Cálculo de Programas
Resolução - Ficha 08

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Exercício 1

$$
\begin{aligned}
& T\ id = id \\
\equiv\ & \{(\mathrm{F1})\} \\
& id \times id = id \\
\equiv\ & \{\text{Def-}\times\} \\
& \langle id \cdot \pi_1, id \cdot \pi_2 \rangle = id \\
\equiv\ & \{\text{Natural-id (twice)}\} \\
& \langle \pi_1, \pi_2 \rangle = id \\
\equiv\ & \{\text{Reflexão-}\times\} \\
& id = id
\end{aligned}
$$

$$
\begin{aligned}
& T\ (f \cdot g) = T\ f \cdot T\ g \\
\equiv\ & \{(\mathrm{F1})\} \\
& (f \cdot g) \times (f \cdot g) = (f \times f) \cdot (g \times g) \\
\equiv\ & \{\text{Def-}\times\text{ (twice)}\} \\
& \langle f \cdot g \cdot \pi_1, f \cdot g \cdot \pi_2 \rangle = \langle f \cdot \pi_1, f \cdot \pi_2 \rangle \cdot (g \times g) \\
\equiv\ & \{\text{Fusão-}\times\} \\
& \langle f \cdot g \cdot \pi_1, f \cdot g \cdot \pi_2 \rangle = \langle f \cdot \pi_1 \cdot (g \times g), f \cdot \pi_2 \cdot (g \times g) \rangle \\
\equiv\ & \{\text{Natural-}\pi_1\text{, Natural-}\pi_2\} \\
& \langle f \cdot g \cdot \pi_1, f \cdot g \cdot \pi_2 \rangle = \langle f \cdot g \cdot \pi_1, f \cdot g \cdot \pi_2 \rangle
\end{aligned}
$$

$$
\begin{aligned}
& \mu \cdot T\ u = \mu \cdot u \\
\equiv\ & \{\text{Def. } \mu\text{, Def. T u}\} \\
& (\pi_1 \times \pi_2) \cdot (\langle id, id \rangle \times \langle id, id \rangle) = (\pi_1 \times \pi_2) \cdot \langle id, id \rangle \\
\equiv\ & \{\text{Def-}\times\text{, Absorção-}\times\} \\
& (\pi_1 \times \pi_2) \cdot \langle \langle id, id \rangle \cdot \pi_1, \langle id, id \rangle \cdot \pi_2 \rangle = \langle \pi_1, \pi_2 \rangle \\
\equiv\ & \{\text{Absorção-}\times\text{, Reflexão-}\times\} \\
& \langle \pi_1 \cdot \langle id, id \rangle \cdot \pi_1, \pi_2 \cdot \langle id, id \rangle \cdot \pi_2 \rangle = \langle \pi_1, \pi_2 \rangle \\
\equiv\ & \{\text{Cancelamento-}\times\} \\
& \langle \pi_1, \pi_2 \rangle = \langle \pi_1, \pi_2 \rangle
\end{aligned}
$$

