Como as cargas estão sempre dentro da região limitada V, suas trajetórias devem ser tais que as velocidades tangenciam a fronteira de V, com componentes normais nulas exatamente sobre a fronteira. Podemos imaginar que tais trajetórias podem ser decompostas, aproximadamente, em movimentos periódicos tangentes e normais à fronteira de V. Como estamos supondo partículas não relativísticas, suas velocidades são muito menores do que c e, portanto, as frequências dos movimentos que compõem suas trajetórias devem ser tais que satisfazem

$$\omega r' \quad \ll \quad c.$$

Em termos de sua transformada de Fourier temporal, a densidade de carga em t' pode ser escrita como

$$\rho\left(\mathbf{r}', t - \frac{r}{c} + \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c}\right) \quad = \quad \int_{-\infty}^{+\infty} d\omega\, \tilde{\rho}\left(\mathbf{r}', \omega\right) \exp\left[-i\omega\left(t - \frac{r}{c}\right)\right] \exp\left(-i\omega \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c}\right).$$

Portanto, para velocidades não relativísticas,

$$\exp\left(-i\omega \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c}\right) \quad \approx \quad 1 - i\omega \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c},$$

até ordem

$$\frac{v}{c},$$

onde v é a velocidade máxima na distribuição de cargas em movimento. Assim,

$$\begin{aligned}
\rho\left(\mathbf{r}', t - \frac{r}{c} + \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c}\right) \quad &\approx \quad \int_{-\infty}^{+\infty} d\omega\, \tilde{\rho}\left(\mathbf{r}', \omega\right) \exp\left[-i\omega\left(t - \frac{r}{c}\right)\right]\left(1 - i\omega \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c}\right) \\
&= \quad \rho\left(\mathbf{r}', t - \frac{r}{c}\right) - i \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c} \int_{-\infty}^{+\infty} d\omega\, \tilde{\rho}\left(\mathbf{r}', \omega\right) \omega \exp\left[-i\omega\left(t - \frac{r}{c}\right)\right] \\
&= \quad \rho\left(\mathbf{r}', t - \frac{r}{c}\right) - i \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c} \int_{-\infty}^{+\infty} d\omega\, \tilde{\rho}\left(\mathbf{r}', \omega\right) i \frac{\partial}{\partial t} \exp\left[-i\omega\left(t - \frac{r}{c}\right)\right] \\
&= \quad \rho\left(\mathbf{r}', t - \frac{r}{c}\right) + \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c} \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} d\omega\, \tilde{\rho}\left(\mathbf{r}', \omega\right) \exp\left[-i\omega\left(t - \frac{r}{c}\right)\right] \\
&= \quad \rho\left(\mathbf{r}', t - \frac{r}{c}\right) + \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c} \frac{\partial}{\partial t} \rho\left(\mathbf{r}', t - \frac{r}{c}\right).
\end{aligned}$$

Agora, a Eq. (2) fica

$$\rho\left(\mathbf{r}', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right) \quad = \quad \sum_{n=0}^{\infty} \frac{1}{n!} \left(\Delta t'\right)^n \frac{\partial^n}{\partial t^n}\left[\rho\left(\mathbf{r}', t - \frac{r}{c}\right) + \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c} \frac{\partial}{\partial t} \rho\left(\mathbf{r}', t - \frac{r}{c}\right)\right],$$

que, juntamente com a Eq. (1), fornece o integrando para o cálculo do potencial escalar:

$$\begin{aligned}
\frac{\rho\left(\mathbf{r}', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)}{|\mathbf{r} - \mathbf{r}'|} \quad &\approx \quad \frac{1}{r} \sum_{m=0}^{\infty} \alpha_m \left(\frac{r'}{r}\right)^m \sum_{n=0}^{\infty} \frac{1}{n!} \left(\Delta t'\right)^n \frac{\partial^n}{\partial t^n} \rho\left(\mathbf{r}', t - \frac{r}{c}\right) \\
&+ \quad \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{rc} \sum_{m=0}^{\infty} \alpha_m \left(\frac{r'}{r}\right)^m \sum_{n=0}^{\infty} \frac{1}{n!} \left(\Delta t'\right)^n \frac{\partial^{n+1}}{\partial t^{n+1}} \rho\left(\mathbf{r}', t - \frac{r}{c}\right),
\end{aligned}$$

