Com isso,

$$\begin{aligned}
\frac{dW}{dt_{\text{ret}}} &= \int_{4\pi} d\Omega \left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right) c\frac{1}{4\pi}\mathbf{E}_{\text{rad}}\cdot\mathbf{E}_{\text{rad}}R^2 \\
&= \int_{4\pi} d\Omega \left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right) c\frac{1}{4\pi}\left|q\frac{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]}{Rc^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}\right|^2 R^2,
\end{aligned}$$

isto é,

$$\frac{dW}{dt_{\text{ret}}} = \frac{q^2}{4\pi c^3}\int_{4\pi} d\Omega\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]\right|^2}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^5},$$

como anteriormente.

O numerador do integrando pode ser escrito como

$$\begin{aligned}
\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]\right|^2 &= \left|\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\left(\hat{\mathbf{R}}\cdot\mathbf{a}\right)-\mathbf{a}\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)\right|^2 \\
&= \left|\hat{\mathbf{R}}-\boldsymbol{\beta}\right|^2\left(\hat{\mathbf{R}}\cdot\mathbf{a}\right)^2 - 2\mathbf{a}\cdot\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\left(\hat{\mathbf{R}}\cdot\mathbf{a}\right)\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right) \\
&+ \left(\mathbf{a}\cdot\mathbf{a}\right)\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2,
\end{aligned}$$

isto é,

$$\begin{aligned}
\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]\right|^2 &= \left(1-2\hat{\mathbf{R}}\cdot\boldsymbol{\beta}+\beta^2\right)\left(\hat{\mathbf{R}}\cdot\mathbf{a}\right)^2 \\
&- \left(\hat{\mathbf{R}}\cdot\mathbf{a}\right)^2\left(2-2\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)+2\left(\mathbf{a}\cdot\boldsymbol{\beta}\right)\left(\mathbf{a}\cdot\hat{\mathbf{R}}\right)\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right) \\
&+ a^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2,
\end{aligned}$$

ou seja,

$$\begin{aligned}
\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]\right|^2 &= -\left(1-\beta^2\right)\left(\hat{\mathbf{R}}\cdot\mathbf{a}\right)^2 \\
&+ 2\left(\mathbf{a}\cdot\boldsymbol{\beta}\right)\left(\mathbf{a}\cdot\hat{\mathbf{R}}\right)\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right) \\
&+ a^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2.
\end{aligned}$$

Assim,

$$\begin{aligned}
\frac{dW}{dt_{\text{ret}}} &= \frac{q^2}{4\pi c^3}\int_{4\pi} d\Omega\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]\right|^2}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^5} \\
&= -\frac{q^2\left(1-\beta^2\right)}{4\pi c^3}\int_{4\pi} d\Omega\frac{\left(\mathbf{a}\cdot\hat{\mathbf{R}}\right)^2}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^5}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{2q^2\left(\mathbf{a}\cdot\boldsymbol{\beta}\right)}{4\pi c^3}\int_{4\pi} d\Omega \frac{\left(\mathbf{a}\cdot\hat{\mathbf{R}}\right)}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^4} \\
&+ \quad \frac{q^2 a^2}{4\pi c^3}\int_{4\pi} d\Omega \frac{1}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}.
\end{aligned}
$$

As integrais acima são sobre os ângulos que definem a direção e o sentido do versor $\hat{\mathbf{R}}$, enquanto $\boldsymbol{\beta}$ é um vetor fixo na integração. Podemos, portanto, escolher o eixo z ao longo de $\boldsymbol{\beta}$ e integrar sobre os ângulos θ e φ. Logo,

$$
\begin{aligned}
\int_{4\pi} d\Omega \frac{1}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3} &= \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \frac{\operatorname{sen}\theta}{\left(1-\beta\cos\theta\right)^3} \\
&= 2\pi \int_{-1}^{1} du \frac{1}{\left(1-\beta u\right)^3} \\
&= \frac{\pi}{\beta}\left[\frac{1}{\left(1-\beta\right)^2}-\frac{1}{\left(1+\beta\right)^2}\right] \\
&= \frac{4\pi}{\left(1-\beta^2\right)^2}.
\end{aligned}
$$

Também podemos escrever

$$
\begin{aligned}
\int_{4\pi} d\Omega \frac{\mathbf{a}\cdot\hat{\mathbf{R}}}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^4} &= \sum_{k=1}^{3} \mathbf{a}\cdot\hat{\mathbf{x}}_k \int_{4\pi} d\Omega \frac{\hat{\mathbf{x}}_k\cdot\hat{\mathbf{R}}}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^4} \\
&= \sum_{k=1}^{3} \frac{\mathbf{a}\cdot\hat{\mathbf{x}}_k}{3}\frac{\partial}{\partial\beta_k}\int_{4\pi} d\Omega \frac{1}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}.
\end{aligned}
$$

