System geometry and variables

Consider the pendulum link (stem + body + motor) of total mass $M_{\text{total}} = M_{\text{stem}} + M_{\text{body}} + M_{\text{motor}}$, pivoted at the stem's base. Let ϕ be the tilt angle of the pendulum from vertical (positive CCW about the horizontal tilt axis). Let ψ be the rotation angle of the inertia wheel (driven by the motor) relative to the pendulum. The wheel has mass M_{wheel} and (outer/inner) radii $R_{\text{out}} = D_{\text{wheel}_{\text{out}}}/2$, $R_{\text{in}} = D_{\text{wheel}_{\text{in}}}/2$. Its moment of inertia about its horizontal axis is that of a thin ring: $I_{\text{wheel}} = \frac{1}{4}M_{\text{wheel}}(R_{\text{out}}^2 + R_{\text{in}}^2)$.

Assuming no slipping and no front-back motion, the only restoring torque is from gravity on the pendulum's center of mass (COM). The COM height above the pivot is $L_{\text{com}} = \frac{M_{\text{stem}}\frac{H_{\text{stem}}}{2} + M_{\text{body}}\left(H_{\text{stem}} + \frac{H_{\text{body}}}{2}\right) + M_{\text{motor}}\left(H_{\text{stem}} + H_{\text{body}} + \frac{D_{\text{motor}}}{2}\right)}{M_{\text{total}}}$.

The total inertia of the pendulum about the tilt axis is $I_{\text{pendulum}} = \frac{1}{3}M_{\text{stem}}H_{\text{stem}}^2 + \left(\frac{1}{3}M_{\text{body}}H_{\text{body}}^2 + M_{\text{body}}H_{\text{stem}}^2\right) + M_{\text{motor}}(H_{\text{stem}} + H_{\text{body}} + \frac{D_{\text{motor}}}{2})^2$, where the first term is a slender-rod formula for the stem, the second combines the body's inertia about its base and its offset, and the third treats the motor mass as a point at its center. (Any small rotor inertia of the motor can be lumped into the reaction wheel inertia if needed.)

Nonlinear equations of motion

Using Lagrange's equations (or Newton's laws) yields coupled torques on the pendulum and wheel. Let τ be the torque exerted by the motor on the wheel (positive when the wheel spins CCW). By action–reaction, the wheel exerts $-\tau$ on the pendulum. Including gravity, the nonlinear equations are:

- **Pendulum:** $I_{\text{pendulum}}\ddot{\phi} = M_{\text{total}}gL_{\text{com}}\sin\phi - \tau$.
- **Wheel:** $I_{\text{wheel}}\ddot{\psi} = \tau$.

Here I_{pendulum} and I_{wheel} are as defined above, and $M_{\text{total}}gL_{\text{com}}\sin\phi$ is the gravitational torque (positive for $\phi > 0$ as the pendulum tips further). In words, the wheel's motor torque τ accelerates the wheel and produces an equal-and-opposite torque on the pendulum.

Linearization about upright

For control design we linearize around the upright equilibrium $\phi = 0$. For small ϕ, $\sin\phi \approx \phi$. Neglecting damping and friction, the linearized equations are: $I_{\text{pendulum}}\ddot{\phi} = M_{\text{total}} g L_{\text{com}} \phi - \tau, I_{\text{wheel}}\ddot{\psi} = \tau$. Equivalently, one can write $I_{\text{pendulum}}\ddot{\phi} - M_{\text{total}} g L_{\text{com}} \phi + \tau = 0, I_{\text{wheel}}\ddot{\psi} - \tau = 0$. In this form the gravitational term $M_{\text{total}} g L_{\text{com}}, \phi$ acts as a linear “negative stiffness” (destabilizing torque) about the upright position. These coupled linear equations match the standard reaction-wheel inverted-pendulum model.

Including motor dynamics

With supply voltage V and armature resistance R, the motor torque is $\tau = K_t i$ and the back-EMF is $K_e\dot{\psi}$, yielding (neglecting inductance) $i = \frac{V - K_e\dot{\psi}}{R}, \tau = K_t i \approx \frac{K_t}{R}V - \frac{K_t K_e}{R}\dot{\psi}$. Using control input $u \in [-1, 1]$ to represent the normalized voltage (so $V = V_{\max} u$), we may simply write $\tau = K_u u$ for some constant K_u (and possibly include a small feedback term in $\dot{\psi}$ if needed). The essential model remains $I_{\text{pendulum}}\ddot{\phi} = M_{\text{total}} g L_{\text{com}} \phi - K_u u, I_{\text{wheel}}\ddot{\psi} = K_u u$. Thus, in linear state-space form one can take states $x = [\phi, \dot{\phi}, \psi, \dot{\psi}]^T$ and write:

$$\begin{pmatrix} \dot{\phi} \\ \ddot{\phi} \\ \dot{\psi} \\ \ddot{\psi} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{M_{\text{total}} g L_{\text{com}}}{I_{\text{pendulum}}} & 0 & 0 & -\frac{1}{I_{\text{pendulum}}} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \dot{\phi} \\ \psi \\ \dot{\psi} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{K_u}{I_{\text{pendulum}}} \\ 0 \\ \frac{K_u}{I_{\text{wheel}}} \end{pmatrix} u.$$

Any equivalent linear form is acceptable. In summary, the **symbolic linear model** is:

- $I_{\text{pendulum}}\ddot{\phi} = M_{\text{total}} g L_{\text{com}}, \phi - \tau$
- $I_{\text{wheel}}\ddot{\psi} = \tau$

with $\tau = K_u u$ as the control torque.