lim_{x\to-\infty} \frac{|x|\cdot\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}}{x\left(1+\frac{1}{x}\right)} = \lim_{x\to-\infty}\frac{-\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}}{1+\frac{1}{x}} = -1$$

c) $$\lim_{x\to+\infty}\frac{2x^2-3x-5}{\sqrt{x^4+1}} = \lim_{x\to+\infty}\frac{2x^2-3x-5}{\sqrt{x^4\left(1+\frac{1}{x^4}\right)}} =$$

$$= \lim_{x\to+\infty}\frac{2x^2-3x-5}{x^2\cdot\sqrt{1+\frac{1}{x^4}}} = \lim_{x\to+\infty}\frac{2x^2}{x^2} = \lim_{x\to+\infty} 2 = 2$$

d) $$\lim_{x\to-\infty}\frac{2x^2-3x-5}{\sqrt{x^4+1}}$$

Já vimos, item *c*, que $\lim\limits_{x\to+\infty}\frac{2x^2-3x-5}{\sqrt{x^4+1}} = \lim\limits_{x\to+\infty} 2$, isto é, não depende de *x*.

Portanto, $\lim\limits_{x\to-\infty} 2 = 2$.

e) $$\lim_{x\to+\infty}\frac{x^2}{1+x\sqrt{x}} = \lim_{x\to+\infty}\frac{x^2}{x\sqrt{x}\left(\frac{1}{x\sqrt{x}}+1\right)} = \lim_{x\to+\infty}\frac{x}{\sqrt{x}\left(\frac{1}{x\sqrt{x}}+1\right)} =$$

$$= \lim_{x\to+\infty}\frac{\sqrt{x}}{\frac{1}{x\sqrt{x}}+1} = \lim_{x\to+\infty}\sqrt{x} = +\infty$$

f) $$\lim_{x\to-\infty}\frac{x+\sqrt[3]{x}}{x^2+1} = \lim_{x\to-\infty}\frac{x\left(1+\frac{\sqrt[3]{x}}{x}\right)}{x^2\left(1+\frac{1}{x^2}\right)} =$$

$$= \lim_{x\to-\infty}\frac{1+\frac{1}{\sqrt[3]{x^2}}}{x\left(1+\frac{1}{x^2}\right)} = \lim_{x\to-\infty}\frac{1}{x} = 0$$

g) $$\lim_{x\to-\infty}\frac{x}{\sqrt[3]{x^3-1\,000}} = \lim_{x\to-\infty}\frac{x}{\sqrt[3]{x^3\left[1-\left(\frac{10}{x}\right)^3\right]}} =$$

$$= \lim_{x\to-\infty}\frac{x}{x\sqrt[3]{1-\left(\frac{10}{x}\right)^3}} = \lim_{x\to-\infty}\frac{1}{\sqrt[3]{1-\left(\frac{10}{x}\right)^3}} = 1$$

h) $$\lim_{x\to+\infty}\frac{\sqrt[3]{x^2+1}}{x+1} = \lim_{x\to+\infty}\frac{\sqrt[3]{x^2\left(1+\frac{1}{x^2}\right)}}{x\left(1+\frac{1}{x}\right)} =$$

$$= \lim_{x\to+\infty}\frac{\sqrt[3]{x^2}\sqrt[3]{1+\frac{1}{x^2}}}{x\left(1+\frac{1}{x}\right)} = \lim_{x\to+\infty}\frac{\sqrt[3]{1+\frac{1}{x^2}}}{\sqrt[3]{x}\left(1+\frac{1}{x}\right)} = 0$$

84. a) $$\lim_{x\to+\infty}\left(\sqrt{x^2+3x+4}-x\right) = \lim_{x\to+\infty}\frac{(\sqrt{x^2+3x+4}-x)(\sqrt{x^2+3x+4}+x)}{\sqrt{x^2+3x+4}+x} =$$

$$= \lim_{x\to+\infty}\frac{3x+4}{\sqrt{x^2+3x+4}+x} = \lim_{x\to+\infty}\frac{x\left(3+\frac{4}{x}\right)}{\sqrt{x^2\left(1+\frac{3}{x}+\frac{4}{x^2}\right)}+x} =$$

$$= \lim_{x\to+\infty}\frac{x\left(3+\frac{4}{x}\right)}{|x|\cdot\sqrt{1+\frac{3}{x}+\frac{4}{x^2}}+x} = \lim_{x\to+\infty}\frac{x\left(3+\frac{4}{x}\right)}{x\left(\sqrt{1+\frac{3}{x}+\frac{4}{x^2}}+1\right)} = \frac{3}{2}$$

b) $$\lim_{x\to-\infty}\left(\sqrt{x^2+3x+4}-x\right) = \lim_{x\to-\infty}\frac{(\sqrt{x^2+3x+4}-x)(\sqrt{x^2+3x+4}+x)}{\sqrt{x^2+3x+4}+x} =$$

$$= \lim_{x\to-\infty}\frac{x\left(3+\frac{4}{x}\right)}{|x|\cdot\sqrt{1+\frac{3}{x}+\frac{4}{x^2}}+x} = \lim_{x\to-\infty}\frac{x\left(3+\frac{4}{x}\right)}{x\left(-\sqrt{1+\frac{3}{x}+\frac{4}{x^2}}+1\right)} = +\infty$$

c) $$\lim_{x\to+\infty}\left(\sqrt{x+4}-\sqrt{x-2}\right) = \lim_{x\to+\infty}\frac{(\sqrt{x+4}-\sqrt{x-2})(\sqrt{x+4}+\sqrt{x-2})}{\sqrt{x+4}+\sqrt{x-2}} =$$

$$= \lim_{x\to+\infty}\frac{6}{\sqrt{x}\left(\sqrt{1+\frac{4}{x}}+\sqrt{1-\frac{2}{x}}\right)} = 0$$

d) $$\lim_{x\to+\infty}\left(\sqrt{x^2-x+1}-x\right) = \lim_{x\to+\infty}\frac{(\sqrt{x^2-x+1}-x)(\sqrt{x^2-x+1}+x)}{\sqrt{x^2-x+1}+x} =$$

$$= \lim_{x\to+\infty}\frac{x\left(-1+\frac{1}{x}\right)}{|x|\cdot\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}+x} = \lim_{x\to+\infty}\frac{-1+\frac{1}{x}}{\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}+1} = \frac{-1}{2}$$

e) $$\lim_{x\to+\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right) = \lim_{x\to+\infty}\frac{(\sqrt{x^2+1}-\sqrt{x^2-1})(\sqrt{x^2+1}+\sqrt{x^2-1})}{\sqrt{x^2+1}+\sqrt{x^2-1}}$$

$$= \lim_{x\to+\infty}\frac{2}{|x|\cdot\left(\sqrt{1+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x^2}}\right)} = 0$$

f) $\lim_{x\to+\infty}(\sqrt{x^2-4x+5}-\sqrt{x^2-3x+4}) =$

$$= \lim_{x\to+\infty}\frac{(\sqrt{x^2-4x+5}-\sqrt{x^2-3x+4})(\sqrt{x^2-4x+5}+\sqrt{x^2-3x+4})}{\sqrt{x^2-4x+5}+\sqrt{x^2-3x+4}} =$$

$$= \lim_{x\to+\infty}\frac{x\left(-1+\frac{1}{x}\right)}{|x|\cdot\left(\sqrt{1-\frac{4}{x}+\frac{5}{x^2}}+\sqrt{1-\frac{3}{x}+\frac{4}{x^2}}\right)} = \frac{-1}{2}$$

g) $\lim_{x\to+\infty}(x-\sqrt{x^2+4}) = \lim_{x\to+\infty}\frac{(x-\sqrt{x^2+4})(x+\sqrt{x^2+4})}{(x+\sqrt{x^2+4})} =$

$$= \lim_{x\to+\infty}\frac{-4}{x+|x|\cdot\sqrt{1+\frac{4}{x^2}}} = \lim_{x\to+\infty}\frac{-4}{x\left(1+\sqrt{1+\frac{4}{x^2}}\right)} = 0$$

h) $\lim_{x\to+\infty}(\sqrt{x^2+ax+b}-x) = \lim_{x\to+\infty}\frac{(\sqrt{x^2+ax+b}-x)(\sqrt{x^2+ax+b}+x)}{\sqrt{x^2+ax+b}+x} =$

$$= \lim_{x\to+\infty}\frac{ax+b}{\sqrt{x^2\left(1+\frac{a}{x}+\frac{b}{x^2}\right)}+x} = \lim_{x\to+\infty}\frac{x\left(a+\frac{b}{x}\right)}{x\cdot\left(\sqrt{1+\frac{a}{x}+\frac{b}{x^2}}+1\right)} = \frac{a}{2}$$

85. a) $\lim_{x\to+\infty}\frac{x+\sqrt[3]{x^3-5x^2-2}}{\sqrt[3]{x^3+1}} = \lim_{x\to+\infty}\frac{x\left(1+\sqrt[3]{1-\frac{5}{x}-\frac{2}{x^3}}\right)}{x\left(\sqrt[3]{1+\frac{1}{x^3}}\right)} = 2$

b) $\lim_{x\to+\infty}\frac{\sqrt{x}-\sqrt{x+1}}{\sqrt{x+2}-\sqrt{x+3}} = \lim_{x\to+\infty}\frac{\sqrt{x}\left(1-\sqrt{1+\frac{1}{x}}\right)}{\sqrt{x}\left(\sqrt{1+\frac{2}{x}}-\sqrt{1+\frac{3}{x}}\right)} =$

$$= \lim_{x\to+\infty}\frac{\left(1-\sqrt{1+\frac{1}{x}}\right)\left(1+\sqrt{1+\frac{1}{x}}\right)\left(\sqrt{1+\frac{2}{x}}+\sqrt{1+\frac{3}{x}}\right)}{\left(\sqrt{1+\frac{2}{x}}-\sqrt{1+\frac{3}{x}}\right)\left(1+\sqrt{1+\frac{1}{x}}\right)\left(\sqrt{1+\frac{2}{x}}+\sqrt{1+\frac{3}{x}}\right)} =$$

$$= \lim_{x\to+\infty}\frac{-\frac{1}{x}\left(\sqrt{1+\frac{2}{x}}+\sqrt{1+\frac{3}{x}}\right)}{\frac{-1}{x}\left(1+\sqrt{1+\frac{1}{x}}\right)} = 1$$

c) $\lim_{x\to+\infty}\frac{\sqrt{x^2+2x+4}-x}{x-\sqrt{x^2-x+1}} =$

$$= \lim_{x\to+\infty}\frac{(\sqrt{x^2+2x+4}-x)(\sqrt{x^2+2x+4}+x)(x+\sqrt{x^2-x+1})}{(x-\sqrt{x^2-x+1})(x+\sqrt{x^2-x+1})(\sqrt{x^2+2x+4}+x)} =$$

$$= \lim_{x\to+\infty}\frac{(2x+4)(x+\sqrt{x^2-x+1})}{(x-1)(\sqrt{x^2+2x+4}+x)} =$$

$$= \lim_{x\to+\infty}\frac{2x^2\cdot\left(1+\frac{2}{x}\right)\left(1+\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}\right)}{x^2\cdot\left(1-\frac{1}{x}\right)\left(\sqrt{1+\frac{2}{x}+\frac{4}{x^2}}+1\right)} = 2$$

86. a) $\lim_{x\to+\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}\right) = \lim_{x\to+\infty}\sqrt{x+\sqrt{x\left(1+\frac{1}{\sqrt{x}}\right)}} =$

$$= \lim_{x\to+\infty}\sqrt{x+\sqrt{x}\cdot\sqrt{1+\frac{1}{\sqrt{x}}}} = \lim_{x\to+\infty}\sqrt{x\cdot\left[1+\frac{1}{\sqrt{x}}\cdot\sqrt{1+\frac{1}{\sqrt{x}}}\right]} = +\infty$$

b) $\lim_{x\to+\infty}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{x}$

Pelo item anterior, $\lim_{x\to+\infty}\sqrt{x+\sqrt{x+\sqrt{x}}} = \lim_{x\to+\infty}\sqrt{x} = +\infty$.

Então, vem:

$$\lim_{x\to+\infty}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{x} = \lim_{x\to+\infty}\frac{\sqrt{x}}{x} = \lim_{x\to+\infty}\frac{1}{\sqrt{x}} = 0.$$

c) $\lim_{x\to+\infty}\frac{\sqrt{x}+\sqrt[3]{x}+\sqrt[4]{x}}{\sqrt{4x+1}} = \lim_{x\to+\infty}\frac{\sqrt{x}\left(1+\frac{1}{\sqrt[6]{x}}+\frac{1}{\sqrt[4]{x}}\right)}{\sqrt{4x\left(1+\frac{1}{4x}\right)}} =$

$$= \lim_{x\to+\infty}\frac{\sqrt{x}\left(1+\frac{1}{\sqrt[6]{x}}+\frac{1}{\sqrt[4]{x}}\right)}{2\sqrt{x}\cdot\sqrt{1+\frac{1}{4x}}} = \frac{1}{2}$$

87. a) Dado um número real $M > 0$, temos:

$\forall M > 0, \exists \sqrt{M} > 0 \mid x > \sqrt{M} \Rightarrow x^2 > M$

então:

$\lim_{x\to+\infty} x^2 = +\infty$.

b) Dado um número real $M > 0$, temos:

$\forall M > 0, \exists -\sqrt{M} < 0 \mid x < -\sqrt{M} \Rightarrow x^2 > M$

então:

$\lim_{x\to-\infty} x^2 = +\infty$.

88. a) Dado um número real $M > 0$, temos:

$\forall M > 0,\ \exists\ \sqrt[3]{M} > 0 \mid x > \sqrt[3]{M} \Rightarrow x^3 > M$

então:

$$\lim_{x \to +\infty} x^3 = +\infty.$$

b) Dado um número real $M < 0$, temos:

$\forall M < 0,\ \exists\ \sqrt[3]{M} < 0 \mid x < \sqrt[3]{M} \Rightarrow x^3 > M$

então:

$$\lim_{x \to -\infty} x^3 = -\infty.$$

Capítulo IV – Complementos sobre limites

90. a) $$\lim_{x\to 0} \frac{\text{sen } 3x}{2x} = \lim_{x\to 0} \left(\frac{3}{2} \cdot \frac{\text{sen } 3x}{3x}\right) = \frac{3}{2}$$

b) $$\lim_{x\to 0} \frac{\text{sen } 2x}{\text{sen } x} = \lim_{x\to 0} \left(2 \cdot \frac{\text{sen } 2x}{2x} \cdot \frac{x}{\text{sen } x}\right) = 2$$

c) $$\lim_{x\to 0} \frac{\text{sen } ax}{bx} = \lim_{x\to 0} \left(\frac{a}{b} \cdot \frac{\text{sen } ax}{ax}\right) = \frac{a}{b}$$

d) $$\lim_{x\to 0} \frac{\text{sen } ax}{\text{sen } bx} = \lim_{x\to 0} \left(\frac{a}{b} \cdot \frac{\text{sen } ax}{ax} \cdot \frac{bx}{\text{sen } bx}\right) = \frac{a}{b}$$

e) $$\lim_{x\to 0} \frac{\text{tg } 2x}{3x} = \lim_{x\to 0} \left(\frac{2}{3} \cdot \frac{\text{sen } 2x}{2x} \cdot \frac{1}{\cos 2x}\right) =$$
$$= \lim_{x\to 0} \left(\frac{2}{3} \cdot \frac{\text{sen } 2x}{2x} \cdot \frac{1}{2\cos^2 x - 1}\right) = \frac{2}{3}$$

f) $$\lim_{x\to 0} \frac{\text{tg } ax}{bx} = \lim_{x\to 0} \left(\frac{a}{b} \cdot \frac{\text{sen } ax}{ax} \cdot \frac{1}{\cos ax}\right)$$
$$= \lim_{x\to 0} \left(\frac{a}{b} \cdot \frac{\text{sen } ax}{ax} \cdot \frac{1}{2\cos^2 ax - 1}\right) = \frac{a}{b}$$

g) $$\lim_{x\to 0} \frac{1 - \cos x}{x} = \lim_{x\to 0} \left(\frac{(1 - \cos x)(1 + \cos x)}{x(1 + \cos x)}\right) =$$
$$= \lim_{x\to 0} \left(\frac{1 - \cos^2 x}{x(1 + \cos x)}\right) = \lim_{x\to 0} \left(\frac{(\text{sen } x)^2}{x^2} \cdot \frac{x}{1 + \cos x}\right) = 0$$

h) $$\lim_{x\to 0} \frac{1 - \sec x}{x^2} = \lim_{x\to 0} \frac{\cos x - 1}{x^2 \cos x} = \lim_{x\to 0} \frac{(\cos x - 1)(\cos x + 1)}{x^2 \cos x (\cos x + 1)} =$$
$$= \lim_{x\to 0} \left(\frac{-\text{sen}^2 x}{x^2} \cdot \frac{1}{\cos x (\cos x + 1)}\right) = \frac{-1}{2}$$

i) $$\lim_{x\to 0} \frac{\text{tg } x + \text{sen } x}{x} = \lim_{x\to 0} \left(\frac{\text{sen } x}{x} \cdot \frac{1}{\cos x} + \frac{\text{sen } x}{x}\right) = 2$$

j) $$\lim_{x\to 0} \frac{1 - \cos x}{x \cdot \text{sen } x} = \lim_{x\to 0} \frac{(1 - \cos x)(1 + \cos x)}{x \cdot \text{sen } x\,(1 + \cos x)} =$$
$$= \lim_{x\to 0} \left(\frac{\text{sen}^2 x}{x^2} \cdot \frac{x}{\text{sen } x} \cdot \frac{1}{1 + \cos x}\right) = \frac{1}{2}$$

92. a) $$\lim_{x\to a} \frac{\cos x - \cos a}{x - a} = \lim_{x\to a} \frac{-2 \cdot \text{sen } \frac{x + a}{2} \cdot \text{sen } \frac{x - a}{2}}{x - a} =$$
$$= \lim_{x\to a} \left(-\frac{\text{sen } \frac{x - a}{2}}{\frac{x - a}{2}} \cdot \text{sen } \frac{x + a}{2}\right) = -\text{sen } \frac{a + a}{2} = -\text{sen } a$$

b) $$\lim_{x\to a} \frac{\text{tg } x - \text{tg } a}{x - a} = \lim_{x\to a} \frac{\frac{\text{sen } (x - a)}{\cos x \cdot \cos a}}{x - a} =$$
$$= \lim_{x\to a} \left(\frac{\text{sen } (x - a)}{x - a} \cdot \frac{1}{\cos x \cdot \cos a}\right) = \frac{1}{\cos^2 a} = \sec^2 a$$

c) $$\lim_{x\to a} \frac{\sec x - \sec a}{x - a} = \lim_{x\to a} \frac{\frac{\cos a - \cos x}{\cos x \cdot \cos a}}{x - a} =$$
$$= \lim_{x\to a} \frac{-2 \cdot \text{sen } \frac{x + a}{2} \cdot \text{sen } \frac{x - a}{2}}{(x - a)(\cos x \cdot \cos a)} = \lim_{x\to a} \left(-\frac{\text{sen } \frac{x - a}{2}}{\frac{x - a}{2}} \cdot \frac{\text{sen } \frac{x + a}{2}}{\cos x \cdot \cos a}\right) =$$
$$= \frac{\text{sen } a}{\cos^2 a} = \sec a \cdot \text{tg } a$$

d) $$\lim_{x\to \frac{\pi}{4}} \frac{\text{sen } x - \cos x}{1 - \text{tg } x} = \lim_{x\to \frac{\pi}{4}} \frac{\text{sen } x - \cos x}{1 - \frac{\text{sen } x}{\cos x}} =$$
$$= \lim_{x\to \frac{\pi}{4}} \frac{-\cos x\,(\cos x - \text{sen } x)}{\cos x - \text{sen } x} = -\cos \frac{\pi}{4} = \frac{-\sqrt{2}}{2}$$

e) $$\lim_{x\to 0} \frac{\text{tg } x - \text{sen } x}{\text{sen}^2 x} = \lim_{x\to 0} \frac{\frac{\text{sen } x}{\cos x} - \text{sen } x}{\text{sen}^2 x} =$$
$$= \lim_{x\to 0} \frac{\text{sen } x\,(1 - \cos x)}{\text{sen}^2 x \cos x} = \lim_{x\to 0} \frac{(1 - \cos x)(1 + \cos x)}{\text{sen } x \cos x\,(1 + \cos x)} =$$
$$= \lim_{x\to 0} \frac{\text{sen}^2 x}{\text{sen } x \cos x\,(1 + \cos x)} = \lim_{x\to 0} \frac{\text{sen } x}{\cos x\,(1 + \cos x)} = 0$$

f) $$\lim_{x\to 0} \frac{\mathrm{sen}\, 3x - \mathrm{sen}\, 2x}{\mathrm{sen}\, x} = \lim_{x\to 0} \frac{(3\,\mathrm{sen}\, x - 4\,\mathrm{sen}^3 x) - 2\,\mathrm{sen}\, x \cos x}{\mathrm{sen}\, x} =$$
$$= \lim_{x\to 0} (3 - 4\,\mathrm{sen}^2 x - 2\cos x) = 3 - 2 = 1$$

g) $$\lim_{x\to 0} \frac{\cos 2x - \cos 3x}{x^2} = \lim_{x\to 0} \frac{-2\,\mathrm{sen}\,\frac{2x+3x}{2}\,\mathrm{sen}\,\frac{2x-3x}{2}}{x^2} =$$
$$= \lim_{x\to 0} \frac{-2\,\mathrm{sen}\,\frac{5x}{2}\,\mathrm{sen}\left(\frac{-x}{2}\right)}{x^2} = \lim_{x\to 0} \frac{2\,\mathrm{sen}\,\frac{5x}{2}\,\mathrm{sen}\,\frac{x}{2}}{x^2} =$$
$$= \lim_{x\to 0} \frac{1}{2}\cdot\frac{5}{2}\cdot 2\,\frac{\mathrm{sen}\,\frac{5x}{2}}{\frac{5x}{2}}\cdot\frac{\mathrm{sen}\,\frac{x}{2}}{\frac{x}{2}} = \frac{5}{2}$$

h) $$\lim_{x\to 0} \frac{\mathrm{sen}\,(x+a) - \mathrm{sen}\, a}{x} = \lim_{x\to 0} \frac{\mathrm{sen}\, x \cos a + \mathrm{sen}\, a \cos x - \mathrm{sen}\, a}{x} =$$
$$= \lim_{x\to 0} \frac{\mathrm{sen}\, x \cos a + \mathrm{sen}\, a (\cos x - 1)}{x} =$$
$$= \lim_{x\to 0} \left(\frac{\mathrm{sen}\, x}{x}\cdot \cos a + \mathrm{sen}\, a \cdot \frac{(\cos x - 1)(\cos x + 1)}{x\,(\cos x + 1)}\right) =$$
$$= \lim_{x\to 0} \left(\frac{\mathrm{sen}\, x}{x}\cdot \cos a - \frac{x\cdot \mathrm{sen}\, a}{(\cos x + 1)}\cdot\frac{\mathrm{sen}^2 x}{x^2}\right) = \cos a$$

i) $$\lim_{x\to 0} \frac{\cos (x+a) - \cos a}{x} = \lim_{x\to 0} \frac{\cos x \cdot \cos a - \mathrm{sen}\, x\cdot \mathrm{sen}\, a - \cos a}{x} =$$
$$= \lim_{x\to 0} \left(\cos a\, \frac{(\cos x - 1)}{x} - \frac{\mathrm{sen}\, x}{x}\cdot \mathrm{sen}\, a\right) =$$
$$= \lim_{x\to 0} \left[\frac{(\cos x - 1)(\cos x + 1)}{x(\cos x + 1)}\cdot \cos a - \frac{\mathrm{sen}\, x}{x}\cdot \mathrm{sen}\, a\right] =$$
$$= \lim_{x\to 0} \left(-x\cdot\frac{\mathrm{sen}^2 x}{x^2}\cdot\frac{1}{\cos x + 1}\cdot \cos a - \frac{\mathrm{sen}\, x}{x}\cdot \mathrm{sen}\, a\right) = -\mathrm{sen}\, a$$

j) $$\lim_{x\to \pi} \frac{1 - \mathrm{sen}\,\frac{x}{2}}{\pi - x} = \lim_{x\to \pi} \frac{\mathrm{sen}\,\frac{\pi}{2} - \mathrm{sen}\,\frac{x}{2}}{\pi - x} =$$
$$= \lim_{x\to \pi} \frac{2\,\mathrm{sen}\,\frac{\frac{\pi}{2} - \frac{x}{2}}{2}\cdot \cos \frac{\frac{\pi}{2} + \frac{x}{2}}{2}}{\pi - x} = \lim_{x\to \pi} \frac{\mathrm{sen}\,\frac{\pi - x}{4}\cdot \cos\frac{\pi + x}{4}}{2\cdot\frac{\pi - x}{4}} =$$
$$= \lim_{x\to \pi} \frac{1}{2}\cdot\frac{\mathrm{sen}\,\frac{\pi - x}{4}}{\frac{\pi - x}{4}}\cdot \cos\frac{\pi + x}{4} = 0$$

k) $$\lim_{x\to \frac{\pi}{3}} \frac{1 - 2\cos x}{\pi - 3x} = \lim_{x\to \frac{\pi}{3}} \frac{2\left(\frac{1}{2} - \cos x\right)}{\pi - 3x} = \lim_{x\to \frac{\pi}{3}} \frac{2\left(\cos\frac{\pi}{3} - \cos x\right)}{\pi - 3x} =$$
$$= \lim_{x\to \frac{\pi}{3}} 2\cdot\frac{-2\,\mathrm{sen}\,\frac{\frac{\pi}{3} + x}{2}\,\mathrm{sen}\,\frac{\frac{\pi}{3} - x}{2}}{\pi - 3x} =$$
$$= \lim_{x\to \frac{\pi}{3}} -4\,\frac{\mathrm{sen}\,\frac{\pi + 3x}{6}\,\mathrm{sen}\,\frac{\pi - 3x}{6}}{6\left(\frac{\pi - 3x}{6}\right)} =$$
$$= \lim_{x\to \frac{\pi}{3}} -\frac{2}{3}\cdot\frac{\mathrm{sen}\,\frac{\pi - 3x}{6}}{\frac{\pi - 3x}{6}}\cdot \mathrm{sen}\,\frac{\pi + 3x}{6} = -\frac{2}{3}\cdot\frac{\sqrt{3}}{2} = \frac{-\sqrt{3}}{3}$$

l) $$\lim_{x\to 1} \frac{1 - x^2}{\mathrm{sen}\, \pi x} = \lim_{x\to 1}\left(\frac{1}{\mathrm{sen}\,\pi x} - \frac{x^2}{\mathrm{sen}\,\pi x}\right) =$$
$$= \lim_{x\to 1}\left(\frac{1}{\pi x}\cdot\frac{\pi x}{\mathrm{sen}\,\pi x} - \frac{\pi x}{\mathrm{sen}\,\pi x}\cdot\frac{x}{\pi}\right) = 0$$

m) $$\lim_{x\to \frac{\pi}{4}} \frac{\cos 2x}{\cos x - \mathrm{sen}\, x} = \lim_{x\to \frac{\pi}{4}} \frac{\cos^2 x - \mathrm{sen}^2 x}{\cos x - \mathrm{sen}\, x} = \lim_{x\to \frac{\pi}{4}} (\cos x + \mathrm{sen}\, x) = \sqrt{2}$$

n) $$\lim_{x\to 0} \frac{1 - \cos^3 x}{\mathrm{sen}^2 x} = \lim_{x\to 0} \frac{(1 - \cos x)(\cos^2 x + \cos x + 1)}{\mathrm{sen}^2 x} =$$
$$= \lim_{x\to 0} \frac{(1 - \cos x)(1 + \cos x)(\cos^2 x + \cos x + 1)}{\mathrm{sen}^2 x\cdot(1 + \cos x)} =$$
$$= \lim_{x\to 0} \left(\frac{\cos^2 x + \cos x + 1}{1 + \cos x}\right) = \frac{3}{2}$$

o) $$\lim_{x\to 0} \frac{\mathrm{sen}\, ax - \mathrm{sen}\, bx}{x} = \lim_{x\to 0} \frac{2\,\mathrm{sen}\,\frac{(a-b)x}{2}\cos\frac{(a+b)x}{2}}{x} =$$
$$= \lim_{x\to 0} \left[(a - b)\cdot\frac{\mathrm{sen}\left(\frac{a-b}{2}\right)x}{\left(\frac{a-b}{2}\right)x}\cdot\cos\left(\frac{a+b}{2}\right)x\right] = a - b$$

p) $$\lim_{x\to 0} \frac{\cos ax - \cos bx}{x} = \lim_{x\to 0} \frac{-2\cdot\mathrm{sen}\left(\frac{a+b}{2}\right)x\cdot\mathrm{sen}\left(\frac{a-b}{2}\right)x}{x} =$$

$$= \lim_{x\to 0} \left(-\left(\frac{a-b}{2}\right)x \cdot (a+b) \frac{\operatorname{sen}\left(\frac{a+b}{2}\right)x}{\left(\frac{a+b}{2}\right)x} \cdot \frac{\operatorname{sen}\left(\frac{a-b}{2}\right)x}{\left(\frac{a-b}{2}\right)x} \right) = 0$$

q) $$\lim_{x\to 0} \frac{x - \operatorname{sen} 2x}{x + \operatorname{sen} 3x} = \lim_{x\to 0} \frac{x\left(1 - \frac{\operatorname{sen} 2x}{x}\right)}{x\left(1 + \frac{\operatorname{sen} 3x}{x}\right)} =$$

$$= \lim_{x\to 0} \frac{1 - 2\,\frac{\operatorname{sen} 2x}{2x}}{1 + 3\,\frac{\operatorname{sen} 3x}{3x}} = \frac{-1}{4}$$

r) $$\lim_{x\to 0} \frac{1 - \cos x}{x^2} = \lim_{x\to 0} \frac{(1 - \cos x)(1 + \cos x)}{x^2(1 + \cos x)} =$$

$$= \lim_{x\to 0} \left(\frac{\operatorname{sen}^2 x}{x^2} \cdot \frac{1}{1 + \cos x}\right) = \frac{1}{2}$$

s) $$\lim_{x\to 1} \frac{\cos \frac{\pi x}{2}}{1 - x} = \lim_{x\to 1} \frac{\operatorname{sen}\left(\frac{\pi}{2} - \frac{\pi x}{2}\right)}{1 - x} =$$

$$= \lim_{x\to 1} \frac{\operatorname{sen} \frac{\pi}{2}(1 - x)}{1 - x} = \lim_{x\to 1} \frac{\frac{\pi}{2} \cdot \operatorname{sen} \frac{\pi}{2}(1 - x)}{\frac{\pi}{2}(1 - x)} = \frac{\pi}{2}$$

t) $$\lim_{x\to 0} \frac{\sqrt{1 + \operatorname{sen} x} - \sqrt{1 - \operatorname{sen} x}}{x} =$$

$$= \lim_{x\to 0} \frac{(\sqrt{1 + \operatorname{sen} x} - \sqrt{1 - \operatorname{sen} x})(\sqrt{1 + \operatorname{sen} x} + \sqrt{1 - \operatorname{sen} x})}{x(\sqrt{1 + \operatorname{sen} x} + \sqrt{1 - \operatorname{sen} x})} =$$

$$= \lim_{x\to 0} 2 \cdot \frac{\operatorname{sen} x}{x} \cdot \frac{1}{\sqrt{1 + \operatorname{sen} x} + \sqrt{1 - \operatorname{sen} x}} = 1$$

93. a) $\lim\limits_{x\to 0} x \cdot \operatorname{sen} \frac{1}{x}$

Fazendo $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$ e para $x \to 0$, $y \to +\infty$.

Temos:

$$\lim_{x\to 0} x \cdot \operatorname{sen} \frac{1}{x} = \lim_{y\to +\infty} \frac{\operatorname{sen} y}{y} = 0.$$

b) $\lim\limits_{x\to +\infty} x \cdot \operatorname{sen} \frac{1}{x}$

Fazendo $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$ e para $x \to +\infty$, $y \to 0$.

