Lista de exercícios 14.

1 Refaça detalhadamente o cálculo da integral $I(C)$, definida no exemplo dessa aula:

Vamos então partir da integral definida em aula. no caso de uma ressonância. $w \approx w_1$.

$$G(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dw \frac{4\pi N_{n_1} e^2}{m(w_1^2 - w^2 - i\gamma_1 w)} \exp(-iw\tau)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dw \frac{w_p^2}{w_1^2 - w^2 - i\gamma_1 w} \exp(-iw\tau)$$

$w_p \rightarrow$ frequência de plasma.

$$I(C) = \oint_C dz \frac{w_p^2}{w_1^2 - z^2 - i\gamma_1 z} \exp(-iz\tau)$$

Para calcular essa integral temos que definir os polos do integrando
Temos que buscar pontos no qual

$$w_1^2 - z^2 - i\gamma_1 z = 0$$ $\rightarrow$ Bhaskara!

$$\Rightarrow z^2 + i\gamma_1 z - w_1^2 = 0$$

logo temos os polos em

$$z_\pm = \frac{-i\gamma_1 \pm \sqrt{4w_1^2 - \gamma_1^2}}{2}$$

É fácil perceber que só vamos ter polos no semi-plano inferior dos complexos.

Aplicando o Teorema dos Resíduos:

$$\oint f(z)\, dz = 2\pi i \sum \text{Res}$$

Temos portanto,

$$I(C) = \oint_C \frac{w_p^2}{w_1^2 - z^2 - i\gamma_1 z} \exp(-iz\tau) = -\oint_C \frac{w_p^2}{z^2 + i\gamma_1 z - w_1} \exp(-iz\tau)$$

$$= 2\pi i \frac{w_p^2 \exp(-iz_-\tau)}{z_- - z_+} + 2\pi i \frac{w_p^2 \exp(-iz_+\tau)}{z_+ - z_-}$$

$$= 2\pi i\, w_p^2 \frac{\exp(-iz_+\tau) - \exp(-iz_-\tau)}{z_+ - z_-} = ⊛$$

Substituindo $z_+$ e $z_-$ pelos polos calculados:

$$⊛ = 2\pi i\, w_p^2 \frac{\left[\exp\left(-i\frac{-i\gamma_1 + \sqrt{4w_1^2 - \gamma_1^2}}{2}\tau\right) - \exp\left(-i\frac{-i\gamma_1 - \sqrt{4w_1^2 - \gamma_1^2}}{2}\tau\right)\right.}{\frac{-i\gamma_1 + \sqrt{4w_1^2 - \gamma_1^2} - i\gamma_1 + \sqrt{4w_1^2 - \gamma_1^2}}{2}}$$

$$= \frac{4\pi i\, w_p^2}{2\sqrt{4w_1^2 - \gamma_1^2}} \left[\exp\left(\frac{-\gamma_1 - i\sqrt{4w_1^2 - \gamma_1^2}}{2}\tau\right) - \exp\left(\frac{-\gamma_1 + \sqrt{4w_1^2 - \gamma_1^2}}{2}\tau\right)\right]$$

$$= 2\pi i \frac{w_p^2}{\sqrt{4w_1^2 - \gamma_1}} \exp\left(-\frac{\gamma_1}{2}\tau\right)\left[\exp\left(-i\sqrt{w_1^2 - \frac{\gamma_1^2}{4}}\,\tau\right) - \exp\left(i\sqrt{w_1^2 - \frac{\gamma_1^2}{4}}\,\tau\right)\right]$$

$$= 4\pi \frac{w_p^2}{\sqrt{4w_1^2 - \gamma_1^2}} \exp\left(-\frac{\gamma_1}{2}\tau\right) \sin\left(\sqrt{w_1^2 - \frac{\gamma_1^2}{4}}\,\tau\right)$$

Finalmente, chegando em

$$G(\tau) = \frac{w_p^2}{\sqrt{w_1^2 - \frac{\gamma_1^2}{4}}} \exp\left(-\frac{\gamma_1}{2}\tau\right) \sin\left(\sqrt{w_1^2 - \frac{\gamma_1^2}{4}}\,\tau\right)$$

(2) Queremos mostrar $G(\tau) \to 4\pi\sigma$ com $\tau \to \infty$ em um condutor.

Para isso, vamos considerar a definição de $G(\tau)$, uma vez que

$$G(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dw\, [4\pi \chi_c(w)] \exp(-iw\tau)$$

Para considerarmos um sistema condutor temos que consider um modelo de $N$ moléculas.

