Queremos calcular o campo elétrico $\vec{E}$ a partir de:

$$\phi(\vec{r},t) = \frac{q}{(R - \vec{R}\cdot\vec{\beta})} \qquad \vec{A}(\vec{r},t) = \frac{q\vec{\beta}}{(R - \vec{R}\cdot\vec{\beta})}$$

Uma vez que $\vec{E}$ pode ser descrito por:

$$\vec{E}(\vec{r},t) = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

Algumas notações foram definidas em aula! Então serão usadas nessa conta.

Vamos primeiro trabalhar com:

$$-\vec{\nabla}\phi = -\vec{\nabla}\left(\frac{q}{(R - \vec{R}\cdot\vec{\beta})}\right) = -q\,\vec{\nabla}\left(\frac{1}{(R - \vec{R}\cdot\vec{\beta})}\right)$$

$$\vec{\nabla}\left(\frac{1}{(R - \vec{R}\cdot\vec{\beta})}\right) = \frac{\partial_j R - (\partial_j \vec{R})\cdot\vec{\beta} - \vec{R}\cdot(\partial_j\vec{\beta})}{(R - \vec{R}\cdot\vec{\beta})^2} \qquad \text{I}$$

Algumas derivadas precisam ser feitas:

$$\partial_j R = \frac{1}{2R}\partial_j R^2 = \frac{1}{2R}\partial_j(\vec{R}\cdot\vec{R}) = \frac{2\vec{R}}{2R}\cdot\partial_j\vec{R} = \hat{R}\cdot\partial_j\vec{R}$$

$$\partial_j \vec{R} = \partial_j(\vec{r} - \vec{r}_0(t_R)) = \hat{e}_j - \frac{\partial \vec{r}_0(t_R)}{\partial t_R}\cdot\partial t_R = \hat{e}_j - \vec{v}\cdot\partial t_R$$

$$\partial_j t_R = \partial_j\left(t - \frac{R}{c}\right) = -\frac{1}{c}\partial_j R = -\frac{1}{c}\hat{R}\cdot\partial_j\vec{R} = -\frac{1}{c}\hat{R}\cdot(\hat{e}_j - \vec{v}\partial$$

$$= -\frac{\hat{R}\cdot\hat{e}_j}{c} + \vec{\beta}\cdot\hat{R}\,\partial t_R \Rightarrow \partial t_R(1 - \hat{R}\cdot\vec{\beta}) = \frac{-\hat{R}\cdot\hat{e}_j}{c(1-\hat{R}\cdot\vec{\beta})}$$

logo

$$\partial_j t_R = \frac{-\hat{R}\cdot\hat{e}_j}{c(1-\hat{R}\cdot\vec{\beta})} \Rightarrow \partial_j\vec{R} = \hat{e}_j + \frac{\vec{v}}{c}\frac{\hat{R}\cdot\hat{e}_j}{(1-\hat{R}\cdot\vec{\beta})} = \hat{e}_j + \vec{\beta}\frac{\hat{R}\cdot\hat{e}_j}{(1-\hat{R}\cdot\vec{\beta})}$$

e

$$\partial_j R = \hat{R}\cdot\partial_j \vec{R} = \hat{e}_j\cdot\hat{R} + \frac{(\vec{\beta}\cdot\hat{R})(\hat{e}_j\cdot\hat{R})}{(1-\hat{R}\cdot\vec{\beta})}$$

substituindo em (I):

$$-q\vec{\nabla}\left(\frac{1}{(R-\vec{R}\cdot\vec{\beta})}\right) = \frac{q}{R^2(1-\hat{R}\cdot\vec{\beta})^2}\left[\hat{R}\cdot\hat{e}_j + \frac{(\hat{R}\cdot\vec{\beta})(\hat{R}\cdot\hat{e}_j)}{(1-\hat{R}\cdot\vec{\beta})} - \beta\cdot\hat{e}_j - \frac{(\vec{\beta}\cdot\vec{\beta})(\vec{R}\cdot\vec{e}_j)}{(1-\hat{R}\cdot\vec{\beta})} + \dots\right.$$

$$\left.\dots + \frac{(\vec{a}\cdot\vec{R})(\hat{R}\cdot\hat{e}_j)}{c^2(1-\hat{R}\cdot\vec{\beta})}\right],$$

uma vez que,

$$\partial_j\vec{\beta} = \frac{\partial_j\vec{v}}{c} = \frac{1}{c}\frac{\partial\vec{v}}{\partial t_R}\partial_j t_R = -\vec{a}\,\frac{(\hat{R}\cdot\hat{e}_j)}{c^2(1-\hat{R}\cdot\vec{\beta})}.$$

simplificando,

$$-\vec{\nabla}\phi = \frac{q}{R^2(1-\hat{R}\cdot\vec{\beta})^3}\,\hat{e}_j\left[(1-\beta^2)\hat{R} - \vec{\beta}(1-\hat{R}\cdot\vec{\beta}) + \frac{\vec{a}\cdot\vec{R}}{c^2}\hat{R}\right]$$