que é válida para distribuições cujas cargas têm velocidades pequenas quando comparadas com a velocidade da luz. Para simplificar, vamos manter apenas termos até a primeira ordem em

$$\frac{r'}{r}.$$

Como $\Delta t'$ é proporcional a

$$\frac{r'^2}{r},$$

conforme calculamos acima, quando multiplicado por

$$\frac{1}{r}$$

resulta em um termo de segunda ordem em

$$\frac{r'}{r},$$

que desprezamos. Assim, as somas em n somente contribuem com o termo em que $n = 0$ e temos

$$\begin{aligned}
\frac{\rho\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|} &\approx \frac{1}{r}\sum_{m=0}^{\infty}\alpha_m\left(\frac{r'}{r}\right)^m \rho\left(\mathbf{r}', t - \frac{r}{c}\right) \\
&+ \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{rc}\sum_{m=0}^{\infty}\alpha_m\left(\frac{r'}{r}\right)^m \frac{\partial}{\partial t}\rho\left(\mathbf{r}', t - \frac{r}{c}\right).
\end{aligned}$$

Na primeira soma em m, tanto o termo com $m = 0$ como o termo com $m = 1$ contribuem, mas a segunda soma somente contribui com o termo com $m = 0$ para a expansão até a primeira ordem em

$$\frac{r'}{r},$$

do quociente acima. Assim,

$$\begin{aligned}
\frac{\rho\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|} &\approx \frac{1}{r}\rho\left(\mathbf{r}', t - \frac{r}{c}\right) + \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{r^2}\rho\left(\mathbf{r}', t - \frac{r}{c}\right) \\
&+ \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{rc}\frac{\partial}{\partial t}\rho\left(\mathbf{r}', t - \frac{r}{c}\right).
\end{aligned}$$

O potencial escalar pode ser escrito, finalmente, como

$$\begin{aligned}
\phi(\mathbf{r}, t) &= \int_V d^3r' \frac{\rho\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|} \\
&\approx \frac{1}{r}\int_V d^3r' \rho\left(\mathbf{r}', t - \frac{r}{c}\right) + \frac{1}{r^2}\hat{\mathbf{r}}\cdot\left[\int_V d^3r' \mathbf{r}' \rho\left(\mathbf{r}', t - \frac{r}{c}\right)\right] \\
&+ \frac{1}{rc}\hat{\mathbf{r}}\cdot\left[\frac{\partial}{\partial t}\int_V d^3r' \mathbf{r}' \rho\left(\mathbf{r}', t - \frac{r}{c}\right)\right].
\end{aligned}$$

Como as cargas estão confinadas na região V e há conservação de carga, segue que

$$\int_V d^3r'\rho\left(\mathbf{r}', t-\frac{r}{c}\right) \quad = \quad Q,$$

onde Q é a carga líquida total em V. Também reconhecemos as outras integrais como sendo o dipolo elétrico da distribuição

$$\mathbf{p}\left(t-\frac{r}{c}\right) \quad = \quad \int_V d^3r'\mathbf{r}'\rho\left(\mathbf{r}', t-\frac{r}{c}\right).$$

Logo,

$$\phi\left(\mathbf{r}, t\right) \quad \approx \quad \frac{Q}{r}+\frac{\hat{\mathbf{r}}\cdot\mathbf{p}\left(t-\frac{r}{c}\right)}{r^2}+\frac{\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)}{rc},$$

quando

$$r \quad \gg \quad V^{1/3},$$

onde

$$\dot{\mathbf{p}}\left(t-\frac{r}{c}\right) \quad = \quad \frac{\partial}{\partial t}\mathbf{p}\left(t-\frac{r}{c}\right).$$