Então:

$$\lim_{x\to +\infty} x \cdot \operatorname{sen} \frac{1}{x} = \lim_{y\to 0} \frac{\operatorname{sen} y}{y} = 1.$$

c) $$\lim_{x\to 1} (1 - x) \cdot \operatorname{tg} \frac{\pi x}{2} = \lim_{x\to 1} (1 - x) \cdot \frac{\operatorname{sen} \frac{\pi x}{2}}{\cos \frac{\pi x}{2}} =$$

$$= \underbrace{\lim_{x\to 1} \frac{1 - x}{\cos \frac{\pi x}{2}}}_{\frac{2}{\pi}} \cdot \lim_{x\to 1} \operatorname{sen} \frac{\pi x}{2} = \frac{2}{\pi}$$

(ver ex. 92s)

d) $$\lim_{x\to 0} \operatorname{cotg} 2x \cdot \operatorname{cotg}\left(\frac{\pi}{2} - x\right) = \lim_{x\to 0} \operatorname{cotg} 2x \cdot \operatorname{tg} x =$$

$$= \lim_{x\to 0} \frac{\cos 2x}{\operatorname{sen} 2x} \cdot \frac{\operatorname{sen} x}{\cos x} = \lim_{x\to 0} \frac{\cos^2 x - \operatorname{sen}^2 x}{\cos x} \cdot \frac{1}{2\cos x} =$$

$$= \lim_{x\to 0} \left(\frac{1}{2} - \frac{1}{2}\operatorname{tg}^2 x\right) = \frac{1}{2}$$

97. a) $$\lim_{x\to 2} 3^{\frac{x^2-4}{x-2}} = 3^{\lim\limits_{x\to 2} \frac{x^2-4}{x-2}} = 3^{\lim\limits_{x\to 2} (x+2)} = 3^4 = 81$$

b) $$\lim_{x\to 1} \left(\frac{1}{2}\right)^{\frac{1-x^2}{x-1}} = \left(\frac{1}{2}\right)^{\lim\limits_{x\to 1} \frac{1-x^2}{x-1}} = \left(\frac{1}{2}\right)^{\lim\limits_{x\to 1} [-(1+x)]} = \left(\frac{1}{2}\right)^{-2} = 4$$

c) $$\lim_{x\to 1} 2^{\frac{x^3-3x+2}{x^2+x-2}} = 2^{\lim\limits_{x\to 1} \frac{x^3-3x+2}{x^2+x-2}} = 2^{\lim\limits_{x\to 1} \frac{(x-1)(x^2+x-2)}{x^2+x-2}} =$$

$$= 2^{\lim\limits_{x\to 1} (x-1)} = 2^0 = 1$$

d) $$\lim_{x\to 2} \left(\frac{1}{3}\right)^{\frac{x^3-6x^2+11x-6}{x^2-3x+2}} = \left(\frac{1}{3}\right)^{\lim\limits_{x\to 2} \frac{x^3-6x^2+11x-6}{x^2-3x+2}} =$$

$$= \left(\frac{1}{3}\right)^{\lim\limits_{x\to 2} \frac{(x-1)(x-2)(x-3)}{(x-1)(x-2)}} = \left(\frac{1}{3}\right)^{\lim\limits_{x\to 2} (x-3)} = \left(\frac{1}{3}\right)^{-1} = 3$$

e) $$\lim_{x\to 1} e^{\frac{x-1}{\sqrt{x}-1}} = e^{\lim\limits_{x\to 1} \frac{x-1}{\sqrt{x}-1}} = e^{\lim\limits_{x\to 1} \frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)}} = e^{\lim\limits_{x\to 1} (\sqrt{x}+1)} = e^2$$

f) $\lim\limits_{x\to4}\left(\dfrac{1}{e}\right)^{\frac{x^2-5x+4}{\sqrt{x}-2}} = \left(\dfrac{1}{e}\right)^{\lim\limits_{x\to4}\frac{x^2-5x+4}{\sqrt{x}-2}} =$

$= \left(\dfrac{1}{e}\right)^{\lim\limits_{x\to4}\frac{(x-1)(x-4)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)}} = \left(\dfrac{1}{e}\right)^{\lim\limits_{x\to4}(x-1)(\sqrt{x}+2)} = \left(\dfrac{1}{e}\right)^{12} = e^{-12}$

101. a) $\lim\limits_{x\to-1}\log_3\dfrac{x^2+3x+2}{x^2+5x+4} = \lim\limits_{x\to-1}\log_3\dfrac{(x+1)(x+2)}{(x+1)(x+4)} =$

$= \lim\limits_{x\to-1}\log_3\left(\dfrac{x+2}{x+4}\right) = \log_3\dfrac{1}{3} = -1$

b) $\lim\limits_{x\to0}\log\dfrac{x-x^3}{x^2+x} = \lim\limits_{x\to0}\log\dfrac{x(1-x)(1+x)}{x\,(x+1)} = \lim\limits_{x\to0}\log(1-x) = 0$

c) $\lim\limits_{x\to3}\ell n\dfrac{x-3}{\sqrt{x+1}-2} = \lim\limits_{x\to3}\ell n\dfrac{(x-3)(\sqrt{x+1}+2)}{(\sqrt{x+1}-2)(\sqrt{x+1}+2)} =$
$= \lim\limits_{x\to3}\ell n(\sqrt{x+1}+2) = \ell n\,4$

d) $\lim\limits_{x\to-2}\log\dfrac{3-\sqrt{1-4x}}{\sqrt{6+x}-2} =$

$= \lim\limits_{x\to-2}\log\dfrac{(3-\sqrt{1-4x})(3+\sqrt{1-4x})(\sqrt{6+x}+2)}{(\sqrt{6+x}-2)(\sqrt{6+x}+2)(3+\sqrt{1-4x})} =$

$= \lim\limits_{x\to-2}\log\dfrac{4(2+x)(\sqrt{6+x}+2)}{(2+x)(3+\sqrt{1-4x})} = \lim\limits_{x\to-2}\log\dfrac{4(\sqrt{6+x}+2)}{3+\sqrt{1-4x}} = \log\dfrac{8}{3}$

103. a) $\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{x}\right)^{3x} = \lim\limits_{x\to+\infty}\left[\left(1+\dfrac{1}{x}\right)^x\right]^3 = e^3$

b) $\lim\limits_{x\to-\infty}\left(1+\dfrac{1}{x}\right)^{x+2} = \lim\limits_{x\to-\infty}\left[\left(1+\dfrac{1}{x}\right)^x\cdot\left(1+\dfrac{1}{x}\right)^2\right] = e\cdot1^2 = e$

c) $\lim\limits_{x\to+\infty}\left(1+\dfrac{4}{x}\right)^x$

Fazendo $\dfrac{x}{4} = y$, temos:

$\lim\limits_{x\to+\infty}\left(1+\dfrac{4}{x}\right)^x = \lim\limits_{y\to+\infty}\left(1+\dfrac{1}{y}\right)^{4y} = \lim\limits_{y\to+\infty}\left[\left(1+\dfrac{1}{y}\right)^y\right]^4 = e^4$

d) $\lim\limits_{x\to-\infty}\left(1+\dfrac{2}{x}\right)^{3x}$

Fazendo $\dfrac{x}{2} = y$, temos:

$\lim\limits_{x\to-\infty}\left[\left(1+\dfrac{2}{x}\right)^x\right]^3 = \lim\limits_{y\to-\infty}\left[\left(1+\dfrac{1}{y}\right)^{2y}\right]^3 = \lim\limits_{y\to-\infty}\left[\left(1+\dfrac{1}{y}\right)^y\right]^6 = e^6$

e) $\lim\limits_{x\to-\infty}\left(1+\dfrac{3}{x}\right)^{\frac{x}{4}}$

Fazendo $\dfrac{x}{3} = y$, temos:

$\lim\limits_{x\to-\infty}\left[\left(1+\dfrac{3}{x}\right)^x\right]^{\frac14} = \lim\limits_{y\to-\infty}\left[\left(1+\dfrac{1}{y}\right)^{3y}\right]^{\frac14} = \lim\limits_{y\to-\infty}\left[\left(1+\dfrac{1}{y}\right)^y\right]^{\frac34} = e^{\frac34}$

f) $\lim\limits_{x\to+\infty}\left(1+\dfrac{a}{x}\right)^x$

Fazendo $\dfrac{x}{a} = y$, temos:

$\lim\limits_{x\to+\infty}\left(1+\dfrac{a}{x}\right)^x = \lim\limits_{y\to+\infty}\left(1+\dfrac{1}{y}\right)^{ay} = \lim\limits_{y\to+\infty}\left[\left(1+\dfrac{1}{y}\right)^y\right]^a = e^a$

g) $\lim\limits_{x\to-\infty}\left(1+\dfrac{a}{x}\right)^{bx}$

Fazendo $\dfrac{x}{a} = y$, temos:

$\lim\limits_{x\to-\infty}\left[\left(1+\dfrac{a}{x}\right)^x\right]^b = \lim\limits_{y\to-\infty}\left[\left(1+\dfrac{1}{y}\right)^{ay}\right]^b = \lim\limits_{y\to-\infty}\left[\left(1+\dfrac{1}{y}\right)^y\right]^{ab} = e^{ab}$

h) $\lim\limits_{x\to+\infty}\left(\dfrac{x}{x+1}\right)^x = \lim\limits_{x\to+\infty}\left(\dfrac{1}{\frac{x+1}{x}}\right)^x = \lim\limits_{x\to+\infty}\left(\dfrac{1}{\left(1+\frac{1}{x}\right)^x}\right) = \dfrac{1}{e}$

104. a) $\lim\limits_{x\to+\infty}\left(1-\dfrac{1}{x}\right)^x = \lim\limits_{x\to+\infty}\left[1+\left(\dfrac{-1}{x}\right)\right]^x$

Fazendo $\dfrac{-1}{x} = y$, para $x\to+\infty$, $y\to0$.

Assim:

$\lim\limits_{x\to+\infty}\left[1+\left(\dfrac{-1}{x}\right)\right]^x = \lim\limits_{y\to0}(1+y)^{-\frac{1}{y}} = \lim\limits_{y\to0}\left[(1+y)^{\frac{1}{y}}\right]^{-1} = e^{-1} = \dfrac{1}{e}$

b) $\lim\limits_{x\to-\infty}\left(1-\dfrac{2}{x}\right)^x = \lim\limits_{x\to-\infty}\left[1+\left(\dfrac{-2}{x}\right)^x\right]$

Fazendo $\dfrac{-2}{x} = y$, para $x\to-\infty$, $y\to0$.

$\lim\limits_{x\to-\infty}\left[1+\left(\dfrac{-2}{x}\right)^x\right] = \lim\limits_{y\to0}\left[(1+y)^{-\frac{2}{y}}\right] = \lim\limits_{y\to0}\left[(1+y)^{\frac{1}{y}}\right]^{-2} = e^{-2} = \dfrac{1}{e^2}$

c) $\lim\limits_{x\to-\infty}\left(1-\dfrac{1}{x}\right)^{3x} = \lim\limits_{x\to-\infty}\left[\left(1+\left(\dfrac{-1}{x}\right)\right)^x\right]^3$

Fazendo $\frac{-1}{x} = y$, para $x \to -\infty$, $y \to 0$.

$$\lim_{x\to-\infty}\left[\left(1+\left(\frac{-1}{x}\right)\right)^x\right]^3 = \lim_{y\to0}\left[(1+y)^{-\frac{1}{y}}\right]^3 = \lim_{y\to0}\left[(1+y)^{\frac{1}{y}}\right]^{-3} = e^{-3} = \frac{1}{e^3}$$

d) $\lim_{x\to+\infty}\left(1-\frac{3}{x}\right)^{2x} = \lim_{x\to+\infty}\left[\left(1+\left(\frac{-3}{x}\right)\right)^x\right]^2$

Fazendo $\frac{-3}{x} = y$, para $x \to +\infty$, $y \to 0$.

$$\lim_{x\to+\infty}\left[\left(1+\left(\frac{-3}{x}\right)\right)^x\right]^2 = \lim_{y\to0}\left[(1+y)^{-\frac{3}{y}}\right]^2 = \lim_{y\to0}\left[(1+y)^{\frac{1}{y}}\right]^{-6} = e^{-6} = \frac{1}{e^6}$$

106. a) $\lim_{x\to+\infty}\left(\frac{x+4}{x-3}\right)^x = \lim_{x\to+\infty}\left(\frac{\frac{x+4}{x}}{\frac{x-3}{x}}\right)^x = \lim_{x\to+\infty}\left(\frac{1+\frac{4}{x}}{1-\frac{3}{x}}\right)^x =$

$$= \frac{\lim_{x\to+\infty}\left(1+\frac{4}{x}\right)^x}{\lim_{x\to+\infty}\left(1-\frac{3}{x}\right)^x} = \frac{e^4}{\frac{1}{e^3}} = e^7$$

b) $\lim_{x\to-\infty}\left(\frac{x+2}{x+1}\right)^x = \lim_{x\to-\infty}\left(\frac{\frac{x+2}{x}}{\frac{x+1}{x}}\right)^x = \lim_{x\to-\infty}\left(\frac{1+\frac{2}{x}}{1+\frac{1}{x}}\right)^x =$

$$= \frac{\lim_{x\to-\infty}\left(1+\frac{2}{x}\right)^x}{\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x} = \frac{e^2}{e} = e$$

c) $\lim_{x\to-\infty}\left(\frac{x-3}{x+2}\right)^x = \lim_{x\to-\infty}\left(\frac{\frac{x-3}{x}}{\frac{x+2}{x}}\right)^x = \lim_{x\to-\infty}\frac{\left(1-\frac{3}{x}\right)^x}{\left(1+\frac{2}{x}\right)^x} =$

$$= \frac{\lim_{x\to-\infty}\left(1-\frac{3}{x}\right)^x}{\lim_{x\to-\infty}\left(1+\frac{2}{x}\right)^x} = \frac{\frac{1}{e^3}}{e^2} = \frac{1}{e^5}$$

d) $\lim_{x\to+\infty}\left(\frac{x-4}{x-1}\right)^{x+3} = \lim_{x\to+\infty}\left(\frac{\frac{x-4}{x}}{\frac{x-1}{x}}\right)^{x+3} = \lim_{x\to+\infty}\left(\frac{1-\frac{4}{x}}{1-\frac{1}{x}}\right)^{x+3} =$

$$= \lim_{x\to+\infty}\left(\frac{1-\frac{4}{x}}{1-\frac{1}{x}}\right)^x \cdot \lim_{x\to+\infty}\left(\frac{1-\frac{4}{x}}{1-\frac{1}{x}}\right)^3 =$$

$$= \frac{\lim_{x\to+\infty}\left(1-\frac{4}{x}\right)^x}{\lim_{x\to+\infty}\left(1-\frac{1}{x}\right)^x}\cdot 1^3 = \frac{\frac{1}{e^4}}{\frac{1}{e}} = \frac{1}{e^3}$$

e) $\lim_{x\to+\infty}\left(\frac{x^2+1}{x^2-3}\right)^{x^2}$

Fazendo $x^2 = y$, $x \to +\infty$, $y \to +\infty$.

$$\lim_{y\to+\infty}\left(\frac{y+1}{y-3}\right)^y = \lim_{y\to+\infty}\left(\frac{\frac{y+1}{y}}{\frac{y-3}{y}}\right)^y = \lim_{y\to+\infty}\left(\frac{1+\frac{1}{y}}{1-\frac{3}{y}}\right)^y =$$

$$= \frac{\lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^y}{\lim_{y\to+\infty}\left(1-\frac{3}{y}\right)^y} = \frac{e}{\frac{1}{e^3}} = e^4$$

107. a) $\lim_{x\to+\infty}\left(\frac{2x+3}{2x+1}\right)^x = \lim_{x\to+\infty}\left(\frac{2x\left(1+\frac{3}{2x}\right)}{2x\left(1+\frac{1}{2x}\right)}\right)^x =$

$$= \lim_{x\to+\infty}\left(\frac{1+\frac{\frac{3}{2}}{x}}{1+\frac{\frac{1}{2}}{x}}\right)^x = \frac{\lim_{x\to+\infty}\left(1+\frac{\frac{3}{2}}{x}\right)^x}{\lim_{x\to+\infty}\left(1+\frac{\frac{1}{2}}{x}\right)^x} = \frac{e^{\frac{3}{2}}}{e^{\frac{1}{2}}} = e$$

b) $\lim_{x\to-\infty}\left(\frac{2x-1}{2x+1}\right)^x = \lim_{x\to-\infty}\left(\frac{2x\left(1-\frac{1}{2x}\right)}{2x\left(1+\frac{1}{2x}\right)}\right)^x =$

$$= \lim_{x\to-\infty}\left(\frac{1-\frac{\frac{1}{2}}{x}}{1+\frac{\frac{1}{2}}{x}}\right)^x = \frac{\lim_{x\to-\infty}\left(1-\frac{\frac{1}{2}}{x}\right)^x}{\lim_{x\to-\infty}\left(1+\frac{\frac{1}{2}}{x}\right)^x} = \frac{\left(\frac{1}{e}\right)^{\frac{1}{2}}}{e^{\frac{1}{2}}} = \frac{1}{e^1} = e^{-1}$$

c) $\lim\limits_{x\to-\infty}\left(\dfrac{3x+2}{3x-1}\right)^{2x} = \lim\limits_{x\to-\infty}\left(\dfrac{3x\left(1+\dfrac{2}{3x}\right)}{3x\left(1-\dfrac{1}{3x}\right)}\right)^{2x} =$

$$= \lim_{x\to-\infty}\left[\left(\frac{1+\dfrac{\frac{2}{3}}{x}}{1-\dfrac{\frac{1}{3}}{x}}\right)^{x}\right]^2 = \frac{\lim\limits_{x\to-\infty}\left[\left(1+\dfrac{\frac{2}{3}}{x}\right)^x\right]^2}{\lim\limits_{x\to-\infty}\left[\left(1-\dfrac{\frac{1}{3}}{x}\right)^x\right]^2} = \frac{\left(e^{\frac{2}{3}}\right)^2}{\left(\dfrac{1}{e^{\frac{1}{3}}}\right)^2} = (e^1)^2 = e^2$$

108. a) $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{x}$

Fazendo $e^{2x} - 1 = w \Rightarrow x = \dfrac{1}{2}\ell n\,(w+1)$ e para $x \to 0,\ w \to 0.$

$$\lim_{x\to 0}\frac{e^{2x}-1}{x} = \lim_{w\to 0}\frac{w}{\frac{1}{2}\ell n\,(w+1)} = \lim_{w\to 0}\frac{2}{\frac{1}{w}\ell n\,(w+1)} =$$

$$= \lim_{w\to 0}\frac{2}{\ell n\,(w+1)^{\frac{1}{w}}} = \frac{\lim\limits_{w\to 0} 2}{\lim\limits_{w\to 0}\left[\ell n\,(1+w)^{\frac{1}{w}}\right]} =$$

$$= \frac{\lim\limits_{w\to 0} 2}{\ell n\left[\lim\limits_{w\to 0}(1+w)^{\frac{1}{w}}\right]} = \frac{2}{\ell n\, e} = 2$$

b) $\lim\limits_{x\to 0}\dfrac{2^{3x}-1}{x}$

Fazendo $2^{3x} - 1 = w \Rightarrow x = \dfrac{\ell n\,(w+1)}{3\,\ell n\,2}$ e para $x \to 0,\ w \to 0.$

$$\lim_{x\to 0}\frac{2^{3x}-1}{x} = \lim_{w\to 0}\frac{3w\,\ell n\,2}{\ell n\,(w+1)} = \lim_{w\to 0}\frac{3\,\ell n\,2}{\frac{1}{w}\ell n\,(1+w)} =$$

$$= \frac{\lim\limits_{w\to 0} 3\,\ell n\,2}{\lim\limits_{w\to 0}\left[\ell n\,(1+w)^{\frac{1}{w}}\right]} = \frac{\lim\limits_{w\to 0} 3\,\ell n\,2}{\ell n\left[\lim\limits_{w\to 0}(1+w)^{\frac{1}{w}}\right]} = 3\,\ell n\,2$$

c) $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{e^{3x}-1} = \lim\limits_{x\to 0}\dfrac{e^{2x}-1}{x}\cdot\dfrac{x}{e^{3x}-1} =$

$$= \left(\lim_{x\to 0}\frac{e^{2x}-1}{x}\right)\cdot\left(\lim_{x\to 0}\frac{1}{\dfrac{e^{3x}-1}{x}}\right) = \underset{\text{(item } a)}{2}\cdot\underset{\text{(análogo ao item } a)}{\frac{1}{3}} = \frac{2}{3}$$

d) $\lim\limits_{x\to 0}\dfrac{3^{2x}-1}{2^{5x}-1} = \lim\limits_{x\to 0}\left(\dfrac{3^{2x}-1}{x}\cdot\dfrac{1}{\dfrac{2^{5x}-1}{x}}\right) =$

$$= \left(\lim_{x\to 0}\frac{3^{2x}-1}{x}\right)\cdot\left(\lim_{x\to 0}\frac{1}{\dfrac{2^{5x}-1}{x}}\right) = \underset{\text{(item } b)}{2\,\ell n\,3}\cdot\underset{\text{(análogo ao item } b)}{\frac{1}{5\,\ell n\,2}} = \frac{2\,\ell n\,3}{5\,\ell n\,2}$$

e) $\lim\limits_{x\to 2}\dfrac{e^x - e^2}{x-2}$

Fazendo $e^x - e^2 = y \Rightarrow x = \ell n(y + e^2)$ e, para $x \to 2,\ y \to 0.$

$$\lim_{x\to 2}\frac{e^x-e^2}{x-2} = \lim_{y\to 0}\frac{y}{\ell n\,(y+e^2) - 2} = \lim_{y\to 0}\frac{y}{\ell n\,(y+e^2) - \ell n\,e^2} =$$

$$= \lim_{y\to 0}\frac{y}{\ell n\left(\dfrac{y+e^2}{e^2}\right)} = \lim_{y\to 0}\frac{y}{\ell n\left(1+\dfrac{y}{e^2}\right)}$$

Fazendo $\dfrac{y}{e^2} = w$, para $y \to 0,\ w \to 0.$

Então, vem:

$$\lim_{w\to 0}\frac{we^2}{\ell n\,(1+w)} = \lim_{w\to 0}\frac{e^2}{\frac{1}{w}\ell n\,(1+w)} = \lim_{w\to 0}\frac{e^2}{\ell n\,(1+w)^{\frac{1}{w}}} =$$

$$= \frac{\lim\limits_{w\to 0} e^2}{\ell n\left[\lim\limits_{w\to 0}(1+w)^{\frac{1}{w}}\right]} = e^2$$

f) $\lim\limits_{x\to a}\dfrac{e^x - e^a}{x-a}$

Idêntico ao exercício anterior, em que $a = 2.$

Portanto, $\lim\limits_{x\to a}\dfrac{e^x - e^a}{x-a} = e^a.$

g) $\lim\limits_{x\to 0}\dfrac{2^x - 2^a}{x-a}$

Fazendo $2^x - 2^a = y$, para $x \to a,\ y \to 0$ e $x = \dfrac{\ell n\,(y+2^a)}{\ell n\,2}.$

$$\lim_{y\to 0}\frac{y}{\dfrac{\ell n\,(y+2^a)}{\ell n\,2} - a} = \lim_{y\to 0}\frac{y\,\ell n\,2}{\ell n\,(y+2^a) - \ell n\,2^a} =$$

$$= \lim_{y\to 0}\frac{y\,\ell n\,2}{\ell n\left(\dfrac{y+2^a}{2^a}\right)} = \lim_{y\to 0}\frac{\ell n\,2}{\dfrac{1}{y}\ell n\left(\dfrac{y}{2^a}+1\right)} = \lim_{y\to 0}\frac{\ell n\,2}{\ell n\left(\dfrac{y}{2^a}+1\right)^{\frac{1}{y}}}$$

Fazendo $\frac{y}{2^a} = w$, para $y \to 0$, $w \to 0$.

$$\lim_{w\to 0} \frac{\ell n\, 2}{\ell n \left[(w+1)^{\frac{1}{w}}\right]^{\frac{1}{2^a}}} = \frac{\lim_{w\to 0}(\ell n\, 2)}{\ell n \left\{\lim_{w\to 0}\left[(1+w)^{\frac{1}{w}}\right]^{\frac{1}{2^a}}\right\}} =$$

$$= \frac{\ell n\, 2}{\frac{1}{2^a}} = 2^a \cdot \ell n\, 2$$

109. a) $\lim_{x\to 0} \frac{\ell n\,(1+x)}{x} = \lim_{x\to 0} \frac{1}{x} \ell n\,(1+x) = \lim_{x\to 0} \ell n\,(1+x)^{\frac{1}{x}} =$

$= \ell n \left[\lim_{x\to 0} (1+x)^{\frac{1}{x}}\right] = \ell n\, e = 1$

b) $\lim_{x\to 0} \frac{\log\,(1+x)}{x} = \lim_{x\to 0} \frac{1}{x} \cdot \log\,(1+x) = \lim_{x\to 0} \log\,(1+x)^{\frac{1}{x}} =$

$= \log \left[\lim_{x\to 0} (1+x)^{\frac{1}{x}}\right] = \log e$

c) $\lim_{x\to 0} \frac{\ell n\,(1+2x)}{x} = \lim_{x\to 0} \frac{1}{x} \ell n\,(1+2x) = \lim_{x\to 0} \ell n\,(1+2x)^{\frac{1}{x}}$

Fazendo $2x = y$, para $x \to 0$, $y \to 0$.

$\lim_{y\to 0} \ell n\,(1+y)^{\frac{2}{y}} = \ell n \left\{\lim_{y\to 0} \left[(1+y)^{\frac{1}{y}}\right]^2\right\} = \ell n\, e^2 = 2$

d) $\lim_{x\to 0} \frac{\log\,(1+3x)}{x} = \lim_{x\to 0} \frac{1}{x} \log\,(1+3x) = \lim_{x\to 0} \log\,(1+3x)^{\frac{1}{x}}$

Fazendo $3x = y$, para $x \to 0$, $y \to 0$.

$\lim_{y\to 0} \log\,(1+y)^{\frac{3}{y}} = \log \left\{\lim_{y\to 0} \left[(1+y)^{\frac{1}{y}}\right]^3\right\} = \log e^3 =$

$= 3 \log e = 3 \frac{\ell n\, e}{\ell n\, 10} = \frac{3}{\ell n\, 10}$

110. $\lim_{x\to 0} \sqrt[x]{1-2x} = \lim_{x\to 0} (1-2x)^{\frac{1}{x}}$

Fazendo $2x = y$, para $x \to 0$, $y \to 0$.

$\lim_{y\to 0} (1-y)^{\frac{2}{y}} = \lim_{y\to 0} \left[(1-y)^{\frac{1}{y}}\right]^2 = \lim_{y\to 0} \left\{[1+(-y)]^{\frac{1}{y}}\right\}^2 = \frac{1}{e^2}$

Capítulo V — Continuidade

112. a) $f(0) = 3$

$$\left.\begin{aligned} \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} 3 = 3 \\ \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} 2 = 2 \end{aligned}\right\} \Rightarrow \nexists \lim_{x\to 0} f(x)$$

$f(x)$ não é contínua em $x = 0$.

b) $f(-2) = 4$

$$\left.\begin{aligned} \lim_{x\to -2^+} f(x) &= \lim_{x\to -2^+} \frac{x^2-4}{x+2} = \lim_{x\to -2^+} (x-2) = -4 \\ \lim_{x\to -2^-} f(x) &= \lim_{x\to -2^-} \frac{x^2-4}{x+2} = \lim_{x\to -2^-} (x-2) = -4 \end{aligned}\right\} \Rightarrow \lim_{x\to -2} f(x) = -4$$

$\lim_{x\to -2} f(x) \neq f(-2)$. Portanto, $f(x)$ não é contínua em $x = -2$.

c) $f(1) = -2$

$$\left.\begin{aligned} \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} \frac{1-x^2}{x-1} = \lim_{x\to 1^+} [-(1+x)] = -2 \\ \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} \frac{1-x^2}{x-1} = \lim_{x\to 1^-} [-(1+x)] = -2 \end{aligned}\right\} \Rightarrow \lim_{x\to 1} f(x) = -2$$

$\lim_{x\to 1} f(x) = f(1) \Rightarrow f(x)$ é contínua em $x = 1$.

d) $f(-1) = 1$

$$\left.\begin{aligned} \lim_{x\to -1^+} f(x) &= \lim_{x\to -1^+} \frac{x^3+1}{x+1} = \lim_{x\to -1^+} (x^2-x+1) = 3 \\ \lim_{x\to -1^-} f(x) &= \lim_{x\to -1^-} \frac{x^3+1}{x+1} = \lim_{x\to -1^-} (x^2-x+1) = 3 \end{aligned}\right\} \Rightarrow \lim_{x\to -1} f(x) = 3$$

$\lim_{x\to -1} f(x) \neq f(-1) \Rightarrow f(x)$ não é contínua em $x = -1$.

113. a) $f(-2) = 3 \cdot (-2) + 2 = -4$

$$\left.\begin{aligned} \lim_{x\to -2^+} f(x) &= \lim_{x\to -2^+} (3x+2) = -4 \\ \lim_{x\to -2^-} f(x) &= \lim_{x\to -2^-} (-2x) = +4 \end{aligned}\right\} \Rightarrow \nexists \lim_{x\to -2} f(x)$$

Portanto, $f(x)$ não é contínua em $x = -2$.

b) $f(1) = 1^2 + 4 \cdot 1 - 5 = 0$

$$\left.\begin{array}{l} \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (x^2 - 3x + 2) = 0 \\ \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (x^2 + 4x - 5) = 0 \end{array}\right\} \Rightarrow \lim_{x\to 1} f(x) = 0$$

$\lim\limits_{x\to 1} f(x) = f(1) \Rightarrow f(x)$ é contínua em $x = 1$.

c) $f(4) = 2$

$$\left.\begin{array}{l} \lim\limits_{x\to 4^+} f(x) = \lim\limits_{x\to 4^+} (3x - 10) = 2 \\ \lim\limits_{x\to 4^-} f(x) = \lim\limits_{x\to 4^-} (10 - 2x) = 2 \end{array}\right\} \Rightarrow \lim_{x\to 4} f(x) = 2$$

$\lim\limits_{x\to 4} f(x) = f(4) \Rightarrow f(x)$ é contínua em $x = 4$.

d) $f(1) = 2$

$$\left.\begin{array}{l} \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (2x^2 - 3x + 2) = 1 \\ \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (2 - x^2) = 1 \end{array}\right\} \Rightarrow \lim_{x\to 1} f(x) = 1$$

$\lim\limits_{x\to 1} f(x) = f(1) \neq f(x)$ não é contínua em $x = 1$.

114. a) $f(0) = 1$

$$\left.\begin{array}{l} \lim\limits_{x\to 0^+} \dfrac{\text{sen } x}{x} = 1 \\ \lim\limits_{x\to 0^-} \dfrac{\text{sen } x}{x} = 1 \end{array}\right\} \lim_{x\to 0} \frac{\text{sen } x}{x} = 1$$

$\lim\limits_{x\to 0} f(x) = f(0) \Rightarrow f(x)$ é contínua em $x = 0$.

b) $f(0) = 0$

$$\lim_{x\to 0^+} \frac{1 - \cos x}{x} = \lim_{x\to 0^+} \frac{(1 - \cos x)(1 + \cos x)}{x(1 + \cos x)} = \lim_{x\to 0^+} \frac{1 - \cos^2 x}{x(1 + \cos x)} =$$

$$= \lim_{x\to 0^+} \frac{\text{sen}^2 x}{x(1 + \cos x)} = \lim_{x\to 0^+} \frac{\text{sen}^2 x}{x^2} \cdot \frac{x}{(1 + \cos x)} = 0 = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x)$$

$\lim\limits_{x\to 0} f(x) = f(0) \Rightarrow f(x)$ é contínua em $x = 0$.

c) $f(0) = 1$

$$\lim_{x\to 0^+} \frac{1 - \cos x}{\text{sen } x} = \lim_{x\to 0^+} \frac{(1 - \cos x)(1 + \cos x)}{\text{sen } x(1 + \cos x)} = \lim_{x\to 0^+} \frac{\text{sen}^2 x}{\text{sen } x(1 + \cos x)} =$$

$$= \lim_{x\to 0^+} \frac{\text{sen } x}{1 + \cos x} = 0 = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x)$$

$\lim\limits_{x\to 0} f(x) \neq f(0) \Rightarrow f(x)$ não é contínua em $x = 0$.

d) $f(0) = 1$

$$\lim_{x\to 0^+} \frac{x - \text{sen } x}{x + \text{sen } x} = \lim_{x\to 0^+} \frac{x\left(1 - \dfrac{\text{sen } x}{x}\right)}{x\left(1 + \dfrac{\text{sen } x}{x}\right)} = \lim_{x\to 0^+} \frac{\left(1 - \dfrac{\text{sen } x}{x}\right)\left(1 + \dfrac{\text{sen } x}{x}\right)}{\left(1 + \dfrac{\text{sen } x}{x}\right)^2} =$$

$$= \lim_{x\to 0^+} \frac{1 - \left(\dfrac{\text{sen } x}{x}\right)^2}{\left(1 + \dfrac{\text{sen } x}{x}\right)^2} = 0 = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x)$$

$\lim\limits_{x\to 0} f(x) \neq f(0) \Rightarrow f(x)$ não é contínua em $x = 0$.