Exercício 2

$$loop\ (a, b) = (2 + b, 2 - a) \equiv loop = ((2+) \times (2-)) \cdot swap$$

$$\begin{aligned}
&\langle f, g\rangle = for\ loop\ (4, -2)\\
\equiv\ &\{\text{Def. for}\}\\
&\langle f, g\rangle = (\![\underline{(4, -2)}, loop]\!)\\
\equiv\ &\{\text{Def. loop}\}\\
&\langle f, g\rangle = (\![\underline{4} \times \underline{-2}, (2+) \times (2-) \cdot swap]\!)\\
\equiv\ &\{\text{Def-}\times\ \text{(twice), Fusão-}\times\}\\
&\langle f, g\rangle = (\![\langle \underline{4} \cdot \pi_1, \underline{-2} \cdot \pi_2\rangle, \langle (2+) \cdot \pi_1 \cdot swap, (2-) \cdot \pi_2 \cdot swap\rangle]\!)\\
\equiv\ &\{\text{Lei da Troca}\}\\
&\langle f, g\rangle = (\!\langle [\underline{4} \cdot \pi_1, (2+) \cdot \pi_1 \cdot swap], [\underline{-2} \cdot \pi_2, (2-) \cdot \pi_2 \cdot swap]\rangle\!)\\
\equiv\ &\{\text{Fokkinga}\}\\
&\begin{cases} f \cdot [zero, succ] = [\underline{4} \cdot \pi_1, (2+) \cdot \pi_1 \cdot swap] \cdot (id + \langle f, g\rangle) \\ g \cdot [zero, succ] = [\underline{-2} \cdot \pi_2, (2-) \cdot \pi_2 \cdot swap] \cdot (id + \langle f, g\rangle) \end{cases}\\
\equiv\ &\{\text{Fusão-+, Absorção-+, Eq-+}\}\\
&\begin{cases} f \cdot zero = \underline{4} \cdot \pi_1 \\ f \cdot succ = (2+) \cdot \pi_1 \cdot swap \\ g \cdot zero = \underline{-2} \cdot \pi_2 \\ g \cdot succ = (2-) \cdot \pi_2 \cdot swap \end{cases}\\
\equiv\ &\{\text{pointwise}\}\\
&\begin{cases} f\ 0 = 4 \\ f\ (n+1) = 2 + g\ n \\ g\ 0 = -2 \\ g\ (n+1) = 2 - f\ n \end{cases}
\end{aligned}$$

Exercício 3

$$\begin{aligned}
&\langle\langle f, g\rangle, j\rangle = (\!\langle\langle h, k\rangle, l\rangle\!)\\
\equiv\ &\{\text{Fokkinga}\}\\
&\begin{cases} \langle f, g\rangle \cdot in = \langle h, k\rangle \cdot F\ \langle\langle f, g\rangle, j\rangle \\ j \cdot in = l \cdot F\ \langle\langle f, g\rangle, j\rangle \end{cases}\\
\equiv\ &\{\text{Fusão-}\times\ \text{(twice)}\}\\
&\begin{cases} \langle f \cdot in, g \cdot in\rangle = \langle h \cdot F\ \langle\langle f, g\rangle, j\rangle, k \cdot F\ \langle\langle f, g\rangle, j\rangle\rangle \\ j \cdot in = l \cdot F\ \langle\langle f, g\rangle, j\rangle \end{cases}\\
\equiv\ &\{\text{Eq-}\times\}\\
&\begin{cases} f \cdot in = h \cdot F\ \langle\langle f, g\rangle, j\rangle \\ g \cdot in = k \cdot F\ \langle\langle f, g\rangle, j\rangle \\ j \cdot in = l \cdot F\ \langle\langle f, g\rangle, j\rangle \end{cases}
\end{aligned}$$

Exercício 4

$$\begin{cases} impar\ 0 = false \\ impar\ (n+1) = par\ n \\ par\ 0 = true \\ par\ (n+1) = impar\ n \end{cases}$$

$$\equiv \{\text{pointfree}\}$$

$$\begin{cases} impar \cdot zero = False \\ impar \cdot succ = par \\ par \cdot zero = True \\ par \cdot succ = impar \end{cases}$$

$$\equiv \{\text{Eq-+}\}$$

$$\begin{cases} [impar \cdot zero, impar \cdot succ] = [False, par] \\ [par \cdot zero, par \cdot succ] = [True, impar] \end{cases}$$

$$\equiv \{\text{Fusão-+, Cancelamento-+ (twice)}\}$$

$$\begin{cases} impar \cdot in = [False, \pi_2 \cdot \langle impar, par \rangle] \\ par \cdot in = [True, \pi_1 \cdot \langle impar, par \rangle] \end{cases}$$

$$\equiv \{\text{Natural-id (twice), Absorção-+}\}$$

$$\begin{cases} impar \cdot in = [False, \pi_2] \cdot (id + \langle impar, par \rangle) \\ par \cdot in = [True, \pi_1] \cdot (id + \langle impar, par \rangle) \end{cases}$$

Podemos então concluir que $h = [False, \pi_2]$ e $k = [True, \pi_1]$.

