Methods of Stochastic Modelling
(Métodos de Modelação Estocástica)

Modelação e Desempenho de Redes e Serviços

Prof. Amaro de Sousa (asou@ua.pt)

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Random experiment

- In a random experiment, the *sample space*, S, is the set of all possible results of the experiment
- Any subset E of the sample space S is named an event

- Given two events E and F :
  - The *union of the events*, $E \cup F$, is the set of possible results that belong to at least one of them
  - The *intersection of the events*, EF, is the set of possible results that belong to both events simultaneously

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- When $EF = \varnothing$ ($\varnothing$ is the empty set), we say that the events are mutually exclusive
- The *complement of an event E*, E^c, is the set of all possible results (i.e., all results in the sample space S) that do not belong to E

Probabilities defined over events

- For each possible event E of a sample space S, the assignment of a real value $P(E)$ can represent the occurrence probability of event E, if it satisfies the following conditions:

  (1) $0 \leq P(E) \leq 1$

  (2) $P(S) = 1$

  (3) For any set of mutually exclusive events E_1, E_2, E_3, ...

$$P\left(\bigcup_i E_i\right) = \sum_i P(E_i)$$

- Corollaries:

  $P(E) + P(E^c) = 1$

  $P(E \cup F) = P(E) + P(F) - P(EF)$

Conditional probabilities

- Given two event E and F, the *conditional probability of E knowing that F has occured*, represented by $P(E|F)$, is given by

$$P(E|F) = P(EF) / P(F)$$

- Two events E and F are named *independent events* if

$$P(EF) = P(E)P(F)$$

- If E and F are independent events, then:

$$P(E|F) = P(EF) / P(F) = P(E)P(F) / P(F) = P(E)$$

$$P(F|E) = P(FE) / P(E) = P(F)P(E) / P(E) = P(F)$$

meaning that the occurrence probability of one event does not change if we know that the other event has occurred.

Bayes Rule

Consider a set of events F_1, F_2, …, F_n that are mutually exclusive, and their union is the sample space S of a random experiment. Consider any other event E.

The probability of event E is given by:

$$P(E) = \sum_{i=1}^{n} P(EF_i) = \sum_{i=1}^{n} P(E \mid F_i)P(F_i)$$

Moreover, knowing that event E has occurred, the occurrence probability of event F_j (with j = 1, 2, …, n) is given by:

$$P(F_j \mid E) = \frac{P(EF_j)}{P(E)} = \frac{P(E \mid F_j)P(F_j)}{P(E)} = \frac{P(E \mid F_j)P(F_j)}{\sum_{i=1}^{n} P(E \mid F_i)P(F_i)}$$

Example 1 – conditional probabilities

In a multiple-choice test, a student knows the answer with probability p (and, of course, guesses it with probability $1 - p$). When guessing the answer, the student answers correctly with probability $1/m$, where m is the number of multiple-choice answers.

Determine the probability of the student *(i)* to answer correctly each question and *(ii)* to know the answer when he answers correctly the question.

Events:
E – the student answers correctly
F_1 – the student knows the answer
F_2 – the student does not know the answer

(i) $P(E) = P(E|F_1)P(F_1) + P(E|F_2)P(F_2)$

$= 1 \times p + 1/m \times (1 - p) =$

$= p + (1 - p)/m$

(ii) $P(F_1|E) = P(E|F_1)P(F_1) / P(E)$

$= 1 \times p / [p + (1 - p)/m] =$

$= p\, m / [1 + (m - 1)\, p]$

If $p = 50\%$ and $m = 4$, then *(i)* $P(E) = 62.5\%$ and *(ii)* $P(F_1|E) = 80\%$

Example 2 – conditional probabilities

In a wireless link between two hosts, the probability of the transmitted data packets being received with errors is 0.1% in normal link conditions or 10% with external interferences. The probability of external interferences is 2%. In reception, the hosts are able detect if each data packet is or is not received with errors.

Determine: *(i)* the probability of a data packet being received with errors and *(ii)* the probability of the link being with interference when a data packet is received with errors.