115. a) $f(2) = 0$

$$\left.\begin{array}{l} \lim\limits_{x\to 2^+} \dfrac{3}{x - 2} = \lim\limits_{x\to 2^+} \dfrac{3(x + 2)}{x^2 - 4} = +\infty \\ \lim\limits_{x\to 2^-} \dfrac{3}{x - 2} = \lim\limits_{x\to 2^-} \dfrac{3(x + 2)}{x^2 - 4} = +\infty \end{array}\right\} \Rightarrow \lim_{x\to 2} f(x) = +\infty$$

$\lim\limits_{x\to 2} f(x) \neq f(x) \Rightarrow f(2)$ não é contínua em $x = 2$.

b) $f(1) = 2$

$$\lim_{x\to 1^+} \frac{x^2 - 1}{(x - 1)^2} = \lim_{x\to 1^+} \frac{x + 1}{x - 1} = +\infty = \lim_{x\to 1^-} f(x) = \lim_{x\to 1} f(x)$$

$\lim\limits_{x\to 1} f(x) \neq f(1) \Rightarrow f(x)$ não é contínua em $x = 1$.

c) $f(1) = 1$

$$\left.\begin{array}{l} \lim\limits_{x\to 1^+} \dfrac{1}{|x - 1|} = \lim\limits_{x\to 1^+} \dfrac{1}{x - 1} = +\infty \\ \lim\limits_{x\to 1^-} \dfrac{1}{|x - 1|} = \lim\limits_{x\to 1^-} \dfrac{1}{-x + 1} = +\infty \end{array}\right\} \Rightarrow \lim_{x\to 1} f(x) = +\infty$$

$\lim\limits_{x\to 1} f(x) \neq f(1) \Rightarrow f(x)$ não é contínua em $x = 1$.

116. a) $f(2) = a$

$$\left.\begin{array}{l} \lim\limits_{x\to 2^+} \dfrac{x^2 - 5x + 6}{x - 2} = \lim\limits_{x\to 2^+} (x - 3) = -1 \\ \lim\limits_{x\to 2^-} \dfrac{x^2 - 5x + 6}{x - 2} = \lim\limits_{x\to 2^-} (x - 3) = -1 \end{array}\right\} \Rightarrow \lim_{x\to 2} f(x) = -1$$

$f(x)$ é contínua em $x = 2$ se $\lim\limits_{x\to 2} f(x) = f(2) \Rightarrow a = -1.$

b) $f(1) = a$

$$\left.\begin{array}{l} \lim\limits_{x\to 1^+} \dfrac{x - 1}{1 - x^3} = \lim\limits_{x\to 1^+} \dfrac{-1}{x^2 + x + 1} = \dfrac{-1}{3} \\ \lim\limits_{x\to 1^-} \dfrac{x - 1}{1 - x^3} = \lim\limits_{x\to 1^-} \dfrac{-1}{x^2 + x + 1} = \dfrac{-1}{3} \end{array}\right\} \Rightarrow \lim_{x\to 2} f(x) = \frac{-1}{3}$$

$f(x)$ é contínua em $x = 1$ se $\lim\limits_{x\to 1} f(x) = f(1) \Rightarrow a = \dfrac{-1}{3}.$

c) $f(4) = 3 \cdot 4 + a = 12 + a$

$$\left.\begin{array}{l} \lim\limits_{x\to 4^+} \dfrac{\sqrt{x} - 2}{x - 4} = \lim\limits_{x\to 4^+} \dfrac{1}{\sqrt{x} + 2} = \dfrac{1}{4} \\ \lim\limits_{x\to 4^-} (3x + a) = 12 + a \end{array}\right\} \Rightarrow 12 + a = \frac{1}{4} \Rightarrow a = \frac{-47}{4}$$

$\lim\limits_{x\to 4} f(x)$ existe se $\lim\limits_{x\to 4^+} f(x) = \lim\limits_{x\to 4^-}$ e, nessas condições, $a = \dfrac{-47}{4}.$

A função $f(x)$ é contínua em $x = 4$ se $\lim\limits_{x\to 4} f(x) = f(4) \Rightarrow a = \dfrac{-47}{4}.$

d) $f(0) = a$

$$\lim_{x\to 0^+} \frac{\sqrt{x + 2} - \sqrt{2}}{x} = \lim_{x\to 0^+} \frac{(\sqrt{x + 2} - \sqrt{2})(\sqrt{x + 2} + \sqrt{2})}{x(\sqrt{x + 2} + \sqrt{2})}$$

$$= \lim_{x\to 0^+} \frac{1}{\sqrt{x + 2} + \sqrt{2}} = \frac{\sqrt{2}}{4}$$

$$\lim_{x\to 0^-} (3x^2 - 4x + a) = a$$

$\lim\limits_{x\to 0} f(x)$ existe se $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^-} f(x) \Rightarrow a = \dfrac{\sqrt{2}}{4}.$

Para $a = \dfrac{\sqrt{2}}{4}$, $\lim\limits_{x\to 0} f(x) = f(0)$ e a função é contínua.

e) $f(0) = a$

$$\lim_{x\to 0^+} \frac{\sqrt[3]{x + 1} - 1}{x} = \lim_{x\to 0^+} \frac{(\sqrt[3]{x + 1} - 1)(\sqrt[3]{(x + 1)^2} + \sqrt[3]{x + 1} + 1)}{x(\sqrt[3]{(x + 1)^2} + \sqrt[3]{x + 1} + 1)} =$$

$$= \lim_{x\to 0^+} \frac{1}{\sqrt[3]{(x + 1)^2} + \sqrt[3]{x + 1} + 1} = \frac{1}{3} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x)$$

Portanto, $\lim\limits_{x\to 0} f(x) = f(0) \Rightarrow a = \dfrac{1}{3}.$

117. $f(0) = \cos a$

$$\lim_{x\to 0^+} \frac{\operatorname{tg} x}{\operatorname{sen} 2x} = \lim_{x\to 0^+} \frac{\dfrac{\operatorname{sen} x}{\cos x}}{2 \operatorname{sen} x \cos x} = \lim_{x\to 0^+} \frac{1}{2\cos^2 x} = \frac{1}{2} =$$

$$= \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x)$$

$f(x)$ é contínua em $x = 0$ se $\lim\limits_{x\to 0} f(x) = f(0) \Rightarrow \cos a = \dfrac{1}{2} \Rightarrow$

$\Rightarrow a = \pm\dfrac{\pi}{3} + 2k\pi,\ k \in \mathbb{Z}$

Capítulo VI — Derivadas

118. $f'(2) = \lim\limits_{x\to 2} \dfrac{f(x) - f(2)}{x - 2} = \lim\limits_{x\to 2} \dfrac{3x + 1 - 7}{x - 2} = \lim\limits_{x\to 2} 3 = 3$

119. $f'(1) = \lim\limits_{x\to 1} \dfrac{x^2 + 2x + 5 - 8}{x - 1} = \lim\limits_{x\to 1} \dfrac{(x + 3)(x - 1)}{(x - 1)} = \lim\limits_{x\to 1} (x + 3) = 4$

120. $f'(-1) = \lim\limits_{x\to -1} \dfrac{x^3 + 1}{x + 1} = \lim\limits_{x\to -1} (x^2 - x + 1) = 3$

121. $f'(1) = \lim\limits_{\Delta x\to 0} \dfrac{f(1 + \Delta x) - f(1)}{\Delta x} = \lim\limits_{\Delta x\to 0} \dfrac{|1 + \Delta x| - |1|}{\Delta x} =$

$= \lim\limits_{\Delta x\to 0} \dfrac{1 + \Delta x - 1}{\Delta x} = \lim\limits_{\Delta x\to 0} \dfrac{\Delta x}{\Delta x} = \lim\limits_{\Delta x\to 0} 1 = 1$

122. $f'(0) = \lim\limits_{\Delta x\to 0} \dfrac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim\limits_{\Delta x\to 0} \dfrac{|\Delta x|}{\Delta x}$

Temos, então:

$$\left.\begin{array}{l} \lim\limits_{\Delta x\to 0^+} \dfrac{|\Delta x|}{\Delta x} = \lim\limits_{\Delta x\to 0^+} \dfrac{\Delta x}{\Delta x} = 1 \\ \lim\limits_{\Delta x\to 0^-} \dfrac{|\Delta x|}{\Delta x} = \lim\limits_{\Delta x\to 0^-} \dfrac{-\Delta x}{\Delta x} = -1 \end{array}\right\} \Rightarrow \nexists \lim_{\Delta x\to 0} \frac{|\Delta x|}{\Delta x} \Rightarrow \nexists f'(0)$$

123. $f'\left(\dfrac{\pi}{4}\right) = \lim\limits_{\Delta x\to 0} \dfrac{\cos\left(\dfrac{\pi}{4} + \Delta x\right) - \cos\dfrac{\pi}{4}}{\Delta x} =$

$$= \lim_{\Delta x \to 0} \frac{-2\,\text{sen}\left(\frac{\pi}{4} + \frac{\Delta x}{2}\right) \cdot \text{sen}\left(\frac{\Delta x}{2}\right)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \left[-\text{sen}\left(\frac{\pi}{4} + \frac{\Delta x}{2}\right) \cdot \frac{\text{sen}\,\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \right] = -\text{sen}\,\frac{\pi}{4} = \frac{-\sqrt{2}}{2}$$

124. $f'(1) = \lim_{x \to 1} \frac{\sqrt{x} - \sqrt{1}}{x - 1} = \lim_{x \to 1} \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x - 1)(\sqrt{x} + 1)} = \lim_{x \to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{2}$

125. $f'(2) = \lim_{x \to 2} \frac{\sqrt[3]{x} - \sqrt[3]{2}}{x - 2} = \lim_{x \to 2} \frac{(\sqrt[3]{x} - \sqrt[3]{2})(\sqrt[3]{x^2} + \sqrt[3]{2x} + \sqrt[3]{4})}{(x - 2)(\sqrt[3]{x^2} + \sqrt[3]{2x} + \sqrt[3]{4})} =$

$= \lim_{x \to 2} \frac{1}{\sqrt[3]{x^2} + \sqrt[3]{2x} + \sqrt[3]{4}} = \frac{1}{3\sqrt[3]{4}}$

126. $f'(0) = \lim_{x \to 0} \frac{\sqrt[5]{x} - \sqrt[5]{0}}{x - 0} = \lim_{x \to 0} \frac{\sqrt[5]{x}}{x} = \lim_{x \to 0} \frac{1}{\sqrt[5]{x^4}} = +\infty$

Portanto, não existe *f'(0)*.

127. $f(x) = x \cdot |x| = \begin{cases} x^2, & \text{se } x \geqslant 0 \\ -x^2, & \text{se } x < 0 \end{cases}$

Assim:

$$\left.\begin{array}{l} \lim_{x \to 0^+} \frac{x \cdot |x| - 0}{x - 0} = \lim_{x \to 0^+} \frac{x^2}{x} = \lim_{x \to 0^+} x = 0 \\ \lim_{x \to 0^-} \frac{x \cdot |x| - 0}{x - 0} = \lim_{x \to 0^-} \left(\frac{-x^2}{x}\right) = \lim_{x \to 0^-} (-x) = 0 \end{array}\right\} \Rightarrow f'(0) = 0$$

130. a) $f(x) = x + 1$

$x_0 = 3 \Rightarrow f(3) = 4 \Rightarrow P(3, 4)$

$f'(3) = \lim_{x \to 3} \frac{x + 1 - 4}{x - 3} = \lim_{x \to 3} \frac{x - 3}{x - 3} = 1$

$y - 4 = 1(x - 3) \Rightarrow y = x + 1$

b) $f(x) = x^2 - 2x$

$x_0 = 1 \Rightarrow f(1) = -1 \Rightarrow P(1, -1)$

$f'(1) = \lim_{x \to 1} \frac{x^2 - 2x + 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)^2}{x - 1} = \lim_{x \to 1} (x - 1) = 0$

$y + 1 = 0(x - 1) \Rightarrow y = -1$

c) $f(x) = \text{sen}\,x$

$x_0 = 0 \Rightarrow f(0) = \text{sen}\,0 = 0 \Rightarrow P(0, 0)$

$f'(0) = \lim_{x \to 0} \frac{\text{sen}\,x - 0}{x - 0} = \lim_{x \to 0} \frac{\text{sen}\,x}{x} = 1$

$y - 0 = 1(x - 0) \Rightarrow y = x$

d) $f(x) = \frac{1}{x}$

$x_0 = 1 \Rightarrow f(1) = \frac{1}{1} = 1 \Rightarrow P(1, 1)$

$f'(1) = \lim_{x \to 1} \frac{\frac{1}{x} - 1}{x - 1} = \lim_{x \to 1} \frac{\frac{1}{x} - 1}{x\left(1 - \frac{1}{x}\right)} = \lim_{x \to 1} \frac{-1}{x} = -1$

$y - 1 = -1(x - 1) \Rightarrow y = -x + 2$

e) $f(x) = \sqrt{x}$

$x_0 = 4 \Rightarrow f(4) = 2 \Rightarrow P(4, 2)$

$f'(4) = \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \to 4} \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{(x - 4)(\sqrt{x} + 2)} = \lim_{x \to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{4}$

$y - 2 = \frac{1}{4}(x - 4) \Rightarrow y = \frac{1}{4}x + 1$

f) $f(x) = \sqrt[3]{x^2}$

$x_0 = 2\sqrt{2} \Rightarrow f(2\sqrt{2}) = 2 \Rightarrow P(2\sqrt{2}, 2)$

$f'(2\sqrt{2}) = \lim_{x \to 2\sqrt{2}} \frac{\sqrt[3]{x^2} - 2}{x - 2\sqrt{2}} =$

$= \lim_{x \to 2\sqrt{2}} \frac{(\sqrt[3]{x^2} - \sqrt[3]{8})(\sqrt[3]{x^4} + \sqrt[3]{8x^2} + \sqrt[3]{64})(x + 2\sqrt{2})}{(x - 2\sqrt{2})(x + 2\sqrt{2})(\sqrt[3]{x^4} + \sqrt[3]{8x^2} + \sqrt[3]{64})} =$

$= \lim_{x \to 2\sqrt{2}} \frac{(x^2 - 8)(x + 2\sqrt{2})}{(x^2 - 8)(\sqrt[3]{x^4} + \sqrt[3]{8x^2} + \sqrt[3]{64})} = \frac{4\sqrt{2}}{4 + 4 + 4} = \frac{\sqrt{2}}{3}$

Então: $y - 2 = \frac{\sqrt{2}}{3}(x - 2\sqrt{2}) \Rightarrow y = \frac{\sqrt{2}}{3}x + \frac{2}{3}$.

132. $s'(t_0) = s'(3) = \lim_{t \to 3} \frac{\frac{1}{t} - \frac{1}{3}}{t - 3} = \lim_{t \to 3} \frac{-1}{3t} = \frac{-1}{9}$

velocidade $= \frac{-1}{9}$ m/s.

134. $v'(t_0) = v'(5) = \lim_{t \to 5} \frac{5t^2 + 3t + 2 - 142}{t - 5} = \lim_{t \to 5} \frac{5\left(t + \frac{28}{5}\right)(t - 5)}{t - 5} =$

$= \lim_{t \to 5} 5\left(t + \frac{28}{5}\right) = 53 \Rightarrow$ aceleração $=$ *53 m/s²*.

136. a) $f(x) = c \cdot x^n$

$$\frac{\Delta y}{\Delta x} = \frac{c(x + \Delta x)^n - cx^n}{\Delta x} =$$

$$= \frac{c\left[\binom{n}{0}x^n + \binom{n}{1}x^{n-1}\Delta x + \binom{n}{2}x^{n-2} \cdot (\Delta x)^2 + \dots + \binom{n}{n}(\Delta x)^n\right] - cx^n}{\Delta x} =$$

$$= c \cdot \left[\binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2} \cdot \Delta x + \binom{n}{3}x^{n-3} \cdot (\Delta x)^2 + \dots + \binom{n}{n}(\Delta x)^{n-1}\right]$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = c \cdot \binom{n}{1}x^{n-1} = c \cdot nx^{n-1}$$

b) $g(x) = \text{tg } x$

$$\frac{\Delta y}{\Delta x} = \frac{\text{tg}(x + \Delta x) - \text{tg } x}{\Delta x} = \frac{\frac{\text{sen } \Delta x}{\cos (x + \Delta x) \cdot \cos x}}{\Delta x} =$$

$$= \frac{\text{sen } \Delta x}{\Delta x} \cdot \frac{1}{\cos (x + \Delta x) \cdot \cos x}$$

$$g'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{1}{\cos^2 x} = \sec^2 x$$

c) $h(x) = \sec x = \dfrac{1}{\cos x}$

$$\frac{\Delta y}{\Delta x} = \frac{\frac{1}{\cos (x + \Delta x)} - \frac{1}{\cos x}}{\Delta x} = \frac{\frac{\cos x - \cos (x + \Delta x)}{\cos (x + \Delta x) \cdot \cos x}}{\Delta x} =$$

$$= \frac{\frac{-2 \text{ sen}\left(\frac{2x + \Delta x}{2}\right) \cdot \text{sen}\left(\frac{-\Delta x}{2}\right)}{\cos (x + \Delta x) \cdot \cos x}}{\Delta x} = \frac{\text{sen } \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \frac{\text{sen}\left(\frac{2x + \Delta x}{2}\right)}{\cos (x + \Delta x) \cdot \cos x}$$

$$h'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{\text{sen } x}{\cos^2 x} = \frac{\text{sen } x}{\cos x} \cdot \frac{1}{\cos x} = \text{tg } x \cdot \sec x$$

138. $f(x) = e^x$
$x_0 = 2 \Rightarrow f(x_0) = e^2 \Rightarrow P(2, e^2)$
$f'(x) = e^x \Rightarrow f'(2) = e^2$
$y - e^2 = e^2(x - 2) \Rightarrow e^2x - y - e^2 = 0$

140. $s = t^4$
a) $v(t) = s'(t) = 4t^3 \Rightarrow v(2) = 32$ m/s
b) $a(t) = v'(t) = 12t^2 \Rightarrow a(3) = 108$ m/s^2
c) $v(t) = 108 \Rightarrow 4t^3 = 108 \Rightarrow t = 3$s
d) $a(t) = 48 \Rightarrow 12t^2 = 48 \Rightarrow t = 2$s

Capítulo VII – Regras de derivação

144. a) $f(x) = (3x^2 + x)(1 + x + x^3)$
$f'(x) = (6x + 1)(1 + x + x^3) + (3x^2 + x)(3x^2 + 1) =$
$= 15x^4 + 4x^3 + 9x^2 + 8x + 1$

b) $f(x) = x^2(x + x^4)(1 + x + x^3)$
$f'(x) = 2x(x + x^4)(1 + x + x^3) + x^2(1 + 4x^3)(1 + x + x^3) + x^2(x + x^4)(1 + 3x^2)$
$f'(x) = (2x^2 + 2x^3 + 4x^5 + 2x^6 + 2x^8) + (x^2 + x^3 + 5x^5 + 4x^6 + 4x^8) +$
$+ (x^3 + 3x^5 + x^6 + 3x^8)$
$f'(x) = 9x^8 + 7x^6 + 12x^5 + 4x^3 + 3x^2$

c) $f(x) = (2 + 3x + x^2)^5$
$f'(x) = 5(x^2 + 3x + 2)^4(2x + 3)$

d) $f(x) = (2x + 3)^{52}$
$f'(x) = 52(2x + 3)^{51} \cdot 2 = 104(2x + 3)^{51}$

e) $f(x) = x^3 \cdot e^x$
$f'(x) = 3x^2 \cdot e^x + x^3 \cdot e^x = (x^3 + 3x^2)e^x$

f) $f(x) = x \cdot e^x + \cos x$
$f'(x) = 1 \cdot e^x + x \cdot e^x - \text{sen } x = e^x(1 + x) - \text{sen } x$

g) $f(x) = x^4 \cdot a^{2x}$
$f'(x) = 4x^3 \cdot a^{2x} + x^4 \cdot 2a^{2x} \cdot \ell n \ a = 2a^{2x}x^3(2 + x \ln a)$

h) $f(x) = e^{3x}$
$f'(x) = 3e^{3x}$

i) $f(x) = e^{5x+1} = e^{5x} \cdot e$
$f'(x) = 5e^{5x} \cdot e + e^{5x} \cdot 0 = 5 \cdot e^{5x+1}$

j) $f(x) = \cos^5 x$
$f'(x) = 5 \cos^4 x \cdot (-\text{sen } x) = -5 \cos^4 x \cdot \text{sen } x$

k) $f(x) = \underbrace{\text{sen}^7 x}_{u} \cdot \underbrace{\cos^3 x}_{v}$

$$f'(x) = \underbrace{7 \text{ sen}^6 x \cdot \cos x}_{u'} \cdot \underbrace{\cos^3 x}_{v} + \underbrace{\text{sen}^7 x}_{u} \cdot \underbrace{3 \cos^2 x \cdot (-\text{ sen } x)}_{v'}$$

$$f'(x) = 7 \text{ sen}^6 x \cos^4 x - 3 \text{ sen}^8 x \cos^2 x$$
$$f'(x) = \text{sen}^6 x \cos^2 x(7 \cos^2 x - 3 \text{ sen}^2 x)$$

l) $f(x) = a \text{ sen } x + b \cos x$
$f'(x) = a \cos x - b \text{ sen } x$

145. $f(x) = (\text{sen } x + e^x)^2 \cdot (\cos x + x^3)^3$
$u = (\text{sen } x + e^x)^2 \Rightarrow u' = 2(\text{sen } x + e^x)(\cos x + e^x)$
$v = (\cos x + x^3)^3 \Rightarrow v' = 3(\cos x + x^3)^2(-\text{sen } x + 3x^2)$
$f'(x) = u'v + uv'$
$f'(x) = 2(\text{sen } x + e^x)(\cos x + e^x)(\cos x + x^3)^3 +$
$+ 3(\text{sen } x + e^x)^2(\cos x + x^3)^2(-\text{sen } x + 3x^2)$
$f'(0) = 2 \cdot 1 \cdot 2 \cdot 1 + 3 \cdot 1 \cdot 1 \cdot 0 = 4$

146. $f(x) = (3 \text{ sen } x + 4 \cos x)^5$; $x_0 = \pi$
$f(\pi) = (3 \text{ sen } \pi + 4 \cos \pi)^5 = (-4)^5 = -1\,024$
Portanto, o ponto de tangência é $P(\pi, -1\,024)$.
$f'(x) = 5 \cdot (3 \text{ sen } x + 4 \cos x)^4 \cdot (3 \cos x - 4 \text{ sen } x)$
$f'(\pi) = 5 \cdot (-4)^4 \cdot (-3) = -3\,840$
Então: $y - (-1\,024) = -3\,840(x - \pi) \Rightarrow y = -3\,840x + (3\,840\pi - 1\,024)$.

147. $s = a \cdot \underbrace{e^{-t}}_{x} \cdot \underbrace{\cos t}_{y}$

$$\left.\begin{array}{l} s' = v = a(xy' + x'y) \\ \begin{array}{l|l} x = e^{-t} & y = \cos t \\ x' = -e^{-t} & y' = -\text{sen } t \end{array} \end{array}\right\} \begin{array}{l} \Rightarrow v = a\,[e^{-t}(-\text{sen } t) + (-e^{-t}) \cos t] \Rightarrow \\ \Rightarrow v = -a \cdot e^{-t} \cdot (\text{sen } t + \cos t) \end{array}$$

$$\left.\begin{array}{l} v' = \alpha = -a(xw' + x'w) \\ \begin{array}{l|l} x = e^{-t} & w = \text{sen } t + \cos t \\ x' = -e^{-t} & w' = \cos t - \text{sen } t \end{array} \end{array}\right\} \begin{array}{l} \Rightarrow \alpha = -a[e^{-t}(\cos t - \text{sen } t) + \\ \quad + (-e^{-t})(\text{sen } t + \cos t)] \Rightarrow \\ \Rightarrow \alpha = 2ae^{-t} \text{ sen } t \end{array}$$

149. a) $f(x) = \dfrac{2}{x^7} = 2x^{-7}$

$f'(x) = 2 \cdot (-7) \cdot x^{-8} \Rightarrow f'(x) = \dfrac{-14}{x^8}$

b) $f(x) = 3x^{-5}$
$f'(x) = 3 \cdot (-5) \cdot x^{-6} \Rightarrow f'(x) = -15 \cdot x^{-6}$

c) $f(x) = \dfrac{1}{x^2 + x + 1}$

$f'(x) = \dfrac{0 \cdot (x^2 + x + 1) - 1 \cdot (2x + 1)}{(x^2 + x + 1)^2} \Rightarrow f'(x) = \dfrac{-(2x + 1)}{(x^2 + x + 1)^2}$

d) $f(x) = \dfrac{x + 1}{x - 1}$

$f'(x) = \dfrac{1(x - 1) - 1(x + 1)}{(x - 1)^2} = \dfrac{-2}{(x - 1)^2}$

e) $f(x) = \dfrac{x + 3}{x - 1} + \dfrac{x + 2}{x + 1}$

$f'(x) = \dfrac{1(x - 1) - 1(x + 3)}{(x - 1)^2} + \dfrac{1(x + 1) - 1(x + 2)}{(x + 1)^2}$

$f'(x) = \dfrac{-4}{(x - 1)^2} + \dfrac{-1}{(x + 1)^2} = \dfrac{-4(x + 1)^2 - (x - 1)^2}{[(x - 1)(x + 1)]^2} = \dfrac{-(5x^2 + 6x + 5)}{(x^2 - 1)^2}$

f) $f(x) = \dfrac{x^2 + 3x + 1}{x - 2}$

$f'(x) = \dfrac{(2x + 3)(x - 2) - (x^2 + 3x + 1) \cdot 1}{(x - 2)^2} = \dfrac{x^2 - 4x - 7}{(x - 2)^2}$

g) $f(x) = \dfrac{x^2 \cdot \text{sen } x}{e^x}$

$$\begin{array}{l|l} u(x) = x^2 \cdot \text{sen } x & v(x) = e^x \\ u'(x) = 2x \cdot \text{sen } x + x^2 \cdot \cos x & v'(x) = e^x \end{array}$$

$f'(x) = \dfrac{(2x \cdot \text{sen } x + x^2 \cdot \cos x) \cdot e^x - e^x \cdot x^2 \cdot \text{sen } x}{(e^x)^2}$

$f'(x) = \dfrac{e^x[2x \text{ sen } x + x^2 \cos x - x^2 \text{ sen } x]}{(e^x)^2} = \dfrac{x(2 \text{ sen } x + x \cos x - x \text{ sen } x)}{e^x}$

h) $f(x) = \dfrac{\cos x}{x \cdot e^x}$

$$\begin{array}{l|l} u(x) = \cos x & u(x) = x \cdot e^x \\ u'(x) = -\text{sen } x & u'(x) = e^x + x \cdot e^x \end{array}$$

$f'(x) = \dfrac{-xe^x \text{ sen } x - \cos x(e^x + xe^x)}{(x \cdot e^x)^2}$

$f'(x) = \dfrac{-e^x(x \text{ sen } x + x \cos x + \cos x)}{x^2 \cdot (e^x)^2}$

$f'(x) = -\dfrac{x(\text{sen } x + \cos x) + \cos x}{x^2 \cdot e^x}$

150. a) $f(x) = \text{cotg } x = \dfrac{\cos x}{\text{sen } x}$

$f'(x) = \dfrac{-\text{sen } x \cdot \text{sen } x - \cos x \cdot \cos x}{\text{sen}^2 x} = -\dfrac{\text{sen}^2 x + \cos^2 x}{\text{sen}^2 x}$

$$f'(x) = -\frac{1}{\text{sen}^2 x} = -\text{cossec}^2 x$$

b) $f(x) = \sec x = \dfrac{1}{\cos x}$

$$f'(x) = \frac{0 \cdot \cos x - 1 \cdot (-\text{sen } x)}{\cos^2 x} = \frac{\text{sen } x}{\cos^2 x} = \frac{\text{sen } x}{\cos x} \cdot \frac{1}{\cos x} = \text{tg } x \cdot \sec x$$

c) $f(x) = \text{cossec } x = \dfrac{1}{\text{sen } x}$

$$f'(x) = \frac{0 \cdot \text{sen } x - 1 \cdot \cos x}{\text{sen}^2 x} = \frac{-\cos x}{\text{sen}^2 x} = -\frac{\cos x}{\text{sen } x} \cdot \frac{1}{\text{sen } x} = -\text{cotg } x \cdot \text{cossec } x$$

d) $f(x) = \text{tg}^2 x$

$f'(x) = 2 \cdot \text{tg } x \cdot (\text{tg } x)'$

$$(\text{tg } x)' = \left(\frac{\text{sen } x}{\cos x}\right)' = \frac{\cos x \cdot \cos x - \text{sen } x(-\text{sen } x)}{\cos^2 x} = \frac{\cos^2 x + \text{sen}^2 x}{\cos^2 x} =$$

$$= \frac{1}{\cos^2 x}$$

Portanto: $f'(x) = 2 \cdot \text{tg } x \cdot \dfrac{1}{\cos^2 x} = \dfrac{2 \text{ sen } x}{\cos^3 x}$

ou $f'(x) = 2 \text{ tg } x \cdot \dfrac{1}{\cos^2 x} = 2 \cdot \text{tg } x \cdot \sec^2 x.$

e) $f(x) = \sec x - \text{tg } x$

$f'(x) = (\sec x)' - (\text{tg } x)'$

$(\sec x)' = \text{tg } x \cdot \sec x$ (item *b*)

$(\text{tg } x)' = \sec^2 x$ (item *d*)

Portanto: $f'(x) = \text{tg } x \cdot \sec x - \sec^2 x = \sec x(\text{tg } x - \sec x).$

f) $f(x) = \underbrace{(x^2 + 1)}_{u} \cdot \underbrace{\text{tg } x}_{v}$

$$\left.\begin{array}{l|l} u = x^2 + 1 & v = \text{tg } x \\ u' = 2x & v' = \sec^2 x \text{ (item } d\text{)} \end{array}\right\} \Rightarrow f'(x) = u'v + uv' \Rightarrow$$

$$\Rightarrow f'(x) = 2x \cdot \text{tg } x + (x^2 + 1) \cdot \sec^2 x$$

g) $f(x) = \dfrac{\text{tg } x}{\text{sen } x + \cos x}$

$$\left.\begin{array}{l|l} u = \text{tg } x & v = \text{sen } x + \cos x \\ u' = \sec^2 x \text{ (item } d\text{)} & v' = \cos x - \text{sen } x \end{array}\right\} \Rightarrow$$

$$\Rightarrow f'(x) = \frac{\sec^2 x(\text{sen } x + \cos x) - \text{tg } x(\cos x - \text{sen } x)}{(\text{sen } x + \cos x)^2}$$

h) $f(x) = \left(\dfrac{e^x}{\text{tg } x}\right)^2 \Rightarrow f'(x) = \left[2 \cdot \left(\dfrac{e^x}{\text{tg } x}\right)\right]\left(\dfrac{e^x}{\text{tg } x}\right)'$

$$\left(\frac{e^x}{\text{tg } x}\right)' = \frac{e^x \cdot \text{tg } x - e^x \cdot \sec^2 x}{(\text{tg } x)^2}$$

Portanto: $f'(x) = 2 \cdot \left(\dfrac{e^x}{\text{tg } x}\right)\dfrac{e^x(\text{tg } x - \sec^2 x)}{(\text{tg } x)^2}$

$$f'(x) = \frac{2e^{2x}}{\text{tg}^3 x}(\text{tg } x - \sec^2 x).$$

151. $f(x) = \dfrac{1}{x} + e^x \qquad (x_0 = -1)$

$f(-1) = -1 + \dfrac{1}{e} = \dfrac{-e + 1}{e} \Rightarrow P\left(-1, \dfrac{1 - e}{e}\right)$

$f'(x) = \dfrac{-1}{x^2} + e^x \Rightarrow f'(-1) = -1 + \dfrac{1}{e} \Rightarrow f'(-1) = \dfrac{1 - e}{e}$

Então, temos:

$y - \left(\dfrac{1 - e}{e}\right) = \left(\dfrac{1 - e}{e}\right)(x + 1) \Rightarrow y = \left(\dfrac{1 - e}{e}\right)(x + 2).$

152. $f(x) = \dfrac{1}{x^2} + e^{-x} + \sec^2 x \qquad \left(x_0 = \dfrac{\pi}{4}\right)$

$$\left.\begin{array}{l} u = \dfrac{1}{x^2} \Rightarrow u' = \dfrac{-2}{x^3} \\ v = e^{-x} \Rightarrow v' = -e^{-x} \\ w = \sec^2 x \Rightarrow w' = 2 \cdot \sec x \cdot (\sec x)' = 2 \text{ tg } x \cdot \sec^2 x \end{array}\right\} \Rightarrow$$

$$\Rightarrow f'(x) = \frac{-2}{x^3} - e^{-x} + 2 \text{ tg } x \cdot \sec^2 x$$

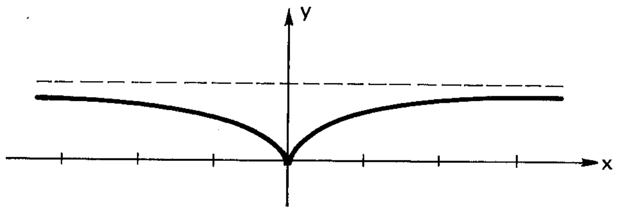
$$f'\left(\frac{\pi}{4}\right) = \frac{-2}{\left(\frac{\pi}{4}\right)^3} - e^{-\frac{\pi}{4}} + 2 \cdot 1 \cdot 2 = \frac{-128}{\pi^3} - \frac{1}{e^{\frac{\pi}{4}}} + 4$$

153. $y = \dfrac{x^2}{x^2 + 1}$

a) $y' = \dfrac{2x(x^2 + 1) - 2x^3}{(x^2 + 1)^2} \Rightarrow y' = \dfrac{2x}{(x^2 + 1)^2}$

b) $\lim\limits_{x \to +\infty} \dfrac{x^2}{x^2 + 1} = \lim\limits_{x \to +\infty} \dfrac{x^2}{x^2\left(1 + \dfrac{1}{x^2}\right)} = \lim\limits_{x \to +\infty} \dfrac{1}{1 + \dfrac{1}{x^2}} = 1$

c) Ponto de tangência: (x_0, y_0).



$$f(x) = \frac{x^2}{x^2 + 1} \Rightarrow f(x_0) = \frac{x_0^2}{x_0^2 + 1} \Rightarrow P\left(x_0, \frac{x_0^2}{x_0^2 + 1}\right)$$

$$f'(x) = \frac{2x}{(x^2 + 1)^2} \Rightarrow f'(x_0) = \frac{2x_0}{(x_0^2 + 1)^2}$$

Equação da reta tangente:

$$y - y_0 = f'(x)(x - x_0) \Rightarrow y - \frac{x_0^2}{x_0^2 + 1} = \frac{2x_0}{(x_0^2 + 1)}(x - x_0) \Rightarrow$$

$$\Rightarrow y = \frac{2x_0}{(x_0^2 + 1)^2}(x - x_0) + \frac{x_0^2}{x_0^2 + 1}$$

Como a reta tangente deve passar por *(0, 0)*, então vem:

$$0 = \frac{2x_0}{(x_0^2 + 1)^2}(-x_0) + \frac{x_0^2}{x_0^2 + 1}$$

$$\frac{2x_0^2}{(x_0^2 + 1)^2} = \frac{x_0^2}{x_0^2 + 1}$$

Essa igualdade vale para $x_0 = 0$, o que implica $y_0 = 0$.
Se $x_0 \neq 0$, então temos:

$$\frac{2}{x_0^2 + 1} = 1 \Rightarrow x_0^2 = 1 \Rightarrow x_0 = \pm 1 \Rightarrow y_0 = \frac{1}{2}.$$

Então, os pontos do gráfico em que a tangente passa pela origem são: *(0, 0)*, $\left(-1, \frac{1}{2}\right)$ e $\left(1, \frac{1}{2}\right)$.