Para entendermos esse resultado, calculemos:

$$
\begin{aligned}
\frac{\partial}{\partial\beta_k}\left(\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right) &= \frac{\partial}{\partial\beta_k}\left[\sum_{l=1}^{3}\left(\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_l\right)\left(\hat{\mathbf{x}}_l\cdot\boldsymbol{\beta}\right)\right] \\
&= \frac{\partial}{\partial\beta_k}\left[\sum_{l=1}^{3}\left(\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_l\right)\beta_l\right] \\
&= \sum_{l=1}^{3}\left(\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_l\right)\frac{\partial}{\partial\beta_k}\beta_l \\
&= \sum_{l=1}^{3}\left(\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_l\right)\delta_{lk} \\
&= \hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_k.
\end{aligned}
$$

Com isso, vemos que

$$
\begin{aligned}
\int_{4\pi} d\Omega \frac{\hat{\mathbf{x}}_k \cdot \hat{\mathbf{R}}}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^4} &= \int_{4\pi} d\Omega \frac{\frac{\partial}{\partial \beta_k}\left(\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^4} \\
&= -\int_{4\pi} d\Omega \frac{\frac{\partial}{\partial \beta_k}\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^4} \\
&= \frac{1}{3}\frac{\partial}{\partial \beta_k}\int_{4\pi} d\Omega \frac{1}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}.
\end{aligned}
$$

Então, usando o resultado da integral calculada logo acima, resulta em

$$
\begin{aligned}
\int_{4\pi} d\Omega \frac{\mathbf{a}\cdot\hat{\mathbf{R}}}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^4} &= \sum_{k=1}^{3}\frac{4\pi}{3}\mathbf{a}\cdot\hat{\mathbf{x}}_k \frac{\partial}{\partial \beta_k}\left[\frac{1}{\left(1-\beta^2\right)^2}\right] \\
&= \sum_{k=1}^{3}\frac{4\pi}{3}\mathbf{a}\cdot\hat{\mathbf{x}}_k \left[\frac{2}{\left(1-\beta^2\right)^3}\right]\frac{\partial \beta^2}{\partial \beta_k} \\
&= \sum_{k=1}^{3}\frac{4\pi}{3}\mathbf{a}\cdot\hat{\mathbf{x}}_k \left[\frac{2}{\left(1-\beta^2\right)^3}\right] 2\beta_k,
\end{aligned}
$$

isto é,

$$
\int_{4\pi} d\Omega \frac{\mathbf{a}\cdot\hat{\mathbf{R}}}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^4} = \frac{16\pi\mathbf{a}\cdot\boldsymbol{\beta}}{3\left(1-\beta^2\right)^3}.
$$

Finalmente, a integral restante pode ser calculada de forma análoga e obtemos

$$
\begin{aligned}
\int_{4\pi} d\Omega \frac{\left(\mathbf{a}\cdot\hat{\mathbf{R}}\right)^2}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^5} &= \sum_{k=1}^{3}\sum_{l=1}^{3}\mathbf{a}\cdot\hat{\mathbf{x}}_k\mathbf{a}\cdot\hat{\mathbf{x}}_l \int_{4\pi} d\Omega \frac{\left(\hat{\mathbf{x}}_k\cdot\hat{\mathbf{R}}\right)\left(\hat{\mathbf{x}}_l\cdot\hat{\mathbf{R}}\right)}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^5} \\
&= \sum_{k=1}^{3}\sum_{l=1}^{3}\frac{\mathbf{a}\cdot\hat{\mathbf{x}}_k\mathbf{a}\cdot\hat{\mathbf{x}}_l}{12}\frac{\partial}{\partial \beta_k}\frac{\partial}{\partial \beta_l}\int_{4\pi} d\Omega \frac{1}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3} \\
&= \sum_{k=1}^{3}\sum_{l=1}^{3}\frac{\pi\mathbf{a}\cdot\hat{\mathbf{x}}_k\mathbf{a}\cdot\hat{\mathbf{x}}_l}{3}\frac{\partial}{\partial \beta_k}\frac{\partial}{\partial \beta_l}\left[\frac{1}{\left(1-\beta^2\right)^2}\right].
\end{aligned}
$$