$$\langle impar, par \rangle = for\ swap\ (False, True)$$

$$\equiv \{\text{Def. for}\}$$

$$\langle impar, par \rangle = (\![[False \times True, swap]]\!)$$

$$\equiv \{\text{Def. swap, Def-}\times\}$$

$$\langle impar, par \rangle = (\![[\langle False \cdot \pi_1, True \cdot \pi_2 \rangle, \langle \pi_2, \pi_1 \rangle]]\!)$$

$$\equiv \{\text{Lei da Troca}\}$$

$$\langle impar, par \rangle = (\![\langle [False \cdot \pi_1, \pi_2], [True \cdot \pi_2, \pi_1] \rangle]\!)$$

$$\equiv \{\text{Fokkinga}\}$$

$$\begin{cases} impar \cdot [zero, succ] = [False\pi_1, \pi_2] \cdot (id + \langle impar, par \rangle) \\ par \cdot [zero, succ] = [True \cdot \pi_2, \pi_1] \cdot (id + \langle impar, par \rangle) \end{cases}$$

$$\equiv \{\text{Fusão-+, Absorção-+}\}$$

$$\begin{cases} [impar \cdot zero, impar \cdot succ] = [False \cdot \pi_1, \pi_2 \cdot \langle impar, par \rangle] \\ [par \cdot zero, par \cdot succ] = [True \cdot \pi_2, \pi_1 \cdot \langle impar, par \rangle] \end{cases}$$

$$\equiv \{\text{Cancelamento-}\times\text{ (twice), Eq-+, pointwise}\}$$

$$\begin{cases} impar\ 0 = False \\ impar\ (n+1) = par\ n \\ par\ 0 = True \\ par\ (n+1) = impar\ n \end{cases}$$

Exercício 5

$$\begin{cases} insg\ 0 = \ [\ ] \\ insg\ (n+1) = fsuc + 1 : insg\ n \\ fsuc\ 0 = 1 \\ fsuc\ (n+1) = fsuc\ n + 1 \end{cases}$$

$$\equiv \{\text{pointwise}\}$$

$$\begin{cases} insg \cdot zero = nil \\ insg \cdot succ = cons \cdot \langle fsuc, insg \rangle \\ fsuc \cdot zero = one \\ fsuc \cdot succ = (1+) \cdot fsuc \end{cases}$$

$$\equiv \{\text{Cancelamento-}\times\}$$

$$\begin{cases} insg \cdot zero = nil \\ insg \cdot succ = cons \cdot \langle fsuc, insg \rangle \\ fsuc \cdot zero = one \\ fsuc \cdot succ = (1+) \cdot \pi_1 \cdot \langle fsuc, insg \rangle \end{cases}$$

$$\equiv \{\text{Eq-+, Fusão-+, Absorção-+}\}$$

$$\begin{cases} insg \cdot in = [nil, cons] \cdot (id + \langle fsuc, insg \rangle) \\ fsuc \cdot in = [one, (1+) \cdot \pi_1] \cdot (id + \langle fsuc, insg \rangle) \end{cases}$$

$$\equiv \{\text{Fokkinga}\}$$

$$\langle fsuc, insg \rangle = (\!|\, \langle in_*, [one, (1+) \cdot \pi_1] \rangle \,|\!)$$

$$
\begin{aligned}
& insgfor = for\ \langle (1+) \cdot \pi_1, cons \rangle\ \langle one, nil \rangle \\
\equiv\ & \{\text{Def. for}\} \\
& insgfor = (\!|\, [\langle one, nil \rangle, \langle (1+) \cdot \pi_1, cons \rangle] \,|\!) \\
\equiv\ & \{\text{Lei da Troca}\} \\
& insgfor = (\!|\, \langle [one, (1+) \cdot \pi_1], [nil, cons] \rangle \,|\!) \\
\equiv\ & \{insg = \pi_2 \cdot insgfor \implies insgfor = \langle f, insg \rangle\} \\
& \langle f, insg \rangle = (\!|\, \langle [one, (1+) \cdot \pi_1], [nil, cons] \rangle \,|\!) \\
\equiv\ & \{\text{Fokkinga}\} \\
& \begin{cases} f \cdot [zero, succ] = [one, (1+) \cdot \pi_1] \cdot F\ \langle f, insg \rangle \\ insg \cdot [zero, succ] = [nil, cons] \cdot F\ \langle f, insg \rangle \end{cases} \\
\equiv\ & \{\text{Functor dos Naturais, Fusão-+, Absorção-+, Eq-+}\} \\
& \begin{cases} f \cdot zero = one \\ f \cdot succ = (1+) \cdot \pi_1 \cdot \langle f, insg \rangle \\ insg \cdot zero = nil \\ insg \cdot succ = cons \cdot \langle f, insg \rangle \end{cases} \\
\equiv\ & \{\text{pointwise}\} \\
& \begin{cases} f\ 0 = 1 \\ f\ (n+1) = 1 + f\ n \\ insg\ 0 = [\,] \\ insg\ (n+1) = f\ n\ :\ insg\ n \end{cases}
\end{aligned}
$$