155. a) $F(x) = \text{sen } 4x$

$y = f(x) = 4x$ e $z = g(y) = \text{sen } y \Rightarrow y' = 4$ e $z' = \cos y$

$F'(x) = y' \cdot z' = 4 \cdot \cos y = 4 \cdot \cos 4x$

b) $F(x) = \dfrac{\cos 7x}{x}$

$$\left.\begin{array}{l} y = f(x) = 7x \Rightarrow y' = f'(x) = 7 \\ z = g(y) = \cos y \Rightarrow z' = g'(y) = -\text{sen } y \end{array}\right\} \Rightarrow (\cos 7x)' = f'(x) \cdot g'(y) = 7 \cdot (-\text{sen } y) = -7 \cdot \text{sen } 7x$$

$$F'(x) = \frac{(-7 \text{ sen } 7x)x - 1 \cdot \cos 7x}{x^2}$$

$$F'(x) = \frac{-(7x \text{ sen } 7x + \cos 7x)}{x^2}$$

c) $F(x) = a \cdot \text{sen } bx$

$$\left.\begin{array}{l} y = f(x) = bx \Rightarrow y' = f'(x) = b \\ z = g(y) = \text{sen } y \Rightarrow z' = g'(y) = \cos y \end{array}\right\} \Rightarrow (\text{sen } bx)' = b \cos bx$$

$F'(x) = a \cdot b \cdot \cos bx$

d) $F(x) = \cos(3x^2 + x + 5)$

$y = f(x) = 3x^2 + x + 5 \Rightarrow y' = f'(x) = 6x + 1$

$z = g(y) = \cos y \Rightarrow z' = g'(y) = -\text{sen } y$

$F'(x) = f'(x) \cdot g'(y) = (6x + 1)(-\text{sen } y)$

$F'(x) = -(6x + 1) \cdot \text{sen}(3x^2 + x + 5)$

e) $F(x) = \text{sen } e^x$

$y = f(x) = e^x \Rightarrow y' = f'(x) = e^x$

$z = g(y) = \text{sen } y \Rightarrow z' = g'(y) = \cos y$

$F'(x) = f'(x) \cdot g'(y) = e^x \cdot \cos y = e^x \cdot \cos e^x$

f) $F(x) = x + 3 \cdot \text{tg } 4x$

$$\left.\begin{array}{l} y = f(x) = 4x \Rightarrow y' = f'(x) = 4 \\ z = g(y) = \text{tg } y \Rightarrow z' = g'(y) = \sec^2 y \end{array}\right\} \Rightarrow (\text{tg } 4x)' = 4 \sec^2 4x$$

$F'(x) = 1 + 12 \sec^2 4x$

g) $F(x) = a^{\text{sen } x}$

$y = f(x) = \text{sen } x \Rightarrow y' = f'(x) = \cos x$

$z = g(y) = a^y \Rightarrow z' = g'(y) = a^y \cdot \ln a$

$F'(x) = f'(x) \cdot g'(y) = \cos x \cdot a^y \cdot \ln a$

$F'(x) = a^{\text{sen } x} \cdot \cos x \cdot \ln a$

h) $F(x) = \text{cotg}(3x - 1)$

$y = f(x) = 3x - 1 \Rightarrow y' = f'(x) = 3$

$z = g(y) = \text{cotg } y \Rightarrow z' = g'(y) = -\text{cossec}^2 y$

$F'(x) = f'(x) \cdot g'(y) = -3 \text{ cossec}^2 (3x - 1)$

i) $F(x) = a^{x^2 + 5x + 1}$

$y = f(x) = x^2 + 5x + 1 \Rightarrow y' = f'(x) = 2x + 5$

$z = g(y) = a^y \Rightarrow z' = g'(y) = a^y \cdot \ln a$

$F'(x) = f'(x) \cdot g'(y) = (2x + 5) \cdot a^{x^2 + 5x + 1} \cdot \ln a$

j) $F(x) = \text{tg}(\cos x)$

$y = f(x) = \cos x \Rightarrow y' = f'(x) = -\text{sen } x$

$z = g(y) = \text{tg } y \Rightarrow z' = g'(y) = \dfrac{1}{\cos^2 y} = \sec^2 y$

$F'(x) = f'(x) \cdot g'(y) = -\text{sen } x \cdot \sec^2(\cos x)$

k) $F(x) = tg^3\, 2x$

$y = f(x) = 2x \Rightarrow f'(x) = 2$

$z = g(y) = tg^3\, y \Rightarrow g'(y) = 3\, tg^2\, y \cdot sec^2\, y$

$F'(x) = f'(x) \cdot g'(y) = 6\, tg^2\, 2x \cdot sec^2\, 2x$

l) $F(x) = e^{sen\, 2x}$

$y = f(x) = 2x \Rightarrow y' = f'(x) = 2$

$z = g(y) = sen\, y \Rightarrow z' = g'(y) = cos\, y$

$t = h(z) = e^z \Rightarrow t' = h'(z) = e^z$

$F'(x) = f'(x) \cdot g'(y) \cdot h'(z) = 2e^{sen\, 2x} \cdot cos\, 2x$

156. $f(x) = cos^3\, x + sen^3\, x \qquad \left(x_0 = \frac{\pi}{2}\right)$

$f'(x) = 3\, cos^2\, x \cdot (-sen\, x) + 3\, sen^2\, x \cdot cos\, x$

$f'\left(\frac{\pi}{2}\right) = 3 \cdot 0 \cdot (-1) + 3 \cdot 1 \cdot 0 \Rightarrow f'\left(\frac{\pi}{2}\right) = 0$

157. $f(x) = e^{x^2 + 5x}$

$y = g(x) = x^2 + 5x \Rightarrow y' = g'(x) = 2x + 5$

$z = h(y) = e^y \Rightarrow z' = h'(y) = e^y$

$f'(x) = g'(x) \cdot h'(y) = (2x + 5) \cdot e^{x^2 + 5x}$

$f'(-1) = (2(-1) + 5)e^{1^2 + 5(-1)} \Rightarrow f'(-1) = 3 \cdot e^{-4}$

158. $f(x) = \dfrac{e^x + e^{-x}}{2} \qquad (x = -2)$

$f(-2) = \dfrac{e^{-2} + e^2}{2} \Rightarrow P\left(-2, \dfrac{e^{-2} + e^2}{2}\right)$

Derivando, vem:

$f'(x) = \dfrac{e^x - e^{-x}}{2} \Rightarrow f'(-2) = \dfrac{e^{-2} - e^2}{2}$.

A equação da reta é:

$$y - \left(\frac{e^{-2} + e^2}{2}\right) = \left(\frac{e^{-2} - e^2}{2}\right)(x + 2)$$

$$y = \left(\frac{e^{-2} - e^2}{2}\right)x + \frac{3e^{-2} - e^2}{2}.$$

161. a) $f(x) = \dfrac{1}{x} + \ell n\, x \Rightarrow f'(x) = \dfrac{-1}{x^2} + \dfrac{1}{x}$

b) $f(x) = x^n \cdot \ell n\, x$

$f'(x) = n \cdot x^{n-1} \cdot \ell n\, x + x^n \cdot \dfrac{1}{x}$

$f'(x) = n \cdot x^{n-1} \cdot \ell n\, x + x^{n-1} \Rightarrow f'(x) = x^{n-1} \cdot (n \cdot \ell n\, x + 1)$

c) $f(x) = (ax + b) \cdot \ell n\, x$

$f'(x) = a \cdot \ell n\, x + (ax + b) \cdot \dfrac{1}{x}$

d) $f(x) = sen\, x \cdot \ell n\, x$

$f'(x) = cos\, x \cdot \ell n\, x + \dfrac{sen\, x}{x}$

e) $f(x) = \dfrac{\ell n\, x}{cos\, x} \Rightarrow f'(x) = \dfrac{\dfrac{cos\, x}{x} + (\ell n\, x) \cdot sen\, x}{cos^2\, x} \Rightarrow$

$\Rightarrow f'(x) = \dfrac{cos\, x + x \cdot \ell n\, x \cdot sen\, x}{x \cdot cos^2\, x}$

f) $f(x) = \ell n\, (ax^2 + bx + c)$

$f'(x) = \dfrac{2ax + b}{ax^2 + bx + c}$

g) $f(x) = \ell n\, (sen\, x)$

$f'(x) = \dfrac{cos\, x}{sen\, x} = cotg\, x$

h) $f(x) = log_a\, log_b\, x$

$y = log_b\, x \Rightarrow y' = \dfrac{1}{x\, \ell n\, b}$

$z = log_a\, y \Rightarrow z' = \dfrac{1}{y\, \ell n\, a}$

$f'(x) = \dfrac{1}{x\, \ell n\, b} \cdot \dfrac{1}{y\, \ell n\, a} \Rightarrow f'(x) = \dfrac{1}{x \cdot log_b\, x \cdot \ell n\, a \cdot \ell n\, b}$

Observando que $log_b\, x \cdot \ell n\, b = \ell n\, x$, vem:

$f'(x) = \dfrac{1}{x \cdot \ell n\, x \cdot \ell n\, a}$.

162. e) $f(x) = \sqrt[3]{x\sqrt{x}} = \sqrt[3]{\sqrt{x^3}} = \sqrt[6]{x^3} = \sqrt{x} = x^{\frac{1}{2}}$

$f'(x) = \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$

g) $f(x) = \sqrt{ax + b} = (ax + b)^{\frac{1}{2}}$

$f'(x) = \dfrac{1}{2}(ax + b)^{-\frac{1}{2}} \cdot a \Rightarrow f'(x) = \dfrac{a}{2\sqrt{ax + b}}$

i) $f(x) = \sqrt{a + b\sqrt{x}} = (a + b\sqrt{x})^{\frac{1}{2}}$

$$f'(x) = \frac{1}{2}(a + b\sqrt{x})^{-\frac{1}{2}} \cdot \frac{b}{2\sqrt{x}} = \frac{b}{4\sqrt{x}(a + b\sqrt{x})}$$

$$f'(x) = \frac{b}{4\sqrt{ax + bx^{\frac{3}{2}}}}$$

j) $$f(x) = \left(\frac{ax^2 + bx + c}{cx^2 + bx + a}\right)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2} \cdot \left(\frac{ax^2 + bx + c}{cx^2 + bx + a}\right)^{-\frac{1}{2}} \cdot$$
$$\cdot \frac{(2ax + b)(cx^2 + bx + a) - (2cx + b)(ax^2 + bx + c)}{(cx^2 + bx + a)^2}$$

$$f'(x) = \frac{1}{2} \cdot \frac{(cx^2 + bx + a)^{\frac{1}{2}}}{(ax^2 + bx + c)^{\frac{1}{2}}} \cdot \frac{b(a - c)x^2 + 2(a + c)(a - c)x + b(a - c)}{(cx^2 + bx + a)^2}$$

$$f'(x) = \frac{(a - c)[bx^2 + 2(a + c)x + b]}{2\sqrt{(ax^2 + bx + c)(cx^2 + bx + a)^3}}$$

n) $$f(x) = \underbrace{(x^2 + 1)}_{u} \cdot \underbrace{\sqrt{3x + 2}}_{v}$$

$$\left.\begin{array}{l} u = (x^2 + 1) \\ u' = (2x) \end{array}\right| \begin{array}{l} v = \sqrt{3x + 2} \\ v' = \frac{1}{2} \cdot \frac{1}{\sqrt{3x + 2}} \cdot 3 \end{array}$$

$$f'(x) = (x^2 + 1) \cdot \frac{3}{2\sqrt{3x + 2}} + 2x\sqrt{3x + 2} = \frac{3(x^2 + 1) + 2x \cdot 2(3x + 2)}{2\sqrt{3x + 2}}$$

$$f'(x) = \frac{15x^2 + 8x + 3}{2\sqrt{3x + 2}}$$

q) $f(x) = \cos \sqrt{x}$

$$\left.\begin{array}{l} y = \sqrt{x} \Rightarrow y' = \frac{1}{2\sqrt{x}} \\ z = \cos y \Rightarrow z' = -\operatorname{sen} y \end{array}\right\} \Rightarrow f'(x) = y' \cdot z' = \frac{-\operatorname{sen} \sqrt{x}}{2\sqrt{x}}$$

s) $$f(x) = \ell n \sqrt{\frac{1 + x}{1 - x}} = \ell n \left(\frac{1 + x}{1 - x}\right)^{\frac{1}{2}} = \frac{1}{2} \ell n \left(\frac{1 + x}{1 - x}\right)$$

$$y = \frac{1 + x}{1 - x} \Rightarrow y' = \frac{2}{(1 - x)^2}$$

$$z = \ell n\, y \Rightarrow z' = \frac{1}{y}$$

$$f'(x) = \frac{1}{2} y' \cdot z' = \frac{1}{2} \cdot \frac{2}{(1 - x)^2} \cdot \frac{1}{\frac{1 + x}{1 - x}}$$

$$f'(x) = \frac{1 - x}{(1 - x)^2(1 + x)} = \frac{1}{1 - x^2}$$

t) $f(x) = \log_a \sqrt{1 + x}$

$$y = \sqrt{1 + x} \Rightarrow y' = \frac{1}{2\sqrt{1 + x}}$$

$$z = \log_a y \Rightarrow z' = \frac{1}{y \ell n\, a}$$

$$f'(x) = y' \cdot z' = \frac{1}{2\sqrt{1 + x}} \cdot \frac{1}{\sqrt{1 + x} \cdot \ell n\, a}$$

$$f'(x) = \frac{1}{2(1 + x)\ell n\, a}$$

163. a) $f(x) = \operatorname{arc\,sen} 3x$

$$y = 3x \Rightarrow y' = 3$$

$$z = \operatorname{arc\,sen} y \Rightarrow z' = \frac{1}{\sqrt{1 - y^2}} \qquad f'(x) = \frac{3}{\sqrt{1 - 9x^2}}$$

b) $f(x) = \operatorname{arc\,cos} x^3$

$$\left.\begin{array}{l} y = x^3 \Rightarrow y' = 3x^2 \\ z = \operatorname{arc\,cos} y \Rightarrow z' = \frac{-1}{\sqrt{1 - y^2}} \end{array}\right\} \Rightarrow f'(x) = \frac{-3x^2}{\sqrt{1 - x^6}}$$

c) $f(x) = \operatorname{arc\,tg} \frac{1}{x}$

$$\left.\begin{array}{l} y = \frac{1}{x} \Rightarrow y' = \frac{-1}{x^2} \\ z = \operatorname{arc\,tg} y \Rightarrow z' = \frac{1}{1 + y^2} \end{array}\right\} \begin{array}{l} \Rightarrow f'(x) = \frac{-x^2}{x^2(x^2 + 1)} \Rightarrow \\ \Rightarrow f'(x) = \frac{-1}{x^2 + 1} \end{array}$$

g) $$f(x) = \frac{\sqrt[3]{x}}{\operatorname{arc\,sen} x}$$

$$\left.\begin{array}{l} u = \sqrt[3]{x} \Rightarrow u' = \frac{1}{3\sqrt[3]{x^2}} \\ v = \operatorname{arc\,sen} x \Rightarrow v' = \frac{1}{\sqrt{1 - x^2}} \end{array}\right\} \Rightarrow$$

$$\Rightarrow f'(x) = \frac{\frac{1}{3\sqrt[3]{x^2}} \cdot \operatorname{arc\,sen} x - \sqrt[3]{x}\, \frac{1}{\sqrt{1 - x^2}}}{(\operatorname{arc\,sen} x)^2}$$

$$\Rightarrow f'(x) = \frac{\dfrac{\sqrt{1-x^2}\cdot \text{arc sen } x - \sqrt[3]{x}\cdot 3\sqrt[3]{x^2}}{3\sqrt[3]{x^2}\sqrt{1-x^2}}}{(\text{arc sen } x)^2} =$$

$$= \frac{\sqrt{1-x^2}\cdot \text{arc sen } x - 3x}{3\sqrt[6]{x^4(1-x^2)^3}\cdot(\text{arc sen } x)^2}$$

j) $\left.\begin{array}{l} f(x) = x\cdot \text{arc sen } x^2 - e^{x^3} \\ a = x\cdot \text{arc sen } x^2 \\ b = e^{x^3} \end{array}\right\} \Rightarrow f'(x) = a' - b'$

$a = x\cdot \text{arc sen } x^2$

$u = x \Rightarrow u' = 1$

$v = \text{arc sen } x^2$

$\left.\begin{array}{l} y = x^2 \Rightarrow y' = 2x \\ z = \text{arc sen } y \Rightarrow z' = \dfrac{1}{\sqrt{1-y^2}} \end{array}\right\} \Rightarrow v' = y'\cdot z' = \dfrac{2x}{\sqrt{1-x^4}}$

Então:

$a' = uv' + u'v \Rightarrow a' = \dfrac{2x^2}{\sqrt{1-x^4}} + \text{arc sen } x^2$

$b = e^{x^3} \Rightarrow \left\{\begin{array}{l} p = x^3 \Rightarrow p' = 3x^2 \\ q = e^p \Rightarrow q' = e^p \end{array}\right. \Rightarrow b' = p'\cdot q' \Rightarrow b' = 3x^2\cdot e^{x^3}$

Portanto: $f'(x) = \dfrac{2x^2}{\sqrt{1-x^4}} + \text{arc sen } x^2 - 3x^2\cdot e^{x^3}$.

k) $f(x) = \text{arc cos } \dfrac{\sqrt{x}}{e^x}$

$y = \dfrac{\sqrt{x}}{e^x} \Rightarrow y' = \dfrac{\dfrac{1}{2\sqrt{x}}\cdot e^x - \sqrt{x}\cdot e^x}{(e^x)^2} = \dfrac{e^x\left(\dfrac{1}{2\sqrt{x}} - \sqrt{x}\right)}{(e^x)^2} = \dfrac{1-2x}{2e^x\sqrt{x}}$

$z = \text{arc cos } y \Rightarrow z' = \dfrac{-1}{\sqrt{1-y^2}} = \dfrac{-1}{\sqrt{1-\dfrac{x}{e^{2x}}}} = \dfrac{-e^x}{\sqrt{e^{2x}-x}}$

$f'(x) = y'\cdot z' = \dfrac{1-2x}{2e^x\sqrt{x}}\cdot\dfrac{-e^x}{\sqrt{e^{2x}-x}} = \dfrac{2x-1}{2\sqrt{xe^{2x}-x^2}}$

l) $f(x) = \ell n \dfrac{\text{arc sen } x}{\text{arc cos } x}$

$y = \dfrac{\text{arc sen } x}{\text{arc cos } x}$

$\left.\begin{array}{l} u = \text{arc sen } x \Rightarrow u' = \dfrac{1}{\sqrt{1-x^2}} \\ v = \text{arc cos } x \Rightarrow v' = \dfrac{-1}{\sqrt{1-x^2}} \end{array}\right\} \Rightarrow y' = \dfrac{\text{arc cos } x + \text{arc sen } x}{\sqrt{1-x^2}\cdot(\text{arc cos } x)^2}$

$f'(x) = \dfrac{1}{y}\cdot y' = \dfrac{\text{arc cos } x}{\text{arc sen } x}\cdot\dfrac{\text{arc cos } x + \text{arc sen } x}{\sqrt{1-x^2}\cdot(\text{arc cos } x)^2}$

$f'(x) = \dfrac{\text{arc sen } x + \text{arc cos } x}{\sqrt{1-x^2}\cdot \text{arc sen } x\cdot \text{arc cos } x}$

164. $f(x) = x\cdot\sqrt{x+1} \quad (x_0 = 3)$

$f(3) = 3\cdot 2 = 6 \Rightarrow P(3, 6)$, ponto de tangência

$f'(x) = \sqrt{x+1} + \dfrac{x}{2\sqrt{x+1}} \Rightarrow f'(x) = \dfrac{3x+2}{2\sqrt{x+1}} \Rightarrow f'(3) = \dfrac{11}{4}$

$y - 6 = \dfrac{11}{4}(x-3) \Rightarrow y = \dfrac{11}{4}x - \dfrac{9}{4}$

165. Uma reta é paralela ao eixo dos x se o seu coeficiente angular é igual a zero, isto é, $f'(x_0) = 0$.

Assim, vamos derivar $f(x)$ e obter o ponto x_0.

$f(x) = \dfrac{x+1}{\sqrt{x-1}}$

$\left.\begin{array}{l} u = x+1 \Rightarrow u' = 1 \\ v = \sqrt{x-1} \Rightarrow v' = \dfrac{1}{2\sqrt{x-1}} \end{array}\right\} \Rightarrow f'(x) = \dfrac{u'v - uv'}{v^2} = \dfrac{x-3}{2\sqrt{(x-1)^3}}$

$f'(x_0) = 0 \Leftrightarrow x_0 - 3 = 0$ ou $x_0 = 3$

Então, $f(x_0) = f(3) = 2\sqrt{2}$.

O ponto procurado é $(x_0, y_0) = (3, 2\sqrt{2})$.

170. a) $f(x) = (\text{sen } x)^{(x^2)}$

$u = \text{sen } x \Rightarrow u' = \cos x$

$v = x^2 \Rightarrow v' = 2x$

$f'(x) = (\text{sen } x)^{(x^2)}\cdot\left[2x\cdot\ell n \text{ sen } x + x^2\cdot\dfrac{\cos x}{\text{sen } x}\right]$

$f'(x) = (\text{sen } x)^{(x^2)}\cdot[2x\cdot\ell n \text{ sen } x + x^2\cdot \text{cotg } x]$

b) $f(x) = x^{(x^3)}$

$u = x \Rightarrow u' = 1$

$v = x^3 \Rightarrow v' = 3x^2$

$f'(x) = x^{(x^3)} \cdot \left[3x^2 \cdot \ell n\ x + x^3 \cdot \frac{1}{x}\right]$

$f'(x) = x^{(x^3)} \cdot [3x^2 \cdot \ell n\ x + x^2] \Rightarrow f'(x) = x^{(x^3+2)} \cdot [3\,\ell n\ x + 1]$

c) $f(x) = (x)^{(e^x)}$

$\left.\begin{array}{l} u = x \Rightarrow u' = 1 \\ v = e^x \Rightarrow v' = e^x \end{array}\right\} \Rightarrow f'(x) = (x)^{(e^x)} \cdot \left[e^x \cdot \ell n\ x + e^x \cdot \frac{1}{x}\right] \Rightarrow$

$\Rightarrow f'(x) = (x)^{(e^x)} \cdot e^x \cdot \left[\ell n\ x + \frac{1}{x}\right]$

d) $f(x) = (e^x)^{(\text{tg } 3x)}$

$\left.\begin{array}{l} u = e^x \Rightarrow u' = e^x \\ v = \text{tg } 3x \Rightarrow v' = 3\sec^2 3x \end{array}\right\} \Rightarrow$

$\Rightarrow f'(x) = (e^x)^{(\text{tg } 3x)} \cdot \left[3\sec^2 3x \cdot \ell n\ e^x + \text{tg } 3x \cdot \frac{e^x}{e^x}\right] \Rightarrow$

$\Rightarrow f'(x) = (e^x)^{(\text{tg } 3x)} \cdot [3x \cdot \sec^2 3x + \text{tg } 3x]$

172. a) $v(t) = S'(t) = -a\omega \text{ sen } (\omega t + \varphi)$

b) $v(0) = -a\omega \text{ sen } \varphi$

c) $\alpha(t) = v'(t) = -a\omega^2 \cos (\omega t + \varphi)$

d) $\alpha(1) = -a\omega^2 \cos (\omega + \varphi)$

173. $y = A \text{ sen } kx$

$y' = Ak \cos kx \Rightarrow y'(0) = Ak = 12$ ① $(A > 0 \Rightarrow k > 0)$

$y'' = -Ak^2 \text{ sen } kx$

$y'' + 4y = 0 \Leftrightarrow -Ak^2 \text{ sen } kx + 4A \text{ sen } kx = 0 \Rightarrow k^2 = 4 \Rightarrow k = 2$ ②

Substituindo ② em ①, vem: $A = 6$.

Então, $A = 6$ e $k = 2$.

Capítulo VIII — Estudo da variação das funções

175. $f\left(\frac{1}{2}\right) = 0$ e $f(1) = 0$

$f'(x) = \frac{6x^2 - 16x + 9}{(3x - 4)^2}$

Como $f(x)$ não é contínua para $x = \frac{4}{3}$ e $\frac{4}{3} \notin I$, então vale o teorema de Rolle: existe $c \in I$ tal que $f'(c) = 0$.

176. $f(-3) = (-3) + 3 = 0$

$f(7) = 7 - 7 = 0$

$f'(x) = \begin{cases} 1 & \text{se } x < 2 \\ -1 & \text{se } x > 2 \end{cases}$

Como $f(x)$ não é derivável para $x = 2$, que pertence ao intervalo I, não vale o teorema de Rolle: não existe $c \in I$ tal que $f'(c) = 0$.

177. $f(-1) = 1 - 1 = 0$

$f(1) = 1 - 1 = 0$

$f(x) = \begin{cases} 1 + x & \text{se } x < 0 \\ 1 - x & \text{se } x \geqslant 0 \end{cases}$

$f'(x) = \begin{cases} 1 & \text{se } x < 0 \\ -1 & \text{se } x > 0 \end{cases}$

Como $f(x)$ não é derivável para $x = 0$, que pertence ao intervalo I, não vale o teorema de Rolle: não existe $c \in I$ tal que $f'(c) = 0$.

179. $f(x) = x^2 - 6x + 8$, $I = [2, 4]$

$f(x)$ é contínua e derivável em $\mathbb{R}$ e, portanto, em I.

$\left.\begin{array}{l} f(2) = 0 \\ f(4) = 0 \end{array}\right. \Rightarrow f(2) = f(4)$

Seja $c \in I$, tal que $f'(c) = 0$.

$f'(x) = 2x - 6 \Rightarrow f'(c) = 2c - 6 = 0 \Rightarrow c = 3$

180. $f(x) = x^3 - 2x^2 - x + 2$, $I = [-1, 2]$

$f(x)$ é contínua e derivável em $\mathbb{R}$ e, portanto, em I.

$f(-1) = f(2) = 0$

Seja $c \in I$, tal que $f'(c) = 0$.

$f'(x) = 3x^2 - 4x - 1 \Rightarrow f'(c) = 3c^2 - 4c - 1 = 0 \Rightarrow c = \frac{2 \pm \sqrt{7}}{3}$

181. $f(x) = x^3 - 16x$, $I = [0, 4]$

$f(x)$ é contínua e derivável em $\mathbb{R}$ e, portanto, em I.

$f(0) = f(4) = 0$

Seja $c \in I$, tal que $f'(c) = 0$.

$f'(x) = 3x^2 - 16 \Rightarrow f'(c) = 3c^2 - 16 \Rightarrow c = \frac{4\sqrt{3}}{3}$

183. $f(x) = x^2 + 2x - 1$, $I = [0, 1]$

$f(x)$ é contínua e derivável em $\mathbb{R}$ e, portanto, em I, e $f'(x) = 2x + 2$. Ainda $f(0) = -1$ e $f(1) = 2$, isto é, $f(0) \neq f(1)$.

Então:

$f'(c) = \frac{f(1) - f(0)}{1 - 0} \Rightarrow 2c + 2 = 3 \Rightarrow c = \frac{1}{2}$.

184. $f(x) = \sqrt[3]{x^2}$, $I = [0, 1]$

$f(x)$ é contínua e derivável em $[0, 1]$ e $f'(x) = \frac{2}{3\sqrt[3]{x}}$.

$$\left.\begin{array}{l} f(0) = 0 \\ f(1) = 1 \end{array}\right\} \Rightarrow f(0) \neq f(1) \text{ (2º caso)}$$

$$f'(c) = \frac{2}{3\sqrt[3]{c}} = \frac{1-0}{1-0} = 1 \Rightarrow c = \frac{8}{27}$$

185. $f(x) = \sqrt{100 - x^2}$, $I = [-8, 6]$

$f(x)$ é contínua e derivável em I e $f'(x) = \dfrac{-x}{\sqrt{100 - x^2}}$.

$$\left.\begin{array}{l} f(-8) = 6 \\ f(6) = 8 \end{array}\right. \Rightarrow f'(c) = \frac{-c}{\sqrt{100 - c^2}} = \frac{8-6}{6+8} = \frac{1}{7}$$

$$-7c = \sqrt{100 - c^2} \Rightarrow c = \pm\sqrt{2}$$

186. $f(x) = \dfrac{x^2 + 4x}{x - 1}$, $I = [2, 6]$

$f(x)$ é contínua e derivável em $I = [2, 6]$ e $f'(x) = \dfrac{x^2 - 2x - 4}{(x - 1)^2}$.

$f(6) = 12 = f(2)$

Então: $f'(c) = \dfrac{c^2 - 2c - 4}{(c - 1)^2} = \dfrac{12 - 12}{6 - 2} = 0 \Rightarrow c^2 - 2c - 4 = 0 \Rightarrow$

$\Rightarrow c = 1 \pm \sqrt{5}$.

Como $c = 1 - \sqrt{5} \notin I$, então $c = 1 + \sqrt{5}$ é a solução.