Mas,

$$
\frac{\partial}{\partial \beta_k}\frac{\partial}{\partial \beta_l}\left[\frac{1}{\left(1-\beta^2\right)^2}\right] = \frac{\partial}{\partial \beta_k}\left\{2\beta_l\frac{\partial}{\partial \beta^2}\left[\frac{1}{\left(1-\beta^2\right)^2}\right]\right\}
$$

$$
\begin{aligned}
&= \frac{\partial}{\partial \beta_k}\left[\frac{4\beta_l}{\left(1-\beta^2\right)^3}\right] \\
&= \frac{4\delta_{kl}}{\left(1-\beta^2\right)^3} + 4\beta_l \frac{\partial}{\partial \beta_k}\left[\frac{1}{\left(1-\beta^2\right)^3}\right]
\end{aligned}
$$

e, portanto,

$$
\begin{aligned}
\frac{\partial}{\partial \beta_k}\frac{\partial}{\partial \beta_l}\left[\frac{1}{\left(1-\beta^2\right)^2}\right] &= \frac{4\delta_{kl}}{\left(1-\beta^2\right)^3} + 8\beta_k\beta_l \frac{\partial}{\partial \beta^2}\left[\frac{1}{\left(1-\beta^2\right)^3}\right] \\
&= \frac{4\delta_{kl}}{\left(1-\beta^2\right)^3} + \frac{24\beta_k\beta_l}{\left(1-\beta^2\right)^4}.
\end{aligned}
$$

Com isso,

$$
\begin{aligned}
\int_{4\pi} d\Omega \frac{\left(\mathbf{a}\cdot\hat{\mathbf{R}}\right)^2}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^5} &= \sum_{k=1}^{3}\sum_{l=1}^{3}\frac{\pi \mathbf{a}\cdot\hat{\mathbf{x}}_k \mathbf{a}\cdot\hat{\mathbf{x}}_l}{3}\frac{4\delta_{kl}}{\left(1-\beta^2\right)^3} \\
&+ \sum_{k=1}^{3}\sum_{l=1}^{3}\frac{\pi \mathbf{a}\cdot\hat{\mathbf{x}}_k \mathbf{a}\cdot\hat{\mathbf{x}}_l}{3}\frac{24\beta_k\beta_l}{\left(1-\beta^2\right)^4} \\
&= \frac{4\pi a^2}{3\left(1-\beta^2\right)^3} + \frac{8\pi\left(\mathbf{a}\cdot\boldsymbol{\beta}\right)^2}{\left(1-\beta^2\right)^4}.
\end{aligned}
$$

Utilizando esses resultados, concluímos que a potência irradiada pela partícula carregada é dada por

$$
\begin{aligned}
\frac{dW}{dt_{\mathrm{ret}}} &= -\frac{q^2\left(1-\beta^2\right)}{4\pi c^3}\frac{4\pi a^2}{3\left(1-\beta^2\right)^3} - \frac{q^2\left(1-\beta^2\right)}{4\pi c^3}\frac{8\pi\left(\mathbf{a}\cdot\boldsymbol{\beta}\right)^2}{\left(1-\beta^2\right)^4} \\
&+ \frac{2q^2\mathbf{a}\cdot\boldsymbol{\beta}}{4\pi c^3}\frac{16\pi\mathbf{a}\cdot\boldsymbol{\beta}}{3\left(1-\beta^2\right)^3} \\
&+ \frac{q^2a^2}{4\pi c^3}\frac{4\pi}{\left(1-\beta^2\right)^2},
\end{aligned}
$$

isto é,

$$
\begin{aligned}
\frac{dW}{dt_{\mathrm{ret}}} &= \frac{2q^2}{3c^3}\left[\frac{\left(\mathbf{a}\cdot\boldsymbol{\beta}\right)^2}{\left(1-\beta^2\right)^3} + \frac{a^2}{\left(1-\beta^2\right)^2}\right] \\
&= \frac{2q^2}{3c^3}\left[\frac{\left(\mathbf{a}\cdot\boldsymbol{\beta}\right)^2 + a^2 - a^2\beta^2}{\left(1-\beta^2\right)^3}\right].
\end{aligned}
$$

Podemos simplificar ainda mais essa expressão observando que

$$
\begin{aligned}
\left(\mathbf{a}\times\boldsymbol{\beta}\right)^2 &= \left(\mathbf{a}\times\boldsymbol{\beta}\right)\cdot\left(\mathbf{a}\times\boldsymbol{\beta}\right) \\
&= \mathbf{a}\cdot\left[\boldsymbol{\beta}\times\left(\mathbf{a}\times\boldsymbol{\beta}\right)\right] \\
&= \mathbf{a}\cdot\left[\mathbf{a}\beta^2 - \boldsymbol{\beta}\mathbf{a}\cdot\boldsymbol{\beta}\right] \\
&= a^2\beta^2 - \left(\mathbf{a}\cdot\boldsymbol{\beta}\right)^2
\end{aligned}
$$

e, portanto,

$$\frac{dW}{dt_{\text{ret}}} = \frac{2q^2}{3c^3}\left[\frac{a^2-(\mathbf{a}\times\boldsymbol{\beta})^2}{(1-\beta^2)^3}\right].$$