Exercício 6

$$\begin{cases} f_1\ [\,] = [\,] \\ f_1\ (h : t) = h\ :\ f_2\ t \\ f_2\ [\,] = [\,] \\ f_1\ (h : t) = f_1\ t \end{cases}$$

$\equiv$ {Def. composição, pointfree}

$$\begin{cases} f_1 \cdot nil = nil \\ f_1 \cdot cons = cons \cdot (id \times f_2) \\ f_2 \cdot nil = nil \\ f_2 \cdot cons = f_1 \cdot \pi_2 \end{cases}$$

$\equiv$ {Eq-+, Fusão-+}

$$\begin{cases} f_1 \cdot [nil, cons] = [nil, cons \cdot (id \times f_2)] \\ f_2 \cdot [nil, cons] = [nil, f_1 \cdot \pi_2] \end{cases}$$

$\equiv$ {Def. in, Natural-id (2*), Natural-π_2, Cancelamento-$\times$}

$$\begin{cases} f_1 \cdot in = [nil \cdot id, cons \cdot (id \times \pi_2 \cdot \langle f_1, f_2 \rangle)] \\ f_2 \cdot in = [nil \cdot id, \pi_2 \cdot (id \times \pi_1 \cdot \langle f_1, f_2 \rangle)] \end{cases}$$

$\equiv$ {Functor-$\times$}

$$\begin{cases} f_1 \cdot in = [nil \cdot id, cons \cdot (id \times \pi_2) \cdot (id \times \langle f_1, f_2 \rangle)] \\ f_2 \cdot in = [nil \cdot id, \pi_2 \cdot (id \times \pi_1) \cdot (id \times \langle f_1, f_2 \rangle)] \end{cases}$$

$\equiv$ {Absorção-+, Functor de listas}

$$\begin{cases} f_1 \cdot in = [nil, cons \cdot (id \times \pi_2)] \cdot F\ \langle f_1, f_2 \rangle \\ f_2 \cdot in = [nil, \pi_2 \cdot (id \times \pi_1)] \cdot F\ \langle f_1, f_2 \rangle \end{cases}$$

$\equiv$ {Fokkinga}

$$\langle f_1, f_2 \rangle = (\!|\, \langle in_* \cdot (id + (id \times \pi_2)), [nil, \pi_2 \cdot (id \times \pi_1)] \rangle \,|\!)$$

$\equiv$ {Natural-π_2}

$$\langle f_1, f_2 \rangle = (\!|\, \langle in_* \cdot (id + (id \times \pi_2)), [nil, \pi_1 \cdot \pi_2] \rangle \,|\!)$$

$$\begin{tikzcd} A^* \arrow[r, bend left, "out"] \arrow[d, "\langle f_1, f_2 \rangle"'] & a + A \times A^* \arrow[l, bend left, "in"] \arrow[d, "id + id \times \langle f_1, f_2 \rangle"] \\ A^* \times A^* & 1 + A \times (A^* \times A^*) \arrow[l, "g"] \end{tikzcd}$$

Diagrama do gene do catamorfismo:

$$\begin{tikzcd} & 1 + A \times (A^* \times A^*) \arrow[dl, "id + (id \times \pi_2)"'] \arrow[dr, "id + \pi_2"] & \\ 1 + A \times A^* \arrow[d, "in_*"] & & 1 + (A^* \times A^*) \arrow[d, "{[nil, \pi_1]}"'] \\ A^* & & A^* \end{tikzcd}$$

A função f_1 seleciona os elementos de uma lista nas posições pares, e a função f_2 seleciona os elementos de uma lista nas posições ímpares.