Events: E – the packet is received with errors
F_1 – the link is in the normal state
F_2 – the link is with interference

(i) $P(E) = P(E|F_1)P(F_1) + P(E|F_2)P(F_2)$

$= 0.001\times(1 - 0.02) + 0.1\times0.02$

$= 0.00298 = 0.298\%$

(ii) $P(F_2|E) = P(E|F_2)P(F_2) / P(E)$

$= 0.1 \times 0.02 / 0.00298$

$= 0.671 = 67.1\%$

Random variables

- A <u>*random variable X is a function*</u> that assigns a real value to each possible result in *S* of a random experiment.

- The <u>*distribution function*</u> (or *cumulative distribution function*) of a random variable *X* is defined as:

$$F(x) = P(X \leq x) \quad , -\infty < x < +\infty$$

- <u>*Properties*</u> of the distribution function:

(1) $0 \leq F(x) \leq 1$ for all values of x

(2) if $x_1 \leq x_2$ than $F(x_1) \leq F(x_2)$ (non-decreasing function)

(3) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$

(4) $P(a < X \leq b) = F(b) - F(a)$, for $a < b$

Discrete random variables

- X is <u>*a discrete random variable*</u> if it can only take a finite number of values or a countable number of distinct separate values $x_1, x_2, \ldots, x_i, \ldots$

- The <u>*probability function*</u> (or *mass probability function*) of a discrete random variable X is defined as:

$$f(x_i) = P(X = x_i) \qquad \text{for all values of } i = 1,2,3,\ldots$$

- It is mandatory that: $$\sum_{i=1}^{\infty} f(x_i) = 1$$

- Consequently, the <u>*distribution function*</u> of X becomes:

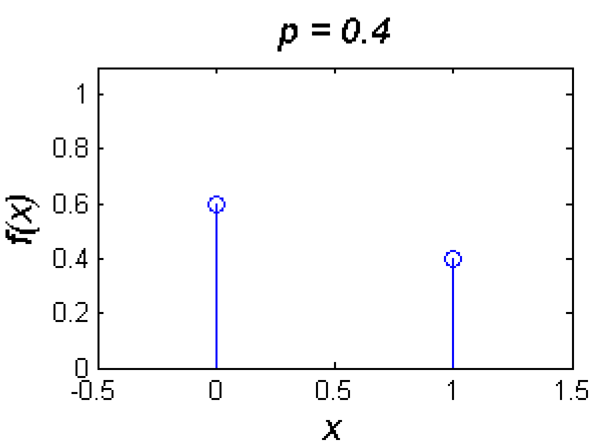
$$F(x) = \sum_{x_i \le x} f(x_i) \qquad , -\infty < x < +\infty$$

Examples of discrete random variables

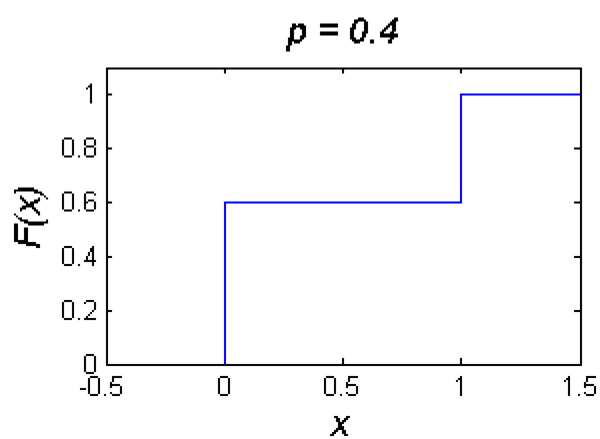
<u>*Bernoulli random variable*</u> represents an experiment that can be a success with probability p or a failure with probability $1 - p$.