A outra parte do campo é calculada por $-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}$. No sistema que estamos utilizando portanto temos:

$$-\frac{1}{c}\frac{\partial A}{\partial t} = -\frac{1}{c}\frac{\partial A}{\partial t_R}\cdot\frac{\partial t_R}{\partial t}$$

$$\frac{\partial t_R}{\partial t} = \frac{\partial}{\partial t}\left(t-\frac{R}{c}\right) = 1-\frac{1}{c}\frac{\partial}{\partial t}R = 1-\frac{1}{c}\frac{\partial R}{\partial t_R}\cdot\frac{\partial t_R}{\partial t} \Rightarrow$$

$$\frac{\partial t_R}{\partial t} + \frac{1}{c}\frac{\partial R}{\partial t_R}\cdot\frac{\partial t_R}{\partial t} = 1 \Rightarrow \frac{\partial t_R}{\partial t} = \frac{1}{\left(1+\frac{1}{c}\frac{\partial R}{\partial t_R}\right)}$$

② 

Olhando agora para

$$\frac{\partial \vec{A}}{\partial t_R} = \frac{\partial}{\partial t_R}\left(\frac{q\vec{\beta}}{R - \vec{R}\cdot\vec{\beta}}\right) = \frac{q\frac{\partial}{\partial t_R}\vec{v}}{cR(1-\hat{R}\cdot\vec{\beta})} - \frac{q\vec{\beta}}{R^2(1-\hat{R}\cdot\vec{\beta})^2}\left(\frac{\partial}{\partial t_R}R - \frac{\partial \vec{R}}{\partial t_R}\cdot\vec{\beta} - \frac{\partial \vec{\beta}}{\partial t_R}\cdot\vec{R}\right) \quad \text{II}$$

Brincando um pouco com as derivadas:

$$\frac{\partial v}{\partial t_R} = \vec{a} \Rightarrow \frac{\partial \beta}{\partial t_R} = \frac{\vec{a}}{c}$$

$$\frac{\partial R}{\partial t_R} = \frac{1}{2R}\frac{\partial}{\partial t_R}R^2 = \frac{1}{2R}\frac{\partial}{\partial t_R}\vec{R}\cdot\vec{R} = \frac{2\vec{R}}{2R}\cdot\frac{\partial \vec{R}}{\partial t_R} = \hat{R}\cdot\frac{\partial \vec{R}}{\partial t_R}$$

$$\frac{\partial \vec{R}}{\partial t_R} = \frac{\partial}{\partial t_R}(\vec{r} - \vec{r}_0(t_R)) = -\vec{v}$$

logo

$$\frac{\partial \vec{R}}{\partial t_R} = -\vec{v} \Rightarrow \frac{\partial R}{\partial t_R} = \hat{R}\cdot\frac{\partial \vec{R}}{\partial t_R} = -\hat{R}\cdot\vec{v}$$

substituindo em II:

$$\frac{\partial \vec{A}}{\partial t_R} = q\left[\frac{\vec{a}}{cR(1-\hat{R}\cdot\vec{\beta})} - \frac{\vec{\beta}}{R^2(1-\hat{R}\cdot\vec{\beta})^2}\left(\vec{v}\cdot\vec{\beta} - \hat{R}\cdot\vec{v} - \frac{\vec{R}\cdot\vec{a}}{c}\right)\right]$$

como queremos

$$\frac{\partial A}{\partial t} = \frac{\partial A}{\partial t_R}\cdot\frac{\partial t_R}{\partial t} \quad ; \quad \frac{\partial t_R}{\partial t} = \frac{1}{\left(1+\frac{1}{c}\frac{\partial R}{\partial t_R}\right)} = \frac{1}{(1-\hat{R}\cdot\vec{\beta})}$$