Exercício 7

$$
\begin{aligned}
& H\ (g \cdot h) = (H\ g) \cdot (H\ h) \\
\equiv\ & \{\text{Functor-H (3*)}\} \\
& F\ (g \cdot h) + G\ (g \cdot h) = (F\ g + G\ g) \cdot (F\ h + G\ h) \\
\equiv\ & \{\text{Functor-F (3*), Functor-G (3*)}\} \\
& id + (g \cdot h) = (id + g) \cdot (id + h) \\
\equiv\ & \{\text{Natural-id, Functor-+}\} \\
& (id \cdot id) + (g \cdot h) = (id \cdot id) + (g \cdot h)
\end{aligned}
$$

$$
\begin{aligned}
& H\ id = id \\
\equiv\ & \{\text{Functor-H}\} \\
& F\ id + G\ id = id \\
\equiv\ & \{\text{Functor-F, Functor-G}\} \\
& id + id = id \\
\equiv\ & \{\text{Functor-id-+}\} \\
& id = id
\end{aligned}
$$

$$
\begin{aligned}
& K\ (g \cdot h) = (K\ g) \cdot (K\ h) \\
\equiv\ & \{\text{Functor-K}\} \\
& G\ (g \cdot h) \times F\ (g \cdot h) = (G\ g \times F\ g) \cdot (G\ h \times F\ h) \\
\equiv\ & \{\text{Functor-F (3*), Functor-G (3*)}\} \\
& (g \cdot h) \times id = (g \times id) \cdot (h \times id) \\
\equiv\ & \{\text{Natural-id, Functor-}\times\} \\
& (g \cdot h) \times (id \cdot id) = (g \cdot h) \times (id \cdot id)
\end{aligned}
$$

$$
\begin{aligned}
& K\ id = id \\
\equiv\ & \{\text{Functor-K}\} \\
& G\ id \times F\ id = id \\
\equiv\ & \{\text{Functor-F, Functor-G}\} \\
& id \times id = id \\
\equiv\ & \{\text{Functor-id-}\times\} \\
& id = id
\end{aligned}
$$

Exercício 8

$$
\begin{aligned}
& H\ id = id \\
\equiv\ & \{\text{Definição do Functor H}\} \\
& (F \cdot G)\ id = id \\
\equiv\ & \{\text{Composição de Functores}\} \\
& F\ (G\ id) = id \\
\equiv\ & \{\text{Functor-id-G}\} \\
& F\ id = id \\
\equiv\ & \{\text{Functor-id-F}\} \\
& id = id
\end{aligned}
$$

$$
\begin{aligned}
& H\ (f \cdot g) = (H\ f) \cdot (H\ g) \\
\equiv\ & \{\text{Definição do Functor H}\} \\
& (F \cdot G)\ (f \cdot g) = ((F \cdot G)\ f) \cdot ((F \cdot G)\ g) \\
\equiv\ & \{\text{Composição de Functores}\} \\
& F\ (G\ (f \cdot g)) = (F\ (G\ f)) \cdot (F\ (G\ g)) \\
\equiv\ & \{\text{Functor-G, Functor-F}\} \\
& F\ (G\ f \cdot G \cdot g) = F\ (G\ f \cdot G \cdot g)
\end{aligned}
$$

Exercício 9

$unzip\ xs = (map\ \pi_1\ xs, map\ \pi_2\ xs)$

$\equiv$ {pointfree}

$unzip = \langle map\ \pi_1, map\ \pi_2 \rangle$

$\equiv$ {Def-map-cata}

$unzip = \langle (\!| in \cdot B\ (\pi_1, id) |\!), (\!| in \cdot B\ (\pi_2, id) |\!) \rangle$

