Cálculo de Programas
Resolução - Ficha 08

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Exercício 1

$$
\begin{array}{ll}
 & T\ id = id \\
\equiv & \quad \{\ \text{(F1)}\ \} \\
 & id \times id = id \\
\equiv & \quad \{\ \text{def-}\times\ \} \\
 & \langle id \cdot \pi_1, id \cdot \pi_2 \rangle = id \\
\equiv & \quad \{\ \text{natural-}id\ \text{(twice)}\ \} \\
 & \langle \pi_1, \pi_2 \rangle = id \\
\equiv & \quad \{\ \text{reflexão-}\times\ \} \\
 & id = id \\
\Box &
\end{array}
$$

$$
\begin{array}{ll}
 & T\ (f \cdot g) = (T\ f) \cdot (T\ g) \\
\equiv & \quad \{\ \text{(F1)}\ \} \\
 & (f \cdot g) \times (f \cdot g) = (f \times f) \cdot (g \times g) \\
\equiv & \quad \{\ \text{def-}\times\ \text{(twice)}\ \} \\
 & \langle f \cdot g \cdot \pi_1, f \cdot g \cdot \pi_2 \rangle = \langle f \cdot \pi_1, f \cdot \pi_2 \rangle \cdot (g \times g) \\
\equiv & \quad \{\ \text{fusão-}\times\ \} \\
 & \langle f \cdot g \cdot \pi_1, f \cdot g \cdot \pi_2 \rangle = \langle f \cdot \pi_1 \cdot (g \times g), f \cdot \pi_2 \cdot (g \times g) \rangle \\
\equiv & \quad \{\ \text{natural-}\pi_1\text{, natural-}\pi_2\ \} \\
 & \langle f \cdot g \cdot \pi_1, f \cdot g \cdot \pi_2 \rangle = \langle f \cdot g \cdot \pi_1, f \cdot g \cdot \pi_2 \rangle \\
\Box &
\end{array}
$$

$$
\begin{array}{ll}
 & \mu \cdot (T\ u) = \mu \cdot u \\
\equiv & \quad \{\ \text{def. } \mu \text{, def. } T\ u\ \} \\
 & (\pi_1 \times \pi_2) \cdot (\langle id, id \rangle \times \langle id, id \rangle) = (\pi_1 \times \pi_2) \cdot \langle id, id \rangle \\
\equiv & \quad \{\ \text{def-}\times\text{, absorção-}\times\text{, natural-}id\ \} \\
 & (\pi_1 \times \pi_2) \cdot \langle \langle id, id \rangle \cdot \pi_1\ (\langle id, id \rangle \cdot \pi_2) = \langle \pi_1, \pi_2 \rangle, \cdot \rangle \\
\equiv & \quad \{\ \text{absorção-}\times\text{, reflexão-}\times\ \} \\
 & \langle \pi_1 \cdot \langle id, id \rangle \cdot \pi_1, \pi_2 \cdot \langle id, id \rangle \cdot \pi_2 \rangle = \langle \pi_1, \pi_2 \rangle \\
\equiv & \quad \{\ \text{cancelamento-}\times\ \} \\
 & \langle \pi_1, \pi_2 \rangle = \langle \pi_1, \pi_2 \rangle \\
\Box &
\end{array}
$$

Exercício 2

$$
\begin{array}{ll}
 & loop\ (a, b) = (2 + b, 2 - a) \\
\equiv & \quad \{\ \text{pointfree}\ \} \\
 & loop = ((2+) \times (2-)) \cdot swap
\end{array}
$$

$\langle f, g \rangle =$ for $loop$ $4, -2$

$$\equiv \quad \{ \text{ def. for } \}$$

$$\langle f, g\rangle = (\![\underline{(4,-2)}, loop]\!)$$

$$\equiv \quad \{ \text{ def. } loop \}$$

$$\langle f, g\rangle = (\![\langle \underline{4}, \underline{-2} \rangle, (2+) \times (2-) \cdot swap]\!)$$

$$\equiv \quad \{ \text{ def-}\times\text{, fusão-}\times \}$$

$$\langle f, g\rangle = (\![\langle \underline{4}, \underline{-2} \rangle, \langle (2+) \cdot \pi_1 \cdot swap, (2-) \cdot \pi_2 \cdot swap \rangle]\!)$$