logo

$$\frac{\partial \vec{A}}{\partial t} = q\left[\frac{\vec{a}}{cR(1-\hat{R}\cdot\vec{\beta})} - \frac{\vec{\beta}}{R^2(1-\hat{R}\cdot\vec{\beta})^2}\left(\vec{v}\cdot(\vec{\beta}-\hat{R}) - \frac{\vec{R}\cdot\vec{a}}{c}\right)\right]\frac{1}{(1-\hat{R}\cdot\vec{\beta})}$$

$$\Rightarrow -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} = q\left[\vec{\beta}\frac{(\vec{\beta}-\hat{R})\cdot\vec{\beta}}{R^2(1-\hat{R}\cdot\vec{\beta})^3} - \vec{\beta}\frac{(\hat{R}\cdot\vec{a})}{Rc^2(1-\hat{R}\cdot\vec{\beta})^3} - \frac{\vec{a}}{Rc^2(1-\hat{R}\cdot\vec{\beta})^2}\right]$$

Somando tudo que encontramos até agora:

$$\vec{E}(\vec{r},t) = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\vec{E}(\vec{r},t) = q\left[\frac{(1-\beta^2)\hat{R}}{R^2(1-\hat{R}\cdot\vec{\beta})^3} - \frac{\vec{\beta}(1-\hat{R}\cdot\vec{\beta})}{R^2(1-\hat{R}\cdot\vec{\beta})^3} + \frac{(\hat{R}\cdot\vec{a})\hat{R}}{Rc^2(1-\hat{R}\cdot\vec{\beta})^3} + \dots\right.$$

$$\left.\dots + \vec{\beta}\,\frac{(\vec{\beta}-\hat{R})\vec{\beta}}{R^2(1-\hat{R}\cdot\vec{\beta})^3} - \frac{\vec{\beta}(\hat{R}\cdot\vec{a})}{Rc^2(1-\hat{R}\cdot\vec{\beta})^3} - \frac{\vec{a}}{Rc^2(1-\hat{R}\cdot\vec{\beta})}\right]$$

simplificando os termos proporcionais a $(R^2(1-\hat{R}\cdot\vec{\beta})^3)^{-1}$

$$(1-\beta^2)\hat{R} - \vec{\beta}(1-\hat{R}\cdot\vec{\beta}) + \vec{\beta}(\vec{\beta}-\hat{R})\vec{\beta} = (1-\beta^2)\hat{R} - \vec{\beta}\left[(1-\hat{R}\cdot\vec{\beta}-\vec{\beta}\cdot(\vec{\beta}-\hat{R}))\right]$$

$$= (1-\beta^2)\hat{R} - \vec{\beta}(1-\cancel{\hat{R}\cdot\vec{\beta}}-\vec{\beta}\cdot\vec{\beta}+\cancel{\vec{\beta}\cdot\hat{R}}) = (1-\beta^2)\hat{R} - \vec{\beta}(1-\beta^2)$$

$$= (1-\beta^2)(\hat{R}-\vec{\beta})$$

$$\therefore \frac{(1-\beta^2)(\hat{R}-\vec{\beta})}{R^2(1-\hat{R}\cdot\vec{\beta})^3} = \frac{(1-\beta^2)\hat{R} - \vec{\beta}(1-\hat{R}\cdot\vec{\beta}) + \vec{\beta}(\vec{\beta}-\hat{R})\vec{\beta}}{R^2(1-\hat{R}\cdot\vec{\beta})^3}$$

Nos termos de $(Rc^2(1-\hat{R}\cdot\vec{\beta})^3)$ temos

$$\frac{(\hat{R}\cdot\vec{a})\hat{R} - \vec{\beta}(\hat{R}\cdot\vec{a})}{Rc^2(1-\hat{R}\cdot\vec{\beta})^3} = \frac{(\hat{R}\cdot\vec{a})(\hat{R}-\vec{\beta})}{Rc^2(1-\hat{R}\cdot\vec{\beta})^3}$$

$\therefore$

Finalmente, como queriamos demonstrar

$$\vec{E}(\vec{r},t) = q\left[\frac{(1-\beta^2)(\hat{R}-\vec{\beta})}{R^2(1-\hat{R}\cdot\vec{\beta})^3} + \frac{(\hat{R}\cdot\vec{a})(\hat{R}-\vec{\beta})}{Rc^2(1-\hat{R}\cdot\vec{\beta})^3} - \frac{\vec{a}}{Rc^2(1-\hat{R}\cdot\vec{\beta})^2}\right]$$