**Proof.** This is easily deduced from the decomposition of $\mathtt{E}$ as $[\mathbf{t}]_\times\mathtt{R} = \mathtt{SR}$, where $\mathtt{S}$ is skew-symmetric. We will use the matrices

$$\mathtt{W} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathtt{Z} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{9.13}$$

It may be verified that $\mathtt{W}$ is orthogonal and $\mathtt{Z}$ is skew-symmetric. From Result A4.1-(*p*581), which gives a block decomposition of a general skew-symmetric matrix, the $3 \times 3$ skew-symmetric matrix $\mathtt{S}$ may be written as $\mathtt{S} = k\mathtt{UZU}^\mathsf{T}$ where $\mathtt{U}$ is orthogonal. Noting that, up to sign, $\mathtt{Z} = \mathrm{diag}(1,1,0)\mathtt{W}$, then up to scale, $\mathtt{S} = \mathtt{U}\,\mathrm{diag}(1,1,0)\mathtt{WU}^\mathsf{T}$, and $\mathtt{E} = \mathtt{SR} = \mathtt{U}\,\mathrm{diag}(1,1,0)(\mathtt{WU}^\mathsf{T}\mathtt{R})$. This is a singular value decomposition of $\mathtt{E}$ with two equal singular values, as required. Conversely, a matrix with two equal singular values may be factored as $\mathtt{SR}$ in this way. □

Since $\mathtt{E} = \mathtt{U}\,\mathrm{diag}(1,1,0)\mathtt{V}^\mathsf{T}$, it may seem that $\mathtt{E}$ has six degrees of freedom and not five, since both $\mathtt{U}$ and $\mathtt{V}$ have three degrees of freedom. However, because the two singular values are equal, the SVD is not unique – in fact there is a one-parameter family of SVDs for $\mathtt{E}$. Indeed, an alternative SVD is given by $\mathtt{E} = (\mathtt{U}\,\mathrm{diag}(\mathtt{R}_{2\times2}, 1))\,\mathrm{diag}(1,1,0)(\mathrm{diag}(\mathtt{R}^\mathsf{T}_{2\times2}, 1))\mathtt{V}^\mathsf{T}$ for any $2 \times 2$ rotation matrix $\mathtt{R}$.

### 9.6.2 Extraction of cameras from the essential matrix

The essential matrix may be computed directly from (9.11) using normalized image coordinates, or else computed from the fundamental matrix using (9.12). (Methods of computing the fundamental matrix are deferred to chapter 11). Once the essential matrix is known, the camera matrices may be retrieved from $\mathtt{E}$ as will be described next. In contrast with the fundamental matrix case, where there is a projective ambiguity, the camera matrices may be retrieved from the essential matrix up to scale and a four-fold ambiguity. That is there are four possible solutions, except for overall scale, which cannot be determined.

We may assume that the first camera matrix is $\mathtt{P} = [\mathtt{I} \mid \mathbf{0}]$. In order to compute the second camera matrix, $\mathtt{P}'$, it is necessary to factor $\mathtt{E}$ into the product $\mathtt{SR}$ of a skew-symmetric matrix and a rotation matrix.

**Result 9.18.** *Suppose that the SVD of* $\mathtt{E}$ *is* $\mathtt{U}\,\mathrm{diag}(1,1,0)\mathtt{V}^\mathsf{T}$. *Using the notation of (9.13), there are (ignoring signs) two possible factorizations* $\mathtt{E} = \mathtt{SR}$ *as follows:*

$$\mathtt{S} = \mathtt{UZU}^\mathsf{T} \quad \mathtt{R} = \mathtt{UWV}^\mathsf{T} \;\; or \;\; \mathtt{UW}^\mathsf{T}\mathtt{V}^\mathsf{T}\ . \tag{9.14}$$

