Cálculo de Programas
Resolução - Ficha 08
Eduardo Freitas Fernandes
2026

Exercício 1

$T\ id = id$

$\equiv$ $\quad$ { (F1) }

$id \times id = id$

$\equiv$ $\quad$ { def-$\times$ }

$\langle id \cdot \pi_1, id \cdot \pi_2 \rangle = id$

$\equiv$ $\quad$ { natural-id (twice) }

$\langle \pi_1, \pi_2 \rangle = id$

$\equiv$ $\quad$ { reflexão-$\times$ }

$id = id$

$\square$

$T\ (f \cdot g) = (T\ f) \cdot (T\ g)$

$\equiv$ $\quad$ { (F1) }

$(f \cdot g) \times (f \cdot g) = (f \times f) \cdot (g \times g)$

$\equiv$ $\quad$ { def-$\times$ (twice) }

$\langle f \cdot g \cdot \pi_1, f \cdot g \cdot \pi_2 \rangle = \langle f \cdot \pi_1, f \cdot \pi_2 \rangle \cdot (g \times g)$

$\equiv$ $\quad$ { fusão-$\times$ }

$\langle f \cdot g \cdot \pi_1, f \cdot g \cdot \pi_2 \rangle = \langle f \cdot \pi_1 \cdot (g \times g), f \cdot \pi_2 \cdot (g \times g) \rangle$

$\equiv$ $\quad$ { natural-π_1, natural-π_2 }

$\langle f \cdot g \cdot \pi_1, f \cdot g \cdot \pi_2 \rangle = \langle f \cdot g \cdot \pi_1, f \cdot g \cdot \pi_2 \rangle$

$\square$

$\mu \cdot (T\ u) = \mu \cdot u$

$\equiv$ $\quad$ { def. μ, def. $T\ u$ }

$(\pi_1 \times \pi_2) \cdot (\langle id, id \rangle \times \langle id, id \rangle) = (\pi_1 \times \pi_2) \cdot \langle id, id \rangle$

$\equiv$ $\quad$ { def-$\times$, absorção-$\times$, natural-id }

$(\pi_1 \times \pi_2) \cdot \langle \langle id, id \rangle \cdot \pi_1\ (\langle id, id \rangle \cdot \pi_2) = \langle \pi_1, \pi_2 \rangle, \cdot \rangle$

$\equiv$ $\quad$ { absorção-$\times$, reflexão-$\times$ }

$\langle \pi_1 \cdot \langle id, id \rangle \cdot \pi_1, \pi_2 \cdot \langle id, id \rangle \cdot \pi_2 \rangle = \langle \pi_1, \pi_2 \rangle$

$\equiv$ $\quad$ { cancelamento-$\times$ }

$\langle \pi_1, \pi_2 \rangle = \langle \pi_1, \pi_2 \rangle$

$\square$

Exercício 2

$loop\ (a, b) = (2 + b, 2 - a)$

$\equiv$ $\quad$ { pointfree }

$loop = ((2+) \times (2-)) \cdot swap$

$\langle f, g \rangle = \text{for } loop\ 4, -2$

$$\equiv \qquad \{ \text{ def. for } \}$$

$$\langle f, g \rangle = (\![[\underline{(4, -2)}, loop]]\!)$$

$$\equiv \qquad \{ \text{ def. } loop \}$$

$$\langle f, g \rangle = (\![[\langle \underline{4}, \underline{-2} \rangle, (2+) \times (2-) \cdot swap]]\!)$$

$$\equiv \qquad \{ \text{ def-}\times, \text{ fusão-}\times \}$$

$$\langle f, g \rangle = (\![[\langle \underline{4}, \underline{-2} \rangle, \langle (2+) \cdot \pi_1 \cdot swap, (2-) \cdot \pi_2 \cdot swap \rangle]]\!)$$

$$\equiv \qquad \{ \text{ lei da troca } \}$$

$$\langle f, g \rangle = (\![\langle [\underline{4}, (2+) \cdot \pi_1 \cdot swap], [\underline{-2}, (2-) \cdot \pi_2 \cdot swap] \rangle]\!)$$