Para considerar um meio condutor vamos supor um meio com uma distribuição uniforme de elétrons livres.

$$n_0 \equiv \frac{N_0}{N}$$

Da definição temos

$$\chi_c = \sum_k \frac{N n_k e^2}{m(w_k^2 - w^2 - i\gamma_k w)}$$

$$\chi_c'(w) = \sum_k \frac{N n_k e^2}{m(w_k^2 - w^2 - i\gamma_k w)} + \frac{N n_0 e^2}{m(-w^2 - i\gamma_k w)}$$

$$= \chi_c(w) + \frac{N n_0 e^2}{m(-w^2 - i\gamma_k w)} \quad ; \quad w_0 = 0 \quad \text{elétrons livres.}$$

Finalmente podemos calcular $G(\tau)$ em um condutor:

$$G(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega \left[ 4\pi \chi_c'(\omega) \right] e^{-i\omega\tau}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underbrace{\left[ 4\pi \chi_c(\omega)\right.}_{G'} + \underbrace{\frac{4\pi N n_0 e^2}{m(-\omega^2 - i\gamma_0 \omega)}}_{G''} \left.\right] e^{-i\omega\tau}$$

Podemos separar em duas partes. A primeira é exatamente igual ao item a) onde

$$G'(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega \frac{4\pi N n_1 e^2}{m(\omega_1^2 - \omega^2 - i\gamma_1 \omega)} \exp(-i\omega\tau)$$

$$= \oint_C dz \frac{\omega_p^2}{\omega_1 - z^2 - i\gamma_1 z} \exp(-i\omega\tau)$$

$$= \frac{\omega_p^2}{\sqrt{\omega_1^2 - \frac{\gamma_1^2}{4}}} \exp\left(-\frac{\gamma_1}{2}\tau\right) \sin\left(\sqrt{\omega_1^2 - \frac{\gamma_1^2}{4}}\ \tau\right)$$

Basta então calcular o termo $G''$

$$I(C) = -\oint dz \frac{4\pi N n_0 e^2}{m(z^2 + i\gamma_0 z)} \exp(-iz\tau)$$

$$= -\oint dz \frac{\omega_p^2}{z^2 + i\gamma_0 z} \exp(-iz\tau)$$

com polos em

$z(z + i\gamma_0) = 0$

$z = 0$ e $z = -i\gamma_0$

Aplicando novamente o teorema dos Resíduos:

$$I(c) = 2\pi i \left( \frac{w_0^2}{-z_2} e^0 \right) + 2\pi i \left( \frac{w_0^2}{-i\gamma_0 - z_1} e^{-i(-i\gamma_0)\tau} \right)$$

$$= 2\pi \cancel{i} \left( \frac{w_0^2}{\cancel{i}\gamma_0} - \frac{w_0^2}{\cancel{i}\gamma_0} e^{-\gamma_0 \tau} \right)$$

$$= 2\pi \frac{w_0^2}{\gamma_0} \left( 1 - e^{-\gamma_0 \tau} \right)$$

Por fim,

$$G(\tau) = \frac{w_p^2}{\sqrt{w_1 - \frac{\gamma_1^2}{4}}} \exp\left( -\frac{\gamma_1}{2} \tau \right) \sin\left( \sqrt{w_1^2 - \frac{\gamma_1^2}{4}} \cdot \tau \right)$$

$$+ \frac{1}{2\pi} \frac{2\pi}{\gamma_0} w_0^2 \left( 1 - e^{-\gamma_0 \tau} \right)$$

$$= \frac{w_p^2}{\sqrt{w_1 - \frac{\gamma_1^2}{4}}} \exp\left( -\frac{\gamma_1}{2} \tau \right) \sin\left( \sqrt{w_1^2 - \frac{\gamma_1^2}{4}} \tau \right)$$

$$+ \frac{4\pi N n_0 e^2}{m \gamma_0} \left( 1 - e^{-\gamma_0 \tau} \right)$$

$$\sigma \simeq \frac{N . n_0 e^2}{m \gamma_0} \Rightarrow G(\tau) = G'(\tau) + 4\pi\sigma \left( 1 - \exp(-\gamma_0 \tau) \right)$$

