Cálculo de Programas

Resolução - Ficha 12

Eduardo Freitas Fernandes

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Exercício 1

$$
\begin{array}{ll}
& \mu = id \bullet id \\
\equiv & \qquad \{ \text{ (F6) } \} \\
& \mu = \mu \cdot T\ id \cdot id \\
\equiv & \qquad \{ \text{ natural-}id\text{, functor-}id\text{-}T\text{ (46) } \} \\
& \mu = \mu \cdot id \\
\equiv & \qquad \{ \text{ natural-}id \} \\
& \mu = \mu \\
\square &
\end{array}
$$

$$
\begin{array}{ll}
& (f \cdot g) \bullet h = f \bullet ((T\ g) \cdot h) \\
\equiv & \qquad \{ \text{ (F6) (twice) } \} \\
& \mu \cdot T\ (f \cdot g) \cdot h = \mu \cdot (T\ f) \cdot (T\ g) \cdot h \\
\equiv & \qquad \{ \text{ functor-}T\text{ (45) } \} \\
& \mu \cdot (T\ f) \cdot (T\ g) \cdot h = \mu \cdot (T\ f) \cdot (T\ g) \cdot h \\
\square &
\end{array}
$$

$$
\begin{array}{ll}
& f \bullet u = f \\
\equiv & \qquad \{ \text{ (F6) } \} \\
& \mu \cdot T\ f \cdot u = f \\
\equiv & \qquad \{ \text{ (F4) } \} \\
& \mu \cdot u \cdot f = f \\
\equiv & \qquad \{ \text{ (F2) } \} \\
& id \cdot f = f \\
\equiv & \qquad \{ \text{ natural-}id \} \\
& f = f \\
\square &
\end{array}
$$

$$
\begin{array}{ll}
& f = u \bullet f \\
\equiv & \qquad \{ \text{ (F6) } \} \\
& f = \mu \cdot T\ u \cdot f \\
\equiv & \qquad \{ \text{ (F2) } \} \\
& f = id \cdot f \\
\equiv & \qquad \{ \text{ natural-}id \} \\
& f = f \\
\square &
\end{array}
$$

$$
\begin{array}{ll}
& T\ f = (u \cdot f) \bullet id \\
\equiv & \qquad \{ \text{ (F6) } \} \\
& T\ f = \mu \cdot T\ (u \cdot f) \cdot id \\
\equiv & \qquad \{ \text{ natural-}id\text{, functor-}T\text{ (45) } \} \\
& T\ f = \mu \cdot (T\ u) \cdot (T\ f) \\
\equiv & \qquad \{ \text{ (F2), natural-}id \} \\
& T\ f = T\ f \\
\square &
\end{array}
$$

Exercício 2

$$
\begin{array}{ll}
& discollect = lstr \bullet id \\
\equiv & \quad \{ \text{ composição monádica } \} \\
& discollect = concat \cdot T\ lstr \cdot id \\
\equiv & \quad \{ \text{ natural-}id\text{, def. }concat\text{, absorção-}(\!|\cdot|\!) \} \\
& discollect = (\!| [nil, \mathsf{conc}] \cdot B\ (lstr, id) |\!) \\
\equiv & \quad \{ \text{ universal-cata, def. bi-functor de listas } \} \\
& discollect \cdot [nil, cons] = [nil, \mathsf{conc}] \cdot (id + lstr \times id) \cdot (id + id \times discollect) \\
\equiv & \quad \{ \text{ fusão-}+\text{, absorção-}+\text{ (twice), eq-}+ \} \\
& \left\{ \begin{array}{l} discollect \cdot nil = nil \\ discollect \cdot cons = \mathsf{conc} \cdot (lstr \times id) \cdot (id \times discollect) \end{array} \right. \\
\equiv & \quad \{ \text{ functor-}\times \} \\
& \left\{ \begin{array}{l} discollect \cdot nil = nil \\ discollect \cdot cons = \mathsf{conc} \cdot (lstr \times discollect) \end{array} \right. \\
\equiv & \quad \{ \text{ pointwise, def. conc, def. }lstr \} \\
& \left\{ \begin{array}{l} discollect\ [\,] = [\,] \\ discollect\ ((a, l) : as) = t +\!\!+\ discollect\ as\ \mathbf{where}\ t = [(a, b) \mid b \leftarrow l] \end{array} \right.
\end{array}
$$