$$\equiv \quad \{ \text{ lei da troca } \}$$

$$\langle f, g\rangle = (\!\langle [\underline{4}, (2+) \cdot \pi_1 \cdot swap], [\underline{-2}, (2-) \cdot \pi_2 \cdot swap] \rangle\!)$$

$$\equiv \quad \{ \text{ fokkinga } \}$$

$$\begin{cases} f \cdot [\underline{0}, \mathsf{succ}] = [\underline{4}, (2+) \cdot \pi_1 \cdot swap] \cdot (id + \langle f, g \rangle) \\ g \cdot [\underline{0}, \mathsf{succ}] = [\underline{-2}, (2-) \cdot \pi_2 \cdot swap] \cdot (id + \langle f, g \rangle) \end{cases}$$

$$\equiv \quad \{ \text{ fusão-}+\text{, absorção-}+\text{, eq-}+ \}$$

$$\begin{cases} f \cdot \underline{0} = \underline{4} \\ f \cdot \mathsf{succ} = (2+) \cdot \pi_1 \cdot swap \\ g \cdot \underline{0} = \underline{-2} \\ g \cdot \mathsf{succ} = (2-) \cdot \pi_2 \cdot swap \end{cases}$$

$$\equiv \quad \{ \text{ pointwise } \}$$

$$\begin{cases} f\ 0 = 4 \\ f\ (n+1) = 2 + g\ n \\ g\ 0 = -2 \\ g\ (n+1) = 2 - f\ n \end{cases}$$

Exercício 3

$$\langle \langle f, g \rangle, j \rangle = (\!| \langle \langle h, k \rangle, l \rangle |\!)$$

$$\equiv \quad \{ \text{ fokkinga } \}$$

$$\begin{cases} \langle f, g \rangle \cdot \mathsf{in} = \langle h, k \rangle \cdot \mathsf{F}\ \langle \langle f, g \rangle, h \rangle \\ j \cdot \mathsf{in} = l \cdot \mathsf{F}\ \langle \langle f, g \rangle, h \rangle \end{cases}$$

$$\equiv \quad \{ \text{ fusão-}\times \text{ (twice) } \}$$

$$\begin{cases} \langle f \cdot \mathsf{in}, g \cdot \mathsf{in} \rangle = \langle h \cdot \mathsf{F}\ \langle \langle f, g \rangle, j \rangle, k \cdot \mathsf{F}\ \langle \langle f, g \rangle, j \rangle \rangle \\ j \cdot \mathsf{in} = l \cdot \mathsf{F}\ \langle \langle f, g \rangle, j \rangle \end{cases}$$

$$\equiv \quad \{ \text{ eq-}\times \}$$

$$\begin{cases} f \cdot \mathsf{in} = h \cdot \mathsf{F}\ \langle \langle f, g \rangle, j \rangle \\ g \cdot \mathsf{in} = k \cdot \mathsf{F}\ \langle \langle f, g \rangle, j \rangle \\ j \cdot \mathsf{in} = l \cdot \mathsf{F}\ \langle \langle f, g \rangle, j \rangle \end{cases}$$

Exercício 4

$$\begin{cases} impar\ 0 = false \\ impar\ (n+1) = par\ n \\ par\ 0 = true \\ par\ (n+1) = impar\ n \end{cases}$$

$$\equiv \quad \{ \text{ pointfree } \}$$

$$\begin{cases} impar \cdot \underline{0} = \underline{false} \\ impar \cdot \mathsf{succ} = par \\ par \cdot \underline{0} = \underline{true} \\ par \cdot \mathsf{succ} = impar \end{cases}$$

$$\equiv \quad \{ \text{ eq-+ } \}$$

$$\begin{cases} [impar \cdot \underline{0}, impar \cdot \mathsf{succ}] = [\underline{false}, par] \\ [par \cdot \underline{0}, par \cdot \mathsf{succ}] = [\underline{true}, impar] \end{cases}$$

$$\equiv \quad \{ \text{ fusão-+, cancelamento-+ (twice) } \}$$

$$\begin{cases} impar \cdot \mathsf{in} = [\underline{false}, \pi_2 \cdot \langle impar, par \rangle] \\ par \cdot \mathsf{in} = [\underline{true}, \pi_1 \cdot \langle impar, par \rangle] \end{cases}$$

$$\equiv \quad \{ \text{ natural-}id\text{ (twice), absorção-+ } \}$$

$$\begin{cases} impar \cdot \mathsf{in} = [\underline{false}, \pi_2] \cdot (id + \langle impar, par \rangle) \\ par \cdot \mathsf{in} = [\underline{true}, \pi_1] \cdot (id + \langle impar, par \rangle) \end{cases}$$