A radiação de uma distribuição de cargas em movimento

A radiação eletromagnética é produzida por cargas aceleradas. Nesta postagem vamos ver como esse processo radiativo acontece no caso de um sistema de cargas em movimento. Há algumas considerações que precisam ser feitas, como a hipótese de que a distribuição permanece espacialmente localizada, mas, em essência, a abordagem a seguir é bastante geral para a dinâmica clássica e não relativística das cargas.

A primeira hipótese que fazemos é que as cargas estão sempre localizadas dentro de uma região V limitada do espaço. Consideramos que a distribuição de cargas está tão distante do ponto de observação, $\mathbf{r}$, que temos

$$|\mathbf{r}| \gg |\mathbf{r}'|$$

para todo

$$\mathbf{r}' \in V.$$

Dessa forma, podemos expandir

$$\phi(\mathbf{r}) \text{ e } \mathbf{A}(\mathbf{r})$$

como somas infinitas em potências de

$$\frac{|\mathbf{r}'|}{|\mathbf{r}|}$$

Aqui usamos a notação

$$r=|\mathbf{r}| \text{ e } r'=|\mathbf{r}'|.$$

Temos, assim,

$$\begin{aligned}
\frac{1}{|\mathbf{r}-\mathbf{r}'|} &= \frac{1}{\sqrt{(\mathbf{r}-\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}} \\
&= \frac{1}{\sqrt{r^2-2\mathbf{r}\cdot\mathbf{r}'+(r')^2}} \\
&= \frac{1}{r\sqrt{1-2\frac{\mathbf{r}}{r}\cdot\left(\frac{\mathbf{r}'}{r}\right)+\left(\frac{r'}{r}\right)^2}} \\
&= \frac{1}{r}\left[1-2\frac{\mathbf{r}}{r}\cdot\left(\frac{\mathbf{r}'}{r}\right)+\left(\frac{r'}{r}\right)^2\right]^{-1/2}.
\end{aligned}$$

Definimos, agora,

$$\eta = -2\frac{\mathbf{r}}{r}\cdot\left(\frac{\mathbf{r}'}{r}\right)+\left(\frac{r'}{r}\right)^2$$

e

$$f(\eta) = (1+\eta)^{-1/2}.$$

Claramente vemos que

$$0 \leqslant |\eta| \ll 1,$$

já que

$$r' \ll r$$

por hipótese. Sendo assim, consideramos a série de Taylor em torno de zero para a função f:

$$f(\eta) = 1 - \frac{1}{2}\eta + \frac{3}{8}\eta^2 + \ldots.$$

Vamos aproximar a função f até a segunda ordem em

$$\frac{r'}{r}.$$

Assim, η já é um polinômio dessa ordem e precisamos desprezar termos de ordens superiores em η^2 apenas, já que as outras potências mais altas de η devem ser desprezadas por conterem somente termos de ordens superiores à segunda ordem de

$$\frac{r'}{r}.$$

Portanto,

$$\begin{aligned}
\eta^2 &= \left[-2\frac{\mathbf{r}}{r}\cdot\left(\frac{\mathbf{r}'}{r}\right) + \left(\frac{r'}{r}\right)^2\right]^2 \\
&= 4\left[\frac{\mathbf{r}}{r}\cdot\left(\frac{\mathbf{r}'}{r}\right)\right]^2 - 4\frac{\mathbf{r}}{r}\cdot\left(\frac{\mathbf{r}'}{r}\right)\left(\frac{r'}{r}\right)^2 + \left(\frac{r'}{r}\right)^4 \\
&\approx 4\left[\frac{\mathbf{r}}{r}\cdot\left(\frac{\mathbf{r}'}{r}\right)\right]^2.
\end{aligned}$$

Com isso, obtemos

$$\begin{aligned}
\frac{1}{|\mathbf{r}-\mathbf{r}'|} &= \frac{1}{r}\left\{1 - \frac{1}{2}\left[-2\frac{\mathbf{r}}{r}\cdot\left(\frac{\mathbf{r}'}{r}\right) + \left(\frac{r'}{r}\right)^2\right] + \frac{3}{2}\left[\frac{\mathbf{r}}{r}\cdot\left(\frac{\mathbf{r}'}{r}\right)\right]^2\right\} + \ldots \\
&= \frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3} - \frac{1}{2}\frac{(r')^2}{r^3} + \frac{3}{2}\frac{(\mathbf{r}\cdot\mathbf{r}')^2}{r^5} + \ldots.
\end{aligned}$$