Exercício 3

$$
\begin{array}{ll}
& \mu \cdot \mu = \mu \cdot T\ \mu \\
\equiv & \quad \{ \text{ pointwise, def. comp } \} \\
& \mu\ (\mu\ (((x, y), z), w)) = \mu\ ((\mu \times id)\ (((x, y), z), w)) \\
\equiv & \quad \{ \text{ def.}\mu\text{, def.}\times\text{, def.}id \} \\
& \mu\ ((x, y), z + w) = \mu\ ((x, y + z), w) \\
\equiv & \quad \{ \text{ def.}\mu \} \\
& (x, y + z + w) = (x, y + z + w) \\
\Box &
\end{array}
$$

$$
\begin{array}{ll}
& \mu \cdot u = id \\
\equiv & \quad \{ \text{ pointwise, def. comp } \} \\
& \mu\ (u\ (x, y)) = id\ (x, y) \\
\equiv & \quad \{ \text{ def. }u\text{, def. id } \} \\
& \mu\ ((x, y), 0) = (x, y) \\
\equiv & \quad \{ \text{ def.}\mu \} \\
& (x, y + 0) = (x, y) \\
\Box &
\end{array}
$$

$$
\begin{array}{ll}
& \mu \cdot T\ u = id \\
\equiv & \quad \{ \text{ pointwise, def. comp, def. }T\ u \} \\
& \mu\ ((u \times id)\ (x, y)) = id\ (x, y) \\
\equiv & \quad \{ \text{ def-}\times\text{, def. }u\text{, def. }id \} \\
& \mu\ ((x, 0), y) = (x, y) \\
\equiv & \quad \{ \text{ def.}\mu \} \\
& (x, 0 + y) = (x, y) \\
\Box &
\end{array}
$$

Exercício 4

$$
\begin{array}{ll}
& \mu \cdot T\ u = id \\
\equiv & \quad \{\ \text{def. } \mu\ \} \\
& (\!|\, [id, \mathsf{in} \cdot i_2] \,|\!) \cdot T\ u = id \\
\equiv & \quad \{\ \text{absorção-cata}\ \} \\
& (\!|\, [id, \mathsf{in} \cdot i_2] \cdot B\ (u, id) \,|\!) = id \\
\equiv & \quad \{\ \text{universal-cata, def. bi-functor B, natural-}id\ \} \\
& [id, \mathsf{in} \cdot i_2] \cdot (u + G\ id) \cdot F\ id = \mathsf{in} \\
\equiv & \quad \{\ \text{functor-}id\text{-}F \text{ (twice), absorção-}+\ \} \\
& [u, \mathsf{in} \cdot i_2] = \mathsf{in} \\
\equiv & \quad \{\ \text{def. } u\ \} \\
& [\mathsf{in} \cdot i_1, \mathsf{in} \cdot i_2] = \mathsf{in} \\
\equiv & \quad \{\ \text{fusão-}+\text{, reflexão-}+\ \} \\
& \mathsf{in} = \mathsf{in} \\
\Box &
\end{array}
$$

$$
\begin{array}{ll}
& \mu \cdot u = id \\
\equiv & \quad \{\ \text{def. } \mu\text{, def. } u\ \} \\
& (\!|\, [id, \mathsf{in} \cdot i_2] \,|\!) \cdot \mathsf{in} \cdot i_1 = id \\
\equiv & \quad \{\ \text{cancelamento-cata}\ \} \\
& [id, \mathsf{in} \cdot i_2] \cdot \mathsf{F}\ \mu \cdot i_1 = id \\
\equiv & \quad \{\ \text{base-cata, def. functor F}\ \} \\
& [id, \mathsf{in} \cdot i_2] \cdot (id + G\ \mu) \cdot i_1 = id \\
\equiv & \quad \{\ \text{absorção-}+\ \} \\
& [id, \mathsf{in} \cdot i_2 \cdot G\ \mu] \cdot i_1 = id \\
\equiv & \quad \{\ \text{cancelamento-}+\ \} \\
& id = id \\
\Box &
\end{array}
$$

