Cálculo de Programas
Resolução - Ficha 12

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Exercício 1

$$
\begin{aligned}
& \mu = id \bullet id \\
\equiv & \{ \text{ (F6) } \} \\
& \mu = \mu \cdot T\ id \cdot id \\
\equiv & \{ \text{ Natural-id, Functor-id-T (46) } \} \\
& \mu = \mu \cdot id \\
\equiv & \{ \text{ Natural-id } \} \\
& \mu = \mu
\end{aligned}
$$

$$
\begin{aligned}
& (f \cdot g) \bullet h = f \bullet (T\ g \cdot h) \\
\equiv & \{ \text{ (F6) twice } \} \\
& \mu \cdot T\ (f \cdot g) \cdot h = \mu \cdot (T\ f) \cdot (T\ g) \cdot h \\
\equiv & \{ \text{ Functor-T (45) } \} \\
& \mu \cdot (T\ f) \cdot (T\ g) \cdot h = \mu \cdot (T\ f) \cdot (T\ g) \cdot h
\end{aligned}
$$

$$
\begin{aligned}
& f \bullet u = f \\
\equiv & \{ \text{ (F6) } \} \\
& \mu \cdot T\ f \cdot u = f \\
\equiv & \{ \text{ (F4) } \} \\
& \mu \cdot u \cdot f = f \\
\equiv & \{ \text{ (F2) } \} \\
& id \cdot f = f \\
\equiv & \{ \text{ Natural-id } \} \\
& f = f
\end{aligned}
$$

$$
\begin{aligned}
& f = u \bullet f \\
\equiv & \{ \text{ (F6) } \} \\
& f = \mu \cdot T\ u \cdot f \\
\equiv & \{ \text{ (F2) } \} \\
& f = id \cdot f \\
\equiv & \{ \text{ Natural-id } \} \\
& f = f
\end{aligned}
$$

$$
\begin{aligned}
& T\ f = (u \cdot f) \bullet id \\
\equiv & \{ \text{ (F6) } \} \\
& T\ f = \mu \cdot T\ (u \cdot f) \cdot id \\
\equiv & \{ \text{ Natural-id, Functor-T (45) } \} \\
& T\ f = \mu \cdot (T\ u) \cdot (T\ f) \\
\equiv & \{ \text{ (F2) } \} \\
& T\ f = id \cdot T\ f \\
\equiv & \{ \text{ Natural-id } \} \\
& T\ f = T\ f
\end{aligned}
$$

Exercício 2

$$
\begin{aligned}
& discollect = lstr \bullet id \\
\equiv & \{ \text{ Composição Monádica } \} \\
& discollect = concat \cdot T\ lstr \cdot id \\
\equiv & \{ \text{ Natural-id, Def. concat, Absorção-cata } \} \\
& discollect = (\!| [nil, conc] \cdot B\ (lstr, id) |\!) \\
\equiv & \{ \text{ Universal-cata, Bi-Functor de Listas } \} \\
& discollect \cdot [nil, cons] = [nil, conc] \cdot (id + lstr \times id) \cdot (id + id \times discollect) \\
\equiv & \{ \text{ Fusão-+, Absorção-+ twice, Eq-+ } \} \\
& \begin{cases} discollect \cdot nil = nil \\ discollect \cdot cons = conc \cdot (lstr \times id) \cdot (id \times discollect) \end{cases} \\
\equiv & \{ \text{ Functor-}\times \text{ } \} \\
& \begin{cases} discollect \cdot nil = nil \\ discollect \cdot cons = conc \cdot (lstr \times discollect) \end{cases} \\
\equiv & \{ \text{ pointfree, Def. conc, Def. lstr } \} \\
& \begin{cases} discollect\ [\,] = [\,] \\ discollect\ ((a, l) : as) = t +\!\!+\ discollect\ as\ \textbf{where}\ t = [(a, b) \mid b \leftarrow l] \end{cases}
\end{aligned}
$$

Exercício 3

$$
\begin{aligned}
& \mu \cdot \mu = \mu \cdot T\ \mu \\
\equiv & \{ \text{ pointwise, Def. comp } \} \\
& \mu\ (\mu\ (((x, y), z), w)) = \mu\ ((\mu \times id)\ (((x, y), z), w)) \\
\equiv & \{ \text{ Def. } \mu\text{, Def-}\times\text{, Def. id} \} \\
& \mu\ ((x, y), z + w) = \mu\ ((x, y + z), w) \\
\equiv & \{ \text{ Def. } \mu \text{ } \} \\
& (x, y + z + w) = (x, y + z + w)
\end{aligned}
$$

$$
\begin{aligned}
& \mu \cdot u = id \\
\equiv & \{ \text{ pointwise, Def. comp } \} \\
& \mu\ (u\ (x, y)) = id\ (x, y) \\
\equiv & \{ \text{ Def. } \mu\text{, Def. id } \} \\
& \mu\ ((x, y), 0) = (x, y) \\
\equiv & \{ \text{ Def. } \mu \text{ } \} \\
& (x, y + 0) = (x, y)
\end{aligned}
$$

$$
\begin{aligned}
& \mu \cdot T\ u = id \\
\equiv & \{ \text{ pointwise, Def. comp, Def. T u } \} \\
& \mu\ ((u \times id)\ (x, y)) = id\ (x, y) \\
\equiv & \{ \text{ Def-}\times\text{, Def. u, Def. id } \} \\
& \mu\ ((x, 0) y) = (x, y) \\
\equiv & \{ \text{ Def. } \mu \text{ } \} \\
& (x, 0 + y) = (x, y)
\end{aligned}
$$