If $X = 1$ represents the success and $X = 0$ the failure, the probability function is:

$$f(i) = p^i (1-p)^{1-i}, i = 0,1$$



f(x) – probability function



F(x) – distribution function

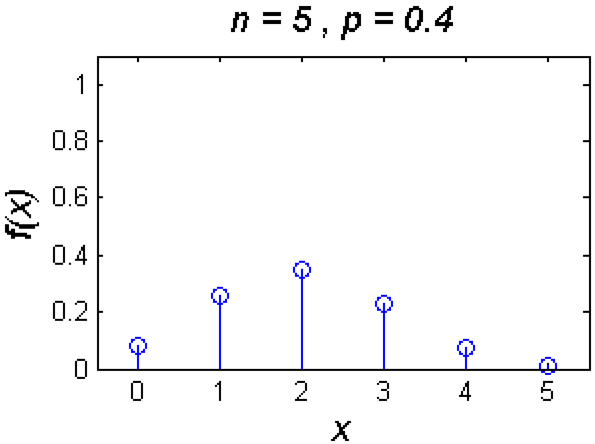
Examples of discrete random variables

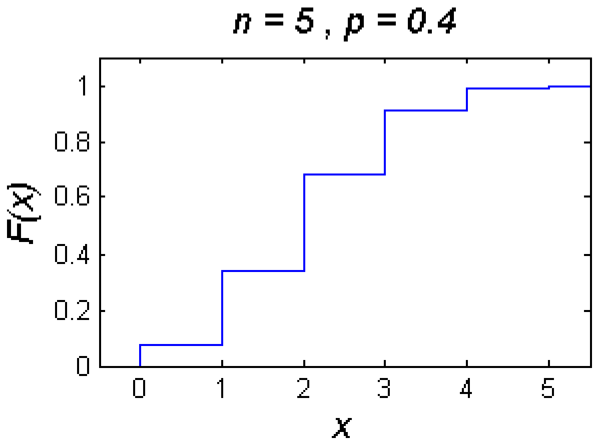
Binomial random variable represents a set of *n* independent Bernoulli experiments, each one that can be a success with probability *p* or a failure with probability 1 – *p*.

If *X* represent the number of successes in *n* experiments, the probability function is:

$$f(i) = \binom{n}{i} p^i (1-p)^{n-i}, i = 0, 1, 2, \ldots, n$$

where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$



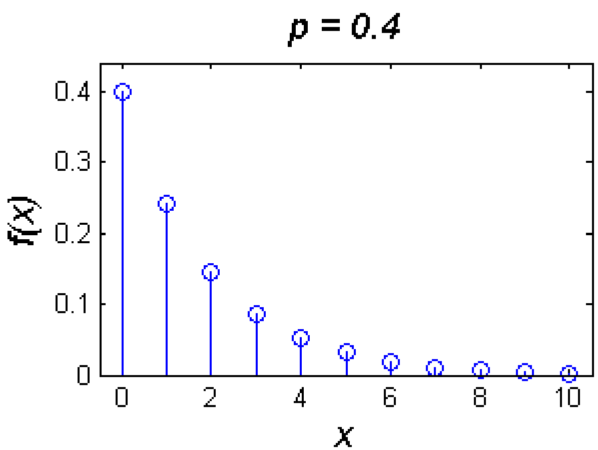


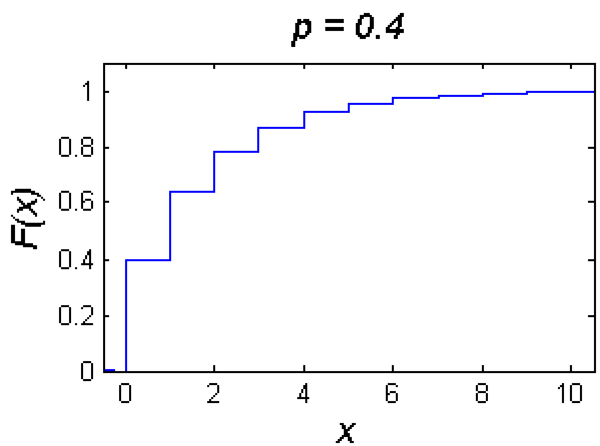
Examples of discrete random variables

Geometric random variable: represents a set independent Bernoulli experiments, all with the same success probability, until an experiment results in a success.

If X represents the number of failures before the success, the probability function is

$$f(i) = (1-p)^i p \quad , i = 0, 1, 2, \ldots$$





If X represents the number of experiments until the success, the probability function is

$$f(i) = (1-p)^{i-1} p \quad , i = 1, 2, \ldots$$

Example 3 – discrete random variables

On a given data link, the BER (*bit error rate*) is 10^{-5} and the errors in the different bits of a data packet are statistically independent.