Podemos então concluir que $h = [\underline{false}, \pi_2]$ e $k = [\underline{true}, \pi_1]$.

$$\langle impar, par \rangle = \mathsf{for}\ swap\ false, true$$

$$\equiv \quad \{ \text{ def. for } \}$$

$$\langle impar, par \rangle = (\![[\langle \underline{false}, \underline{true} \rangle, swap]]\!)$$

$$\equiv \quad \{ \text{ def. } swap \}$$

$$\langle impar, par \rangle = (\![[\langle \underline{false}, \underline{true} \rangle, \langle \pi_2, \pi_1 \rangle]]\!)$$

$$\equiv \quad \{ \text{ lei da troca } \}$$

$$\langle impar, par \rangle = (\![\langle [\underline{false}, \pi_2], [\underline{true}, \pi_1] \rangle]\!)$$

$$\equiv \quad \{ \text{ fokkinga } \}$$

$$\begin{cases} impar \cdot [\underline{0}, \mathsf{succ}] = [\underline{false}, \pi_2] \cdot (id + \langle impar, par \rangle) \\ par \cdot [\underline{0}, \mathsf{succ}] = [\underline{true}, \pi_1] \cdot (id + \langle impar, par \rangle) \end{cases}$$

$$\equiv \quad \{ \text{ fusão-+, absorção-+ } \}$$

$$\begin{cases} [impar \cdot \underline{0}, impar \cdot \mathsf{succ}] = [\underline{false}, \pi_2 \cdot \langle impar, par \rangle] \\ [par \cdot \underline{0}, impar \cdot \mathsf{succ}] = [\underline{true}, \pi_1 \cdot \langle impar, par \rangle] \end{cases}$$

$$\equiv \quad \{ \text{ cancelamento-} \times \text{ (twice), eq-+, pointwise } \}$$

$$\begin{cases} impar\ 0 = false \\ impar\ (n+1) = par\ n \\ par\ 0 = true \\ par\ (n+1) = impar\ n \end{cases}$$

Exercício 5

$$\begin{cases} insg\ 0 = [\,] \\ insg\ (n+1) = fsuc\ n : insg\ n \\ fsuc\ 0 = 1 \\ fsuc\ (n+1) = fsuc\ n + 1 \end{cases}$$

$$\equiv \quad \{ \text{ pointfree } \}$$

$$\begin{cases} insg \cdot \underline{0} = nil \\ insg \cdot \mathsf{succ} = cons \cdot \langle fsuc, insg \rangle \\ fsuc \cdot \underline{0} = \underline{1} \\ fsuc \cdot \mathsf{succ}\ 0 = (1+) \cdot fsuc \end{cases}$$

$\equiv \quad \{ \text{ cancelamento-}\times \ \}$

$$\begin{cases} insg \cdot \underline{0} = nil \\ insg \cdot \mathsf{succ} = cons \cdot \langle fsuc, insg \rangle \\ fsuc \cdot \underline{0} = \underline{1} \\ fsuc \cdot \mathsf{succ}\ 0 = (1+) \cdot \pi_1 \cdot \langle fsuc, insg \rangle \end{cases}$$

$\equiv \quad \{ \text{ eq-}+\text{, fusão-}+\text{, absorção-}+ \ \}$

$$\begin{cases} insg \cdot \mathsf{in} = [nil, cons] \cdot (id + \langle fsuc, insg \rangle) \\ fsuc \cdot \mathsf{in} = [\underline{1}, (1+) \cdot \pi_1] \cdot (id + \langle fsuc, insg \rangle) \end{cases}$$

$\equiv \quad \{ \text{ fokkinga } \}$

$$\langle fsuc, insg \rangle = (\!|\langle \mathsf{in}, [\underline{1}, (1+) \cdot \pi_1] \rangle|\!)$$

$$insgfor = \mathsf{for}\ \langle (1+) \cdot \pi_1, cons \rangle\ \langle \underline{1}, nil \rangle$$

$\equiv \quad \{ \text{ def. for } \}$

$$insgfor = (\!|[\langle \underline{1}, nil \rangle, \langle (1+) \cdot \pi_1, cons \rangle]|\!)$$

$\equiv \quad \{ \text{ lei da troca } \}$

$$insgfor = (\!|\langle [\underline{1}, (1+) \cdot \pi_1], [nil, cons] \rangle|\!)$$