$$\equiv \qquad \{ \text{ fokkinga } \}$$

$$\begin{cases} f \cdot [\underline{0}, \text{succ}] = [\underline{4}, (2+) \cdot \pi_1 \cdot swap] \cdot (id + \langle f, g \rangle) \\ g \cdot [\underline{0}, \text{succ}] = [\underline{-2}, (2-) \cdot \pi_2 \cdot swap] \cdot (id + \langle f, g \rangle) \end{cases}$$

$$\equiv \qquad \{ \text{ fusão-+, absorção-+, eq-+ } \}$$

$$\begin{cases} f \cdot \underline{0} = \underline{4} \\ f \cdot \text{succ} = (2+) \cdot \pi_1 \cdot swap \\ g \cdot \underline{0} = \underline{-2} \\ g \cdot \text{succ} = (2-) \cdot \pi_2 \cdot swap \end{cases}$$

$$\equiv \qquad \{ \text{ pointwise } \}$$

$$\begin{cases} f\, 0 = 4 \\ f\, (n+1) = 2 + g\, n \\ g\, 0 = -2 \\ g\, (n+1) = 2 - f\, n \end{cases}$$

Exercício 3

$$\langle \langle f, g \rangle, j \rangle = (\![\langle \langle h, k \rangle, l \rangle]\!)$$

$$\equiv \qquad \{ \text{ fokkinga } \}$$

$$\begin{cases} \langle f, g \rangle \cdot \text{in} = \langle h, k \rangle \cdot \mathsf{F} \langle \langle f, g \rangle, h \rangle \\ j \cdot \text{in} = l \cdot \mathsf{F} \langle \langle f, g \rangle, h \rangle \end{cases}$$

$$\equiv \qquad \{ \text{ fusão-}\times \text{ (twice) } \}$$

$$\begin{cases} \langle f \cdot \text{in}, g \cdot \text{in} \rangle = \langle h \cdot \mathsf{F} \langle \langle f, g \rangle, j \rangle, k \cdot \mathsf{F} \langle \langle f, g \rangle, j \rangle \rangle \\ j \cdot \text{in} = l \cdot \mathsf{F} \langle \langle f, g \rangle, j \rangle \end{cases}$$

$$\equiv \qquad \{ \text{ eq-}\times \}$$

$$\begin{cases} f \cdot \text{in} = h \cdot \mathsf{F} \langle \langle f, g \rangle, j \rangle \\ g \cdot \text{in} = k \cdot \mathsf{F} \langle \langle f, g \rangle, j \rangle \\ j \cdot \text{in} = l \cdot \mathsf{F} \langle \langle f, g \rangle, j \rangle \end{cases}$$

Exercício 4

$$\begin{cases} impar\, 0 = false \\ impar\, (n+1) = par\, n \\ par\, 0 = true \\ par\, (n+1) = impar\, n \end{cases}$$

$$\equiv \qquad \{ \text{ pointfree } \}$$

$$\begin{cases} impar \cdot \underline{0} = \underline{false} \\ impar \cdot \mathsf{succ} = par \\ par \cdot \underline{0} = \underline{true} \\ par \cdot \mathsf{succ} = impar \end{cases}$$

$\equiv \qquad \{ \text{ eq-+ } \}$

$$\begin{cases} [impar \cdot \underline{0}, impar \cdot \mathsf{succ}] = [\underline{false}, par] \\ [par \cdot \underline{0}, par \cdot \mathsf{succ}] = [\underline{true}, impar] \end{cases}$$

$\equiv \qquad \{ \text{ fusão-+, cancelamento-+ (twice) } \}$

$$\begin{cases} impar \cdot \mathsf{in} = [\underline{false}, \pi_2 \cdot \langle impar, par \rangle] \\ par \cdot \mathsf{in} = [\underline{true}, \pi_1 \cdot \langle impar, par \rangle] \end{cases}$$

$\equiv \qquad \{ \text{ natural-}id \text{ (twice), absorção-+ } \}$

$$\begin{cases} impar \cdot \mathsf{in} = [\underline{false}, \pi_2] \cdot (id + \langle impar, par \rangle) \\ par \cdot \mathsf{in} = [\underline{true}, \pi_1] \cdot (id + \langle impar, par \rangle) \end{cases}$$

Podemos então concluir que $h = [\underline{false}, \pi_2]$ e $k = [\underline{true}, \pi_1]$.