Determine: *(i)* the probability of a data packet of size 100 Bytes to be received without errors and *(ii)* the probability of a data packet of size 1000 Bytes to be received with at least 2 errors.

The number of bits in error on a data packet is a binomial random variable with the probability of success given by the BER value and the number of Bernoulli experiments given by the number of bits of the packet

$$f(i) = \binom{n}{i} p^i (1-p)^{n-i}, i = 0, 1, 2, \ldots, n$$

(i) $$f(0) = \binom{n}{0} p^0 (1-p)^{n-0} = \binom{100 \times 8}{0} \times \left(1-10^{-5}\right)^{100\times8} = 0.992 = 99.2\%$$

(ii) $$1 - f(0) - f(1) = 1 - \binom{n}{0} p^0 (1-p)^{n-0} - \binom{n}{1} p^1 (1-p)^{n-1}$$

$$= 1 - \left(1-10^{-5}\right)^{8000} - 8000 \times 10^{-5} \left(1-10^{-5}\right)^{7999} = 3.034\text{E-}3 = 0.3\%$$

Continuous random variables

- A *random variable X is said continuous* if it exists a non-negative function $f(x)$ such that for any interval of continuous values B:

$$P(X \in B) = \int_B f(x)dx \qquad \int_{-\infty}^{+\infty} f(x)dx = 1$$

$f(x)$ is the *probability density function* of the random variable X

- Therefore: $$P\{a \le X \le b\} = \int_a^b f(x)dx$$

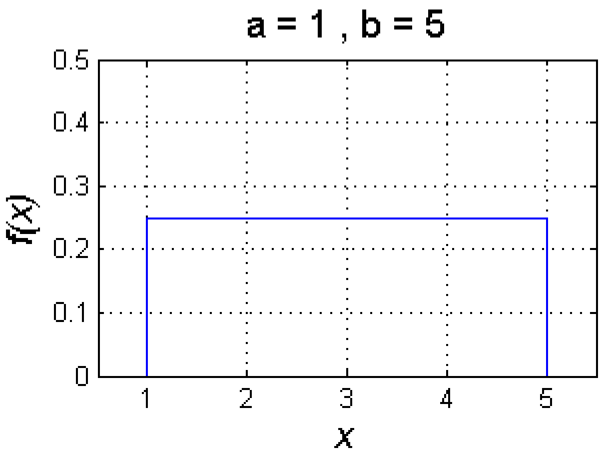
- The *distribution function* of the random variable X becomes:

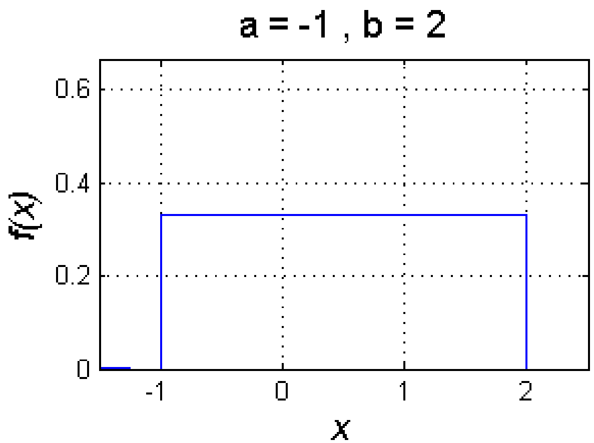
$$F(x) = P(X \in [-\infty, x]) = \int_{-\infty}^{x} f(y)dy$$

Examples of continuous random variables

Random variable with uniform distribution: a random variable is uniformly distributed in the interval [*a*,*b*] if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a} & , a < x < b \\ 0 & , cc \end{cases}$$