188. $f(x) = \dfrac{4}{(x - 3)^2}$ não é contínua para $x = 3$, que está em $I = [1, 6]$.

189. $f(x) = \dfrac{2x - 1}{3x - 4}$ não é contínua para $x = \dfrac{4}{3}$, que está em $I = [1, 2]$.

190. $f'(x) = \begin{cases} 3 & \text{se } x < 1 \\ -3 & \text{se } x > 1 \end{cases}$

$f(x)$ não é derivável para $x = 1$ (embora seja contínua nesse ponto), que está no intervalo $I = [-2, 4]$.

192. a) $f(x) = 2x^3 - 15x^2 - 84x + 13$

$f'(x) = 6x^2 - 30x - 84 \geqslant 0 \Leftrightarrow x \leqslant -2$ ou $x \geqslant 7$

b) $g(x) = 2\cos x - x + 1$

$g'(x) = -2\,\text{sen}\,x - 1$

$g'(x) = -(1 + 2\,\text{sen}\,x) \geqslant 0 \Leftrightarrow 1 + 2\,\text{sen}\,x \leqslant 0 \Rightarrow \text{sen}\,x \leqslant -\dfrac{1}{2} \Rightarrow$

$\Rightarrow \dfrac{7\pi}{6} + 2k\pi \leqslant x \leqslant \dfrac{11\pi}{6} + 2k\pi \ (k \in \mathbb{Z})$

c) $h(x) = \text{sen}\,x - \cos x$

$h'(x) = \cos x + \text{sen}\,x$

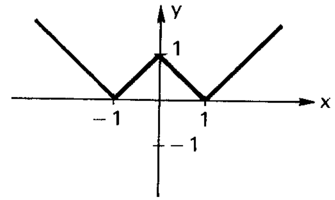
$h'(x) \geqslant 0 \Rightarrow \cos x + \cos\left(\dfrac{\pi}{2} - x\right) \geqslant 0 \Rightarrow$

$\Rightarrow \sqrt{2} \cdot \cos\left(x - \dfrac{\pi}{4}\right) \geqslant 0 \Rightarrow -\dfrac{\pi}{2} + 2k\pi \leqslant x - \dfrac{\pi}{4} \leqslant \dfrac{\pi}{2} + 2k\pi$

Então: $-\dfrac{\pi}{4} + 2k\pi \leqslant x \leqslant \dfrac{3\pi}{4} + 2k\pi$.

d) $i(x) = \big||x| - 1\big|$

Construindo a tabela, podemos ter o gráfico de $i(x)$:



$$i(x) = \begin{cases} -x - 1, & x < -1 \\ x + 1, & -1 \leqslant x \leqslant 0 \\ -x + 1, & 0 < x < 1 \\ x - 1, & x \geqslant 1 \end{cases}$$

Assim, temos: $i'(x) = \begin{cases} -1, & x < -1 \\ 1, & -1 \leqslant x \leqslant 0 \\ -1, & 0 < x < 1 \\ 1, & x \geqslant 1 \end{cases}$

Então: $i'(x) \geqslant 0$ para $-1 \leqslant x \leqslant 0$ ou $x \geqslant 1$.

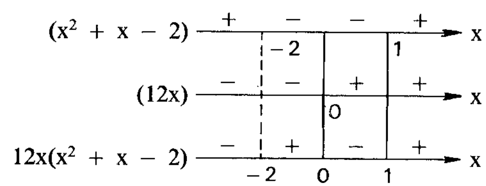
194. a) $f(x) = 3x^4 + 4x^3 - 12x^2 + 1$

$f'(x) = 12x^3 + 12x^2 - 12x$

$f'(x) = 12x(x^2 + x - 2)$

Estudando o sinal de $x^2 + x - 2$, temos: $\begin{cases} x^2 + x - 2 \geqslant 0 \text{ para } x \leqslant -2 \text{ ou } x \geqslant 1 \\ x^2 + x - 2 \leqslant 0 \text{ para } -2 \leqslant x \leqslant 1 \end{cases}$

Assim, temos:



Portanto, $f'(x) \leqslant 0$ para $x \leqslant -2$ ou $0 \leqslant x \leqslant 1$.

b) $g(x) = e^x - x$

$g'(x) = e^x - 1$

$$g'(x) \leqslant 0 \Leftrightarrow e^x - 1 \leqslant 0 \Rightarrow e^x \leqslant 1$$
$$\ell n e^x \leqslant \ell n\, 1$$
$$x \leqslant 0$$

c) $h(x) = \dfrac{3x - 2}{x + 1}$

$h'(x) = \dfrac{3(x + 1) - (3x + 2)}{(x + 1)^2} = \dfrac{5}{(x + 1)^2} \Rightarrow h'(x) \geqslant 0, \forall x \in \mathbb{R} - \{-1\}$

Então, $h(x)$ é sempre crescente.

d) $i(x) = \text{arc sen } x$

$i'(x) = \dfrac{1}{\sqrt{1 - x^2}} \Rightarrow i'(x) \geqslant 0, \forall x \in \mathbb{R} - \{1\}$

Então, $i(x)$ é sempre crescente.

195. $f(x) = x^3 - 9x^2 + 15x - 5$

$f'(x) = 3x^2 - 18x + 15$

$f'(x)$ é crescente para $x \leqslant 1$ ou $x \geqslant 5$.

$f'(x)$ é decrescente para $1 \leqslant x \leqslant 5$.

196. $f(x) = x^4 + 4$

$$f'(x) = 4x^3 + 4 \Rightarrow \begin{cases} f'(x) \geqslant 0 \Rightarrow x^3 \geqslant -1 \Rightarrow x \geqslant -1 \text{ (crescente)} \\ f'(x) \leqslant 0 \Rightarrow x^3 \leqslant -1 \Rightarrow x \leqslant -1 \text{ (decrescente)} \end{cases}$$

197. $f(x) = x^5 - \dfrac{25}{3}x^3 + 20x - 2$

$f'(x) = 5x^4 - 25x^2 + 20$

Fazendo $x^2 = y$, temos $f'(y) = 5y^2 - 25y + 20$.

Calculando as raízes, temos: $y = 1$ ou $y = 4$.

Assim: $x^2 = 1 \Rightarrow x = \pm 1$ e $x^2 = 4 \Rightarrow x = \pm 2$.

	$x < -2$	-2 a -1	-1 a 1	1 a 2	$x > 2$
$(x^2 - 1)$	+	+	−	+	+
$(x^2 - 4)$	+	−	−	−	+
$f'(x)$	+	−	+	−	+

$f(x)$ é crescente se $f'(x) \geqslant 0$: $x \leqslant -2$ ou $-1 \leqslant x \leqslant 1$ ou $x \geqslant 2$.

$f(x)$ é decrescente se $f'(x) \leqslant 0$: $-2 \leqslant x \leqslant -1$ ou $1 \leqslant x \leqslant 2$.

198. $f(x) = x + \dfrac{1}{x}$

$f'(x) = 1 - \dfrac{1}{x^2} = \dfrac{x^2 - 1}{x^2}$

	$x < -1$	-1 a 0	0 a 1	$x > 1$
$(x^2 - 1)$	+	−	−	+
(x^2)	+	+	+	+
$f'(x)$	+	−	−	+

$f(x)$ é crescente se $f'(x) \geqslant 0$: $x \leqslant -1$ ou $x \geqslant 1$.

$f(x)$ é decrescente se $f'(x) \leqslant 0$: $-1 \leqslant x \leqslant 1$ e $x \neq 0$.

199. $f(x) = x\sqrt{9 - x^2}$

$f'(x) = \sqrt{9 - x^2} + \dfrac{x(-2x)}{2\sqrt{9 - x^2}} \Rightarrow f'(x) = \dfrac{9 - 2x^2}{\sqrt{9 - x^2}}$

	$x < -3$	-3 a $-\frac{3\sqrt{2}}{2}$	$-\frac{3\sqrt{2}}{2}$ a $\frac{3\sqrt{2}}{2}$	$\frac{3\sqrt{2}}{2}$ a 3	$x > 3$
$9 - 2x^2$	−	−	+	−	−
$\sqrt{9 - x^2}$		+	+	+	
$f'(x)$		−	+	−	

Observação: $\sqrt{9 - x^2} > 0 \Leftrightarrow 9 - x^2 > 0 \Rightarrow -3 < x < 3$

Se $9 - x^2 < 0$, não há significado.

Então: $f(x)$ é crescente para $\dfrac{-3\sqrt{2}}{2} \leqslant x \leqslant \dfrac{3\sqrt{2}}{2}$.

$f(x)$ é decrescente para $-3 < x \leqslant \dfrac{-3\sqrt{2}}{2}$ ou $\dfrac{3\sqrt{2}}{2} \leqslant x < 3$.

200. $f(x) = 2 - \sqrt[3]{x - 1}$

$f'(x) = \dfrac{-1}{3\sqrt[3]{(x - 1)^2}}$

$\dfrac{-1}{3\sqrt[3]{(x - 1)^2}} \geqslant 0 \Leftrightarrow \sqrt[3]{(x - 1)^2} < 0 \Rightarrow (x - 1)^2 < 0$ (falso)

$\dfrac{-1}{3\sqrt[3]{(x - 1)^2}} \leqslant 0 \Leftrightarrow \sqrt[3]{(x - 1)^2} > 0 \Rightarrow (x - 1)^2 > 0, \forall x \in \mathbb{R}$

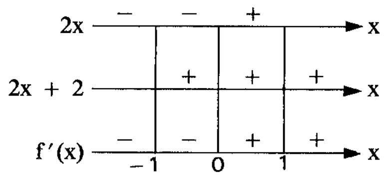
Então, $f(x)$ é decrescente, para todo x real.

201. $f(x) = \begin{cases} 2x + 1, \text{ se } x \leqslant 2 \\ 7 - x, \quad \text{se } x > 2 \end{cases} \Rightarrow f'(x) = \begin{cases} 2, \text{ se } x < 2 \\ -1, \text{ se } x > 2 \end{cases}$

Então, $f(x)$ é crescente para $x \leqslant 2$ e decrescente para $x > 2$.

202. $f(x) = \begin{cases} x^2 - 1, & \text{se } x \leqslant 1 \\ x^2 + 2x - 3, & \text{se } x > 1 \end{cases} \Rightarrow f'(x) = \begin{cases} 2x, & \text{se } x < 1 \\ 2x + 2, & \text{se } x > 1 \end{cases}$

Obtendo as raízes de $f'(x)$, temos:
para $f'(x) = 2x$, $x = 0$ e para $f'(x) = 2x + 2$, $x = -1$.

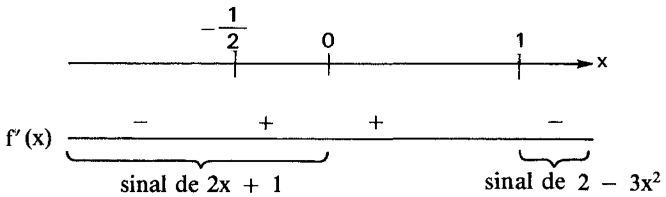


Para $x < 1$, predomina para $f'(x)$ o sinal de $2x$ e para $x > 1$ predomina o sinal de $2x + 2$

Observa-se que em $x = 0$ há mudança de sinal de $f'(x)$.
Então, $f(x)$ é decrescente para $x \leqslant 0$ e crescente para $x \geqslant 0$.

203. $f(x) = \begin{cases} x^2 + x, & \text{se } x \leqslant 0 \\ x, & \text{se } 0 < x \leqslant 1 \\ 2x - x^3, & \text{se } x > 1 \end{cases} \Rightarrow f'(x) = \begin{cases} 2x + 1, & \text{se } x < 0 \\ 1, & \text{se } 0 < x < 1 \\ 2 - 3x^2, & \text{se } x > 1 \end{cases}$

Daí vem a tabela de sinais de $f'(x)$:



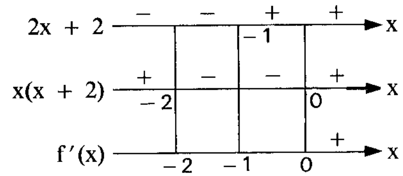
Conclusão:

$f(x)$ é crescente para $-\frac{1}{2} \leqslant x \leqslant 1$.

$f(x)$ é decrescente para $x \leqslant -\frac{1}{2}$ ou $x \geqslant 1$.

204. $f: \mathbb{R}_+ \to \mathbb{R}$, $f(x) = \ell n\, x + \ell n\,(x + 2)$

$$f'(x) = \frac{1}{x} + \frac{1}{x + 2} \Rightarrow f'(x) = \frac{2x + 2}{x(x + 2)}$$



Como o domínio é $\mathbb{R}_+^*$, não interessa o que ocorre quando $x < 0$.
$f(x)$ é crescente para $\forall x$, $x \in \mathbb{R}_+^*$.

205. $f(x) = \begin{cases} -x, & \text{se } x \leqslant -1 \\ x^2, & \text{se } -1 < x \leqslant 1 \\ e^{x-1}, & \text{se } x > 1 \end{cases}$

$f'(x) = \begin{cases} -1, & \text{se } x < -1 \\ 2x, & \text{se } -1 < x < 1 \\ e^{x-1}, & \text{se } x > 1 \end{cases}$

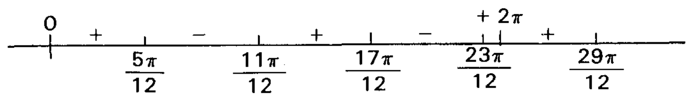
Considerando $f'(x) = 2x$, verifica-se que $f'(x) \leqslant 0$ se $x \leqslant 0$ e $f'(x) \geqslant 0$ se $x \geqslant 0$.
Então, $f'(x) \leqslant 0$ para $x \leqslant 0$ e $f'(x) \geqslant 0$ para $x \geqslant 0$, porque:
$x \leqslant 0 \Rightarrow f'(x) = -1$ ou $f'(x) = 2x$ e
$x \geqslant 0 \Rightarrow f'(x) = 2x$ ou $f'(x) = e^{x-1}$.
Assim, $f(x)$ é crescente em $\mathbb{R}_+$ e decrescente em $\mathbb{R}_-$.

206. $f: [0, 2\pi] \to \mathbb{R}$, $f(x) = 2 + 3 \cdot \text{sen}\left(2x - \frac{\pi}{3}\right)$

$f'(x) = 6 \cos\left(2x - \frac{\pi}{3}\right)$

Determinando as raízes de $f'(x)$, temos:

$\cos\left(2x - \frac{\pi}{3}\right) = 0 \Rightarrow 2x - \frac{\pi}{3} = \frac{\pi}{2} + k\pi \Rightarrow x = \frac{5\pi}{12} + \frac{k\pi}{2}$



Então, $f(x)$ é crescente para $0 \leqslant x \leqslant \frac{5\pi}{12}$ ou $\frac{11\pi}{12} \leqslant x \leqslant \frac{17\pi}{12}$ ou $\frac{23\pi}{12} \leqslant x \leqslant 2\pi$ é decrescente para $\frac{5\pi}{12} \leqslant x \leqslant \frac{11\pi}{12}$ ou $\frac{17\pi}{12} \leqslant x \leqslant \frac{23\pi}{12}$.

207. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{\cos x}$

$f'(x) = -\operatorname{sen} x \cdot e^{\cos x}$

$f(x)$ é crescente se $f'(x) \geqslant 0$, isto é, $-\operatorname{sen} x \cdot e^{\cos x} \geqslant 0 \Rightarrow$

$\Rightarrow \operatorname{sen} x \cdot e^{\cos x} \leqslant 0 \Rightarrow \operatorname{sen} x \leqslant 0 \Rightarrow \pi + 2k\pi \leqslant x \leqslant 2\pi + 2k\pi \ (k \in \mathbb{Z})$

Analogamente, $f(x)$ é decrescente se $\operatorname{sen} x \geqslant 0 \Rightarrow 2k\pi \leqslant x \leqslant \pi + 2k\pi \ (k \in \mathbb{Z})$.

208. $f'(x) = \dfrac{x \cdot \cos x - \operatorname{sen} x}{x^2} = \dfrac{\cos x \cdot (x - \operatorname{tg} x)}{x^2}$

Para todo $x \in \left]0, \dfrac{\pi}{2}\right[$ temos $x < \operatorname{tg} x$; então, $f'(x) < 0$, e, portanto, $f(x)$ é decrescente.

209. $f(x) = 3x^5 + 5x^3 + 1$

$\lim\limits_{x \to -\infty} (3x^5 + 5x^3 + 1) = \lim\limits_{x \to -\infty} \left[x^5\left(3 + \dfrac{5}{x^2} + \dfrac{1}{x^5}\right)\right] = -\infty$

$\lim\limits_{x \to +\infty} (3x^5 + 5x^3 + 1) = \lim\limits_{x \to +\infty} \left[x^5\left(3 + \dfrac{5}{x^2} + \dfrac{1}{x^5}\right)\right] = +\infty$

Isso significa que a função contínua $f(x) = 3x^5 + 5x^3 + 1$ tem imagem $\mathbb{R}$.
Sua derivada $f'(x) = 15x^4 + 15x^2$ é maior ou igual a zero para todo x real, então $f(x)$ é crescente.
Conclusão: seu gráfico só pode cortar o eixo dos x num único ponto.

212. Sejam $x_1 \in I$ e $x_2 \in I$, com $x_1 < x_2$.
Temos:

$x_1 < x_2 \Rightarrow f(x_1) \leqslant f(x_2) \Rightarrow \dfrac{1}{f(x_1)} \geqslant \dfrac{1}{f(x_2)} \Rightarrow h(x_1) \geqslant h(x_2) \Rightarrow$

$\Rightarrow$ h é decrescente em I.

213. Sejam $f(x)$ e $g(x)$ crescentes, isto é:

$x_1 < x_2 \Rightarrow f(x_1) \leqslant f(x_2)$ ① e $x_1 < x_2 \Rightarrow g(x_1) \leqslant g(x_2)$

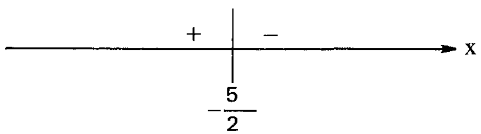
Então, sendo $h = f \circ g$, isto é, $h(x) = (f \circ g)(x) = f(g(x))$, vem:

de ②: $g(x_1) < g(x_2) \Rightarrow f(g(x_1)) \leqslant f(g(x_2)) \Rightarrow h(x_1) \leqslant h(x_2) \Rightarrow$

$\Rightarrow h(x)$ é crescente em I.

214. $f(x) = -x^2 - 5x - 4$

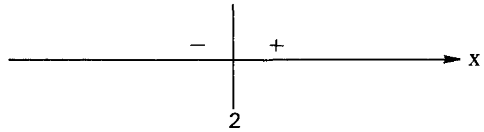
$f'(x) = -2x - 5$, cuja raiz é $x = -\dfrac{5}{2}$



$-\dfrac{5}{2}$ é ponto de máximo.

215. $f(x) = 2x^2 - 8x + 11$

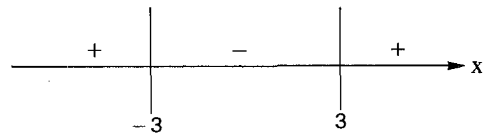
$f'(x) = 4x - 8$, cuja raiz é $x = 2$



$x = 2$ é ponto de mínimo.

216. $f(x) = x^3 - 27x + 1$

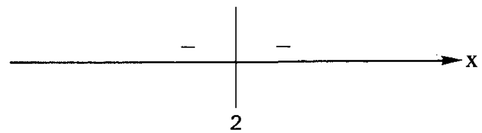
$f'(x) = 3x^2 - 27$, cujas raízes são $x = -3$ e $x = 3$



$x = -3$ é ponto de máximo e $x = 3$ é ponto de mínimo.

217. $f(x) = -x^3 + 6x^2 - 12x + 8$

$f'(x) = -3x^2 + 12x - 12$, cuja raiz é 2



$x = 2$ é ponto de inflexão.

218. $f(x) = (x - 8)^3 \cdot (x - 6)^4$

$f'(x) = 3(x - 8)^2 \cdot (x - 6)^4 + 4(x - 8)^3 \cdot (x - 6)^3$

$f'(x) = (x - 8)^2 \cdot (x - 6)^3[3(x - 6) + 4(x - 8)]$

$f'(x) = (x - 8)^2 \cdot (x - 6)^3 \cdot (7x - 50)$

		6	$\frac{50}{7}$	8	
$(x - 8)^2$	+	+	+	+	x
$(x - 6)^3$	−	+	+	+	x
$(7x - 50)$	−	−	+	+	x
$f'(x)$	+	−	+	+	x

Conclusão: $\begin{cases} x = 6 \text{ é ponto de máximo.} \\ x = \dfrac{50}{7} \text{ é ponto de mínimo.} \end{cases}$

219. $f(x) = \dfrac{1}{x^2 + 5x - 6}$

$f'(x) = \dfrac{-2x - 5}{(x^2 + 5x - 6)^2}$, cuja raiz é $x = \dfrac{-5}{2}$

		$-\frac{5}{2}$		
	$+$		$-$	x

$x = \dfrac{-5}{2}$ é ponto de máximo.

220. $f(x) = \cos 3x$

$f'(x) = -3 \operatorname{sen} 3x$

$$\operatorname{sen} 3x = 0 \Rightarrow \begin{cases} 3x = 0 + 2k\pi \Rightarrow x = \dfrac{2k\pi}{3} \ (k \in \mathbb{Z}) \\ \text{ou} \\ 3x = \pi - 0 + 2k\pi \Rightarrow x = \dfrac{(2k+1)\pi}{3} \ (k \in \mathbb{Z}) \end{cases}$$

	0		$\frac{\pi}{3}$		$\frac{2\pi}{3}$		π		$\frac{4\pi}{3}$		$\frac{5\pi}{3}$		2π		
sen 3x		$+$		$-$		$+$		$-$		$+$		$-$		$+$	x
-3		$-$		$-$		$-$		$-$		$-$		$-$		$-$	x
f'(x)		$-$		$+$		$-$		$+$		$-$		$+$		$-$	x

pontos de máximo: $x = \dfrac{2\pi}{3}$; $x = \dfrac{4\pi}{3}$; $x = 2\pi$; ..., isto é, $x = \dfrac{2k\pi}{3}$

pontos de mínimo: $x = \dfrac{\pi}{3}$; $x = \pi$; $x = \dfrac{5\pi}{3}$; ..., isto é, $x = \dfrac{(2k+1)\pi}{3}$

221. $f(x) = \operatorname{sen}\left(2x - \dfrac{\pi}{4}\right)$

$f'(x) = 2 \cos\left(2x - \dfrac{\pi}{4}\right)$

$$\cos\left(2x - \frac{\pi}{4}\right) = 0 \Rightarrow \begin{cases} 2x - \dfrac{\pi}{4} = \dfrac{\pi}{2} + 2k\pi \Rightarrow x = \dfrac{3\pi}{8} + k\pi \ (k \in \mathbb{Z}) \\ \text{ou} \\ 2x - \dfrac{\pi}{4} = \dfrac{-\pi}{2} + 2k\pi \Rightarrow x = \dfrac{7\pi}{8} + k\pi \ (k \in \mathbb{Z}) \end{cases}$$

	0		$\frac{3\pi}{8}$		$\frac{7\pi}{8}$		$\frac{11\pi}{8}$		$\frac{15\pi}{8}$		$\frac{19\pi}{8}$		$\frac{23\pi}{8}$		
f'(x)		$+$		$-$		$+$		$-$		$+$		$-$		$+$	x

pontos de máximo: $x = \dfrac{3\pi}{8}$; $x = \dfrac{11\pi}{8}$; $x = \dfrac{19\pi}{8}$; ...; isto é, $x = \dfrac{3\pi}{8} + k\pi \ (k \in \mathbb{Z})$

pontos de mínimo: $x = \dfrac{7\pi}{8}$; $x = \dfrac{15\pi}{8}$; $x = \dfrac{23\pi}{8}$; ...; isto é, $x = \dfrac{7\pi}{8} + k\pi \ (k \in \mathbb{Z})$

222. $f(x) = x \cdot \ell n\, x$

$f'(x) = 1 + \ell n\, x$, cuja raiz é $x = e^{-1}$

Consideremos $x_1 < x$; seja $x_1 = e^{-2} \Rightarrow 1 + \ell n\, e^{-2} = 1 - 2 = -1 < 0.$

Consideremos $x_2 > x$; seja $x_2 = e^2 \Rightarrow 1 + \ell n\, e^2 = 1 + 2 = +3 > 0.$

		e^{-1}		
	$-$		$+$	x

$x = e^{-1}$ é ponto de mínimo.

223. $f(x) = \ell n\,(x^2 + 1)$

$f'(x) = \dfrac{2x}{x^2 + 1}$, cuja raiz é $x = 0$

		0		
2x	$-$		$+$	x
$(x^2 + 1)$	$+$		$+$	x
f'(x)	$-$		$+$	x

$x = 0$ é ponto de mínimo.

224. $f(x) = e^{x^3 - 3x}$

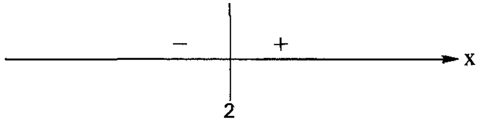
$f'(x) = (3x^2 - 3) \cdot e^{x^3 - 3x}$, cujas raízes são $x = -1$ ou $x = 1$

		-1		1		
$3x^2 - 3$	$+$		$-$		$+$	x
$e^{x^3 - 3x}$	$+$		$+$		$+$	x
f'(x)	$+$		$-$		$+$	x

$x = -1$ é ponto de máximo e $x = 1$ é ponto de mínimo.

226. $f(x) = x^2 - 4x - 1$

$f'(x) = 2x - 4 \Rightarrow f'(x) = 0 \Rightarrow 2x - 4 = 0 \Rightarrow x = 2$



$x = 2$ é ponto de mínimo.

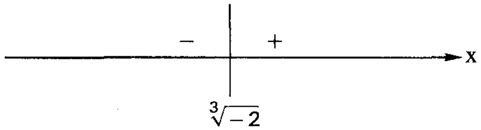
$f(2) = 4 - 8 - 1 = -5 \Rightarrow f(2) = -5$ é o valor mínimo.

227. $f(x) = x^4 + 8x$

$f'(x) = 4x^3 + 8 \Rightarrow 4x^3 + 8 = 0 \Rightarrow x = \sqrt[3]{-2}$

Seja $x_1 = -2$, $x_1 < x$. Então, $f'(x) = -32 + 8 = -24 < 0$.

Seja $x_2 = 2$, $x_2 > x$. Então, $f'(x) = 32 + 8 = 40 > 0$.

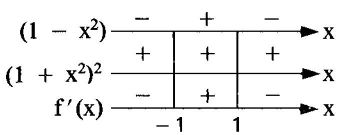


$x = \sqrt[3]{-2}$ é ponto de mínimo.

$f(\sqrt[3]{-2}) = (\sqrt[3]{-2})^4 + 8\sqrt[3]{-2} = 2\sqrt[3]{2} + 8\sqrt[3]{-2}$ é o valor mínimo.

228. $f(x) = \dfrac{x}{1 + x^2}$

$f'(x) = \dfrac{1 - x^2}{(1 + x^2)^2}$, cujas raízes são $x = -1$ e $x = 1$



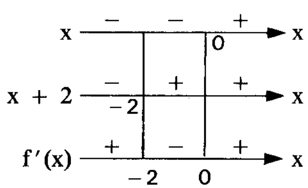
$x = -1$ é ponto de mínimo e
$x = 1$ é ponto de máximo.

$f(-1) = \dfrac{-1}{2}$ é o valor mínimo.

$f(1) = \dfrac{1}{2}$ é o valor máximo.

229. $f(x) = x^2e^x$

$f'(x) = xe^x(x + 2)$, cujas raízes são $x = 0$ e $x = -2$



$x = -2$ é ponto de máximo e
$x = 0$ é ponto de mínimo.

$f(-2) = (-2)^2 \cdot e^{-2} = 4e^{-2}$ é o valor máximo.

$f(0) = 0$ é o valor mínimo.

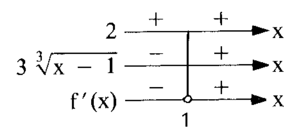
230. $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$f'(x) = \dfrac{4}{(e^x + e^{-x})^2} > 0, \forall x \in \mathbb{R}$

$f(x)$ não tem extremos.

231. $f(x) = (x - 1)^{\frac{2}{3}}$

$f'(x) = \dfrac{2}{3\sqrt[3]{x - 1}}$

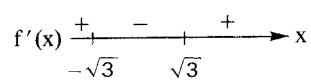


$x = 1$ é ponto de mínimo.

$f(1) = 0$ é o valor mínimo.

232. $f(x) = x^3 - 9x$

$f'(x) = 3x^2 - 9$, cujas raízes são $x = -\sqrt{3}$ e $x = \sqrt{3}$



$x = -\sqrt{3}$ é ponto de máximo.
$x = \sqrt{3}$ é ponto de mínimo.

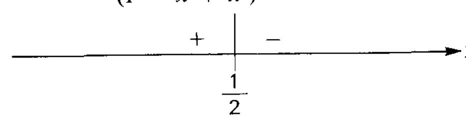
$f(-\sqrt{3}) = (-\sqrt{3})^3 - 9(-\sqrt{3}) = -3\sqrt{3} + 9\sqrt{3} = 6\sqrt{3}$

$f(\sqrt{3}) = (\sqrt{3})^3 - 9(\sqrt{3}) = 3\sqrt{3} - 9\sqrt{3} = -6\sqrt{3}$

Então: $(-\sqrt{3}, 6\sqrt{3})$ é ponto máximo e $(\sqrt{3}, -6\sqrt{3})$ é ponto mínimo.

233. $f(x) = \dfrac{1 + x - x^2}{1 - x + x^2}$

$f'(x) = \dfrac{2(1 - 2x)}{(1 - x + x^2)^2}$, cuja raiz é $x = \dfrac{1}{2}$



$x = \dfrac{1}{2}$ é ponto de máximo.

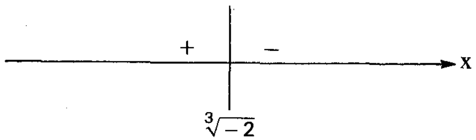
$f\left(\dfrac{1}{2}\right) = \dfrac{5}{3} \Rightarrow \left(\dfrac{1}{2}, \dfrac{5}{3}\right)$ é ponto máximo.

234. $f(x) = \dfrac{1 - x^3}{x^2}$

$f'(x) = \dfrac{-(x^3 + 2)}{x^3} \Rightarrow x = \sqrt[3]{-2}$ é a raiz

Seja $x_1 = -2 < \sqrt[3]{-2} \Rightarrow f'(-2) = \dfrac{-3}{4} < 0.$

Seja $x_2 = -1 > \sqrt[3]{-2} \Rightarrow f'(-1) = 2 > 0.$



$x = \sqrt[3]{-2}$ é ponto de mínimo.

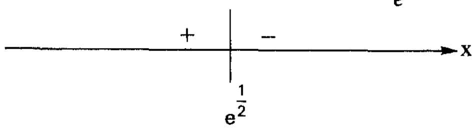
$f(\sqrt[3]{-2}) = \dfrac{1 - (\sqrt[3]{-2})^3}{(\sqrt[3]{-2})^2} = \dfrac{3}{\sqrt[3]{4}} \Rightarrow \left(\sqrt[3]{-2}, \dfrac{3}{\sqrt[3]{4}}\right)$ é ponto mínimo.

235. $f(x) = \dfrac{\ln x}{x^2}$

$f'(x) = \dfrac{1 - 2\ln x}{x^3} \Rightarrow x = e^{\frac{1}{2}}$ é a raiz

Sendo $x_1 = e^{-1} < e^{\frac{1}{2}}$, então $f'(e^{-1}) = 2e^3 > 0.$

Sendo $x_2 = e > e^{\frac{1}{2}}$, então $f'(e) = \dfrac{-1}{e^3} < 0.$



$x = e^{\frac{1}{2}}$ é ponto de máximo.

$f\left(e^{\frac{1}{2}}\right) = \dfrac{\ln e^{\frac{1}{2}}}{\left(e^{\frac{1}{2}}\right)^2} = \dfrac{1}{2e} \Rightarrow \left(e^{\frac{1}{2}}, \dfrac{1}{2e}\right)$ é ponto máximo.

236. $f(x) = x^3 + ax^2 + b$

$f'(x) = 3x^2 + 2ax \Rightarrow 3x^2 + 2ax = 0$ tem raízes $\begin{cases} x = 0 \\ \text{ou} \\ x = \dfrac{-2a}{3} \end{cases}$

Sendo $(1, 5)$ um extremo relativo, então $-\dfrac{2a}{3} = 1$ e $f(1) = 5$. Daí $a = -\dfrac{3}{2}$.

