Cálculo de Programas

Resolução - Ficha 11

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Exercício 1

$$
\begin{aligned}
& mirror = (\!|\, \mathsf{in}_2 \cdot \alpha \,|\!) \\
\equiv \quad & \{ \text{ def. } \mathsf{in}_2\text{, def. } \alpha \ \} \\
& mirror = (\!|\, [Leaf, Fork] \cdot (id + swap) \,|\!) \\
\equiv \quad & \{ \text{ absorção-}+ \ \} \\
& mirror = (\!|\, [Leaf, Fork \cdot swap] \,|\!) \\
\equiv \quad & \{ \text{ universal-cata } \} \\
& mirror \cdot \mathsf{in} = [Leaf, Fork \cdot swap] \cdot (id + (mirror \times mirror)) \\
\equiv \quad & \{ \text{ def. in, fusão-}+\text{, absorção-}+\text{, eq-}+ \ \} \\
& \begin{cases} mirror \cdot Leaf = Leaf \\ mirror \cdot Fork = Fork \cdot swap \cdot (mirror \times mirror) \end{cases} \\
\equiv \quad & \{ \text{ pointwise, Def. comp } \} \\
& \begin{cases} mirror\ (Leaf\ x) = Leaf\ x \\ mirror\ (Fork\ (l, r)) = Fork\ (swap\ (mirror\ l, mirror\ r)) \end{cases} \\
\equiv \quad & \{ \text{ def. } swap \ \} \\
& \begin{cases} mirror\ (Leaf\ x) = Leaf\ x \\ mirror\ (Fork\ (l, r)) = Fork\ (mirror\ r, mirror\ l) \end{cases}
\end{aligned}
$$

Exercício 2

$$
\begin{aligned}
& (\!|\, g \,|\!) \cdot (\!|\, \mathsf{in}_2 \cdot \alpha \,|\!) = (\!|\, g \cdot \alpha \,|\!) \\
\Longleftarrow \quad & \{ \text{ fusão-cata } \} \\
& (\!|\, g \,|\!) \cdot \mathsf{in}_2 \cdot \alpha = g \cdot \alpha \cdot \mathsf{F}\ (\!|\, g \,|\!) \\
\equiv \quad & \{ \text{ (F1) } \} \\
& g \cdot G\ (\!|\, g \,|\!) \cdot \alpha = g \cdot \alpha \cdot \mathsf{F}\ (\!|\, g \,|\!) \\
\Longleftarrow \quad & \{ \text{ leibniz } \} \\
& G\ (\!|\, g \,|\!) \cdot \alpha = \alpha \cdot \mathsf{F}\ (\!|\, g \,|\!) \\
\Longleftarrow \quad & \{ \text{ generalização de } (\!|\, g \,|\!) \text{ em } f \ \} \\
& G\, f \cdot \alpha = \alpha \cdot \mathsf{F}\, f
\end{aligned}
$$

Exercício 3

$$mirror = (\!| g |\!)$$
$$mirror = (\!| \mathsf{in}_2 \cdot \alpha |\!)$$

$$
\begin{array}{ll}
& (\!| g |\!) = (\!| \mathsf{in}_2 \cdot \alpha |\!) \\
\equiv & \quad \{ \text{ def. } \mathsf{in}_2 \text{, def. } \alpha \ \} \\
& (\!| g |\!) = (\!| [Leaf, Fork] \cdot (id + swap) |\!) \\
\equiv & \quad \{ \text{ absorção-+ } \} \\
& (\!| g |\!) = (\!| [Leaf, Fork \cdot swap] |\!)
\end{array}
$$

Podemos então dizer que $g = [Leaf, Fork \cdot swap]$. Precisamos também de provar que $id = (\!| g \cdot \alpha |\!)$.

$$
\begin{array}{ll}
& id = (\!| g \cdot \alpha |\!) \\
\equiv & \quad \{ \text{ def. } g \text{, def. } \alpha \ \} \\
& id = (\!| [Leaf, Fork \cdot swap] \cdot (id + swap) |\!) \\
\equiv & \quad \{ \text{ absorção-+ } \} \\
& id = (\!| [Leaf, Fork \cdot swap \cdot swap] |\!) \\
\equiv & \quad \{ \ swap \cdot swap = id \ \} \\
& id = (\!| [Leaf, Fork] |\!) \\
\equiv & \quad \{ \ [Leaf, Fork] = \mathsf{in}_{LTree} \ \} \\
& id = (\!| \mathsf{in}_{LTree} |\!) \\
\equiv & \quad \{ \text{ reflexão-cata } \} \\
& id = id \\
\Box &
\end{array}
$$

Podemos então provar que *mirror* é o seu próprio isomorfismo:

$$
\begin{array}{ll}
& (\!| g |\!) \cdot (\!| \mathsf{in}_2 \cdot \alpha |\!) = (\!| g \cdot \alpha |\!) \\
\Longleftarrow & \quad \{ \text{ (F3) } \} \\
& G\,f \cdot \alpha = \alpha \cdot \mathsf{F}\,f \\
\equiv & \quad \{ \ G\,f = \mathsf{F}\,f = id + f \times f \text{, def. } \alpha \text{ (twice) } \} \\
& (id + f \times f) \cdot (id + swap) = (id + swap) \cdot (id + f \times f) \\
\equiv & \quad \{ \text{ functor-+ } \} \\
& (id + ((f \times f) \cdot swap)) = (id + (swap \cdot (f \times f))) \\
\Box &
\end{array}
$$

Através da propriedade grátis da função *swap* (i.e: $swap \cdot (f \times g) = (g \times f) \cdot swap$), podemos garantir a veracidade desta propriedade.