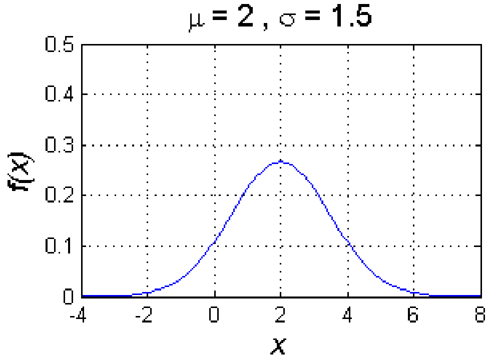




Examples of continuous random variables

<u>Random variable with Normal (ou Gaussian) distribution</u>: a random variable has a normal distribution with average μ and standard deviation σ if its probability density function is:

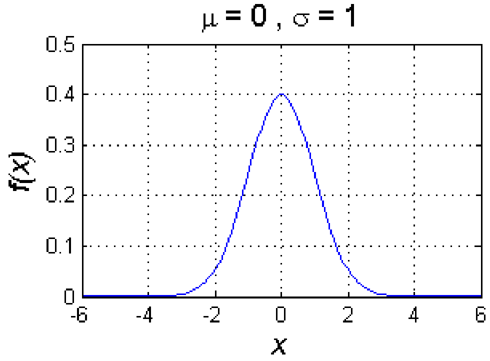
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$



The Standard Normal (or Gaussian) distribution is the normal distribution with average 0 and standard deviation 1.

In this case:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



Average (or expected value) of a random variable

- *Average* (or *expected value*), $E[X]$, of a random variable X:

$$E[X] = \begin{cases} \displaystyle\sum_{j=1}^{\infty} x_j \, f_X(x_j) & \text{if } X \text{ discrete} \\ \displaystyle\int_{-\infty}^{+\infty} x \, f_X(x_j) dx & \text{if } X \text{ continuous} \end{cases}$$

- Important property of the average: $E\left[\sum_{i=1}^{n} c_i \, X_i\right] = \sum_{i=1}^{n} c_i \, E[X_i]$

- Average of a random variable $Y = g(X)$:

$$E[g(X)] = \begin{cases} \displaystyle\sum_{j=1}^{\infty} g(x_j) \, f_X(x_j) & \text{if } X \text{ discrete} \\ \displaystyle\int_{-\infty}^{+\infty} g(x) \, f_X(x_j) dx & \text{if } X \text{ continuous} \end{cases}$$

Variance and standard deviation of a random variable

- *Variance* of a random variable X:

$$Var[X] = E\left[\left(X - E[X]\right)^2\right] = E\left[X^2\right] - E[X]^2$$

2nd moment of X

- Important properties of the variance:

$$Var[X] \geq 0$$

$$Var[cX] = c^2\ Var[X]$$

$$Var\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} Var[X_i] \quad \text{if } X_i \text{ are independent}$$

- *Standard deviation* of a random variable X:

$$\sigma[X] = \sqrt{Var[X]}$$

Example 4

A data link of 10 Mbps supports a flow of data packets whose size is 100 Bytes with probability 10%, 500 Bytes with probability 50% and 1500 Bytes with probability 40%. Consider the random variable X representing the packet transmission time.

Determine: *(i)* the average packet transmission time $E[X]$, *(ii)* the second moment of the packet transmission time $E[X^2]$ and *(iii)* the variance of the packet transmission time $Var[X]$.

(i) $$E[X] = \sum_{j=1}^{\infty} x_j f_X(x_j) = \frac{100 \times 8}{10^7} \times 0.1 + \frac{500 \times 8}{10^7} \times 0.5 + \frac{1500 \times 8}{10^7} \times 0.4$$

$$= 0.688 \times 10^{-3} \text{ sec} = 0.688 \text{ msec}$$

(ii) $$E[X^2] = \sum_{j=1}^{\infty} (x_j)^2 f_X(x_j) = \left(\frac{100 \times 8}{10^7}\right)^2 \times 0.1 + \left(\frac{500 \times 8}{10^7}\right)^2 \times 0.5 + \left(\frac{1500 \times 8}{10^7}\right)^2 \times 0.4$$

$$= 6.5664 \times 10^{-7} \text{ sec}^2$$

Example 4 - continuation

A data link of 10 Mbps supports a flow of data packets whose size is 100 Bytes with probability 10%, 500 Bytes with probability 50% and 1500 Bytes with probability 40%. Consider the random variable *X* representing the packet transmission time.