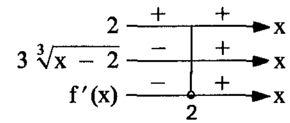
Assim, $f(1) = 1 - \dfrac{3}{2} + b = 5 \Rightarrow b = \dfrac{11}{2}$.

Portanto, $a = \dfrac{-3}{2}$ e $b = \dfrac{11}{2}$.

238. $f(x) = (x - 2)^{\frac{2}{3}}$

Como f é derivável em $[1, 5]$, apliquemos o teorema de Fermat:

$f'(x) = \dfrac{2}{3\sqrt[3]{x - 2}}$



Então, $x = 2$ é ponto de mínimo interior.

Calculemos o valor de f nesse ponto e nos extremos do intervalo $[1, 5]$:

$f(1) = (1 - 2)^{\frac{2}{3}} = \sqrt[3]{(-1)^2} = 1$

$f(2) = (2 - 2)^{\frac{2}{3}} = 0$

$f(5) = (5 - 2)^{\frac{2}{3}} = \sqrt[3]{3^2} = \sqrt[3]{9}$

O valor máximo absoluto de f no intervalo $[1, 5]$ é o maior dos números $f(1)$, $f(2)$ e $f(5)$; portanto, $f(5) = \sqrt[3]{9}$.

O valor mínimo absoluto de f no intervalo $[1, 5]$ é o menor dos números $f(1)$, $f(2)$ e $f(5)$; portanto, $f(2) = 0$.

Assim, $x = 2$ é o ponto de mínimo absoluto e $x = 5$ é o ponto de máximo absoluto.

239. $f(x) = \dfrac{1}{x - 2}$ é contínua no intervalo $[3, 6]$, em que não está $x = 2$.

$f'(x) = \dfrac{-1}{(x - 2)^2}$

então $f'(x) < 0$ para todo x do domínio de f.

Calculando $f(3) = 1$ e $f(6) = \dfrac{1}{4}$, temos que $x = 3$ é ponto de máximo absoluto e $x = 6$ é um ponto de mínimo absoluto em $[3, 6]$, pois $f(x)$ é decrescente.

240. $h = 30 + 20t - 5t^2$

$h' = 20 - 10t = 0 \Rightarrow 20 - 10t = 0 \Rightarrow t = 2$

A altura máxima é atingida no instante $t = 2s$.

241. $s = a \cdot \cos(kt + \ell)$
$v = s' = -ak \cdot \text{sen}(kt + \ell)$
$\alpha = v' = -ak^2 \cos(kt + \ell)$

a) A velocidade v é máxima no instante em que $sen\ (kt + \ell) = -1$, ou seja,

$$kt + \ell = \frac{3\pi}{2} + 2n\pi \Rightarrow t = \frac{3\pi}{2k} - \frac{\ell}{k} + \frac{2n\pi}{k}.$$

A posição em que isso ocorre é

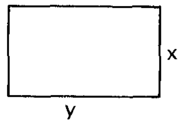
$$s = a \cdot \cos\left(\frac{3\pi}{2} + 2n\pi\right) = 0.$$

b) A aceleração α é mínima no instante em que $cos\ (kt + \ell) = 1$, ou seja,

$$kt + \ell = 2n\pi \Rightarrow t = -\frac{\ell}{k} + \frac{2n\pi}{k}.$$

A posição em que isso ocorre é
$s = a \cdot \cos(2n\pi) = a.$

243.



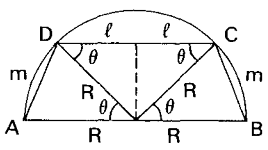
$x + y =$ semiperímetro $\Rightarrow x + y = a \Rightarrow y = a - x$
A área do retângulo é dada por $S = xy$.
Então: $S = x(a - x) \Rightarrow S = -x^2 + ax$.
Derivando: $S' = -2x + a \Rightarrow x = \frac{a}{2}$ é raiz.

Considerando $0 < x < \frac{a}{2}$, $S' = -2x + a > 0$ e para $\frac{a}{2} < x < a$,

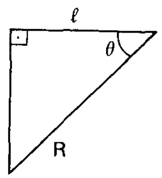
$S' = -2x + a < 0$, isto é, $x = \frac{a}{2}$ é ponto de máximo local.

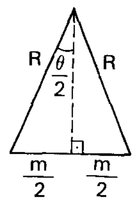
A área máxima do retângulo se dá para $x = \frac{a}{2}$ e $y = a - \frac{a}{2} = \frac{a}{2}$, isto é, quando o retângulo é um quadrado.

244.



perímetro $= 2R + 2\ell + 2m$
$\ell = R \cdot \cos\theta$
$\frac{m}{2} = R \cdot \text{sen}\frac{\theta}{2}$





então:

perímetro $P = 2R + 2R\cos\theta + 4R\,\text{sen}\frac{\theta}{2}$

$$P' = -2R\,\text{sen}\,\theta + 2R\cos\frac{\theta}{2}$$

$$P' = 0 \Rightarrow \text{sen}\,\theta = \cos\frac{\theta}{2} \Rightarrow 2 \cdot \text{sen}\frac{\theta}{2} \cdot \cos\frac{\theta}{2} = \cos\frac{\theta}{2} \Rightarrow$$

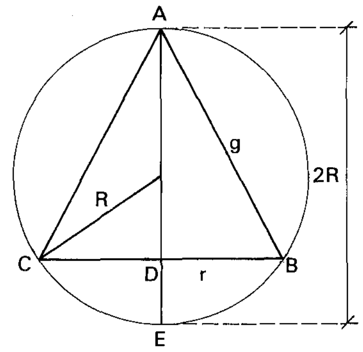
$$\Rightarrow \text{sen}\frac{\theta}{2} = \frac{1}{2} \Rightarrow \frac{\theta}{2} = \frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{3}$$

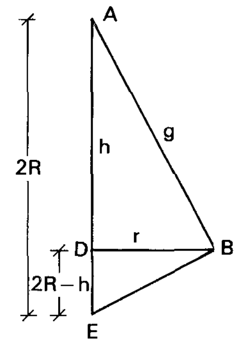
$$\left.\begin{array}{l} 0 < \theta < \frac{\pi}{3} \Rightarrow P' > 0 \\ \frac{\pi}{3} < \theta < \frac{\pi}{2} \Rightarrow P' < 0 \end{array}\right\} \Rightarrow \frac{\pi}{3} \text{ é ponto de máximo local}$$

então o perímetro máximo é:

$$P_{\text{máx}} = P\left(\frac{\pi}{3}\right) = 2R + 2R \cdot \cos\frac{\pi}{3} + 4R \cdot \text{sen}\frac{\pi}{6} = 5R$$

246.





No ΔABE, temos:
$g^2 = 2Rh \Rightarrow g = \sqrt{2Rh}$
e
$r^2 = h(2R - h) \Rightarrow r = \sqrt{h(2R - h)}$.
$A_\ell = \pi rg$
$A_\ell = \pi\sqrt{h(2R - h)} \cdot \sqrt{2Rh} \Rightarrow$
$A_\ell = \pi\sqrt{4R^2h^2 - 2Rh^3}$

Derivando, vem:

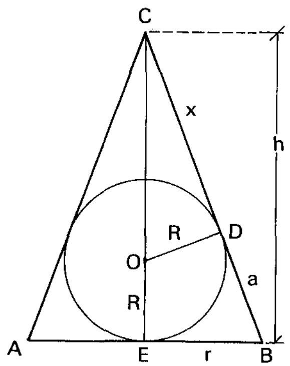
$$A_\ell' = \pi\frac{8R^2h - 6Rh^2}{2\sqrt{4R^2h^2 - 2Rh^3}} = \pi\frac{R(4R - 3h)}{\sqrt{2R(2R - h)}}$$

Determinando os zeros de A_ℓ', vem:

$$A_\ell' = 0 \Leftrightarrow 4R - 3h = 0 \Rightarrow h = \frac{4}{3}R.$$

Sendo $h = \frac{4}{3}R$, vem $A_\ell = \frac{8\sqrt{3}}{9}R^2$ e $r = \frac{2\sqrt{2}}{3}R$.

247.



$V = \frac{1}{3}\pi r^2 h$

Considerando o ΔCOD, temos:

$x^2 = (h - R)^2 - R^2 \Rightarrow x = \sqrt{h^2 - 2Rh}$.

Considerando a semelhança entre os triângulos, $\Delta COD \sim \Delta CEB$, vem:

$$\frac{x}{h} = \frac{R}{r} \Rightarrow \frac{\sqrt{h^2 - 2Rh}}{h} = \frac{R}{r} \Rightarrow$$

$$\Rightarrow \frac{h^2 - 2Rh}{h^2} = \frac{R^2}{r^2} \Rightarrow r^2 = \frac{R^2h}{h - 2R}.$$

Substituindo o valor de r^2 na fórmula do volume, vem:

$$V = \frac{1}{3}\pi\left(\frac{R^2h}{h - 2R}\right) \cdot h \Rightarrow V = \frac{\pi R^2}{3}\left(\frac{h^2}{h - 2R}\right).$$

Derivando, temos:

$$V' = \frac{\pi R^2}{3} \cdot \frac{h(h - 4R)}{(h - 2R)^2}.$$

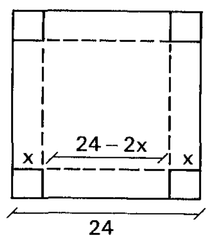
Determinando as raízes de v', vem:

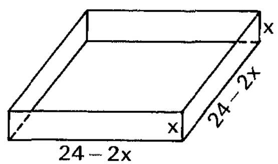
$h(h - 4R) = 0 \Rightarrow h = 0$ ou $h = 4R$.

Como $h = 0 \Rightarrow V = 0$, então devemos ter $h = 4R$.

Sendo $h = 4R$, vem $V = \frac{8}{3}\pi R^3$ e $r = R\sqrt{2}$.

248.



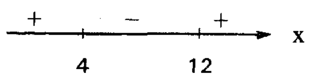


Volume da caixa aberta = área da base × altura

$V = (24 - 2x)^2 \cdot x \Rightarrow V = 4x^3 - 96x^2 + 576x$

Derivando, temos:

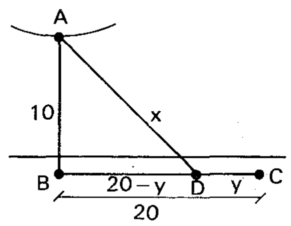
$V' = 12x^2 - 192x + 576$, cujas raízes são: $x = 4$ e $x = 12$.



$x = 4$ é ponto de máximo e
$x = 12$ é ponto de mínimo.

Para o volume maior possível, devemos ter $x = 4$.

249.



$$t_1 = \frac{x}{v_1} \Rightarrow t_1 = \frac{\sqrt{10^2 + (20 - y)^2}}{4}$$

$$t_2 = \frac{y}{v_2} \Rightarrow t_2 = \frac{y}{5}$$

$$t = t_1 + t_2 = \frac{\sqrt{y^2 - 40y + 500}}{4} + \frac{y}{5}$$

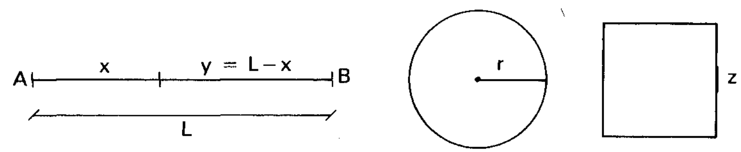
Derivando, temos:

$t' = \dfrac{5y - 100 + 4\sqrt{y^2 - 40y + 500}}{20\sqrt{y^2 - 40y + 500}}$ e, fazendo $t' = 0$, vem:

$4\sqrt{y^2 - 40y + 500} = -5y + 100 \Rightarrow y = 33{,}3$ (rejeitada) ou $y = 6{,}6$.

Então, $DC = y = 6{,}6$ e $BD = 20 - y = 13{,}4$.

250.



$$x = 2\pi r \qquad L - x = 4z$$

$$r = \frac{x}{2\pi} \qquad z = \frac{L - x}{4}$$

$$A_\bullet = \pi r^2 \qquad A_\blacksquare = z^2$$

$$A_\bullet = \pi\frac{x^2}{4\pi^2} \qquad A_\blacksquare = \frac{(L - x)^2}{16}$$

$$A_\bullet = \frac{x^2}{4\pi} \qquad A_\blacksquare = \frac{L^2 - 2Lx + x^2}{16}$$

$$A = A_\bullet + A_\blacksquare = \frac{x^2}{4\pi} + \frac{L^2 - 2Lx + x^2}{16} = \frac{(4 + \pi)x^2 - 2\pi Lx + \pi L^2}{16\pi}$$

Para procurar o valor máximo, devemos obter A':

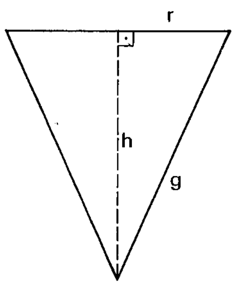
$$A' = \frac{2(4 + \pi)x - 2\pi L}{16\pi}$$

$$A' = \frac{(4 + \pi)x - \pi L}{8\pi}$$

De onde, $A' = 0 \Rightarrow (4 + \pi)x - \pi L = 0 \Rightarrow x = \dfrac{L\pi}{4 + \pi}$

e daí $y = L - x \Rightarrow y = \dfrac{4L}{4 + \pi}$.

251.



$$V = \frac{\pi r^2 h}{3} \Rightarrow r^2 = \frac{3V}{\pi h}$$

$$A_\ell = \pi r g = \pi r\sqrt{r^2 + h^2} =$$
$$= \pi\sqrt{r^4 + r^2h^2} =$$
$$= \pi\sqrt{\frac{9V^2}{\pi^2h^2} + \frac{3Vh}{\pi}} =$$
$$= \sqrt{\frac{9V^2}{h^2} + 3\pi Vh}$$

$$A_\ell' = \frac{-\frac{18V^2}{h^3} + 3\pi V}{2\sqrt{\frac{9V^2}{h^2} + 3\pi Vh}}$$

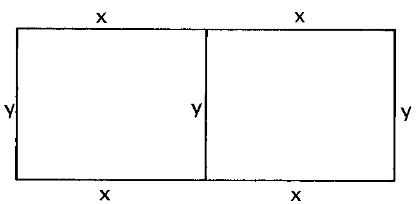
$$A_\ell' = 0 \Rightarrow h^3 = \frac{6V}{\pi}$$

De $V = \frac{\pi r^2 h}{3}$ vem $\frac{V}{h^3} = \frac{\pi}{3} \cdot \frac{r^2}{h^2}$; então:

$$\frac{r^2}{h^2} = \frac{3V}{\pi h^3} = \frac{3V}{6V} = \frac{1}{2}$$

e daí $\frac{r}{h} = \frac{1}{\sqrt{2}}$.

252.

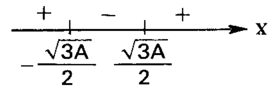


O comprimento da cerca é $L = 4x + 3y$. A área de cada curral é $A = xy$, então $y = \frac{A}{x}$ e daí:

$$L = 4x + \frac{3A}{x}$$

$$L' = 4 - \frac{3A}{x^2}.$$

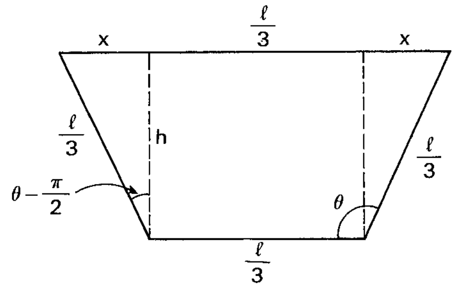
Impondo $L' = 0$, vem $4x^2 - 3A = 0 \Rightarrow x = \frac{\sqrt{3A}}{2}$, que é ponto de mínimo, conforme se vê pela variação de sinal de L':



Calculemos o valor de $L_{mín}$:

$$L_{mín} = 4 \cdot \frac{\sqrt{3A}}{2} + \frac{3A}{\frac{\sqrt{3A}}{2}} = 4\sqrt{3A}.$$

253.



A secção reta da calha é um trapézio de área:

$$A = \frac{h}{2} \cdot \left[2x + 2\frac{\ell}{3}\right] = h\left(x + \frac{\ell}{3}\right).$$

Temos também:

$$h = \frac{\ell}{3} \cdot \cos\left(\theta - \frac{\pi}{2}\right) = \frac{\ell}{3} \cdot \operatorname{sen}\theta$$

$$x = \frac{\ell}{3} \cdot \operatorname{sen}\left(\theta - \frac{\pi}{2}\right) = -\frac{\ell}{3} \cdot \cos\theta$$

então:

$$A = -\frac{\ell^2}{9} \cdot \operatorname{sen}\theta \cdot \cos\theta + \frac{\ell^2}{9} \cdot \operatorname{sen}\theta = \frac{\ell^2}{9}\left(\operatorname{sen}\theta - \frac{\operatorname{sen}2\theta}{2}\right)$$

$$A' = \frac{\ell^2}{9}(\cos\theta - \cos 2\theta) = 0 \Rightarrow \cos 2\theta = \cos\theta \Rightarrow \theta = \frac{2\pi}{3}$$

254. $f(x) = x \cdot (x - 2)^3$

$f'(x) = (x - 2)^2(4x - 2)$

$f''(x) = (x - 2)(12x - 12)$

Calculando os zeros de $f'(x)$, temos: $x = 2$ e $x = \frac{1}{2}$.

Calculando $f''(x)$ nesses pontos, temos:

$f''(2) = 0$

$f''\left(\frac{1}{2}\right) > 0 \Rightarrow \frac{1}{2}$ é ponto de mínimo.

255. $f(x) = \frac{x}{1 + x^2}$

$$f'(x) = \frac{-x^2 + 1}{(1 + x^2)^2}$$

$$f''(x) = \frac{2x(x^2 - 3)}{(1 + x^2)^3}$$

$f'(x) = 0 \Rightarrow x = -1$ ou $x = 1$ (raízes de $f'(x)$)

Substituindo em $f''(x)$, vem:

$f''(-1) = \frac{1}{2} > 0 \Rightarrow x = -1$ é ponto de mínimo.

$f''(1) = -\frac{1}{2} < 0 \Rightarrow x = 1$ é ponto de máximo.

256. $f(x) = x^2e^x$

$f'(x) = xe^x(2 + x)$

$f''(x) = e^x(2 + 4x + x^2)$

Obtendo as raízes de $f'(x)$, vem $x = 0$ e $x = -2$.

Calculando $f''(x)$ nesses pontos, vem:

$f''(0) = 2 > 0 \Rightarrow x = 0$ é ponto de mínimo.

$f''(-2) = -2e^{-2} < 0 \Rightarrow x = -2$ é ponto de máximo.

257. $f(x) = e^x + e^{-x}$

$f'(x) = e^x - e^{-x}$

$f''(x) = e^x + e^{-x}$

Com $f'(x) = 0$ temos, para raiz, $x = 0$.

Calculando $f''(0)$, vem: $f''(0) = e^0 + e^0 = 2 > 0$.

Portanto, $x = 0$ é ponto de mínimo.

258. $f(x) = \log_e (1 + x^2)$

$f'(x) = \frac{2x}{1 + x^2}$; $f'(x) = 0 \Rightarrow x = 0$

$$f''(x) = \frac{2 - 2x^2}{(1 + x^2)^2}$$

Então, $f''(0) = 2 > 0 \Rightarrow x = 0$ é ponto de mínimo.

259. $f(x) = (x - 1)^{\frac{2}{3}}$

$f'(x) = \frac{2}{3\sqrt[3]{x - 1}} \neq 0$

Não é possível determinar os extremos usando o critério da derivada segunda.

260. $f(x) = \dfrac{\ell n \frac{x}{2}}{(\ell n\, x)^2}$

$$f'(x) = \frac{\ell n\, x - 2\,\ell n \frac{x}{2}}{x(\ell n\, x)^3} = \frac{\ell n \frac{4}{x}}{x(\ell n\, x)^3}$$

$f'(x) = \ell n \frac{4}{x} = 0 \Leftrightarrow \frac{4}{x} = 1 \Rightarrow x = 4$

Calculemos $f''(x)$:

$$f''(x) = \frac{-\left[\ell n\, x + \ell n \frac{4}{x}(\ell n\, x + 3)\right]}{x^2(\ell n\, x)^4}.$$

Então: $f''(4) = \frac{-1}{16(\ell n\, 4)^3} < 0 \Rightarrow x = 4$ é ponto de máximo.

Então, determinemos $f(4)$:

$f(4) = \frac{\ell n\, 2}{(\ell n\, 4)^2} = \frac{1}{4\,\ell n\, 2} \Rightarrow P\left(4, \frac{1}{4\,\ell n\, 2}\right)$ é ponto de máximo.

261. $f(x) = -(x - 1)^2$

$f'(x) = -2(x - 1)$; $f'(x) = 0 \Rightarrow x = 1$

$f''(x) = -2 < 0 \Rightarrow x = 1$ é ponto de máximo.

Então: $f(1) = 0$; $f(-2) = -9$; $f(3) = -4$.

Isso significa que:

$x = 1$ é ponto de máximo absoluto.

$x = -2$ é ponto de mínimo absoluto.

262. $f(x) = x^2 - 4x + 8$

$f'(x) = 2x - 4 \Rightarrow x = 2$ é raiz de $f'(x)$.

$f''(x) = 2 > 0 \Rightarrow x = 2$ é ponto de mínimo.

Então: $f(-1) = 13$; $f(2) = 4$; $f(3) = 7$.

Isso significa que:

$x = -1$ é ponto de máximo absoluto.

$x = 2$ é ponto de mínimo absoluto.

263. $f(x) = \begin{cases} x + 2, & \text{se } x < 1 \\ x^2 - 6x + 8, & \text{se } x \geqslant 1 \end{cases}$

$f'(x) = \begin{cases} 1, & \text{se } x < 1 \\ 2x - 6, & \text{se } x \geqslant 1 \end{cases} \Rightarrow x = 3$ é raiz de $f'(x)$

$f''(x) = \begin{cases} 0, & \text{se } x < 1 \\ 2, & \text{se } x \geqslant 1 \end{cases} \Rightarrow x = 3$ é ponto de mínimo

Calculando os extremantes:

$f(-6) = -4$; $f(5) = 3$; $f(1) = 3$; $f(3) = -1$.

Concluímos:

$x = 1$ e $x = 5$ são pontos de máximo absoluto.

$x = -6$ é ponto de mínimo absoluto.

265. Seja $P(x, y)$ um ponto da curva $(x - 3)^2 + (y - 6)^2 = 20$.

Notando que $\left(\dfrac{x - 3}{\sqrt{20}}\right)^2 + \left(\dfrac{y - 6}{\sqrt{20}}\right)^2 = 1$, façamos:

$\dfrac{x - 3}{\sqrt{20}} = \cos\theta$ e $\dfrac{y - 6}{\sqrt{20}} = \operatorname{sen}\theta$.

(Assim teremos a vantagem de lidar com uma só variável: θ.) Temos, então:
$x = 3 + \sqrt{20} \cdot \cos\theta$ e $y = 6 + \sqrt{20} \cdot \operatorname{sen}\theta$.

Calculemos a distância entre $P(x, y)$ e $(-2, -4)$:

$d = \sqrt{(x + 2)^2 + (y + 4)^2} = \sqrt{(5 + \sqrt{20} \cdot \cos\theta)^2 + (10 + \sqrt{20} \cdot \operatorname{sen}\theta)^2} =$

$= \sqrt{145 + 10\sqrt{20} \cdot \cos\theta + 20\sqrt{20} \cdot \operatorname{sen}\theta}$

Para calcularmos os valores de θ que tornam d mínimo (ou máximo), calculamos d' e impomos $d' = 0$. Resulta $-\operatorname{sen}\theta + 2 \cdot \cos\theta = 0 \Rightarrow \operatorname{tg}\theta = 2$.

Daí resultam as alternativas:

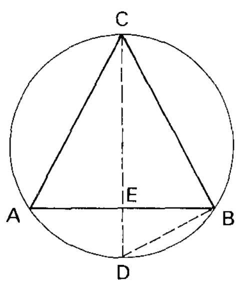
$\operatorname{sen}\theta = \dfrac{2}{\sqrt{5}}$ e $\cos\theta = \dfrac{1}{\sqrt{5}} \Rightarrow P(5, 10)$

ou

$\operatorname{sen}\theta = -\dfrac{2}{\sqrt{5}}$ e $\cos\theta = -\dfrac{1}{\sqrt{5}} \Rightarrow P(1, 2)$

As distâncias desses pontos ao ponto $(-2, -4)$ são, respectivamente, $\sqrt{245}$ e $\sqrt{45}$.
Conclusão: $(1, 2)$ é que está a distância mínima.

267.



A figura ao lado é uma secção da esfera e do cone inscrito, feita por um plano contendo o eixo de simetria do cone.

O volume do cone é dado por:

$V = \dfrac{\pi}{3} r^2 h$, em que $r = BE$ (raio da base do cone) e $h = CE$ (altura do cone).

No triângulo retângulo BCD, temos:

$(BE)^2 = (CE)(ED) \Rightarrow r^2 = h(2R - h) = 2Rh - h^2 \Rightarrow r = \sqrt{2Rh - h^2}$.

Então, em função de h, temos:

$$V = \frac{\pi}{3}(2Rh^2 - h^3).$$

Procuremos o valor máximo para $0 < h < 2R$:

$$V' = \frac{\pi}{3}(4Rh - 3h^2)$$

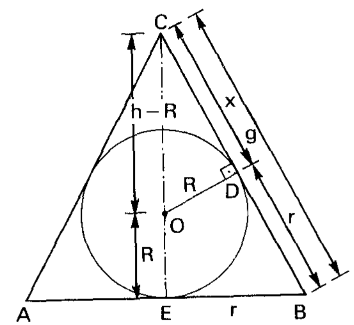
$$V' = 0 \Rightarrow h = \frac{4R}{3}$$

Como $V = 0$ para $h = 0$ ou $h = 2R$ e, sendo $h = \dfrac{4R}{3}$, então:

$V = \dfrac{\pi}{3} \cdot \left(\dfrac{32R^3}{27}\right) \Rightarrow V = \dfrac{32\pi R^3}{81}$; portanto, $h = \dfrac{4R}{3}$ é o ponto de máximo para V e, nesse caso,

$$r = \sqrt{2R \cdot \frac{4R}{3} - \frac{16R^2}{9}} \Rightarrow r = \frac{2\sqrt{2}\,R}{3}.$$

268.



A área total é dada por: $A_t = \pi r(g + r)$ ①

Considerando o ΔCOD, temos:

$x^2 = (h - R)^2 - R^2 \Rightarrow x = \sqrt{h^2 - 2Rh}$.

Como $g = x + r$, então $g = \sqrt{h^2 - 2Rh} + r$ ②

Mas devemos colocar toda a equação em função de uma única variável.

Então, da semelhança dos triângulos ΔCOD e ΔCEB, vem: $\dfrac{x}{h} = \dfrac{R}{r} \Rightarrow \dfrac{\sqrt{h^2 - 2Rh}}{h} = \dfrac{R}{r} \Rightarrow$

$$\Rightarrow \frac{h - 2R}{h} = \frac{R^2}{r^2} \Rightarrow r^2 = \frac{R^2 h}{h - 2R} \Rightarrow r = R \cdot \sqrt{\frac{h}{h - 2R}} \quad ③$$

Então, substituindo ② e ③ em ①, vem:

$$A_t = \pi R\sqrt{\frac{h}{h - 2R}} \cdot \left(\sqrt{h^2 - 2Rh} + R \cdot \sqrt{\frac{h}{h - 2R}} + R \cdot \sqrt{\frac{h}{h - 2R}}\right) \Rightarrow$$

$$\Rightarrow A_t = \pi R \frac{h^2}{h - 2R}.$$

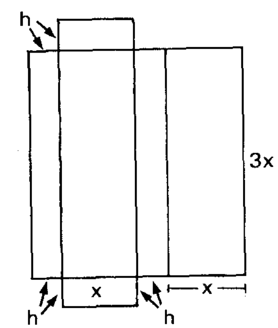
Derivando, vem:

$A_t' = \dfrac{h^2 - 4Rh}{(h - 2R)^2}$ e, com $A_t' = 0$, temos $h = 0$ ou $h = 4R$.

Como $A_t = 0$ se $h = 0$, então $h = 4R$ e $A_t = 8\pi R^2$.

Então: $r = R \cdot \sqrt{\dfrac{h}{h - 2R}} \Rightarrow r = \sqrt{2}\,R$.

269. Uma caixa de dimensões x, $3x$ e h, com tampa, planificada, é mostrada na figura abaixo:



A área de papelão utilizada para montar essa caixa é:
$S = 2(xh) + 2(3x \cdot h) + 2(3x \cdot x) = 6x^2 + 8xh.$

O volume da caixa é $V = (3x)(x)(h)$, então $h = \dfrac{V}{3x^2}$ e daí

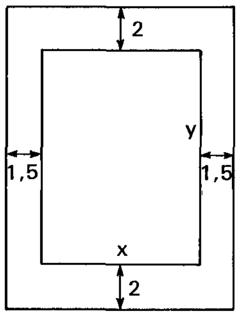
$$S = 6x^2 + \frac{8V}{3x}.$$

$$S' = 12x - \frac{8V}{3x^2}$$

$$S' = 0 \Rightarrow x = \sqrt[3]{\frac{2V}{9}} = \frac{\sqrt[3]{6V}}{3} \Rightarrow h = \frac{\sqrt[3]{6V}}{2}$$

As dimensões procuradas são $\dfrac{\sqrt[3]{6V}}{3}$, $\sqrt[3]{6V}$ e $\dfrac{\sqrt[3]{6V}}{2}$.

270.



$xy = 300 \Rightarrow y = \dfrac{300}{x}$

$(x + 4)(y + 3) = A$ (área total da página)

Então: $A = 3x + 4y + 312.$

Como $y = \dfrac{300}{x}$, vem:

$$A = 3x + 4 \cdot \frac{300}{x} + 312 \Rightarrow$$

$$\Rightarrow A = \frac{3(x^2 + 104x + 400)}{x}.$$

Derivando A, temos:

$$A' = 3\frac{(2x + 104)x - (x^2 + 104x + 400)}{x^2} \Rightarrow A' = 3 \cdot \frac{x^2 - 400}{x^2}.$$

Determinando as raízes de A', vem $x = -20$ ou $x = 20.$

Só interessa o valor $x = 20 \Rightarrow y = 15.$

Então, as dimensões da página são: $\begin{cases} x + 4 = 24 \\ \text{e} \\ y + 3 = 18 \end{cases}$

271. preço de venda, por quilo $= 3\,000 - 30x$ (x é o n.º de dias)
variação do peso, por dia $= 150 + 2{,}5x$
preço de venda do porco: $v = (3\,000 - 30x)(150 + 2{,}5x)$ (1)
custo do porco por dia: 2 000
custo do porco em x dias: $c = 2\,000x$ (2)
lucro = (1) − (2)
$\ell = (3\,000 - 30x)(150 + 2{,}5x) - 2\,000x$
$\ell = -75x^2 + 1\,000x + 450\,000$

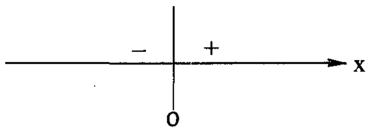
Derivando:
$\ell' = -150x + 1\,000.$
Obtendo a raiz, vem: $x = 6{,}6.$
Se vender no 6.º dia, o preço de um porco será Cr$ 453 300,00.
Se vender no 7.º dia, o preço de um porco será Cr$ 453 325,00.
Então, o lucro será máximo se o fazendeiro aguardar *7* dias para vender os porcos.

272. custo $= a + bx$ (x unidades)
venda $= e - dx$ (cada unidade)
lucro $= \ell = (c - dx)x - (a + bx)$
$\ell = cx - dx^2 - a - bx$
$\ell = -dx^2 + (c - b)x - a$
Derivando, vem:
$\ell' = -2dx + (c - b)$
Obtendo a raiz de ℓ', vem:

$$\ell' = 0 \Leftrightarrow -2dx + (c - b) = 0 \Rightarrow x = \frac{c - b}{2d}.$$

273. $f(x) = x^3 + 9x$
$f'(x) = 3x^2 + 9x$
$f''(x) = 6x$
A raiz de $f''(x) = 0$ é $x = 0.$

Para $x < 0$, concavidade negativa.
Para $x > 0$, concavidade positiva.



Calculando $f'''(x)$, vem:
$f'''(x) = 6 \neq 0.$
Então, $f(0) = 0 \Rightarrow (0, 0)$ é ponto de inflexão.