$\langle impar, par \rangle = \mathsf{for}\ swap\ false, true$

$\equiv \qquad \{ \text{ def. for } \}$

$\langle impar, par \rangle = (\![\langle \underline{false}, \underline{true} \rangle, swap]\!])$

$\equiv \qquad \{ \text{ def. } swap \}$

$\langle impar, par \rangle = (\![\langle \underline{false}, \underline{true} \rangle, \langle \pi_2, \pi_1 \rangle]\!])$

$\equiv \qquad \{ \text{ lei da troca } \}$

$\langle impar, par \rangle = (\!\langle [\underline{false}, \pi_2], [\underline{true}, \pi_1] \rangle\!)$

$\equiv \qquad \{ \text{ fokkinga } \}$

$$\begin{cases} impar \cdot [\underline{0}, \mathsf{succ}] = [\underline{false}, \pi_2] \cdot (id + \langle impar, par \rangle) \\ par \cdot [\underline{0}, \mathsf{succ}] = [\underline{true}, \pi_1] \cdot (id + \langle impar, par \rangle) \end{cases}$$

$\equiv \qquad \{ \text{ fusão-+, absorção-+ } \}$

$$\begin{cases} [impar \cdot \underline{0}, impar \cdot \mathsf{succ}] = [\underline{false}, \pi_2 \cdot \langle impar, par \rangle] \\ [par \cdot \underline{0}, impar \cdot \mathsf{succ}] = [\underline{true}, \pi_1 \cdot \langle impar, par \rangle] \end{cases}$$

$\equiv \qquad \{ \text{ cancelamento-}\times \text{ (twice), eq-+, pointwise } \}$

$$\begin{cases} impar\ 0 = false \\ impar\ (n+1) = par\ n \\ par\ 0 = true \\ par\ (n+1) = impar\ n \end{cases}$$

Exercício 5

$$\begin{cases} insg\ 0 = [\ ] \\ insg\ (n+1) = fsuc\ n : insg\ n \\ fsuc\ 0 = 1 \\ fsuc\ (n+1) = fsuc\ n + 1 \end{cases}$$

$\equiv \qquad \{ \text{ pointfree } \}$

$$\begin{cases} insg \cdot \underline{0} = nil \\ insg \cdot \mathsf{succ} = cons \cdot \langle fsuc, insg \rangle \\ fsuc \cdot \underline{0} = \underline{1} \\ fsuc \cdot \mathsf{succ}\ 0 = (1+) \cdot fsuc \end{cases}$$

$\equiv \qquad \{\ \text{cancelamento-}\times\ \}$

$$\begin{cases} insg \cdot \underline{0} = nil \\ insg \cdot \mathsf{succ} = cons \cdot \langle fsuc, insg \rangle \\ fsuc \cdot \underline{0} = \underline{1} \\ fsuc \cdot \mathsf{succ}\ 0 = (1+) \cdot \pi_1 \cdot \langle fsuc, insg \rangle \end{cases}$$

$\equiv \qquad \{\ \text{eq-+, fusão-+, absorção-+}\ \}$

$$\begin{cases} insg \cdot \mathsf{in} = [nil, cons] \cdot (id + \langle fsuc, insg \rangle) \\ fsuc \cdot \mathsf{in} = [\underline{1}, (1+) \cdot \pi_1] \cdot (id + \langle fsuc, insg \rangle) \end{cases}$$

$\equiv \qquad \{\ \text{fokkinga}\ \}$

$$\langle fsuc, insg \rangle = (\!|\langle \mathsf{in}, [\underline{1}, (1+) \cdot \pi_1]\rangle|\!)$$

$$insgfor = \mathsf{for}\ \langle (1+) \cdot \pi_1, cons \rangle\ \langle \underline{1}, nil \rangle$$

$\equiv \qquad \{\ \text{def. for}\ \}$

$$insgfor = (\!|[\langle \underline{1}, nil \rangle, \langle (1+) \cdot \pi_1, cons \rangle]|\!)$$

$\equiv \qquad \{\ \text{lei da troca}\ \}$

$$insgfor = (\!|\langle [\underline{1}, (1+) \cdot \pi_1], [nil, cons] \rangle|\!)$$