Determine: *(i)* the average packet transmission time $E[X]$, *(ii)* the second moment of the packet transmission time $E[X^2]$ and *(iii)* the variance of the packet transmission time $Var[X]$.

(iii) **1st alternative:** $Var[X] = E\left[\left(X - E[X]\right)^2\right]$

$$Var[X] = \left(\frac{100\times 8}{10^7} - E[X]\right)^2 \times 0.1 + \left(\frac{500\times 8}{10^7} - E[X]\right)^2 \times 0.5 + \left(\frac{1500\times 8}{10^7} - E[X]\right)^2 \times 0.4$$

$$= 1.833 \times 10^{-7}\ \text{sec}^2$$

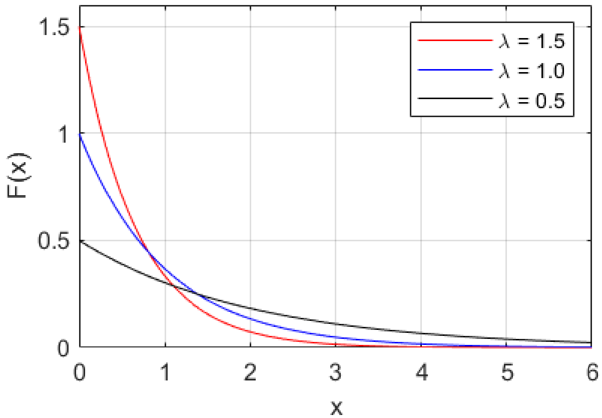
2nd alternative: $Var[X] = E[X^2] - E[X]^2$

$$Var[X] = 6.5664 \times 10^{-7} - (0.688 \times 10^{-3})^2 = 1.833 \times 10^{-7}\ \text{sec}^2$$

Random variable with exponential distribution

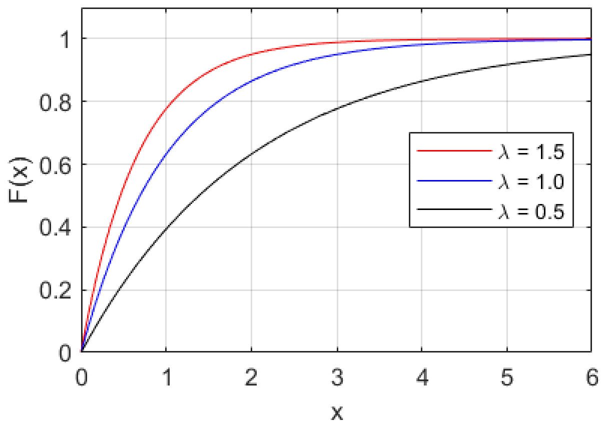
- A continuous random variable *X* following an exponential distribution with parameter λ, $\lambda > 0$, has the probability density function:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



- The distribution function is:

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



Exponential distribution

- The average, variance and standard deviation of an exponentially distributed random variable *X* are only dependent on parameter λ :

$$E[X] = \frac{1}{\lambda} \qquad Var[X] = \left(\frac{1}{\lambda}\right)^2 \qquad \sigma[X] = \frac{1}{\lambda}$$

- The exponential distribution has no memory, meaning that:

$$P\{X > s + t \mid X > t\} = P\{X > s\}$$

- If the random variables X_1 and X_2 are independent and exponentially distributed with averages $1/\lambda_1$ and $1/\lambda_2$ respectively, than:

$$P\{X_1 < X_2\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

Example 5 – exponential distribution

A data link of 10 Mbps supports a flow of data packets whose size is exponentially distributed with an average of 1000 Bytes. Consider the random variable *X* representing the packet transmission time.

Determine: *(i)* the average packet transmission time *E*[*X*], *(ii)* the variance of the packet transmission time *Var*[*X*] and *(iii)* the second moment of the packet transmission time *E*[*X*^{2}].