274. $f(x) = \dfrac{x}{x^2 - 1}$

$$f'(x) = \frac{(x^2 - 1) - 2x^2}{(x^2 - 1)^2} \Rightarrow f'(x) = \frac{-(x^2 + 1)}{(x^2 - 1)^2}$$

$$f''(x) = -\frac{2x(x^2 - 1)^2 - (x^2 + 1)(4x^3 - 4x)}{(x^2 - 1)^2}$$

$$f''(x) = \frac{2x(x^4 + 2x^2 - 3)}{(x^2 - 1)^4} \Rightarrow f''(x) = \frac{2x(x^2 - 1)(x^2 + 3)}{(x^2 - 1)^4} \Rightarrow$$

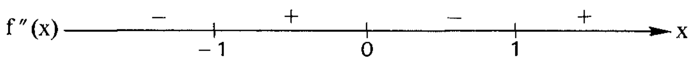
$$\Rightarrow f''(x) = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}$$

Fazendo $f''(x) = 0$, vem $x = 0$ como única raiz real.

Os pontos $x = -1$ e $x = 1$ são pontos de descontinuidade, pois anulam o denominador.
Verifiquemos o sinal de $f''(x)$ nos intervalos referentes aos extremos -1, 0 e 1.

$$f''(-2) = \frac{2(-2)(4+3)}{(4-1)^3} < 0 \qquad f''\left(\frac{1}{2}\right) = \frac{2\left(\frac{1}{2}\right)\left(\frac{1}{4}+3\right)}{\left(\frac{1}{4}-1\right)^3} < 0$$

$$f''\left(-\frac{1}{2}\right) = \frac{2\left(-\frac{1}{2}\right)\left(\frac{1}{4}+3\right)}{\left(\frac{1}{4}-1\right)^3} > 0 \qquad f''(2) = \frac{2(2)(4+3)}{(4-1)^3} > 0$$



Assim, temos: concavidade positiva para $-1 < x < 0$ ou $x > 1$
concavidade negativa para $x < -1$ ou $0 < x < 1$

Como $x = 0$ é raiz de $f''(x)$, então vem $f(0) = \dfrac{0}{0-1} = 0$, que significa $P(0, 0)$ é ponto de inflexão.

275. $f(x) = \sqrt[5]{x-2}$

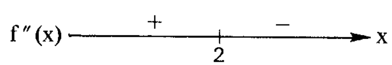
$$f'(x) = \frac{1}{5\sqrt[5]{(x-2)^4}}$$

$$f''(x) = \frac{-4\sqrt[5]{x-2}}{25}$$

Resolvendo $f''(x) = 0$, vem $x = 2$.
Analisando o sinal de $f''(x)$ à esquerda e à direita de 2, vem:

$$f''(1) = \frac{-4}{25}\sqrt[5]{-1} > 0$$

$$f''(3) = \frac{-4}{25}\sqrt[5]{1} < 0$$



Então: para $x < 2$, concavidade positiva
para $x > 2$, concavidade negativa
Sendo 2 raiz de $f''(x)$ e $f(2) = 0$, então $P(2, 0)$ é o ponto de inflexão.

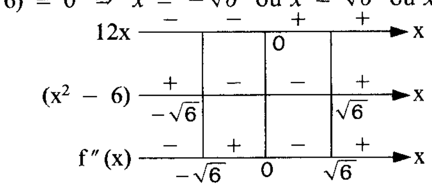
276. $f(x) = \dfrac{2x}{\sqrt[2]{(x^2+4)^3}}$

$$f'(x) = \frac{2(x^2+4)^{\frac{3}{2}} - 2x \cdot 3x(x^2+4)^{\frac{1}{2}}}{\left[(x^2+4)^{\frac{3}{2}}\right]^2} \Rightarrow f'(x) = \frac{-4(x^2-2)}{(x^2+4)^{\frac{5}{2}}}$$

$$f''(x) = \frac{-8x(x^2+4)^{\frac{5}{2}} + 4(x^2-2) \cdot 5x(x^2+4)^{\frac{3}{2}}}{\left[(x^2+4)^{\frac{5}{2}}\right]^2}$$

$$f''(x) = \frac{12x(x^2-6)}{(x^2+4)^{\frac{7}{2}}}$$

Determinando as raízes de $f''(x)$, temos:
$12x(x^2 - 6) = 0 \Rightarrow x = -\sqrt{6}$ ou $x = \sqrt{6}$ ou $x = 0$



Estudando os sinais de $f''(x)$, verificamos que:
para $x < -\sqrt{6}$ ou $0 < x < \sqrt{6}$, concavidade negativa,
para $-\sqrt{6} < x < 0$ ou $x > \sqrt{6}$, concavidade positiva.
Para obter os pontos de inflexão, determinemos $f'''(x)$.

$$f'''(x) = \frac{36(x^2-2)(x^2+4)^{\frac{7}{2}} - 12x(x^2-6) \cdot 7x \cdot (x^2+4)^{\frac{5}{2}}}{\left[(x^2+4)^{\frac{7}{2}}\right]^2}$$

$$f'''(x) = \frac{-48(x^4 - 12x^2 + 6)}{(x^2+4)^{\frac{9}{2}}}$$

Então, temos:

$$f'''(-\sqrt{6}) = \frac{-48(36 - 72 + 6)}{(6+4)^{\frac{9}{2}}} \neq 0$$

$$f'''\sqrt{6} = \frac{-48(36 - 72 + 6)}{(6+4)^{\frac{9}{2}}} \neq 0 \qquad f'''(0) = \frac{-48(0 - 0 + 6)}{(0+4)^{\frac{9}{2}}} \neq 0$$

Assim, $-\sqrt{6}$, $\sqrt{6}$ e 0 são abscissas de pontos de inflexão.
Determinemos as ordenadas desses pontos:

$$f(-\sqrt{6}) = \frac{2(-\sqrt{6})}{\sqrt{10^3}} = \frac{-\sqrt{60}}{50} \Rightarrow P_1\left(-\sqrt{6}, \frac{-\sqrt{60}}{50}\right)$$

$$f(\sqrt{6}) = \frac{2(\sqrt{6})}{\sqrt{10^3}} = \frac{\sqrt{60}}{50} \Rightarrow P_2\left(\sqrt{6}, \frac{\sqrt{60}}{50}\right)$$

$$f(0) = \frac{2 \cdot 0}{\sqrt{4^3}} = 0 \Rightarrow P_3(0, 0)$$

277. $f(x) = \begin{cases} x^2, \text{ se } x < 1 \\ x^3 - 4x^2 + 7x - 3, \text{ se } x \geqslant 1 \end{cases}$

$f'(x) = \begin{cases} 2x, \text{ se } x < 1 \\ 3x^2 - 8x + 7, \text{ se } x \geqslant 1 \end{cases}$

$f''(x) = \begin{cases} 2, \text{ se } x < 1 \\ 6x - 8, \text{ se } x \geqslant 1 \end{cases}$

$f'''(x) = \begin{cases} 0, \text{ se } x < 1 \\ 6, \text{ se } x \geqslant 1 \end{cases}$

Determinando a raiz de $f''(x)$, temos:

$f''(x) = 0 \Leftrightarrow 6x - 8 = 0 \Rightarrow x = \frac{4}{3}$.

Para $x \geqslant 1, f'''\left(\frac{4}{3}\right) \neq 0$, para $1 < x < \frac{4}{3}, f''(x) < 0$ e para $x > \frac{4}{3}, f''(x) > 0$, o que significa que nesses intervalos temos:

para $x < 1$ ou $x > \frac{4}{3}$, concavidade positiva

para $1 < x < \frac{4}{3}$, concavidade negativa

para $x = \frac{4}{3}, f\left(\frac{4}{3}\right) = \frac{43}{27} \Rightarrow P\left(\frac{4}{3}, \frac{43}{27}\right)$ é ponto de inflexão.

para $x = 1$, $f(1) = 1 \Rightarrow \theta(1, 1)$ é ponto de inflexão.

278. $f(x) = \text{sen } x - \cos x$

$f'(x) = \cos x + \text{sen } x$

$f''(x) = -\text{sen } x + \cos x$

A concavidade é positiva se $f''(x) > 0$.

Então: $-\text{sen } x + \cos x > 0 \Rightarrow \frac{5\pi}{4} + 2k\pi < x < \frac{9\pi}{4} + 2k\pi$

279. $f(x) = x^5 - x^4$

$f'(x) = 5x^4 - 4x^3$

$f''(x) = 20x^3 - 12x^2$

A concavidade é negativa se $f''(x) < 0$.

Então: $20x^3 - 12x^2 < 0 \Rightarrow 4x^2(5x - 3) < 0$.

		0		$\frac{3}{5}$	
$4x^2$	+		+		+
$5x - 3$	−		−		+
$f''(x)$	−		−		+

$f''(x) < 0$ para $x < \frac{3}{5}$ e $x \neq 0$.

280. $f(x) = (x^2 - 1)(x^2 - 4)$

$f'(x) = 2x(2x^2 - 5)$

$f''(x) = 12x^2 - 10$

$f'''(x) = 24x$

Obtendo as raízes de $f''(x)$, temos $x = \pm\sqrt{\frac{5}{6}}$.

$f'''\left(\pm\sqrt{\frac{5}{6}}\right) = 24\left(\pm\sqrt{\frac{5}{6}}\right) \neq 0$

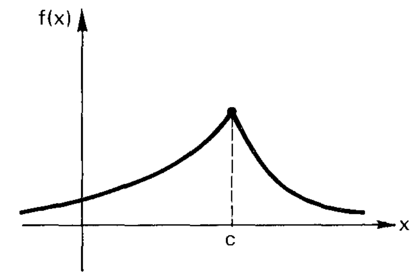
Então, $x = \pm\sqrt{\frac{5}{6}}$ são as abscissas de pontos de inflexão.

Calculando as ordenadas: $f\left(\pm\sqrt{\frac{5}{6}}\right) = \left(\frac{5}{6} - 1\right)\left(\frac{5}{6} - 4\right) = \frac{19}{36}$.

Então, os pontos de inflexão são: $\left(-\sqrt{\frac{5}{6}}, \frac{19}{36}\right)$ e $\left(\sqrt{\frac{5}{6}}, \frac{19}{36}\right)$.

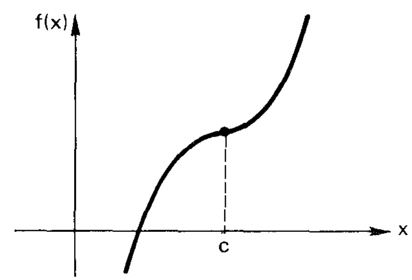
281 $> c$ e $\begin{cases} f'(x) < 0 \Rightarrow \text{função decrescente} \\ f''(x) > 0 \Rightarrow \text{concavidade positiva} \end{cases}$

$x < c$ e $\begin{cases} f'(x) > \ \Rightarrow \text{função crescente} \\ f''(x) > 0 \Rightarrow \text{concavidade positiva} \end{cases}$

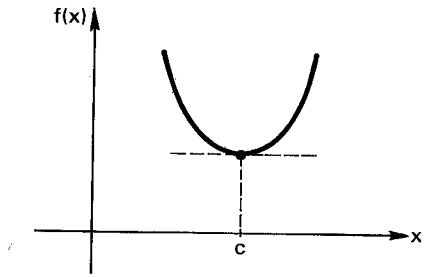


282. $x > c$ e $\begin{cases} f'(x) > 0 \Rightarrow \text{função crescente} \\ f''(x) > 0 \Rightarrow \text{concavidade positiva} \end{cases}$

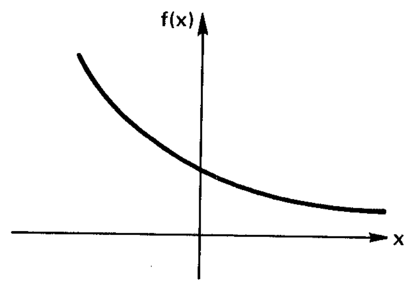
$x < c$ e $\begin{cases} f'(x) > 0 \Rightarrow \text{função crescente} \\ f''(x) < 0 \Rightarrow \text{concavidade negativa} \end{cases}$



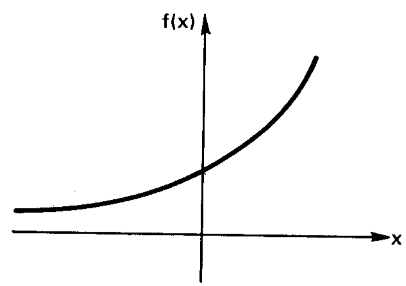
283. $f'(c) = f''(c) = 0$ (a reta tangente ao gráfico é paralela ao eixo x)
$x < c$ e $f''(x) > 0 \Rightarrow$ concavidade positiva
$x > c$ e $f''(x) > 0 \Rightarrow$ concavidade positiva



284. $f(x) > 0 \Rightarrow$ o gráfico está todo acima do eixo dos x.
$f'(x) < 0 \Rightarrow$ função decrescente
$f''(x) > 0 \Rightarrow$ concavidade positiva



285. $f(x) > 0 \Rightarrow$ o gráfico está todo acima do eixo Ox.
$f'(x) > 0 \Rightarrow$ função crescente
$f''(x) > 0 \Rightarrow$ concavidade positiva



286. $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$
$f'(x) = a_1 + 2a_2x + 3a_3x^2$
$f''(x) = 2a_2 + 6a_3x$
$f'''(x) = 6a_3$
Sendo $(0, 3)$ extremo relativo, vem:
1) $f(0) = a_0$ e $f(0) = 3 \Rightarrow a_0 = 3$
2) $x = 0$ é raiz de $f'(x) \Rightarrow f'(0) = 0$ e $f'(0) = a_1 \Rightarrow a_1 = 0$
Sendo $(1, -1)$ ponto de inflexão, temos:
1) $f'''(x) \neq 0 \Rightarrow 6a_3 \neq 0 \Rightarrow a_3 \neq 0$
2) $f''(1) = 0 \Rightarrow 2a_2 + 6a_3 = 0 \Rightarrow a_2 = -3a_3$
3) $f(1) = 3 + a_2 + a_3 = -1 \Rightarrow a_2 = -4 - a_3$
Então, $-4 - a_3 = -3a_3 \Rightarrow a_3 = 2$ e, portanto, $a_2 = -6$.

287. $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$
$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3$
$f''(x) = 2a_2 + 6a_3x + 12a_4x^2$
$f'''(x) = 6a_3 + 24a_4x$
Se o gráfico passa pela origem, então:
$f(0) = 0 \Rightarrow a_0 = 0$.
Se o gráfico é simétrico em relação ao eixo y, então
$f(x) = f(-x)$, $\forall x$, e daí
$a_1x + a_2x^2 + a_3x^3 + a_4x^4 = -a_1x + a_2x^2 - a_3x^3 + a_4x^4, \forall x$
então $a_1 = -a_1$ e $a_3 = -a_3$, ou seja, $a_1 = 0$ e $a_3 = 0$.
Sendo $(1, -1)$ ponto de inflexão, vem:
1) $f'''(x) = 24a_4 \neq 0 \Rightarrow a_4 \neq 0$
2) $f''(1) = 0 \Rightarrow 2a_2 + 12a_4 = 0 \Rightarrow a_2 = -6a_4$
3) $f(1) = a_2 + a_4 = -1 \Rightarrow a_2 = -1 - a_4$
Então: $-1 - a_4 = -6a_4 \Rightarrow a_4 = \dfrac{1}{5} \Rightarrow a_2 = \dfrac{-6}{5}$.

288. $f(x) = 2x^3 - 6x$
a) Seu domínio é $\mathbb{R}$.
b) A função é ímpar, porque:
$f(-x) = 2(-x)^3 - 6(-x) = -2x^3 + 6x = -(2x^3 - 6x) = -f(x)$.
c) Não há pontos de descontinuidade em $\mathbb{R}$.
d) Fazendo $x = 0$, temos $f(0) = 0$.
Fazendo $f(x) = 0$, temos $2x^3 - 6x = 0$, isto é, $x = 0$ ou $x = \pm\sqrt{3}$.
As interseções com os eixos são os pontos $(0, 0)$; $(-\sqrt{3}, 0)$ e $(\sqrt{3}, 0)$.
e) $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} x^3 = +\infty$
$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} x^3 = -\infty$
f) $f'(x) = 6x^2 - 6 = 6(x + 1)(x - 1)$
então: $x \leqslant -1$ ou $x \geqslant 1 \Rightarrow f'(x) \geqslant 0 \Rightarrow f$ crescente
$-1 \leqslant x \leqslant 1 \Rightarrow f'(x) \leqslant 0 \Rightarrow f$ decrescente

g) $f'(x) = 0 \Rightarrow x = -1$ ou $x = 1$

$$f''(x) = 12x \Rightarrow \begin{cases} f''(-1) = -12 < 0 \\ f''(1) = 12 > 0 \end{cases}$$

então f tem um máximo em $x = -1$ e um mínimo em $x = 1$.

h) $f''(x) = 12x$ e, então:

$x < 0 \Rightarrow f''(x) < 0 \Rightarrow$ concavidade negativa

$x > 0 \Rightarrow f''(x) > 0 \Rightarrow$ concavidade positiva

Como o sinal da concavidade muda em $x = 0$, o gráfico tem um ponto de inflexão em $x = 0$.

289. a) O domínio é $\mathbb{R}$.

b) A função não é par nem ímpar porque:

$f(-x) = -4x^3 - x^2 + 24x - 1 \neq f(x)$ e $-f(x)$.

c) A função é contínua em $\mathbb{R}$.

d) $f(0) = -1 \Rightarrow (0, -1) \in$ gáfico

$f(x) = 0 \Rightarrow 4x^3 - x^2 - 24x - 1 = 0 \Rightarrow$ três raízes reais irracionais, sendo uma positiva

e) $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} 4x^3 = +\infty$

$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} 4x^3 = -\infty$

f) $f'(x) = 12x^2 - 2x - 24 = 12\left(x + \frac{4}{3}\right)\left(x - \frac{3}{2}\right)$

$x \leqslant -\frac{4}{3}$ ou $x \geqslant \frac{3}{2} \Rightarrow f'(x) \geqslant 0 \Rightarrow f$ é crescente

$-\frac{4}{3} \leqslant x \leqslant \frac{3}{2} \Rightarrow f'(x) \leqslant 0 \Rightarrow f$ é decrescente

g) $f''(x) = 24x - 2$

$x = -\frac{4}{3} \Rightarrow f'\left(\frac{4}{3}\right) = 0$ e $f''\left(-\frac{4}{3}\right) < 0 \Rightarrow -\frac{4}{3}$ é ponto de máximo

$x = \frac{3}{2} \Rightarrow f'\left(\frac{3}{2}\right) = 0$ e $f''\left(\frac{3}{2}\right) > 0 \Rightarrow \frac{3}{2}$ é ponto de mínimo

h) $x < \frac{1}{12} \Rightarrow f''(x) < 0 \Rightarrow$ concavidade negativa

$x > \frac{1}{12} \Rightarrow f''(x) > 0 \Rightarrow$ concavidade positiva

Como o sinal da concavidade muda em $x = \frac{1}{12}$, este é um ponto de inflexão.

290. a) O domínio é $\mathbb{R}$.

b) A função não é par nem é ímpar porque

$f(-x) = 3x^4 - 4x^3 + 6x^2 - 4 \neq f(x)$ e $-f(x)$.

c) A função é contínua em $\mathbb{R}$.

d) $f(0) = -4 \Rightarrow (0, -4) \in$ gráfico

$f(x) = 0 \Rightarrow 3x^4 + 4x^3 + 6x^2 - 4 = 0 \Rightarrow$ duas raízes reais e duas raízes imaginárias.

e) $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} 3x^4 = +\infty$

$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} 3x^4 = +\infty$

f) $f'(x) = 12x^3 + 12x^2 + 12x = 12(x - 0)(x^2 + x + 1)$

$x \leqslant 0 \Rightarrow f'(x) \leqslant 0 \Rightarrow f$ é decrescente

$x \geqslant 0 \Rightarrow f'(x) \geqslant 0 \Rightarrow f$ é crescente

g) $f''(x) = 36x^2 + 24x + 12$

$x = 0 \Rightarrow f'(0) = 0$ e $f''(0) > 0 \Rightarrow 0$ é ponto de mínimo

h) $\forall x \in \mathbb{R}$, $f''(x) > 0 \Rightarrow$ concavidade positiva.

291. $f(x) = (x - 1)^2(x + 2)^3$

a) Seu domínio é $\mathbb{R}$.

b) A função não é par nem ímpar, pois:

$f(-x) = (-x - 1)^2(-x + 2)^3 = [-(x + 1)]^2[-(x - 2)]^3 = -(x + 1)^2(x - 2)^3$ não é idêntica nem a $f(x)$ nem a $f(-x)$.

c) A função polinomial é contínua em $\mathbb{R}$.

d) Fazendo $x = 0$, temos:

$f(0) = (-1)^2(2)^3 = 8$.

Fazendo $f(x) = 0$, temos $(x - 1)^2(x + 2)^3 = 0$, isto é, $x = 1$ ou $x = -2$.

As interseções com os eixos são os pontos $(0, 8)$; $(1, 0)$ e $(-2, 0)$.

e) $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} x^5 = +\infty$

$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} x^5 = -\infty$

f) $f'(x) = (x - 1)(x + 2)^2(5x + 1)$, então

$x \leqslant -\frac{1}{5}$ ou $x \geqslant 1 \Rightarrow f'(x) \geqslant 0 \Rightarrow f$ é crescente

$-\frac{1}{5} \leqslant x \leqslant 1 \Rightarrow f'(x) \leqslant 0 \Rightarrow f$ é decrescente

g) $f'(x) = 0 \Rightarrow x = 1$ ou $x = -2$ ou $x = \frac{-1}{5}$

$$f''(x) = (x + 2)(20x^2 + 8x - 10) \Rightarrow \begin{cases} f''(1) = 54 > 0 \\ f''(-2) = 0 \\ f''\left(\frac{-1}{5}\right) = \frac{-485}{25} < 0 \end{cases}$$

então f tem um mínimo em $x = 1$ e um máximo em $x = \frac{-1}{5}$.

h) $f''(x) = (x + 2)(20x^2 + 8x - 10)$, então:

		-2		$\frac{-2-3\sqrt{6}}{10}$		$\frac{-2+3\sqrt{6}}{10}$	
$x + 2$	$-$		$+$		$+$		$+$
$20x^2 + 8x - 10$	$+$		$+$		$-$		$+$
$f''(x)$	$-$		$+$		$-$		$+$

para $x < -2$ ou $\frac{-2 - 3\sqrt{6}}{10} < x < \frac{-2 + 3\sqrt{6}}{10} \Rightarrow f''(x) < 0 \Rightarrow$

$\Rightarrow$ concavidade negativa

para $-2 < x < \frac{-2 - 3\sqrt{6}}{10}$ ou $x > \frac{-2 + 3\sqrt{6}}{10} \Rightarrow f''(x) > 0 \Rightarrow$

$\Rightarrow$ concavidade positiva

Como o sinal da concavidade muda em $x = -2$, $x = \frac{-2 - 3\sqrt{6}}{10}$ e $x = \frac{-2 + 3\sqrt{6}}{10}$, o gráfico tem *3* pontos de inflexão.

292. $f(x) = 3x^{\frac{2}{3}} - 2x$

a) O domínio é IR.

b) $f(-x) = 3(-x)^{\frac{2}{3}} - 2(-x) = 3x^{\frac{2}{3}} + 2x$

A função não é par nem ímpar.

c) Não há pontos de descontinuidade.

d) $x = 0 \Rightarrow f(0) = 0$

$f(x) = 0 \Rightarrow x = 0$ ou $x = \frac{27}{8}$

Então, os pontos de interseção com os eixos são: $(0, 0)$ e $\left(\frac{27}{8}, 0\right)$.

e) $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} (-2x) = -\infty$

$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (-2x) = +\infty$

f) $f'(x) = \frac{2}{\sqrt[3]{x}} - 2$

$f'(x) = 0 \Leftrightarrow \frac{2}{\sqrt[3]{x}} - 2 = 0 \Rightarrow \sqrt[3]{x} = 1 \Rightarrow x = 1$

Variação de sinal de $f'(x) = \frac{2 - 2\sqrt[3]{x}}{\sqrt[3]{x}}$:

		0		1	
$2 - 2\sqrt[3]{x}$	$+$		$+$		$-$
$\sqrt[3]{x}$	$-$		$+$		$+$
$f'(x)$	$-$		$+$		$-$

Então: para $0 \leqslant x \leqslant 1$, $f'(x) \geqslant 0 \Rightarrow f$ crescente e
para $x \leqslant 0$ ou $x \geqslant 1$, $f'(x) \leqslant 0 \Rightarrow f$ decrescente

g) $f''(x) = \frac{-2}{3}x^{\frac{2}{3}} \Rightarrow f''(1) = \frac{-2}{3} < 0$, tem máximo em $x = 1$

h) $f''(x) = \frac{-2}{3}x^{\frac{2}{3}} \Rightarrow \begin{cases} x < 0,\ f''(x) < 0 \Rightarrow \text{concavidade negativa} \\ x > 0,\ f''(x) < 0 \Rightarrow \text{concavidade negativa} \end{cases}$

Como o sinal da concavidade não muda em $x = 0$, então $x = 0$ não é ponto de inflexão.

293. a) O domínio é IR.

b) A função não é par nem é ímpar porque:

$f(-x) = -x^{\frac{1}{3}} + 2x^{\frac{4}{3}} \neq f(x)$ e $-f(x)$.

c) A função é contínua em IR.

d) $f(0) = 0 \Rightarrow (0, 0) \in$ gráfico

$f(x) = 0 \Rightarrow x^{\frac{1}{3}} + 2x^{\frac{4}{3}} = 0 \Rightarrow 2\sqrt[3]{x^4} = \sqrt[3]{x} \Rightarrow 8x^4 = -x \Rightarrow$

$\Rightarrow x = 0$ ou $x = -\frac{1}{2} \Rightarrow (0, 0)$ e $\left(-\frac{1}{2}, 0\right)$ estão no gráfico

e) $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} 2x^{\frac{4}{3}} = +\infty$

$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} 2x^{\frac{4}{3}} = +\infty$

f) $f'(x) = \frac{1}{3\sqrt[3]{x^2}} + \frac{8\sqrt[3]{x}}{3} = \frac{1 + 8x}{3\sqrt[3]{x^2}}$

$x \leqslant -\frac{1}{8} \Rightarrow f'(x) \leqslant 0 \Rightarrow f$ decrescente

$x \geqslant -\frac{1}{8} \Rightarrow f'(x) \geqslant 0 \Rightarrow f$ crescente

g) $f''(x) = -\frac{2}{9\sqrt[3]{x^5}} + \frac{8}{9\sqrt[3]{x^2}} = \frac{-2 + 8x}{9\sqrt[3]{x^5}}$

Variação de sinal de $f''(x)$:

		0		$\frac{1}{4}$	
$-2 + 8x$	$-$		$-$		$+$
$9\sqrt[3]{x^5}$	$-$		$+$		$+$
$f''(x)$	$+$		$-$		$+$

$x < 0$ ou $x > \frac{1}{4} \Rightarrow f''(x) > 0 \Rightarrow$ concavidade positiva

$0 < x < \frac{1}{4} \Rightarrow f''(x) < 0 \Rightarrow$ concavidade negativa

pontos de inflexão: $x = 0$ e $x = \frac{1}{4}$.

h) $x = -\frac{1}{8} \Rightarrow f'\left(-\frac{1}{8}\right) = 0$ e $f''\left(-\frac{1}{8}\right) > 0 \Rightarrow -\frac{1}{8}$ é ponto de mínimo

294. a) O domínio é $\mathbb{R}$.

b) A função não é par nem é ímpar porque:

$f(-x) = 1 - (x + 2)^{\frac{1}{3}} \neq f(x)$ e $-f(x)$.

c) A função é contínua em $\mathbb{R}$.

d) $f(0) = 1 - \sqrt[3]{2} \Rightarrow (0, 1 - \sqrt[3]{2}) \in$ gráfico

$f(x) = 0 \Rightarrow \sqrt[3]{x - 2} = -1 \Rightarrow x - 2 = -1 \Rightarrow x = 1 \Rightarrow (1, 0) \in$ gráfico

e) $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} (x - 2)^{\frac{1}{3}} = +\infty$

$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} (x - 2)^{\frac{1}{3}} = -\infty$

f) $f'(x) = \dfrac{1}{3\sqrt[3]{(x - 2)^2}}$

$\forall x \in \mathbb{R}$, $f'(x) > 0 \Rightarrow f$ é crescente em $\mathbb{R}$

g) $f''(x) = -\dfrac{2}{9\sqrt[3]{(x - 2)^5}}$

$x < 2 \Rightarrow f''(x) > 0 \Rightarrow$ concavidade positiva

$x > 2 \Rightarrow f''(x) < 0 \Rightarrow$ concavidade negativa

$x = 2$ é ponto de inflexão.

295. $f(x) = x\sqrt{1 - x}$

a) Domínio: $1 - x \geqslant 0 \Rightarrow x \leqslant 1$.

b) $f(-x) = -x\sqrt{1 + x}$; a função não é par nem ímpar

c) Não há pontos de descontinuidade em $x \leqslant 1$.

d) $x = 0 \Rightarrow f(0) = 0$

$f(x) = 0 \Rightarrow x\sqrt{1 - x} = 0 \Rightarrow \begin{cases} x = 0 \\ \text{ou} \\ x = 1 \end{cases}$

As interseções com os eixos são os pontos $(0, 0)$ e $(1, 0)$.

e) $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} x = -\infty$

Como $x \leqslant 1$, não há $\lim\limits_{x \to +\infty}$.

f) $f'(x) = \dfrac{2 - 3x}{2\sqrt{1 - x}}$

$f'(x) = 0 \Rightarrow x = \frac{2}{3}$

Então, $x \leqslant \frac{2}{3} \Rightarrow f'(x) \geqslant 0 \Rightarrow f$ crescente

$x \geqslant \frac{2}{3} \Rightarrow f'(x) \leqslant 0 \Rightarrow f$ decrescente

g) $f''(x) = \dfrac{3x - 4}{4\sqrt{(1 - x)^3}}$

$f''\left(\frac{2}{3}\right) = \dfrac{3 \cdot \frac{2}{3} - 4}{4\sqrt{\left(1 - \frac{2}{3}\right)^3}} < 0 \Rightarrow f$ tem máximo em $x = \frac{2}{3}$

h) $f''(x) = 0 \Rightarrow x = \frac{4}{3} > 1$, o que significa que não há ponto de inflexão no domínio de f.