$$
\begin{array}{ll}
& \mu \cdot \mu = \mu \cdot T\ \mu \\
\equiv & \quad \{\ \text{def. } \mu\ \} \\
& \mu \cdot \mu = (\!|\, [id, \mathsf{in} \cdot i_2] \,|\!) \cdot T\ \mu \\
\equiv & \quad \{\ \text{absorção-cata, def. } \mu\ \} \\
& \mu \cdot (\!|\, [id, \mathsf{in} \cdot i_2] \,|\!) = (\!|\, [id, \mathsf{in} \cdot i_2] \cdot B\ (\mu, id) \,|\!) \\
\Longleftarrow & \quad \{\ \text{fusão-cata}\ \} \\
& \mu \cdot [id, \mathsf{in} \cdot i_2] = [id, \mathsf{in} \cdot i_2] \cdot B\ (\mu, id) \cdot \mathsf{F}\ \mu \\
\equiv & \quad \{\ \text{fusão-}+\text{, } \mathsf{F}\ f = B\ (id, f), B\ (f, g) = f + G\ g \text{ (twice), absorção-}+\ \} \\
& [\mu, \mu \cdot \mathsf{in} \cdot i_2] = [\mu, \mathsf{in} \cdot i_2 \cdot G\ \mu] \\
\equiv & \quad \{\ \text{eq-}+\ \} \\
& \begin{cases} \mu = \mu \\ \mu \cdot \mathsf{in} \cdot i_2 = \mathsf{in} \cdot i_2 \cdot G\ \mu \end{cases} \\
\equiv & \quad \{\ \text{def. } \mu\text{, cancelamento-cata}\ \} \\
& \begin{cases} true \\ [id, \mathsf{in} \cdot i_2] \cdot \mathsf{F}\ \mu \cdot i_2 = \mathsf{in} \cdot i_2 \cdot G\ \mu \end{cases} \\
\equiv & \quad \{\ \mathsf{F}\ f = id + G\ f\text{, absorção-}+\text{, cancelamento-}+\ \} \\
& \begin{cases} true \\ \mathsf{in} \cdot i_2 \cdot G\ \mu = \mathsf{in} \cdot i_2 \cdot G\ \mu \end{cases} \\
\Box &
\end{array}
$$

Exercício 5

$$A \xrightarrow{\mathsf{in} \cdot i_1} \mathit{LTree}\ A \xleftarrow{(\![[id, \mathsf{in} \cdot i_2]]\!)} \mathit{LTree}\ (\mathit{LTree}\ A)$$

O Functor Base de LTree é $B\ (X, Y) = X + Y \times Y$, logo podemos deduzir o functor G como $G\ Y = Y \times Y$.

Para $G\ Y = 1$ temos o Functor Base de Maybe $B\ (X, Y) = X + 1$.

Para $G\ Y = O \times Y^*$ temos o Functor Base de Árvores de Expressão $B\ (X, Y) = X + O \times Y^*$ (presente na biblioteca *Exp*).

Exercício 6

$$
\begin{array}{ll}
& \mathit{sequence} = (\![[\mathit{return}, \mathit{id}] \cdot (\mathit{nil} + \lfloor \mathit{cons} \rfloor)]\!) \\
\equiv & \qquad \{ \text{ universal-cata } \} \\
& \mathit{sequence} \cdot [\mathit{nil}, \mathit{cons}] = [\mathit{return}, \mathit{id}] \cdot (\mathit{nil} + \lfloor \mathit{cons} \rfloor) \cdot (\mathit{id} + \mathit{id} \times \mathit{sequence}) \\
\equiv & \qquad \{ \text{ fusão-}+\text{, absorção-}+\text{ (twice), eq-}+ \ \} \\
& \begin{cases} \mathit{sequence} \cdot \mathit{nil} = \mathit{return} \cdot \mathit{nil} \\ \mathit{sequence} \cdot \mathit{cons} = \lfloor \mathit{cons} \rfloor \cdot (\mathit{id} \times \mathit{sequence}) \end{cases} \\
\equiv & \qquad \{ \text{ pointwise, def-}\times \ \} \\
& \begin{cases} \mathit{sequence}\ [\,] = \mathit{return}\ [\,] \\ \mathit{sequence}\ (h : t) = \lfloor \mathit{cons} \rfloor\ (h, \mathit{sequence}\ t) \end{cases} \\
\equiv & \qquad \{ \text{ def. } \lfloor f \rfloor \text{, def. } \mathit{cons} \ \} \\
& \begin{cases} \mathit{sequence}\ [\,] = \mathit{return}\ [\,] \\ \mathit{sequence}\ (h : t) = \mathbf{do}\ \{ a \leftarrow h; b \leftarrow \mathit{sequence}\ t; \mathit{return}\ (a : b) \} \end{cases}
\end{array}
$$