Exercício 4: 04/09 Calcule, detalhadamente, a partir dos potenciais de Liénard & Wiechert, o campo elétrico de uma partícula de carga q e massa m percorrendo uma trajetória arbitrária.

Queremos demonstrar o campo $\vec{E}$:

$$\vec{E}(\vec{r},t) = q\left[\frac{(\hat{R}-\vec{\beta})(1-\beta^2)}{R^2(1-\hat{R}\cdot\beta)^3} + \frac{\hat{R}\times\left[(\hat{R}-\vec{\beta})\times\vec{a}\right]}{Rc^2(1-\hat{R}\cdot\vec{\beta})^3}\right] \tag{4.1}$$

partindo de,

$$\phi(\vec{r},t) = \frac{q}{R-\vec{R}\cdot\beta} \tag{4.2}$$

e

$$\vec{A}(\vec{r},t) = \frac{q\vec{\beta}}{R-\vec{R}\cdot\vec{\beta}} \tag{4.3}$$

uma vez que,

$$\vec{E}(\vec{r},t) = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}. \tag{4.4}$$

Vamos calcular primeiro $-\vec{\nabla}\phi$:

$$\begin{aligned}
-\vec{\nabla}\phi &= \vec{\nabla}\left[\frac{q}{R-\vec{R}\cdot\vec{\beta}}\right] = -q\vec{\nabla}\left[(R-\vec{R}\cdot\beta)^{-1}\right] \\
&= q\left[(R-\vec{R}\cdot\vec{\beta})^{-2})\vec{\nabla}(R-\vec{R}\cdot\vec{\beta})\right] \\
&= \frac{q}{(R-\vec{R}\cdot\beta)^2}(\partial_j R - \vec{\beta}\cdot(\partial_j\vec{R}) - \vec{R}\cdot(\partial_j\vec{\beta}))
\end{aligned} \tag{4.5}$$

Vamos agora calcular as derivadas diretamente:

$$\partial_j R = \frac{1}{2R}\partial_j R^2 = \frac{1}{2R}\partial_j(\vec{R}\cdot\vec{R}) = \hat{R}\cdot\partial_j\vec{R} \tag{4.6}$$

$$\partial_j\vec{R} = \partial_j(\vec{r}-\vec{r_0}(t_R)) = \hat{e}_j - \frac{dr_0(t_R)}{dt_R}\partial_j t_R = \hat{e}_j - \vec{v}\partial_j t_R \tag{4.7}$$

$$\begin{aligned}
\partial_j t_R = \partial_j\left(t-\frac{R}{c}\right) &= \partial_j t - \frac{\partial_j R}{c} = -\frac{\partial_j R}{c} = \frac{\hat{R}\cdot\partial\vec{R}}{c} \\
&= \frac{\hat{R}\cdot\hat{e}_j - \hat{R}\cdot\vec{v}\partial_j t_R}{c} = \frac{-\hat{R}\cdot\hat{e}_j}{c} + \vec{\beta}\cdot\hat{R}\partial_j t_R
\end{aligned} \tag{4.8}$$

$$\partial_j t_R - \vec{\beta}\cdot\hat{R}\partial_j t_R = \partial_j t_R(1-\vec{\beta}\cdot\hat{R}) = \frac{-\hat{R}\cdot\hat{e}_j}{c} \implies \partial_j t_R = \frac{-\hat{R}\cdot\hat{e}_j}{c(1-\vec{\beta}\cdot\hat{R})} \tag{4.9}$$

Finalmente,

$$\partial_j t_R = \frac{\hat{R}\cdot\hat{e}_j}{c(1-\vec{\beta}\cdot\hat{R})} \tag{4.10}$$

$$\partial_j \vec{R} = \hat{e}_j + \vec{\beta}\frac{\hat{R}\cdot\hat{e}_j}{c(1-\vec{\beta}\cdot\hat{R})} \tag{4.11}$$

$$\partial_j R = \hat{R}\cdot\hat{e}_j + \frac{(\hat{R}\cdot\vec{\beta})(\hat{R}\cdot\hat{e}_j)}{c(1-\vec{\beta}\cdot\hat{R})} \tag{4.12}$$

substituido em 4.5 temos,

$$\begin{aligned}
-\vec{\nabla}\phi &= \frac{q}{(R-\vec{R}\cdot\beta)^2}(\partial_j R - \vec{\beta}\cdot(\partial_j\vec{R}) - \vec{R}\cdot(\partial_j\vec{\beta})) \\
&= \frac{q}{(R-\vec{R}\cdot\beta)^2}\left[\hat{R}\cdot\hat{e}_j + \frac{(\hat{R}\cdot\vec{\beta})(\hat{R}\cdot\hat{e}_j)}{c(1-\vec{\beta}\cdot\hat{R})} - \vec{\beta}\cdot\hat{e}_j - \frac{(\vec{\beta}\cdot\vec{\beta})(\hat{R}\cdot\hat{e}_j)}{c(1-\vec{\beta}\cdot\hat{R})} + \frac{\vec{a}}{c^2}\frac{(\hat{R}\cdot\hat{e}_j)}{(1-\vec{\beta}\cdot\hat{R})}\right]
\end{aligned} \tag{4.13}$$