$\equiv \qquad \{\ \ \}$

$$\langle f, insg \rangle = (\!|\langle [\underline{1}, (1+) \cdot \pi_1], [nil, cons] \rangle|\!)$$

$\equiv \qquad \{\ \text{fokkinga}\ \}$

$$\begin{cases} f \cdot [\underline{0}, \mathsf{succ}] = [\underline{1}, (1+) \cdot \pi_1] \cdot \mathsf{F}\ \langle f, insg \rangle \\ insg \cdot [\underline{0}, \mathsf{succ}] = [nil, cons] \cdot \mathsf{F}\ \langle f, insg \rangle \end{cases}$$

$\equiv \qquad \{\ \text{def. functor } \mathbb{N}_0\text{, fusão-+, absorção-+, eq-+}\ \}$

$$\begin{cases} f \cdot \underline{0} = \underline{1} \\ f \cdot \mathsf{succ} = (1+) \cdot \pi_1 \cdot \langle f, insg \rangle \\ insg \cdot \underline{0} = nil \\ insg \cdot \mathsf{succ} = cons \cdot \langle f, insg \rangle \end{cases}$$

$\equiv \qquad \{\ \text{pointwise}\ \}$

$$\begin{cases} f\ 0 = 1 \\ f\ (n+1) = 1 + f\ n \\ insg\ 0 = [\ ] \\ insg\ (n+1) = f\ n : insg\ n \end{cases}$$

Exercício 6

$$\begin{cases} f_1\ [\ ] = [\ ] \\ f_1\ (h:t) = h : f_2\ t \\ f_2\ [\ ] = [\ ] \\ f_2\ (h:t) = f_1\ t \end{cases}$$

$\equiv \qquad \{ \text{ def. comp, pointfree } \}$

$$\begin{cases} f_1 \cdot nil = nil \\ f_1 \cdot cons = cons \cdot (id \times f_2) \\ f_2 \cdot nil = nil \\ f_2 \cdot cons = f_1 \cdot \pi_2 \end{cases}$$

$\equiv \qquad \{ \text{ eq-+, fusão-+ } \}$

$$\begin{cases} f_1 \cdot [nil, cons] = [nil, cons \cdot (id \times f_2)] \\ f_2 \cdot [nil, cons] = [nil, f_1 \cdot \pi_2] \end{cases}$$

$\equiv \qquad \{ \text{ def. in, natural-}id\text{ (twice), natural-}\pi_2\text{, cancelamento-}\times \}$

$$\begin{cases} f_1 \cdot \text{in} = [nil \cdot id, cons \cdot (id \times \pi_2 \cdot \langle f_1, f_2 \rangle)] \\ f_2 \cdot \text{in} = [nil \cdot id, \pi_2 \cdot (id \times \pi_1 \cdot \langle f_1, f_2 \rangle)] \end{cases}$$

$\equiv \qquad \{ \text{ functor-}\times \}$

$$\begin{cases} f_1 \cdot \text{in} = [nil \cdot id, cons \cdot (id \times \pi_2) \cdot (id \times \langle f_1, f_2 \rangle)] \\ f_2 \cdot \text{in} = [nil \cdot id, \pi_2 \cdot (id \times \pi_1) \cdot (id \times \langle f_1, f_2 \rangle)] \end{cases}$$

$\equiv \qquad \{ \text{ absorção-+, def. functor de listas, natural-}\pi_2 \}$

$$\begin{cases} f_1 \cdot \text{in} = [nil, cons \cdot (id \times \pi_2)] \cdot \mathsf{F} \langle f_1, f_2 \rangle \\ f_2 \cdot \text{in} = [nil, \pi_1 \cdot \pi_2] \cdot \mathsf{F} \langle f_1, f_2 \rangle \end{cases}$$