Agora devemos calcular uma expansão para o potencial vetorial. Analogamente ao procedimento que seguimos anteriormente, temos

$$\begin{aligned}\frac{\mathbf{J}\left(\mathbf{r}', t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|} \quad &\approx \quad \frac{1}{r}\sum_{m=0}^{\infty}\alpha_m\left(\frac{r'}{r}\right)^m\sum_{n=0}^{\infty}\frac{1}{n!}\left(\Delta t'\right)^n\frac{\partial^n}{\partial t^n}\mathbf{J}\left(\mathbf{r}', t-\frac{r}{c}\right) \\ &+ \quad \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{rc}\sum_{m=0}^{\infty}\alpha_m\left(\frac{r'}{r}\right)^m\sum_{n=0}^{\infty}\frac{1}{n!}\left(\Delta t'\right)^n\frac{\partial^{n+1}}{\partial t^{n+1}}\mathbf{J}\left(\mathbf{r}', t-\frac{r}{c}\right),\end{aligned}$$

que, desprezando ordens superiores à primeira de

$$\frac{r'}{r},$$

resulta em

$$\begin{aligned}\frac{\mathbf{J}\left(\mathbf{r}', t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|} \quad &\approx \quad \frac{1}{r}\mathbf{J}\left(\mathbf{r}', t-\frac{r}{c}\right)+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{r^2}\mathbf{J}\left(\mathbf{r}', t-\frac{r}{c}\right) \\ &+ \quad \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{rc}\frac{\partial}{\partial t}\mathbf{J}\left(\mathbf{r}', t-\frac{r}{c}\right).\end{aligned}$$

Então,

$$\begin{aligned}\mathbf{A}\left(\mathbf{r}, t\right) \quad &\approx \quad \frac{1}{c}\frac{1}{r}\int_V d^3r'\mathbf{J}\left(\mathbf{r}', t-\frac{r}{c}\right)+\frac{1}{c}\frac{1}{r^2}\int_V d^3r'\hat{\mathbf{r}}\cdot\mathbf{r}'\mathbf{J}\left(\mathbf{r}', t-\frac{r}{c}\right) \\ &+ \quad \frac{1}{c}\frac{1}{rc}\frac{\partial}{\partial t}\int_V d^3r'\hat{\mathbf{r}}\cdot\mathbf{r}'\mathbf{J}\left(\mathbf{r}', t-\frac{r}{c}\right).\end{aligned}$$

Consideremos a identidade vetorial

$$\hat{\mathbf{r}} \times \left[\mathbf{r}' \times \mathbf{J}\left(\mathbf{r}', t - \frac{r}{c}\right)\right] = \mathbf{r}'\hat{\mathbf{r}} \cdot \mathbf{J}\left(\mathbf{r}', t - \frac{r}{c}\right) - \mathbf{J}\left(\mathbf{r}', t - \frac{r}{c}\right)\hat{\mathbf{r}} \cdot \mathbf{r}'.$$

Assim,

$$\begin{aligned} \int_V d^3r'\mathbf{J}\left(\mathbf{r}', t - \frac{r}{c}\right)\hat{\mathbf{r}} \cdot \mathbf{r}' &= \int_V d^3r'\mathbf{r}'\hat{\mathbf{r}} \cdot \mathbf{J}\left(\mathbf{r}', t - \frac{r}{c}\right) \\ &- \hat{\mathbf{r}} \times \left[\int_V d^3r'\mathbf{r}' \times \mathbf{J}\left(\mathbf{r}', t - \frac{r}{c}\right)\right]. \end{aligned} \tag{3}$$

Calculemos:

$$\begin{aligned} \int_V d^3r'\mathbf{r}'\hat{\mathbf{r}} \cdot \mathbf{J}\left(\mathbf{r}', t - \frac{r}{c}\right) &= \frac{1}{r}\int_V d^3r'\mathbf{r}'\mathbf{r} \cdot \mathbf{J}\left(\mathbf{r}', t - \frac{r}{c}\right) \\ &= \frac{1}{r}\sum_{m=1}^{3}\int_V d^3r'\mathbf{r}'x_m J_m\left(\mathbf{r}', t - \frac{r}{c}\right) \\ &= \sum_{m=1}^{3}\frac{x_m}{r}\int_V d^3r'\mathbf{r}'J_m\left(\mathbf{r}', t - \frac{r}{c}\right) \\ &= \sum_{m=1}^{3}\frac{x_m}{r}\sum_{n=1}^{3}\hat{\mathbf{x}}_n\int_V d^3r'x'_n J_m\left(\mathbf{r}', t - \frac{r}{c}\right). \end{aligned} \tag{4}$$