**Proof.** That the given factorization is valid is true by inspection. That there are no other factorizations is shown as follows. Suppose $\mathtt{E} = \mathtt{SR}$. The form of $\mathtt{S}$ is determined by the fact that its left null-space is the same as that of $\mathtt{E}$. Hence $\mathtt{S} = \mathtt{UZU}^\mathsf{T}$. The rotation $\mathtt{R}$ may be written as $\mathtt{UXV}^\mathsf{T}$, where $\mathtt{X}$ is some rotation matrix. Then

$$\mathtt{U}\,\mathrm{diag}(1,1,0)\mathtt{V}^\mathsf{T} = \mathtt{E} = \mathtt{SR} = (\mathtt{UZU}^\mathsf{T})(\mathtt{UXV}^\mathsf{T}) = \mathtt{U}(\mathtt{ZX})\mathtt{V}^\mathsf{T}$$

from which one deduces that $\mathtt{ZX} = \mathrm{diag}(1,1,0)$. Since $\mathtt{X}$ is a rotation matrix, it follows that $\mathtt{X} = \mathtt{W}$ or $\mathtt{X} = \mathtt{W}^\mathsf{T}$, as required. □

The factorization (9.14) determines the $\mathbf{t}$ part of the camera matrix $\mathtt{P}'$, up to scale, from $\mathtt{S} = [\mathbf{t}]_\times$. However, the Frobenius norm of $\mathtt{S} = \mathtt{UZU}^\mathsf{T}$ is $\sqrt{2}$, which means that if $\mathtt{S} = [\mathbf{t}]_\times$ *including scale* then $\|\mathbf{t}\| = 1$, which is a convenient normalization for the baseline of the two camera matrices. Since $\mathtt{S}\mathbf{t} = \mathbf{0}$, it follows that $\mathbf{t} = \mathtt{U}(0,0,1)^\mathsf{T} = \mathbf{u}_3$, the last column of $\mathtt{U}$. However, the sign of $\mathtt{E}$, and consequently $\mathbf{t}$, cannot be determined. Thus, corresponding to a given essential matrix, there are four possible choices of the camera matrix $\mathtt{P}'$, based on the two possible choices of $\mathtt{R}$ and two possible signs of $\mathbf{t}$. To summarize:

**Result 9.19.** *For a given essential matrix* $\mathtt{E} = \mathtt{U}\,\mathrm{diag}(1,1,0)\mathtt{V}^\mathsf{T}$, *and first camera matrix* $\mathtt{P} = [\mathtt{I} \mid \mathbf{0}]$, *there are four possible choices for the second camera matrix* $\mathtt{P}'$, *namely*

$$\mathtt{P}' = [\mathtt{UWV}^\mathsf{T} \mid +\mathbf{u}_3] \text{ or } [\mathtt{UWV}^\mathsf{T} \mid -\mathbf{u}_3] \text{ or } [\mathtt{UW}^\mathsf{T}\mathtt{V}^\mathsf{T} \mid +\mathbf{u}_3] \text{ or } [\mathtt{UW}^\mathsf{T}\mathtt{V}^\mathsf{T} \mid -\mathbf{u}_3].$$

### 9.6.3 Geometrical interpretation of the four solutions

It is clear that the difference between the first two solutions is simply that the direction of the translation vector from the first to the second camera is reversed.

The relationship of the first and third solutions in result 9.19 is a little more complicated. However, it may be verified that

$$[\mathtt{UW}^\mathsf{T}\mathtt{V}^\mathsf{T} \mid \mathbf{u}_3] = [\mathtt{UWV}^\mathsf{T} \mid \mathbf{u}_3]\begin{bmatrix} \mathtt{VW}^\mathsf{T}\mathtt{W}^\mathsf{T}\mathtt{V}^\mathsf{T} & \\ & 1 \end{bmatrix}$$

and $\mathtt{VW}^\mathsf{T}\mathtt{W}^\mathsf{T}\mathtt{V}^\mathsf{T} = \mathtt{V}\,\mathrm{diag}(-1,-1,1)\mathtt{V}^\mathsf{T}$ is a rotation through $180^\circ$ about the line joining the two camera centres. Two solutions related in this way are known as a "twisted pair".

The four solutions are illustrated in figure 9.12, where it is shown that a reconstructed point $\mathbf{X}$ will be in front of both cameras in one of these four solutions only. Thus, testing with a single point to determine if it is in front of both cameras is sufficient to decide between the four different solutions for the camera matrix $\mathtt{P}'$.

**Note.** The point of view has been taken here that the essential matrix is a homogeneous quantity. An alternative point of view is that the essential matrix is defined exactly by the equation $\mathtt{E} = [\mathbf{t}]_\times\mathtt{R}$, (i.e. including scale), and is determined only up to indeterminate scale by the equation $\mathbf{x}'^\mathsf{T}\mathtt{E}\mathbf{x} = 0$. The choice of point of view depends on which of these two equations one regards as the defining property of the essential matrix.