$\equiv \quad \{ \quad \}$

$$\langle f, insg \rangle = (\!|\langle [\underline{1}, (1+) \cdot \pi_1], [nil, cons] \rangle|\!)$$

$\equiv \quad \{ \text{ fokkinga } \}$

$$\begin{cases} f \cdot [\underline{0}, \mathsf{succ}] = [\underline{1}, (1+) \cdot \pi_1] \cdot \mathsf{F}\ \langle \cdot, \cdot \rangle\ f\ insg \\ insg \cdot [\underline{0}, \mathsf{succ}] = [nil, cons] \cdot \mathsf{F}\ \langle \cdot, \cdot \rangle\ f\ insg \end{cases}$$

$\equiv \quad \{ \text{ def. functor } \mathbb{N}_0\text{, fusão-}+\text{, absorção-}+\text{, eq-}+ \ \}$

$$\begin{cases} f \cdot \underline{0} = \underline{1} \\ f \cdot \mathsf{succ} = (1+) \cdot \pi_1 \cdot \langle f, insg \rangle \\ insg \cdot \underline{0} = nil \\ insg \cdot \mathsf{succ} = cons \cdot \langle f, insg \rangle \end{cases}$$

$\equiv \quad \{ \text{ pointwise } \}$

$$\begin{cases} f\ 0 = 1 \\ f\ (n+1) = 1 + f\ n \\ insg\ 0 = [\,] \\ insg\ (n+1) = f\ n : insg\ n \end{cases}$$

Exercício 6

$$\begin{cases} f_1\ [\,] = [\,] \\ f_1\ (h:t) = h : f_2\ t \\ f_2\ [\,] = [\,] \\ f_2\ (h:t) = f_1\ t \end{cases}$$

$$\equiv \quad \{ \text{ def. comp, pointfree } \}$$

$$\begin{cases} f_1 \cdot nil = nil \\ f_1 \cdot cons = cons \cdot (id \times f_2) \\ f_2 \cdot nil = nil \\ f_2 \cdot cons = f_1 \cdot \pi_2 \end{cases}$$

$$\equiv \quad \{ \text{ eq-}+\text{, fusão-}+ \}$$

$$\begin{cases} f_1 \cdot [nil, cons] = [nil, cons \cdot (id \times f_2)] \\ f_2 \cdot [nil, cons] = [nil, f_1 \cdot \pi_2] \end{cases}$$

$$\equiv \quad \{ \text{ def. in, natural-}id\text{ (twice), natural-}\pi_2\text{, cancelamento-}\times \}$$

$$\begin{cases} f_1 \cdot \mathsf{in} = [nil \cdot id, cons \cdot (id \times \pi_2 \cdot \langle f_1, f_2 \rangle)] \\ f_2 \cdot \mathsf{in} = [nil \cdot id, \pi_2 \cdot (id \times \pi_1 \cdot \langle f_1, f_2 \rangle)] \end{cases}$$

$$\equiv \quad \{ \text{ functor-}\times \}$$

$$\begin{cases} f_1 \cdot \mathsf{in} = [nil \cdot id, cons \cdot (id \times \pi_2) \cdot (id \times \langle f_1, f_2 \rangle)] \\ f_2 \cdot \mathsf{in} = [nil \cdot id, \pi_2 \cdot (id \times \pi_1) \cdot (id \times \langle f_1, f_2 \rangle)] \end{cases}$$

$$\equiv \quad \{ \text{ absorção-}+\text{, def. functor de listas, natural-}\pi_2 \}$$

$$\begin{cases} f_1 \cdot \mathsf{in} = [nil, cons \cdot (id \times \pi_2)] \cdot \mathsf{F}\ \langle f_1, f_2 \rangle \\ f_2 \cdot \mathsf{in} = [nil, \pi_1 \cdot \pi_2] \cdot \mathsf{F}\ \langle f_1, f_2 \rangle \end{cases}$$

$$\equiv \quad \{ \text{ fokkinga } \}$$

$$\langle f_1, f_2 \rangle = (\!|\, \langle \mathsf{in} \cdot (id \times \pi_2), [nil, \pi_1 \cdot \pi_2] \rangle \,|\!)$$

$$\begin{tikzcd}
A^* \arrow[r, bend left, "\mathsf{out}"] \arrow[d, "\langle f_1, f_2 \rangle"'] & 1 + A \times A^* \arrow[l, bend left, "\mathsf{in}"] \arrow[d, "id + id \times \langle f_1, f_2 \rangle"] \\
A^* \times A^* & 1 + A \times (A^* \times A^*) \arrow[l, bend left, "g"]
\end{tikzcd}$$