$\equiv \qquad \{ \text{ fokkinga } \}$

$$\langle f_1, f_2 \rangle = (\!|\, \langle \text{in} \cdot (id \times \pi_2), [nil, \pi_1 \cdot \pi_2] \rangle \,|\!)$$
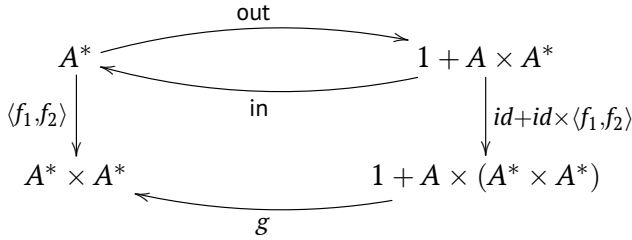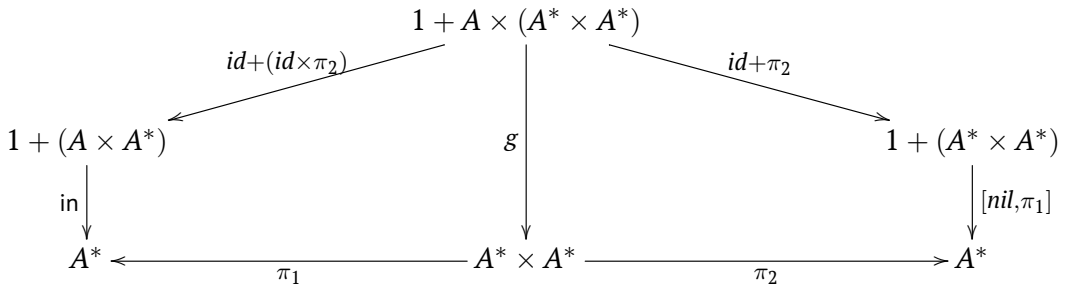

Diagrama do gene do catamorfismo:

A função f_1 seleciona os elementos de uma lista nas posições pares, e a função f_2 seleciona os elementos de uma lista nas posições ímpares.

Exercício 7

$H (g \cdot h) = (H\,g) \cdot (H\,h)$

$\equiv$ { def. functor H (3*) }

$F (g \cdot h) + G (g \cdot h) = (F\,g + G\,g) \cdot (F\,h + G\,h)$

$\equiv$ { def. functor F (3*), def. functor G (3*) }

$id + (g \cdot h) = (id + g) \cdot (id + h)$

$\equiv$ { natural-*id*, functor-+ }

$(id \cdot id) + (g \cdot h) = (id \cdot id) + (g \cdot h)$

$\square$

$K (g \cdot h) = (K\,g) \cdot (K\,h)$

$\equiv$ { def. functor K }

$G (g \cdot h) \times F (g \cdot h) = (G\,g \times F\,g) \cdot (G\,h \times F\,h)$

$\equiv$ { def. functor F (3*), def. functor G (3*) }

$(g \cdot h) \times id = (g \times id) \cdot (h \times id)$

$\equiv$ { natural-*id*, functor-$\times$ }

$(g \cdot h) \times (id \cdot id) = (g \cdot h) \times (id \cdot id)$

$\square$

$H\,id = id$

$\equiv$ { def. functor H }

$F\,id + G\,id = id$

$\equiv$ { def. functor F, def. functor G }

$id + id = id$

$\equiv$ { functor-*id*-+ }

$id = id$

$\square$

$K\,id = id$

$\equiv$ { def. functor K }

$G\,id \times F\,id = id$

$\equiv$ { def. functor F, def. functor G }

$id \times id = id$

$\equiv$ { functor-*id*-$\times$ }

$id = id$

$\square$

Exercício 8

$H\,id = id$

$\equiv$ { def. functor H }

$(F \cdot G)\,id = id$

$\equiv$ { composição de functores }

$F (G\,id) = id$

$\equiv$ { functor-id-G }

$F\,id = id$

$\equiv$ { functor-id-F }

$id = id$

$\square$

$H (f \cdot g) = (H\,f) \cdot (H\,g)$

$\equiv$ { def. functor H }

$(F \cdot G)(f \cdot g) = ((F \cdot G)\,f) \cdot ((F \cdot G)\,g)$

$\equiv$ { composição de functores }

$F (G (f \cdot g)) = (F (G\,f)) \cdot (F (G\,g))$

$\equiv$ { Functor-F, Functor-G }

$F (G\,f \cdot G\,g) = F (G\,f \cdot G\,g)$

$\square$

Exercício 9