Como queremos provar $G(\tau) \to 4\pi\sigma$ com $\tau \to \infty$

$$\lim_{\tau \to \infty} G'(\tau) = 0 \qquad \text{e} \qquad \underbrace{\lim_{\tau \to \infty} \exp(-\gamma_0 \tau) = 0}_{\text{Trivial}}$$

$$\lim_{\tau \to \infty} G'(\tau) = \frac{w_p^2}{\sqrt{w_1^2 - \frac{\gamma_1^2}{4}}} \lim_{\tau \to \infty} e^{-\frac{\gamma_1 \tau}{2}} \cdot \sin\left(\sqrt{w_1^2 - \frac{\gamma_1^2}{4}}\, \tau\right)$$

Como sin é limitada $[-1, +1]$ podemos apenas considerar a exponencial $\lim_{\tau \to \infty} \exp\left(-\frac{\gamma_1}{2}\tau\right) = 0$

$$\therefore \boxed{G(\tau) \to 4\pi\sigma \, , \quad \tau \to \infty}$$

(3) Demonstrar que a continuação analítica da suscetibilidade complexa, para materiais dielétricos, tende a zero no semi-plano complexo superior.

Vamos considerar a definição de $\chi_c$

$$\chi_c(z) = \frac{1}{4\pi} \int_0^\infty d\tau \; G(\tau) \underline{\exp(iz\tau)}$$

$\hookrightarrow$ Analítica no semi-plano complexo.

e como

$$\lim_{\tau \to \infty} G(\tau) = 0$$

é válido para dielétricos, sem condutividade (como mostramos no item anterior). $G(\tau)$ é analítica no eixo real

Escrevendo a susceptibilidade complexa como

$$\chi_c(\omega) = \frac{1}{4\pi} \int_0^{+\infty} d\tau \; G(\tau) \exp(i\omega\tau)$$

$$= \frac{1}{4\pi i\omega} \int_0^{+\infty} d\tau \; G(\tau) \frac{\partial}{\partial \tau} \exp(i\omega\tau)$$

$$= \frac{1}{4\pi i\omega} \left[ \int_0^{+\infty} d\tau \; \frac{\partial}{\partial \tau} [G(\tau) \exp(i\omega\tau)] - \int_0^{+\infty} d\tau \; G'(\tau) \exp(i\omega\tau) \right]$$

$$G'(\tau) \equiv \frac{dG(\tau)}{d\tau}$$

logo

$$\int_0^{+\infty} d\tau \; \frac{\partial}{\partial \tau} [G(\tau) \exp(i\omega\tau)] = \lim_{\eta \to 0^+} \int_\eta^{+\infty} \frac{\partial}{\partial \tau} [G(\tau) \exp(i\omega\tau)]$$

$$= \lim_{\eta \to 0^+} \left[ G(\tau) \exp(i\omega\tau) \Big|_\eta^{+\infty} \right]$$

$$\int_0^{+\infty} d\tau \; \frac{\partial}{\partial \tau} [G(\tau) \exp(i\omega\tau)] = 0$$

Pela continuidade

$$\int_0^{+\infty} d\tau \; \frac{\partial}{\partial \tau} [G(\tau) \exp(i\omega\tau)] = 0$$

Logo escrevemos $\chi_c(\omega)$:

$$\chi_c(\omega) = \frac{i}{4\pi\omega}\int_0^{+\infty} d\tau\; G'(\tau)\exp(i\omega\tau)$$

$$= \frac{1}{4\pi\omega^2}\int_0^{+\infty} d\tau\; G'(\tau)\frac{\partial}{\partial\tau}\exp(i\omega\tau)$$

$$= \frac{1}{4\pi\omega^2}\int_0^{+\infty} d\tau\; \frac{\partial}{\partial\tau}\left[G'(\tau)\exp(i\omega\tau)\right]$$

$$- \frac{1}{4\pi\omega^2}\int_0^{+\infty} d\tau\; G''(\tau)\exp(i\omega\tau)$$

$$= -\frac{G'(0^+)}{4\pi\omega^2} - \frac{i}{4\pi\omega^2}\int_0^{+\infty} d\tau\; G''(\tau)\exp(i\omega\tau)$$

Analogamente, podemos escrever

$$\chi_c(\omega) = -\frac{G'(0^+)}{4\pi\omega^2} + i\frac{G''(0^+)}{4\pi\omega^3} - \frac{G'''(0^+)}{4\pi\omega^4} + \cdots$$

Que na versão analítica nos mostra

$$\chi_c(z) = -\frac{G(0^+)}{4\pi z^2} + i\frac{G''(0^+)}{4\pi z^3} - \frac{G'''(0^+)}{4\pi z^4} + \cdots$$

$$\boxed{\lim_{z\to\infty} \chi_c(z) = 0}$$