(i) $$E[X] = \frac{1000 \times 8}{10^7} = 8 \times 10^{-4} = 0.8 \text{ msec}$$

Capacity of the link in pps (packets per second)

$$E[X] = \frac{1}{\mu} \quad \Leftrightarrow \quad \mu = \frac{1}{E[X]} = \frac{1}{8 \times 10^{-4}} = 1250 \text{ pps}$$

(ii) $$Var[X] = \left(\frac{1}{\mu}\right)^2 = (8 \times 10^{-4})^2 = 6.4 \times 10^{-7} \text{ sec}^2$$

(iii) $$Var[X] = E[X^2] - E[X]^2 \quad \Leftrightarrow \quad E[X^2] = Var[X] + E[X]^2$$

$$E[X^2] = 6.4 \times 10^{-7} + (8 \times 10^{-4})^2 = 1.28 \times 10^{-6} \text{ sec}^2$$

Stochastic process

- A *stochastic process* $\{X(t),\ t \in T\}$ is a set of random variables: for each value $t \in T$, $X(t)$ is a random variable.

- Index t is frequently seen as a time instant. In this interpretation, the random variable $X(t)$ represents the *state* of the stochastic process on time instant t.

- Set T is the *set of all possible indices* of the stochastic process:

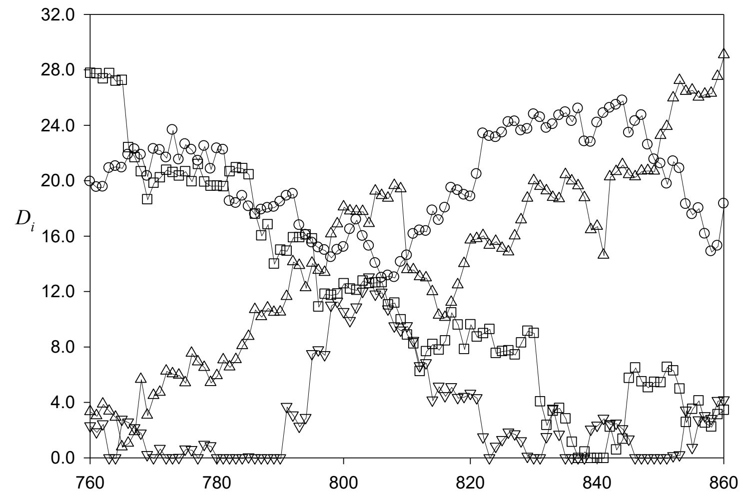
  (1) if T is a countable set, the stochastic process is designated as being in discrete time

  (2) if T is an interval of continuous values, the stochastic process is designated as being in continuous time

- The *state space* of the stochastic process is the set of all possible values that the random variables $X(t)$ can take.

Examples of stochastic processes

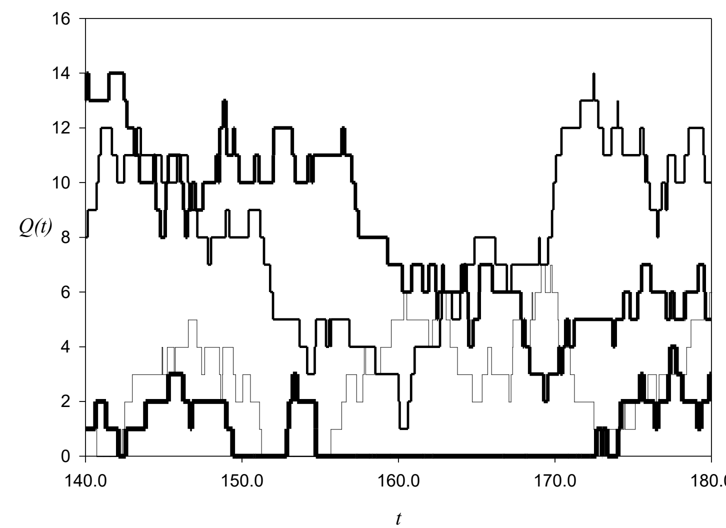
Consider a system with a queue and a server. Clients arrive to the system and are either immediately served (if the server is empty) or go to the queue to wait to be served.



<u>Queuing delay of clients</u>

(1) is a stochastic process in discrete time (1st client, 2nd client, ...)

(2) the state is a continuous value (queuing delay value continuous)



<u>Number of clients in the queue</u>

(1) is a stochastic process in continuous time

(2) the state is a discrete value (0 clients, 1 client, 2 clients, ...)