Notemos que

$$\mathbf{r}' = \sum_{n=1}^{3}\hat{\mathbf{x}}_n x'_n$$

e

$$\boldsymbol{\nabla}' \equiv \sum_{k=1}^{3}\hat{\mathbf{x}}_k\frac{\partial}{\partial x'_k}.$$

Então, segue que

$$\begin{aligned} \boldsymbol{\nabla}'x'_m &= \sum_{k=1}^{3}\hat{\mathbf{x}}_k\frac{\partial x'_m}{\partial x'_k} \\ &= \sum_{k=1}^{3}\hat{\mathbf{x}}_k\delta_{mk} \\ &= \hat{\mathbf{x}}_m, \end{aligned}$$

ou seja,

$$\hat{\mathbf{x}}_m = \boldsymbol{\nabla}'x'_m.$$

Agora, consideremos

$$
\begin{aligned}
\int_V d^3r' x'_n J_m\left(\mathbf{r}', t-\frac{r}{c}\right) &= \int_V d^3r' x'_n \hat{\mathbf{x}}_m \cdot \mathbf{J}\left(\mathbf{r}', t-\frac{r}{c}\right) \\
&= \int_V d^3r' x'_n \left(\boldsymbol{\nabla}' x'_m\right) \cdot \mathbf{J}\left(\mathbf{r}', t-\frac{r}{c}\right) \\
&= \int_V d^3r' \boldsymbol{\nabla}' \cdot \left[x'_n x'_m \mathbf{J}\left(\mathbf{r}', t-\frac{r}{c}\right)\right] \\
&- \int_V d^3r' x'_m \boldsymbol{\nabla}' \cdot \left[x'_n \mathbf{J}\left(\mathbf{r}', t-\frac{r}{c}\right)\right] \\
&= \oint_{S(V)} da'\, x'_m x'_n \hat{\mathbf{n}}' \cdot \mathbf{J}\left(\mathbf{r}', t-\frac{r}{c}\right) \\
&- \int_V d^3r' x'_m \boldsymbol{\nabla}' \cdot \left[x'_n \mathbf{J}\left(\mathbf{r}', t-\frac{r}{c}\right)\right] \\
&= -\int_V d^3r' x'_m \boldsymbol{\nabla}' \cdot \left[x'_n \mathbf{J}\left(\mathbf{r}', t-\frac{r}{c}\right)\right] \\
&= -\int_V d^3r' x'_m \left(\boldsymbol{\nabla}' x'_n\right) \cdot \mathbf{J}\left(\mathbf{r}', t-\frac{r}{c}\right) \\
&- \int_V d^3r' x'_m x'_n \boldsymbol{\nabla}' \cdot \mathbf{J}\left(\mathbf{r}', t-\frac{r}{c}\right) \\
&= -\int_V d^3r' x'_m \hat{\mathbf{x}}_n \cdot \mathbf{J}\left(\mathbf{r}', t-\frac{r}{c}\right) \\
&+ \int_V d^3r' x'_m x'_n \frac{\partial \rho\left(\mathbf{r}', t-\frac{r}{c}\right)}{\partial t} \\
&= -\int_V d^3r' x'_m J_n\left(\mathbf{r}', t-\frac{r}{c}\right) \\
&+ \frac{\partial}{\partial t}\int_V d^3r' x'_m x'_n \rho\left(\mathbf{r}', t-\frac{r}{c}\right).
\end{aligned}
$$