296. a) O domínio é $\mathbb{R} - \{3\}$.

b) A função não é par nem é ímpar porque:

$f(-x) = \dfrac{-x + 1}{-x - 3} \neq f(x)$ e $-f(x)$.

c) A função não é contínua em $x = 3$.

d) $x = 0 \Rightarrow f(0) = -\frac{1}{3} \Rightarrow \left(0, -\frac{1}{3}\right) \in$ gráfico

$f(x) = 0 \Rightarrow \dfrac{x + 1}{x - 3} = 0 \Rightarrow x = -1 \Rightarrow (-1, 0) \in$ gráfico

e) $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} \dfrac{1 + \frac{1}{x}}{1 - \frac{3}{x}} = 1$

$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} \dfrac{1 + \frac{1}{x}}{1 - \frac{3}{x}} = 1$

f) $f'(x) = \dfrac{-4}{(x - 3)^2}$

$\forall x \neq 3$, $f'(x) < 0 \Rightarrow f$ é decrescente no domínio

g) $f''(x) = \dfrac{8}{(x - 3)^3}$

$x < 3 \Rightarrow f''(x) < 0 \Rightarrow$ concavidade negativa

$x > 3 \Rightarrow f''(x) > 0 \Rightarrow$ concavidade positiva

$x = 3$ não é ponto de inflexão, pois não está no domínio de f.

h) $\lim\limits_{x\to 3^-} f(x) = -\infty$ e $\lim\limits_{x\to 3^+} f(x) = +\infty$

297. $f(x) = \dfrac{9x}{x^2 + 9}$

a) Domínio é $\mathbb{R}$.

b) $f(-x) = \dfrac{9(-x)}{(-x)^2 + 9} = \dfrac{-9x}{x^2 + 9} = -f(x) \Rightarrow f$ é ímpar

c) Não há pontos de descontinuidade.

d) $x = 0 \Rightarrow f(0) = 0$

$f(x) = 0 \Rightarrow x = 0$

Então $(0,\ 0)$ é o único ponto de interseção com os eixos.

e) $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \dfrac{9x}{x^2+9} = \lim\limits_{x\to+\infty} \dfrac{9x}{x^2\left(1 + \dfrac{9}{x^2}\right)} = \lim\limits_{x\to+\infty} \dfrac{9}{x^2} = 0$

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{9}{x^2} = 0$

f) $f'(x) = \dfrac{9(-x^2 + 9)}{(x^2 + 9)^2}$

$f'(x) = 0 \Rightarrow -x^2 + 9 = 0 \Rightarrow x = -3$ ou $x = +3$

$x \leqslant -3$ ou $x \geqslant 3 \Rightarrow f'(x) \leqslant 0 \Rightarrow f$ decrescente

$-3 \leqslant x \leqslant 3 \Rightarrow f'(x) \geqslant 0 \Rightarrow f$ crescente

g) $f''(x) = \dfrac{9(2x^3 - 54x)}{(x^2 + 9)^3}$

$f''(-3) = \dfrac{9[2(-3)^3 - 54(-3)]}{(9 + 9)^3} > 0 \Rightarrow -3$ é ponto de mínimo

$f''(3) = \dfrac{9[2(3)^2 - 54(3)]}{(9 + 9)^3} < 0 \Rightarrow 3$ é ponto de máximo

h) $f''(x) = 0 \Rightarrow 2x^3 - 54x = 0 \Rightarrow x = 0$ ou $x = \pm 3\sqrt{3}$

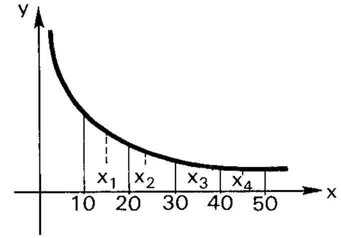
$x < -3\sqrt{3}$ ou $0 < x < 3\sqrt{3} \Rightarrow f''(x) < 0 \Rightarrow$ concavidade negativa

$-3\sqrt{3} < x < 0$ ou $x > 3\sqrt{3} \Rightarrow f''(x) > 0 \Rightarrow$ concavidade positiva

Como o sinal da concavidade muda em $x = -3\sqrt{3}$, $x = 0$ e $x = 3\sqrt{3}$, estes são três pontos de inflexão.

Capítulo IX — Noções de cálculo integral

299.



$x_1 = 15$
$x_2 = 25$
$x_3 = 35$
$x_4 = 45$

$A = \sum\limits_{i=1}^{4} f(\bar{x}_i)\Delta_i x$

$A = \dfrac{200}{15} \cdot 10 + \dfrac{200}{25} \cdot 10 + \dfrac{200}{35} \cdot 10 + \dfrac{200}{45} \cdot 10$

$A = 2\,000\left(\dfrac{1}{15} + \dfrac{1}{25} + \dfrac{1}{35} + \dfrac{1}{45}\right)$

$A = 2\,000 \cdot \dfrac{248}{1\,575} \Rightarrow A = 314{,}9$

301. a) $\int_1^5 (5x + 7)dx$, para $\bar{x}_i = x_{i-1}$

Como a função $5x + 7$ é contínua em $[1,\ 5]$, sabemos pelo teorema 1 que a integral existe. Dividindo $[1,\ 5]$ em n subintervalos iguais de comprimento $\dfrac{4}{n}$, temos:

$x_0 = 1,\ x_1 = 1 + \dfrac{4}{n},\ x_2 = 1 + 2 \cdot \dfrac{4}{n},\ \ldots\ x_{i-1} = 1 + (i - 1)\dfrac{4}{n},$

$x_i = 1 + i\dfrac{4}{n},\ x_n = 5.$

Sendo $\bar{x}_i = x_{i-1}$, vem:

$f(\bar{x}_i) = 5\bar{x}_i + 7 = 5\left(1 + (i - 1)\dfrac{4}{n}\right) + 7 = 5 + \dfrac{20}{n}(i - 1) + 7 =$

$= 12 + \dfrac{20}{n}i - \dfrac{20}{n}$

$f(\bar{x}_i)\Delta_i x = \left(12 + \dfrac{20}{n}i - \dfrac{20}{n}\right) \cdot \dfrac{4}{n}$

$f(\bar{x}_i)\Delta_i x = \dfrac{48}{n} + \dfrac{80}{n^2}i - \dfrac{80}{n^2}.$

Logo,

$\sum\limits_{i=1}^{n} f(\bar{x}_i)\Delta_i x = \sum\limits_{i=1}^{n} \left(\dfrac{48}{n} - \dfrac{80}{n^2} + \dfrac{80}{n^2}i\right) =$

$$= n \cdot \frac{48}{n} - n \cdot \frac{80}{n^2} + \frac{80}{n^2}\sum_{i=1}^{n} i =$$

$$= 48 - \frac{80}{n} + \frac{80}{n^2}\sum_{i=1}^{n} i$$

Como $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, vem:

$$\sum_{i=1}^{n} f(\bar{x}_i)\Delta_i x = 48 - \frac{80}{n} + \frac{80}{n^2}\left(\frac{n(n+1)}{2}\right) = 48 - \frac{80}{n} + 40\left(\frac{n+1}{n}\right).$$

Como $\Delta_1 x = \Delta_2 x = \Delta_3 x = \ldots = \Delta_n x = \frac{4}{n}$, a norma μ será igual a $\frac{4}{n}$; logo, quando μ se aproximar de zero, temos:

1) n cresce arbitrariamente

2) $\frac{80}{n}$ se aproxima de zero

3) $\frac{n+1}{n}$ se aproxima de um

4) $\sum_{i=1}^{n} f(\bar{x}_i)\Delta_i x$ se aproxima arbitrariamente do número $48 + 40$, ou seja,

$$\mu \cong 0 \Rightarrow \sum_{i=1}^{n} f(\bar{x}_i)\Delta_i x \cong 88.$$

b) As considerações iniciais são as mesmas do item a.

Fazendo $\bar{x}_i = 1 + i \cdot \frac{4}{n}$, vem:

$$f(\bar{x}_i) = 5\bar{x}_i + 7 = 5\left(1 + i\frac{4}{n}\right) + 7 = 12 + \frac{20}{n}i$$

$$f(\bar{x}_i) \cdot \Delta_i x = \left(12 + \frac{20}{n}i\right) \cdot \frac{4}{n} = \frac{48}{n} + \frac{80}{n^2}i$$

$$\sum_{i=1}^{n} f(\bar{x}_i)\Delta_i x = \sum_{i=1}^{n}\left(\frac{48}{n} + \frac{80}{n^2}i\right) = n \cdot \frac{48}{n} + \frac{80}{n^2}\sum_{i=1}^{n} i =$$

$$= 48 + \frac{80}{n^2}\left(\frac{n(n+1)}{2}\right) = 48 + 40\left(\frac{n+1}{n}\right)$$

Com as mesmas considerações finais, temos:

$$\mu \cong 0 \Rightarrow \sum_{i=1}^{n} f(\bar{x}_i)\Delta_i x \cong 88.$$

302. $\int_3^6 x^2 dx$

x^2 é contínua em $[3, 6]$, logo a integral existe.

Dividindo $[3, 6]$ em n subintervalos iguais de comprimento $\frac{3}{n}$, temos:

$x_0 = 3$; $x_1 = 3 + \frac{3}{n}$; $x_2 = 3 + 2 \cdot \frac{3}{n}$; $x_3 = 3 + 3 \cdot \frac{3}{n}$; ...;

$x_{i-1} = 3 + (i - 1)\frac{3}{n}$; $x_i = 3 + i \cdot \frac{3}{n}$; $x_n = 6$.

Tomando $\bar{x}_i = x_i = 3 + i \cdot \frac{3}{n}$, vem:

$$f(\bar{x}_i) = \left(3 + i\frac{3}{n}\right)^2 = 9 + \frac{18}{n}i + \frac{9}{n^2}i^2$$

$$f(\bar{x}_i)\Delta_i x = \left(9 + \frac{18}{n}i + \frac{9}{n^2}i^2\right) \cdot \frac{3}{n} = \frac{27}{n} + \frac{54}{n^2}i + \frac{27}{n^3}i^2.$$

Então:

$$\sum_{i=3}^{6} f(\bar{x}_i)\Delta_i x = \sum_{i=3}^{6}\left(\frac{27}{n} + \frac{54}{n^2}i + \frac{27}{n^3}i^2\right) =$$

$$= n \cdot \frac{27}{n} + \frac{54}{n^2}\sum_{i=3}^{6} i + \frac{27}{n^3}\sum_{i=3}^{6} i^2 =$$

$$= 27 + \frac{54}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) =$$

$$= 27 + 27\left(\frac{n+1}{n}\right) + \frac{9}{2}\left(\frac{(n+1)(2n+1)}{n^2}\right)$$

Como $\Delta_1 x = \Delta_2 x = \ldots = \Delta_n x = \frac{3}{n}$, então $\mu = \frac{3}{n}$; logo, quando μ se aproxima arbitrariamente de zero, temos:

1) n cresce arbitrariamente

2) $\frac{n+1}{n}$ se aproxima de 1

3) $\frac{(n+1)(2n+1)}{n^2}$ se aproxima de 2

4) $\sum_{i=3}^{6} f(\bar{x}_i)\Delta_i x$ se aproxima arbitrariamente do número $27 + 27 + 9 = 63$, ou seja,

$$\mu \cong 0 \Rightarrow \sum_{i=3}^{6} f(\bar{x}_i)\Delta_i x_i \cong 63.$$

308. a) $\int_0^1 x dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$

b) $\int_1^2 x^2 dx = \left.\frac{x^3}{3}\right|_1^2 = \left.\frac{1}{3}x^3\right|_1^2 = \frac{1}{3}(8 - 1) = \frac{7}{3}$

c) $\int_0^{\frac{\pi}{4}} \text{sen } x\, dx = \left.-\cos x\right|_0^{\frac{\pi}{4}} = -\left(\cos\frac{\pi}{4} - \cos 0\right) = -\left(\frac{\sqrt{2}}{2} - 1\right) = 1 - \frac{\sqrt{2}}{2}$

d) $\int_0^{\frac{\pi}{4}} \cos x \, dx = \operatorname{sen} x \Big|_0^{\frac{\pi}{4}} = \operatorname{sen} \frac{\pi}{4} - \operatorname{sen} 0 = \frac{\sqrt{2}}{2}$

e) $\int_1^2 \frac{1}{x^2} dx = -\frac{1}{x}\Big|_1^2 = -\left(\frac{1}{2} - 1\right) = -\left(-\frac{1}{2}\right) = \frac{1}{2}$

309. a) $\int_{-1}^1 7 \, dx = 7x\Big|_{-1}^1 = 7 - (-7) = 7 + 7 = 14$

b) $\int_{-1}^1 x^2 \, dx = \frac{x^3}{3}\Big|_{-1}^1 = \frac{1}{3}(1^3 - (-1)^3) = \frac{1}{3}(1 + 1) = \frac{2}{3}$

c) $\int_{-1}^1 x^7 \, dx = \frac{x^8}{8}\Big|_{-1}^1 = \frac{1}{8}(1^8 - (-1)^8) = \frac{1}{8}(1 - 1) = 0$

d) $\int_0^1 (-x^2) \, dx = -\int_0^1 x^2 \, dx = -\frac{x^3}{3}\Big|_0^1 = \frac{-1}{3}(1 - 0) = \frac{-1}{3}$

e) $\int_{-1}^1 2x^4 \, dx = 2\int_{-1}^1 x^4 \, dx = 2\frac{x^5}{5}\Big|_{-1}^1 = \frac{2}{5}(1^5 - (-1)^5) = \frac{4}{5}$

310. a) $\int_0^2 (x^2 - 3x + 5) \, dx = \int_0^2 x^2 \, dx - 3\int_0^2 x \, dx + \int_0^2 5dx =$

$= \frac{x^3}{3}\Big|_0^2 - 3\frac{x^2}{2}\Big|_0^2 + 5x\Big|_0^2 =$

$\frac{1}{3}(8 - 0) - \frac{3}{2}(4 - 0) + 5(2 - 0) = \frac{8}{3} - 6 + 10 = \frac{20}{3}$

b) $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (\operatorname{sen} x + \cos x) \, dx = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \operatorname{sen} x \, dx + \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx =$

$= -\cos x\Big|_{\frac{-\pi}{2}}^{\frac{\pi}{2}} + \operatorname{sen} x\Big|_{\frac{-\pi}{2}}^{\frac{\pi}{2}} = -\left(\cos\frac{\pi}{2} - \cos\left(\frac{-\pi}{2}\right)\right) + \left(\operatorname{sen}\frac{\pi}{2} - \operatorname{sen}\left(\frac{-\pi}{2}\right)\right) =$

$= -(0 - 0) + (1 - (-1)) = 2$

c) $\int_0^{\frac{\pi}{2}} (1 + \cos x) \, dx = \int_0^{\frac{\pi}{2}} dx + \int_0^{\frac{\pi}{2}} \cos x \, dx = x\Big|_0^{\frac{\pi}{2}} + \operatorname{sen} x\Big|_0^{\frac{\pi}{2}} =$

$= \left(\frac{\pi}{2} - 0\right) + \left(\operatorname{sen}\frac{\pi}{2} - \operatorname{sen} 0\right) = \frac{\pi}{2} + 1$

d) $\int_0^1 (x^5 - 1)x \, dx = \int_0^1 (x^6 - x) \, dx = \int_0^1 x^6 \, dx - \int_0^1 x \, dx =$

$= \frac{x^7}{7}\Big|_0^1 - \frac{x^2}{2}\Big|_0^1 = \frac{1}{7}(1 - 0) - \frac{1}{2}(1 - 0) = \frac{1}{7} - \frac{1}{2} = \frac{-5}{14}$

e) $\int_1^4 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx = \int_1^4 x^{\frac{1}{2}} \, dx + \int_1^4 x^{-\frac{1}{2}} \, dx =$

$= \frac{2}{3}x^{\frac{3}{2}}\Big|_1^4 + 2x^{\frac{1}{2}}\Big|_1^4 = \frac{2}{3}\left(4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) + 2\left(4^{\frac{1}{2}} - 1^{\frac{1}{2}}\right) =$

$= \frac{2}{3}(8 - 1) + 2(2 - 1) = \frac{14}{3} + 2 = \frac{20}{3}$

312. a) $f(x) = 4 - x^2, \quad [a, b] = [-2, 2]$

$F(x) = \int(4 - x^2) \, dx = 4x - \frac{x^3}{3}$

$\int_{-2}^2 (4 - x^2) \, dx = F(2) - F(-2) = \frac{16}{3} - \left(\frac{-16}{3}\right) = \frac{32}{3}$

b) $f(x) = x^2 + 7; \quad [a, b] = [0, 3]$

$F(x) = \int(x^2 + 7) \, dx = \frac{x^3}{3} + 7x$

$\int_0^3 (x^2 + 7) \, dx = F(3) - F(0) = 30 - 0 = 30$

c) $f(x) = 3 + \operatorname{sen} x; \quad [a, b] = \left[0, \frac{\pi}{2}\right]$

$F(x) = \int(3 + \operatorname{sen} x) \, dx = 3x - \cos x$

$\int_0^{\frac{\pi}{2}} (3 + \operatorname{sen} x) \, dx = F\left(\frac{\pi}{2}\right) - F(0) = \frac{3\pi}{2} - (-1) = \frac{3\pi}{2} + 1$

d) $f(x) = \sqrt{x} + 1; \ [a, b] = [0, 4]$

$F(x) = \int(\sqrt{x} + 1) \, dx = \frac{2}{3}x^{\frac{3}{2}} + x$

$\int_0^4 f(x) \, dx = F(4) - F(0) = \frac{28}{3} - 0 = \frac{28}{3}$

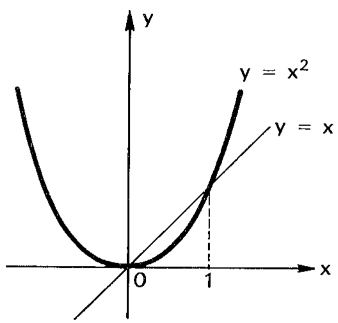
e) $f(x) = \dfrac{1}{1 + x^2}$; $[a, b] = [0, 1]$

$$F(x) = \int \left(\frac{1}{1 + x^2}\right) dx = \text{arc tg } x$$

$$\int_0^1 f(x)\, dx = F(1) - F(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

317. a) $y = x$ e $y = x^2$

$x^2 = x \Rightarrow x^2 - x = 0 \Rightarrow x(x - 1) = 0 \Rightarrow x = 0$ ou $x = 1$



$$F(x) = \int (x - x^2)\, dx = \frac{x^2}{2} - \frac{x^3}{3}$$

$$\int_0^1 (x - x^2)\, dx = F(1) - F(0) =$$

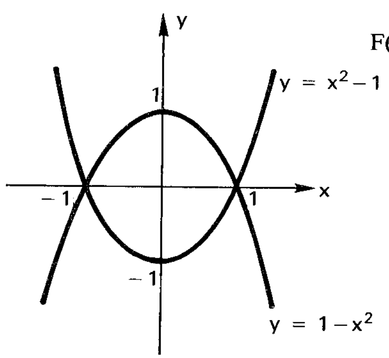
$$= \left(\frac{1}{2} - \frac{1}{3}\right) - 0 = \frac{1}{6}$$

b) $y = x^2 - 1$ e $y = 1 - x^2$

$x^2 - 1 = 1 - x^2 \Rightarrow 2x^2 = 2 \Rightarrow$
$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

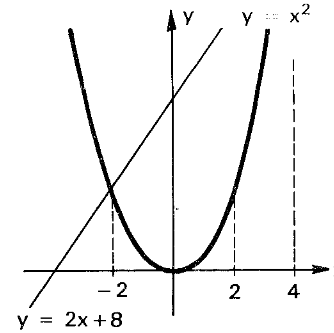
$$F(x) = \int [1 - x^2 - (x^2 - 1)]\, dx =$$

$$= \int (-2x^2 + 2)\, dx = \frac{-2x^3}{3} + 2x$$



$$\int_{-1}^1 (-2x^2 + 2)\, dx = F(1) - F(-1) = \frac{4}{3} - \left(\frac{-4}{3}\right) = \frac{8}{3}$$

c) $y = x^2$ e $y = 2x + 8$



$x^2 = 2x + 8 \Rightarrow x^2 - 2x - 8 = 0 \Rightarrow$
$\Rightarrow x = -2$ ou $x = 4$

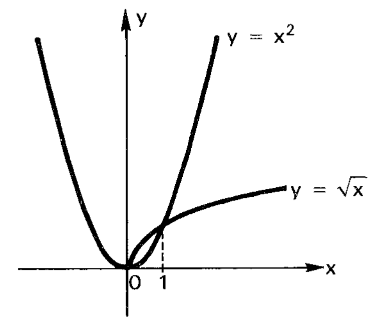
$$F(x) = \int (2x + 8 - x^2)\, dx =$$

$$= x^2 + 8x - \frac{x^3}{3}$$

$$\int_{-2}^4 (2x + 8 - x^2)\, dx = F(4) - F(-2) =$$

$$= \frac{80}{3} - \left(\frac{-28}{3}\right) = \frac{108}{3} = 36$$

d) $y = x^2$ e $y = \sqrt{x} \Rightarrow x^2 = \sqrt{x} \Rightarrow x^4 = x \Rightarrow x^3(x - 1) = 0 \Rightarrow$
$\Rightarrow x = 0$ ou $x = 1$



$$F(x) = \int (\sqrt{x} - x^2)\, dx = \frac{2}{3} x^{\frac{3}{2}} - \frac{x^3}{3}$$

$$\int_0^1 (\sqrt{x} - x^2)\, dx = F(1) - F(0) =$$

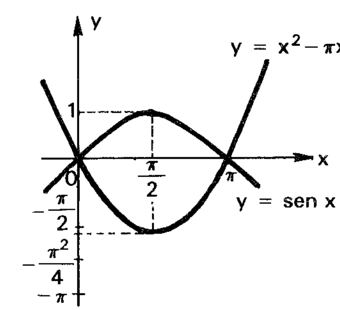
$$= \frac{1}{3} - 0 = \frac{1}{3}$$

e) $y = \text{sen } x$ e $y = x^2 - \pi x$

Analisando a função $y = x^2 - \pi x$, temos:

$x^2 - \pi x = 0 \Rightarrow x = 0$ e $x = \pi$ (raízes)

$$\left.\begin{aligned} x_V &= \frac{-b}{2a} = \frac{+\pi}{2} \\ y_V &= \frac{-\Delta}{4a} = \frac{-\pi^2}{4} \end{aligned}\right\} \Rightarrow V = \left(\frac{\pi}{2}, \frac{-\pi^2}{4}\right)$$



$$F(x) = \int (\text{sen } x - x^2 + \pi x)\, dx =$$

$$= -\cos x - \frac{x^3}{3} + \frac{\pi x^2}{2}$$

$$\int_0^\pi (\text{sen } x - x^2 + \pi x)\, dx = F(\pi) - F(0) = \left(1 + \frac{\pi^3}{6}\right) - (-1) = 2 + \frac{\pi^3}{6}$$

318. a) $F(x) = \int_1^x (5t + 2)\,dt = \left(\frac{5t^2}{2} + 2t\right)\Big|_1^x = \frac{5x^2}{2} + 2x - \frac{9}{2}$

$$\frac{dF}{dx} = \frac{2 \cdot 5x}{2} + 2 \Rightarrow \frac{dF}{dx} = 5x + 2$$

b) $\int_5^x \sqrt{t}\,dt = \frac{2}{3} t^{\frac{3}{2}}\Big|_5^x = \frac{2}{3} x^{\frac{3}{2}} - \frac{2}{3} 5^{\frac{3}{2}} \Rightarrow \frac{dF}{dx} = x^{\frac{1}{2}} = \sqrt{x}$

c) $\int_1^x \sqrt{t}\,dt = \frac{2}{3} t^{\frac{3}{2}}\Big|_1^x = \frac{2}{3} x^{\frac{3}{2}} - \frac{2}{3} 1^{\frac{3}{2}} \Rightarrow \frac{dF}{dx} = x^{\frac{1}{2}} = \sqrt{x}$

320. a) $\left.\begin{array}{l} 3x + 7 = u \\ 3 = u' \end{array}\right\} \Rightarrow \int (3x + 7)^{15} \cdot 3\,dx = \int u^{15} \cdot u'\,dx = \frac{u^{16}}{16} + c = \frac{(3x + 7)^{16}}{16} + c$

b) $\left.\begin{array}{l} 3x = u \\ 3 = u' \end{array}\right\} \Rightarrow \int e^{3x} \cdot 3\,dx = \int e^u \cdot u'\,dx = e^u + c = e^{3x} + c$

c) $\left.\begin{array}{l} 5x = u \\ 5 = u' \end{array}\right\} \Rightarrow \int 5 \cdot \cos 5x\,dx = \int \cos u \cdot u'\,dx = \operatorname{sen} u + c = \operatorname{sen} 5x + c$

d) $\left.\begin{array}{l} 3x + 7 = u \\ 3 = u' \end{array}\right\} \Rightarrow \int \sqrt{u} \cdot u'\,d = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}(3x + 7)^{\frac{3}{2}} + c = \frac{2}{3}\sqrt{(3x + 7)^3} + c$

e) $\left.\begin{array}{l} x + 1 = u \\ 1 = u' \end{array}\right\} \Rightarrow \int u^{-2} \cdot u'\,d = -u^{-1} + c = \frac{-1}{x + 1} + c$

321. a) $\left.\begin{array}{l} x^3 = u \\ 3x^2 = u' \end{array}\right\} \Rightarrow \int e^{x^3} \cdot x^2\,dx = \frac{1}{3}\int e^{x^3} \cdot 3x^2\,dx = \frac{1}{3}\int e^u \cdot u'\,dx =$

$= \frac{1}{3} e^u + c = \frac{1}{3} e^{x^3} + c$

b) $\left.\begin{array}{l} 3x^2 = u \\ 6x = u' \end{array}\right\} \Rightarrow \int x \cdot \cos 3x^2\,dx = \frac{1}{6}\int \cos 3x^2 \cdot 6x\,dx = \frac{1}{6}\int \cos u \cdot u'\,dx =$

$= \frac{1}{6} \operatorname{sen} u + c = \frac{1}{6} \operatorname{sen} 3x^2 + c$

c) $\left.\begin{array}{l} 5x - 1 = u \\ 5 = u' \end{array}\right\} \Rightarrow \int (5x - 1)^{13}\,dx = \frac{1}{5}\int (5x - 1)^{13} \cdot 5\,dx = \frac{1}{5}\int u^{13} \cdot u'\,dx =$

$= \frac{1}{5} \cdot \frac{u^{14}}{14} + c = \frac{(5x - 1)^{14}}{70} + c$

d) $\left.\begin{array}{l} 5x - 1 = u \\ 5 = u' \end{array}\right\} \Rightarrow \int \sqrt{5x - 1}\,dx = \frac{1}{5}\int (5x - 1)^{\frac{1}{2}} \cdot 5\,dx = \frac{1}{5}\int u^{\frac{1}{2}} \cdot u'\,dx =$

$= \frac{1}{5} \cdot \frac{2}{3} \cdot u^{\frac{3}{2}} + c = \frac{2}{15}(5x - 1)^{\frac{3}{2}} + c = \frac{2}{15}\sqrt{(5x - 1)^3} + c$

e) $\left.\begin{array}{l} 3x + 7 = u \\ 3 = u' \end{array}\right\} \Rightarrow \int \frac{1}{(3x + 7)^2}\,dx = \frac{1}{3}\int (3x + 7)^{-2} \cdot 3\,dx =$

$= \frac{1}{3}\int u^{-2} \cdot u'\,dx = \frac{1}{3} \cdot \frac{u^{-1}}{-1} + c = \frac{-1}{3} \cdot (3x + 7)^{-1} + c =$

$= \frac{-1}{3(3x + 7)} + c$

322. a) $\left.\begin{array}{l} 3x = u \\ 3 = u' \end{array}\right\} \Rightarrow \int e^{3x}\,dx = \frac{1}{3}\int e^{3x} \cdot 3\,dx = \frac{1}{3}\int e^u \cdot u'\,dx =$

$= \frac{1}{3} e^u + c = \frac{1}{3} e^{3x} + c$

b) $\left.\begin{array}{l} \operatorname{sen} x = u \\ \cos x = u' \end{array}\right\} \Rightarrow \int (\operatorname{sen} x)^5 \cos x\,dx = \int u^5 \cdot u'\,dx = \frac{u^6}{6} + c =$

$= \frac{(\operatorname{sen} x)^6}{6} + c$

c) $\left.\begin{array}{l} 5x = u \\ 5 = u' \end{array}\right\} \Rightarrow \int \operatorname{sen} 5x\,dx = \frac{1}{5}\int \operatorname{sen} 5x \cdot 5\,dx = \frac{1}{5}\int \operatorname{sen} u \cdot u'\,dx =$

$= \frac{1}{5}(-\cos u) + c = \frac{-\cos 5x}{5} + c$

d) $\left.\begin{array}{l} 3x + 1 = u \\ 3 = u' \end{array}\right\} \Rightarrow \int \cos(3x + 1)\,dx = \frac{1}{3}\int \cos(3x + 1) \cdot 3\,dx =$

$= \frac{1}{3}\int \cos u \cdot u'\,dx = \frac{1}{3}(\operatorname{sen} u) + c = \frac{\operatorname{sen}(3x + 1)}{3} + c$

e) $\left.\begin{array}{l} 3 - 2x = u \\ -2 = u' \end{array}\right\} \Rightarrow \int (3 - 2x)^4\,dx = \frac{-1}{2}\int (3 - 2x)^4(-2)\,dx =$

$= \frac{-1}{2}\int u^4 \cdot u'\,dx = \frac{-1}{2} \cdot \frac{u^5}{5} + c = \frac{-(3 - 2x)^5}{10} + c$

323. a) $\int x \cdot \operatorname{sen} x\,dx$

Fazendo $v(x) = x$ e $u'(x) = \operatorname{sen} x$, então $v'(x) = 1$ e $u(x) = -\cos x$.

Portanto:

$\int x \cdot \operatorname{sen} x\,dx = u(x) \cdot v(x) - \int 1 \cdot (-\cos x)\,dx = -x\cos x + \operatorname{sen} x + c$

b) $\int (3x + 7)\cos x\,dx$

Fazendo $v(x) = 3x + 7$ e $u'(x) = \cos x$, então $v'(x) = 3$ e $u(x) = \operatorname{sen} x$.

$\int v \cdot u'\,dx = u \cdot v - \int u \cdot v'\,dx \Rightarrow \int (3x + 7)\cos x\,dx =$

$= (3x + 7)\operatorname{sen} x - \int \operatorname{sen} x \cdot 3\,dx = (3x + 7)\operatorname{sen} x - 3\int \operatorname{sen} x\,dx =$

$= (3x + 7)\operatorname{sen}x + 3\cos x + c$

c) $\int (2x - 1) \cdot e^x\, dx$

Fazendo $v(x) = 2x - 1$ e $u'(x) = e^x$, então $v'(x) = 2$ e $u(x) = e^x$.

$$\int (2x - 1)e^x\, dx = (2x - 1) \cdot e^x - \int e^x \cdot 2\, dx = (2x - 1)e^x - 2e^x + c =$$

$$= (2x - 3)e^x + c$$

d) $\int (-3x + 1) \cdot \cos 5x\, dx$

$$\text{Fazendo} \begin{cases} v(x) = -3x + 1 \Rightarrow v'(x) = -3 \\ u'(x) = \cos 5x \Rightarrow u(x) = \dfrac{\text{sen } 5x}{5} \end{cases}$$

$$\int (-3x + 1) \cos 5x\, dx = (-3x + 1) \cdot \frac{\text{sen } 5x}{5} - \int \frac{-3}{5} \text{sen } 5x\, dx =$$

$$= \frac{(-3x + 1)}{5} \cdot \text{sen } 5x + \frac{3}{5} \cdot \frac{1}{5} \int \text{sen } 5x \cdot 5\, dx =$$

$$= \frac{-3x + 1}{5} \cdot \text{sen } 5x + \frac{3}{25} (-\cos 5x) + c =$$

$$= \frac{-3x + 1}{5} \cdot \text{sen } 5x - \frac{3 \cos 5x}{25} + c = \frac{5(-3x + 1) \text{ sen } 5x - 3 \cos 5x}{25} + c$$

e) $\int (2x - 3) \cdot e^{1-3x}\, dx$

$$\text{Fazendo} \begin{cases} v(x) = 2x - 3 \Rightarrow v'(x) = 2 \\ u'(x) = e^{1-3x} \Rightarrow u(x) = \int e^{1-3x}\, dx = \dfrac{-1}{3} e^{1-3x} \end{cases}$$

$$\int (2x - 3)e^{1-3x}\, dx = \frac{-1}{3} \cdot e^{1-3x} \cdot (2x - 3) - \int 2\left(\frac{-1}{3}\right)e^{1-3x}\, dx =$$

$$= \frac{-(2x - 3)}{3} e^{1-3x} + \frac{2}{3} \int e^{1-3x}\, dx =$$

$$= \frac{-(2x - 3)}{3} e^{1-3x} - \frac{2}{9} \cdot e^{1-3x} + c =$$

$$= \left(\frac{-2x + 3}{3} - \frac{2}{9}\right) e^{1-3x} + c =$$

$$= \left(-\frac{2}{3}x + \frac{7}{9}\right) e^{1-3x} + c$$

324. Sendo $A(1, 1)$ e $B(2, 3)$ determinemos a reta que passa por esses pontos.

$$\left.\begin{array}{l} 1 = a + b \\ 3 = 2a + b \end{array}\right\} \Rightarrow a = 2 \text{ e } b = -1 \Rightarrow f(x) = 2x - 1$$

$$V = \pi \int_a^b [f(x)]^2\, dx \Rightarrow V = \pi \int_1^2 (2x - 1)^2\, dx = \frac{\pi}{2} \cdot \frac{(2x - 1)^3}{3} \Bigg|_1^2 =$$

$$= \frac{\pi}{6} [3^3 - 1^3] = \frac{13\pi}{3}$$

325. $V = \pi \int_1^3 (x^2)^2\, dx = \pi \int_1^3 x^4\, dx = \pi \dfrac{x^5}{5} \Bigg|_1^3 =$

$$= \frac{\pi}{5} (243 - 1) = \frac{242\pi}{5}$$

326. $V = \pi \int_1^4 \left(\dfrac{1}{x}\right)^2 dx = \pi \int_1^4 x^{-2}\, dx = \pi \dfrac{x^{-1}}{-1} \Bigg|_1^4 =$

$$= -\pi \left[\frac{1}{4} - 1\right] = -\pi \left(\frac{-3}{4}\right) = \frac{3\pi}{4}$$