Exercício 4

$$
\begin{array}{ll}
& (\!| g |\!) \cdot T\,f = (\!| g \cdot B\,(f, id) |\!) \\
\equiv & \quad \{ \text{ def-map-cata } \}
\end{array}
$$

$$
\begin{array}{ll}
& (\!|\, g\, |\!) \cdot (\!|\, \mathsf{in} \cdot B\ (f, id)\, |\!) = (\!|\, g \cdot B\ (f, id)\, |\!) \\
\Longleftarrow & \quad \{\ \text{(F3)}\ \} \\
& G\ f \cdot B\ (f, id) = B\ (f, id) \cdot \mathsf{F}\ f \\
\equiv & \quad \{\ G\ f = B\ (id, f)\ \} \\
& B\ (id, f) \cdot B\ (f, id) = B\ (f, id) \cdot B\ (id, f) \\
\equiv & \quad \{\ \text{functor-}id\text{-F (para bi-functores)}\ \} \\
& B\ (f, f) = B\ (f, f) \\
\Box &
\end{array}
$$

Exercício 5

$$
\begin{array}{ll}
& \mathit{while}\ p\ f\ g = \mathit{tailr}\ ((g + f) \cdot (\neg \cdot p)?) \\
\equiv & \quad \{\ \text{def. } \mathit{tailr}\ \} \\
& \mathit{while}\ p\ f\ g = [\![\, \mathit{join},\ (g + f) \cdot (\neg \cdot p)?\,]\!] \\
\equiv & \quad \{\ [\![f, g]\!] = f \cdot \mathsf{F}\ [\![f, g]\!] \cdot g\ \} \\
& \mathit{while}\ p\ f\ g = \mathit{join} \cdot \mathsf{F}\ (\mathit{while}\ p\ f\ g) \cdot ((g + f) \cdot (\neg \cdot p)?) \\
\equiv & \quad \{\ \text{def. } \mathit{join}\text{, def. functor F}\ \} \\
& \mathit{while}\ p\ f\ g = [id, id] \cdot (id + (\mathit{while}\ p\ f\ g)) \cdot ((g + f) \cdot (\neg \cdot p)?) \\
\equiv & \quad \{\ \text{absorção-}+\text{ (twice)}\ \} \\
& \mathit{while}\ p\ f\ g = [g, (\mathit{while}\ p\ f\ g) \cdot f] \cdot (\neg \cdot p)\ ? \\
\equiv & \quad \{\ \text{def. condicional de McCarthy}\ \} \\
& \mathit{while}\ p\ f\ g = (\neg \cdot p) \rightarrow g, (\mathit{while}\ p\ f\ g) \cdot f \\
\equiv & \quad \{\ \text{pointwise}\ \} \\
& \mathit{while}\ p\ f\ g\ x = \mathbf{if}\ \neg\ (p\ x)\ \mathbf{then}\ g\ x\ \mathbf{else}\ \mathit{while}\ p\ f\ g\ (f\ x)
\end{array}
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Exercício 6

$$
\begin{array}{ll}
& (\mathit{tailr}\ g) \cdot f = \mathit{tailr}\ h \\
\equiv & \quad \{\ \text{def. } \mathit{tailr}\text{, def. hilomorfismo}\ \} \\
& (\!|\, \Delta\, |\!) \cdot [\!(\, g\,)\!] = (\!|\, \Delta\, |\!) \cdot [\!(\, h\,)\!] \\
\Longleftarrow & \quad \{\ \text{leibniz}\ \} \\
& [\!(\, g\,)\!] \cdot f = [\!(\, h\,)\!] \\
\Longleftarrow & \quad \{\ \text{fusão-ana, def. functor}\ \} \\
& g \cdot f = (id + f) \cdot h \\
\Box &
\end{array}
$$

Exercício 7

$$
\begin{array}{ll}
& f \bullet [g, h] = [f \bullet g, f \bullet h] \\
\equiv & \quad \{\ \text{(F9) (3*)}\ \} \\
& \mu \cdot T\ f \cdot [g, h] = [\mu \cdot T\ f \cdot g, \mu \cdot T\ f \cdot h]
\end{array}
$$

$\equiv \quad \{ \text{ fusão-}+ \}$

$$[\mu \cdot T\,f \cdot g, \mu \cdot T\,f \cdot h] = [\mu \cdot T\,f \cdot g, \mu \cdot T\,f \cdot h]$$

$\square$