## 9.7 Closure

### 9.7.1 The literature

The essential matrix was introduced to the computer vision community by Longuet-Higgins [LonguetHiggins-81], with a matrix analogous to $\mathtt{E}$ appearing in the photogrammetry literature, e.g. [VonSanden-08]. Many properties of the essential matrix have been elucidated particularly by Huang and Faugeras [Huang-89], [Maybank-93], and [Horn-90].

The realization that the essential matrix could also be applied in uncalibrated situations, as it represented a projective relation, developed in the early part of the 1990s,

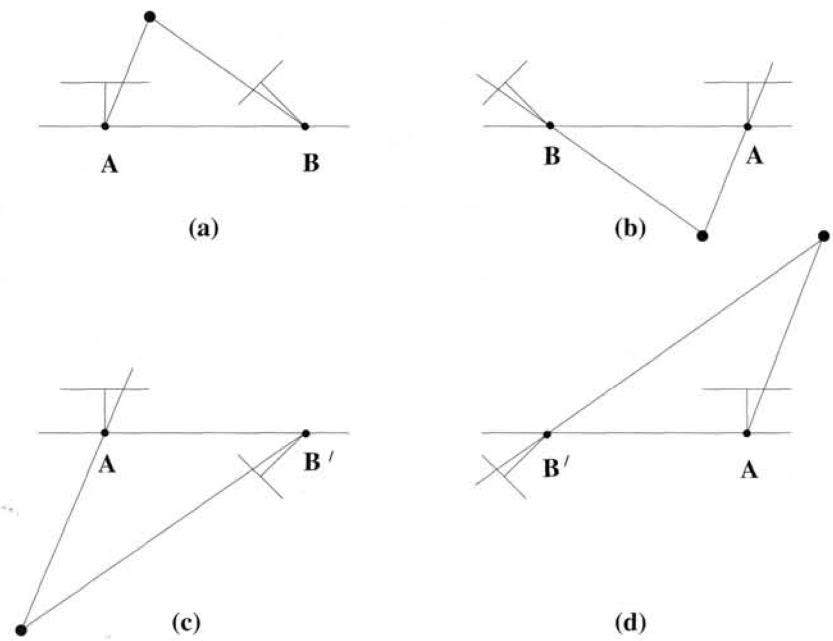


Fig. 9.12. **The four possible solutions for calibrated reconstruction from E.** *Between the left and right sides there is a baseline reversal. Between the top and bottom rows camera B rotates* $180°$ *about the baseline. Note, only in (a) is the reconstructed point in front of both cameras.*

and was published simultaneously by Faugeras [Faugeras-92b, Faugeras-92a], and Hartley *et al.* [Hartley-92a, Hartley-92c].

The special case of pure planar motion was examined by [Maybank-93] for the essential matrix. The corresponding case for the fundamental matrix is investigated by Beardsley and Zisserman [Beardsley-95a] and Viéville and Lingrand [Vieville-95], where further properties are given.

### 9.7.2 Notes and exercises

(i) **Fixating cameras.** Suppose two cameras fixate on a point in space such that their principal axes intersect at that point. Show that if the image coordinates are normalized so that the coordinate origin coincides with the principal point then the $F_{33}$ element of the fundamental matrix is zero.

(ii) **Mirror images.** Suppose that a camera views an object and its reflection in a plane mirror. Show that this situation is equivalent to two views of the object, and that the fundamental matrix is skew-symmetric. Compare the fundamental matrix for this configuration with that of: (a) a pure translation, and (b) a pure planar motion. Show that the fundamental matrix is auto-epipolar (as is (a)).

(iii) Show that if the vanishing line of a plane contains the epipole then the plane is parallel to the baseline.

(iv) **Steiner conic.** Show that the polar of $\mathtt{x}_\mathtt{a}$ intersects the Steiner conic $\mathtt{F}_\mathtt{S}$ at the epipoles (figure 9.10a). Hint, start from $\mathtt{Fe} = \mathtt{F}_\mathtt{S}\mathtt{e} + \mathtt{F}_\mathtt{a}\mathtt{e} = \mathbf{0}$. Since $\mathtt{e}$ lies on