Portanto, substituindo esse resultado na Eq. (4), obtemos

$$
\int_V d^3r' \mathbf{r}' \hat{\mathbf{r}} \cdot \mathbf{J}\left(\mathbf{r}', t-\frac{r}{c}\right) = \sum_{m=1}^{3} \frac{x_m}{r} \sum_{n=1}^{3} \hat{\mathbf{x}}_n \int_V d^3r' x'_n J_m\left(\mathbf{r}', t-\frac{r}{c}\right)
$$

$$
\begin{aligned}
&= -\sum_{m=1}^{3} \frac{x_m}{r} \sum_{n=1}^{3} \hat{\mathbf{x}}_n \int_V d^3r' x'_m J_n \left(\mathbf{r}', t - \frac{r}{c}\right) \\
&+ \sum_{m=1}^{3} \frac{x_m}{r} \sum_{n=1}^{3} \hat{\mathbf{x}}_n \frac{\partial}{\partial t} \int_V d^3r' x'_m x'_n \rho \left(\mathbf{r}', t - \frac{r}{c}\right) \\
&= -\sum_{m=1}^{3} \frac{x_m}{r} \int_V d^3r' x'_m \mathbf{J} \left(\mathbf{r}', t - \frac{r}{c}\right) \\
&+ \sum_{m=1}^{3} \frac{x_m}{r} \frac{\partial}{\partial t} \int_V d^3r' x'_m \mathbf{r}' \rho \left(\mathbf{r}', t - \frac{r}{c}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{r} \int_V d^3r' \mathbf{r} \cdot \mathbf{r}' \mathbf{J} \left(\mathbf{r}', t - \frac{r}{c}\right) \\
&+ \frac{1}{r} \frac{\partial}{\partial t} \int_V d^3r' \mathbf{r} \cdot \mathbf{r}' \mathbf{r}' \rho \left(\mathbf{r}', t - \frac{r}{c}\right) \\
&= -\int_V d^3r' \hat{\mathbf{r}} \cdot \mathbf{r}' \mathbf{J} \left(\mathbf{r}', t - \frac{r}{c}\right) \\
&+ \frac{\partial}{\partial t} \int_V d^3r' \hat{\mathbf{r}} \cdot \mathbf{r}' \mathbf{r}' \rho \left(\mathbf{r}', t - \frac{r}{c}\right).
\end{aligned}
$$

Substituindo esse resultado na Eq. (3), concluímos que

$$
\begin{aligned}
\int_V d^3r' \mathbf{J} \left(\mathbf{r}', t - \frac{r}{c}\right) \hat{\mathbf{r}} \cdot \mathbf{r}' &= -\int_V d^3r' \hat{\mathbf{r}} \cdot \mathbf{r}' \mathbf{J} \left(\mathbf{r}', t - \frac{r}{c}\right) \\
&+ \frac{\partial}{\partial t} \int_V d^3r' \hat{\mathbf{r}} \cdot \mathbf{r}' \mathbf{r}' \rho \left(\mathbf{r}', t - \frac{r}{c}\right) \\
&- \hat{\mathbf{r}} \times \left[\int_V d^3r' \mathbf{r}' \times \mathbf{J} \left(\mathbf{r}', t - \frac{r}{c}\right)\right],
\end{aligned}
$$

ou seja,

$$
\begin{aligned}
\int_V d^3r' \mathbf{J} \left(\mathbf{r}', t - \frac{r}{c}\right) \hat{\mathbf{r}} \cdot \mathbf{r}' &= \left[\frac{1}{2} \int_V d^3r' \mathbf{r}' \times \mathbf{J} \left(\mathbf{r}', t - \frac{r}{c}\right)\right] \times \hat{\mathbf{r}} \\
&+ \frac{1}{2} \frac{\partial}{\partial t} \int_V d^3r' \hat{\mathbf{r}} \cdot \mathbf{r}' \mathbf{r}' \rho \left(\mathbf{r}', t - \frac{r}{c}\right).
\end{aligned}
$$

Definimos o momento de dipolo magnético da distribuição $\mathbf{J}$ como

$$
\mathbf{m}(t) = \frac{1}{2} \int_V d^3r' \mathbf{r}' \times \mathbf{J}(\mathbf{r}', t).
$$

Definimos, também, o objeto

$$
\overleftrightarrow{\boldsymbol{\Upsilon}}(t) = \frac{1}{2} \int_V d^3r' \mathbf{r}' \mathbf{r}' \rho(\mathbf{r}', t),
$$

que é o que se chama díade.