$\equiv$ {Banana-split}

$unzip = (\!| ((in \cdot B\ (\pi_1, id)) \times (in \cdot B\ (\pi_2, id))) \cdot \langle F\ \pi_1, F\ \pi_2 \rangle |\!)$

$\equiv$ {Absorção-$\times$}

$unzip = (\!| \langle in \cdot B\ (\pi_1, id) \cdot F\ \pi_1, in \cdot B\ (\pi_2, id) \cdot F\ \pi_2 \rangle |\!)$

$\equiv$ {Base-cata (twice), Functor-B (para Bi Functores)}

$unzip = (\!| \langle in \cdot B\ (\pi_1, \pi_1), in \cdot B\ (\pi_2, \pi_2) \rangle |\!)$

$\equiv$ {Universal-cata}

$unzip \cdot in = \langle in \cdot B\ (\pi_1, \pi_1), in \cdot B\ (\pi_2, \pi_2) \rangle \cdot F\ unzip$

$\equiv$ {Def. in_*, Def. Bi-Functor de listas}

$unzip \cdot in = \langle [nil, cons] \cdot (id + \pi_1 \times \pi_1), [nil, cons] \cdot (id + \pi_2 \times \pi_2) \rangle \cdot F\ unzip$

$\equiv$ {Absorção-+, Definição do Functor de Listas}

$unzip \cdot in = \langle [nil, cons \cdot (\pi_1 \times \pi_1)], [nil, cons \cdot (\pi_2 \times \pi_2)] \rangle \cdot (id + id \times unzip)$

$\equiv$ {Lei da Troca}

$unzip \cdot in = [\langle nil, nil \rangle, \langle cons \cdot (\pi_1 \times \pi_1), cons \cdot (\pi_2 \times \pi_2) \rangle] \cdot (id + id \times unzip)$

$\equiv$ {Absorção-+, Natural-id}

$unzip \cdot in = [\langle nil, nil \rangle, \langle cons \cdot (\pi_1 \times \pi_1), cons \cdot (\pi_2 \times \pi_2) \rangle \cdot (id \times unzip)]$

$\equiv$ {Def. in, Fusão-+, Eq-+}

$$\begin{cases} unzip \cdot nil = \langle nil, nil \rangle \\ unzip \cdot cons = \langle cons \cdot (\pi_1 \times \pi_1), cons \cdot (\pi_2 \times \pi_2) \rangle \cdot (id \times unzip) \end{cases}$$

$\equiv$ {Absorção-$\times$}

$$\begin{cases} unzip \cdot nil = \langle nil, nil \rangle \\ unzip \cdot cons = (cons \times cons) \cdot \langle \pi_1 \times \pi_1, \pi_2 \times \pi_2 \rangle \cdot (id \times unzip) \end{cases}$$

$\equiv$ {pointwise}

$$\begin{cases} unzip\ [\,] = ([\,], [\,]) \\ unzip\ ((a, b) : xs) = (cons \times cons) \cdot \langle \pi_1 \times \pi_1, \pi_2 \times \pi_2 \rangle \cdot (id \times unzip)\ ((a, b) : xs) \end{cases}$$

$\equiv$ {Definição do produto de funções, Definição de id}

$$\begin{cases} unzip\ [\,] = ([\,], [\,]) \\ unzip\ ((a, b) : xs) = (cons \times cons) \cdot \langle \pi_1 \times \pi_1, \pi_2 \times \pi_2 \rangle\ ((a, b), (as, bs))\ \textbf{where}\ (as, bs) = unzip\ xs \end{cases}$$

$\equiv$ {Definição de split, De. π_1 e π_2}

$$\begin{cases} unzip\ [\,] = ([\,], [\,]) \\ unzip\ ((a, b) : xs) = (cons \times cons)\ ((a, as), (b, bs))\ \textbf{where}\ (as, bs) = unzip\ xs \end{cases}$$

$\equiv$ {Definição do produto de funções, Def. cons}

$$\begin{cases} unzip\ [\,] = ([\,], [\,]) \\ unzip\ ((a, b) : xs) = ((a : as), (b : bs))\ \textbf{where}\ (as, bs) = unzip\ xs \end{cases}$$