Diagrama do gene do catamorfismo:

$$\begin{tikzcd}
 & 1 + A \times (A^* \times A^*) \arrow[dl, "id + (id \times \pi_2)"'] \arrow[dr, "id + \pi_2"] \arrow[dd, "g"'] & \\
1 + (A \times A^*) \arrow[d, "\mathsf{in}"'] & & 1 + (A^* \times A^*) \arrow[d, "{[nil, \pi_1]}"] \\
A^* & A^* \times A^* \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & A^*
\end{tikzcd}$$

A função f_1 seleciona os elementos de uma lista nas posições pares, e a função f_2 seleciona os elementos de uma lista nas posições ímpares.

Exercício 7

$$
\begin{aligned}
& H\ (g \cdot h) = (H\ g) \cdot (H\ h) \\
\equiv & \quad \{ \text{ def. functor H (3*) } \} \\
& F\ (g \cdot h) + G\ (g \cdot h) = (F\ g + G\ g) \cdot (F\ h + G\ h) \\
\equiv & \quad \{ \text{ def. functor F (3*), def. functor G (3*) } \} \\
& id + (g \cdot h) = (id + g) \cdot (id + h) \\
\equiv & \quad \{ \text{ natural-}id\text{, functor-}+ \} \\
& (id \cdot id) + (g \cdot h) = (id \cdot id) + (g \cdot h) \\
\Box &
\end{aligned}
$$

$$
\begin{aligned}
& H\ id = id \\
\equiv & \quad \{ \text{ def. functor H } \} \\
& F\ id + G\ id = id \\
\equiv & \quad \{ \text{ def. functor F, def. functor G } \} \\
& id + id = id \\
\equiv & \quad \{ \text{ functor-}id\text{-}+ \} \\
& id = id \\
\Box &
\end{aligned}
$$

$$
\begin{aligned}
& K\ (g \cdot h) = (K\ g) \cdot (K\ h) \\
\equiv & \quad \{ \text{ def. functor K } \} \\
& G\ (g \cdot h) \times F\ (g \cdot h) = (G\ g \times F\ g) \cdot (G\ h \times F\ h) \\
\equiv & \quad \{ \text{ def. functor F (3*), def. functor G (3*) } \} \\
& (g \cdot h) \times id = (g \times id) \cdot (h \times id) \\
\equiv & \quad \{ \text{ natural-}id\text{, functor-}\times \} \\
& (g \cdot h) \times (id \cdot id) = (g \cdot h) \times (id \cdot id) \\
\Box &
\end{aligned}
$$

$$
\begin{aligned}
& K\ id = id \\
\equiv & \quad \{ \text{ def. functor K } \} \\
& G\ id \times F\ id = id \\
\equiv & \quad \{ \text{ def. functor F, def. functor G } \} \\
& id \times id = id \\
\equiv & \quad \{ \text{ functor-}id\text{-}\times \} \\
& id = id \\
\Box &
\end{aligned}
$$

Exercício 8

$$
\begin{aligned}
& H\ id = id \\
\equiv & \quad \{ \text{ def. functor H } \} \\
& (F \cdot G)\ id = id \\
\equiv & \quad \{ \text{ composição de functores } \} \\
& F\ (G\ id) = id \\
\equiv & \quad \{ \text{ functor-id-G } \} \\
& F\ id = id \\
\equiv & \quad \{ \text{ functor-id-F } \} \\
& id = id \\
\Box &
\end{aligned}
$$

$$
\begin{aligned}
& H\ (f \cdot g) = (H\ f) \cdot (H\ g) \\
\equiv & \quad \{ \text{ def. functor H } \} \\
& (F \cdot G)\ (f \cdot g) = ((F \cdot G)\ f) \cdot ((F \cdot G)\ g) \\
\equiv & \quad \{ \text{ composição de functores } \} \\
& F\ (G\ (f \cdot g)) = (F\ (G\ f)) \cdot (F\ (G\ g)) \\
\equiv & \quad \{ \text{ Functor-F, Functor-G } \} \\
& F\ (G\ f \cdot G\ g) = F\ (G\ f \cdot G\ g) \\
\Box &
\end{aligned}
$$

Exercício 9