Exercício 4

$$
\begin{aligned}
& \mu \cdot T\ u = id \\
\equiv & \ \{\ \text{Def. } \mu\ \} \\
& (\![[id, in \cdot i_2]]\!) \cdot T\ u = id \\
\equiv & \ \{\ \text{Absorção-cata}\ \} \\
& (\![[id, in \cdot i_2] \cdot B\ (u, id)]\!) = id \\
\equiv & \ \{\ \text{Universal-cata, Def. Bi-Functor}\ \} \\
& [id, in \cdot i_2] \cdot (u + G\ id) \cdot F\ id = in \\
\equiv & \ \{\ \text{Functor-id-F twice, Absorção-+}\ \} \\
& [u, in \cdot i_2] = in \\
\equiv & \ \{\ \text{Def. u}\ \} \\
& [in \cdot i_1, in \cdot i_2] = in \\
\equiv & \ \{\ \text{Fusão-+, Reflexão-+}\ \} \\
& in = in
\end{aligned}
$$

$$
\begin{aligned}
& \mu \cdot u = id \\
\equiv & \ \{\ \text{Def. } \mu \text{, Def. u}\ \} \\
& (\![id, in \cdot i_2]\!) \cdot in \cdot i_1 = id \\
\equiv & \ \{\ \text{Cancelamento-cata}\ \} \\
& [id, in \cdot i_2] \cdot F\ \mu \cdot i_1 = id \\
\equiv & \ \{\ \text{Base-cata, Def. Functor}\ \} \\
& [id, in \cdot i_2] \cdot (id + G\ \mu) \cdot i_1 = id \\
\equiv & \ \{\ \text{Absorção-+}\ \} \\
& [id, in \cdot i_2 \cdot G\ \mu] \cdot i_1 = id \\
\equiv & \ \{\ \text{Cancelamento-+}\ \} \\
& id = id
\end{aligned}
$$

$$
\begin{aligned}
& \mu \cdot \mu = \mu \cdot T\ \mu \\
\equiv & \ \{\ \text{Def. } \mu\ \} \\
& \mu \cdot \mu = (\![[id, in \cdot i_2]]\!) \cdot T\ \mu \\
\equiv & \ \{\ \text{Absorção-cata, Def. } \mu\ \} \\
& \mu \cdot (\![[id, in \cdot i_2]]\!) = (\![[id, in \cdot i_2] \cdot B\ (\mu, id)]\!) \\
\Longleftarrow & \ \{\ \text{Fusão-cata}\ \} \\
& \mu \cdot [id, in \cdot i_2] = [id, in \cdot i_2] \cdot B\ (\mu, id) \cdot F\ \mu \\
\equiv & \ \{\ \text{Fusão-+, } F\ f = B\ (id, f),\ B\ (f, g) = f + G\ g \text{ (twice), Absorção-+}\ \} \\
& [\mu, \mu \cdot in \cdot i_2] = [\mu, in \cdot i_2 \cdot G\ \mu] \\
\equiv & \ \{\ \text{Eq-+}\ \} \\
& \begin{cases} \mu = \mu \\ \mu \cdot in \cdot i_2 = in \cdot i_2 \cdot G\ \mu \end{cases} \\
\equiv & \ \{\ \mu = (\![\ldots]\!) \text{, Cancelamento-cata}\ \} \\
& \begin{cases} \mu = \mu \\ [id, in \cdot i_2] \cdot F\ \mu \cdot i_2 = in \cdot i_2 \cdot G\ \mu \end{cases} \\
\equiv & \ \{\ F\ f = id + G\ f \text{, Absorção-+, Cancelamento-+}\ \} \\
& \begin{cases} \mu = \mu \\ in \cdot i_2 \cdot G\ \mu = in \cdot i_2 \cdot G\ \mu \end{cases}
\end{aligned}
$$

Exercício 5

$$A \xrightarrow{in \cdot i_1} LTree\ a \xleftarrow{(\![[id, in \cdot i_2]]\!)} LTree\ (LTree\ A)$$

O Functor Base de LTree é $B\ (X, Y) = X + Y \times Y$, logo podemos deduzir o functor G como $G\ Y = Y \times Y$.

Para $G\ Y = 1$ temos o Functor Base de Maybe $B\ (X, Y) = X + 1$.

Para $G\ Y = O \times Y^*$ temos o Functor Base de Árvores de Expressão $B\ (X, Y) = X + O \times Y^*$ (presente na biblioteca `Exp.hs`).

Exercício 6

$$
\begin{array}{ll}
& sequence = (\![[return, id] \cdot (nil + \lfloor cons \rfloor)]\!) \\
\equiv & \{ \text{ Universal-cata } \} \\
& sequence \cdot [nil, cons] = [return, id] \cdot (nil + \lfloor cons \rfloor) \cdot (id + id \times sequence) \\
\equiv & \{ \text{ Fusão-+, Absorção-+ twice, Eq-+ } \} \\
& \begin{cases} sequence \cdot nil = return \cdot nil \\ sequence \cdot cons = \lfloor cons \rfloor \cdot (id \times sequence) \end{cases} \\
\equiv & \{ \text{ pointwise, Def-}\times \text{ } \} \\
& \begin{cases} sequence\ [\,] = return\ [\,] \\ sequence\ (h : t) = \lfloor cons \rfloor\ (h, sequence\ t) \end{cases} \\
\equiv & \{ \text{ Def. } \lfloor f \rfloor \text{, Def. cons } \} \\
& \begin{cases} sequence\ [\,] = return\ [\,] \\ sequence\ (h : t) = \mathbf{do}\ \{a \leftarrow h;\ b \leftarrow sequence\ y;\ return\ (a : b)\} \end{cases}
\end{array}
$$