Usando

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r},$$

também podemos escrever

$$\begin{aligned}
\frac{1}{|\mathbf{r}-\mathbf{r}'|} &= \frac{1}{r} + \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{r^2} + \frac{3(\hat{\mathbf{r}}\cdot\mathbf{r}')^2 - (r')^2}{2r^3} + \ldots \\
&= \frac{1}{r}\sum_{m=0}^{\infty}\alpha_m\left(\frac{r'}{r}\right)^m, \qquad (1)
\end{aligned}$$

com

$$\begin{aligned}\alpha_0 &= 1, \\ \alpha_1 &= \hat{\mathbf{r}}\cdot\hat{\mathbf{r}}', \\ &\text{etc.}\end{aligned}$$

De forma análoga, temos

$$\begin{aligned}|\mathbf{r}-\mathbf{r}'| &= r\sqrt{1-2\frac{\mathbf{r}}{r}\cdot\left(\frac{\mathbf{r}'}{r}\right)+\left(\frac{r'}{r}\right)^2} \\ &= r\left\{1-\frac{\mathbf{r}}{r}\cdot\left(\frac{\mathbf{r}'}{r}\right)+\frac{1}{2}\left(\frac{r'}{r}\right)^2+\frac{1}{4}\left(-\frac{1}{2}\right)\left[-2\frac{\mathbf{r}}{r}\cdot\left(\frac{\mathbf{r}'}{r}\right)+\left(\frac{r'}{r}\right)^2\right]^2+\dots\right\} \\ &= r-\hat{\mathbf{r}}\cdot\mathbf{r}'+r\frac{1}{2}\left(\frac{r'}{r}\right)^2-r\frac{1}{2}\left(\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{r}\right)^2+\dots \\ &= r-\hat{\mathbf{r}}\cdot\mathbf{r}'+r'\left[\frac{1-(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')^2}{2}\left(\frac{r'}{r}\right)+\dots\right].\end{aligned}$$

Assim, escrevemos

$$\rho\left(\mathbf{r}',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right) = \rho\left(\mathbf{r}',t-\frac{r}{c}+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}-\frac{r'}{c}\left[\frac{1-(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')^2}{2}\left(\frac{r'}{r}\right)+\dots\right]\right).$$

Nesse ponto definimos

$$t' = t-\frac{r}{c}+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}$$

e

$$\Delta t' = -\frac{r'}{c}\left[\frac{1-(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')^2}{2}\left(\frac{r'}{r}\right)+\dots\right].$$

Vemos, portanto, que $\Delta t'$ é da ordem de

$$\frac{r'}{r}.$$

e, portanto, podemos expandir a densidade de carga como uma série de potências em $\Delta t'$ como segue:

$$\begin{aligned}\rho\left(\mathbf{r}',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right) &= \rho\left(\mathbf{r}',t'+\Delta t'\right) \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\Delta t'\right)^n\frac{\partial^n}{\partial t'^n}\rho\left(\mathbf{r}',t'\right).\end{aligned}$$

Notamos também que

$$\frac{\partial\rho\left(\mathbf{r}',t'\right)}{\partial t'} = \frac{\partial}{\partial\left(t-\frac{r}{c}+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right)}\rho\left(\mathbf{r}',t-\frac{r}{c}+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right)$$

e, como

$$
\begin{aligned}
\frac{\partial}{\partial t} &= \frac{\partial\left(t-\frac{r}{c}+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right)}{\partial t}\frac{\partial}{\partial\left(t-\frac{r}{c}+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right)} \\
&= \frac{\partial}{\partial\left(t-\frac{r}{c}+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right)},
\end{aligned}
$$

podemos escrever

$$
\frac{\partial\rho\left(\mathbf{r}',t'\right)}{\partial t'} = \frac{\partial\rho\left(\mathbf{r}',t-\frac{r}{c}+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right)}{\partial t}
$$

e, portanto,

$$
\begin{aligned}
\rho\left(\mathbf{r}',t-\frac{\left|\mathbf{r}-\mathbf{r}'\right|}{c}\right) &= \rho\left(\mathbf{r}',t'+\Delta t'\right) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\Delta t'\right)^{n}\frac{\partial^{n}}{\partial t^{n}}\rho\left(\mathbf{r}',t-\frac{r}{c}+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right). \qquad (2)
\end{aligned}
$$