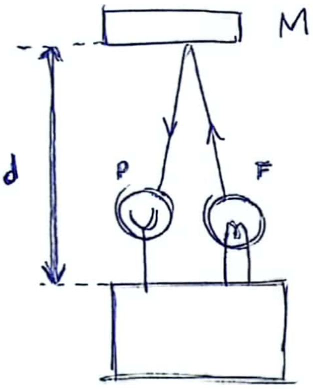
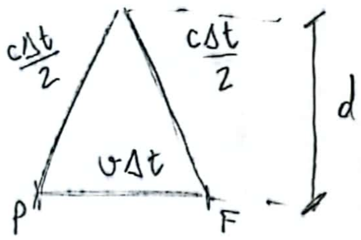
Problema 11.4

a)





Pela geometria:

$$d^2 + \left(\frac{v\Delta t}{2}\right)^2 = \left(\frac{c\Delta t}{2}\right)^2$$

$$\Rightarrow \quad \Delta t^2\left(\left(\frac{c}{2}\right)^2 - \left(\frac{v}{2}\right)^2\right) = d^2 \Rightarrow (\Delta t)^2 = \frac{4d^2}{(c^2-v^2)}$$

$$\Delta t = 2d \frac{1}{\sqrt{c^2-v^2}} = 2d \frac{1}{\sqrt{c^2\left(1-\frac{v^2}{c^2}\right)}}$$

$$\Delta t = \frac{2d}{c\sqrt{1-\frac{v^2}{c^2}}}$$

Porém, pelo enunciado podemos interpretar $\Delta t' = \frac{2d}{c}$ logo é fácil visualizar que

$$\Delta t = \frac{2d}{c} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \Delta t = \gamma \Delta t'$$

b) Temos que considerar um contração na direção do movimento, onde teremos

$$d = d_0\sqrt{1-\frac{v^2}{c^2}} = \frac{d_0}{\gamma}$$

$$d = c\Delta t_1 + v\Delta t_1$$

com Δt_1 sendo relativo a

$$d = c\Delta t_1 + v\Delta t_1 = \Delta t_1(c+v)$$

$$\Delta t_1 = \frac{d}{(c+v)} = \frac{d_0\sqrt{1-\frac{v^2}{c^2}}}{c+v} = \frac{d_0}{c}\frac{\sqrt{1-\frac{v^2}{c^2}}}{1+\frac{v}{c}}$$

No caso de Δt_2

$$c\Delta t_2 = d + v\Delta t_2 \Rightarrow \Delta t_2 (c - v) = d$$

$$\Delta t_2 = \frac{d}{(c-v)} = \frac{d_0\sqrt{1-\frac{v^2}{c^2}}}{c\,(1-v/c)}$$

Nos relacoes de Δt_1 e Δt_2 onde

$$\Delta t = \Delta t_1 + \Delta t_2$$

$$= \frac{d_0}{c}\,\frac{\sqrt{1-v^2/c^2}}{1+v/c} + \frac{d_0}{c}\,\frac{\sqrt{1-v^2/c^2}}{1-v/c}$$

$$= \frac{d_0}{c}\sqrt{1 - v^2/c^2}\left(\frac{1}{1+v/c} + \frac{1}{1-v/c}\right)$$

$$= \frac{d_0}{c}\sqrt{1-v^2/c^2}\left(\frac{2}{1-v^2/c^2}\right)$$

$$\Delta t = \frac{2d_0}{c}\,\frac{\sqrt{1-v^2/c^2}}{1-\frac{v^2}{c^2}} = \frac{2d_0}{c}\,\frac{1}{\sqrt{1-v^2/c^2}} = \frac{2d_0}{c}\gamma = \Delta t'\gamma$$

Problema 11.3

Considerando as transformações na direção x com velocidades v_1 e v_2.

$$A_1 = \begin{pmatrix} \gamma_1 & -\gamma_1\beta_1 & 0 & 0 \\ -\gamma_1\beta_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} \gamma_2 & -\gamma_2\beta_2 & 0 & 0 \\ -\gamma_2\beta_2 & \gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

com $\beta_i = v_i/c \qquad \gamma_i = \dfrac{1}{\sqrt{1-\beta_i^2}}$

$$A_2 \circ A_1 = \begin{pmatrix} \gamma_1\gamma_2(1+\beta_2\beta_1) & -\gamma_1\gamma_2(\beta_2+\beta_1) & 0 & 0 \\ -\gamma_1\gamma_2(\beta_1+\beta_2) & \gamma_1\gamma_2(1+\beta_2\beta_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

com $$\gamma_1\gamma_2(1+\beta_2\beta_1) = \frac{1+\beta_1\beta_2}{(1-\beta_1^2)(1-\beta_2^2)} = \frac{1}{\sqrt{\dfrac{(1-\beta_1^2)(1-\beta_2^2)}{1+\beta_1\beta_2}}}$$

$$= \frac{1}{\sqrt{1-\dfrac{(1+\beta_1\beta_2)^2-(1-\beta_1^2)(1-\beta_2^2)}{(1+\beta_1\beta_2)^2}}} = \frac{1}{\sqrt{1-\left(\dfrac{\beta_1+\beta_2}{1+\beta_1\beta_2}\right)}}$$

$$= \frac{1}{\sqrt{1-\dfrac{1}{c^2}\left(\dfrac{v_1+v_2}{1+\dfrac{v_1v_2}{c^2}}\right)^2}} = \frac{1}{\sqrt{1-\dfrac{v_3^2}{c^2}}} = \frac{1}{\sqrt{1-\beta_3^2}} = \gamma_3$$

$$-\gamma_1\gamma_2(\beta_1+\beta_2) = -\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\left[\gamma_1\gamma_2(1+\beta_1\beta_2)\right] = -\frac{v_3}{c}\gamma_3 = -\beta_3\gamma_3$$

Logo,

$$A_2 \circ A_1 = \begin{pmatrix} \gamma_3 & -\beta_3\gamma_3 & 0 & 0 \\ -\beta_3\gamma_3 & \gamma_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

que nos dá

$$\boxed{v_3 = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}}$$

Problema 11.5

As componentes da velocidade da partícula paralelos a $\vec{v}$ e K' e K,

$$u'_{\parallel} = \frac{dx'(t')}{dt'} \quad e \quad u_{\parallel} = \frac{dx(t)}{dt}$$

são relacionadas por

$$u_{\parallel} = \frac{u'_{\parallel} + v}{1 + \frac{\vec{v} \cdot \vec{u}'}{c^2}} \qquad \vec{u}_{\perp} = \frac{\vec{u}_{\perp}'}{\gamma_0 \left(1 + \frac{\vec{v} \cdot \vec{v}'}{c^2}\right)}$$

$$a_{\parallel}(t) = \frac{du_{\parallel}}{dt} = \frac{d}{dt}\left(\frac{u'_{\parallel} + v}{1 + \frac{u'_{\parallel} v}{c^2}}\right)$$

$$= \frac{1}{\left(1 + \frac{u'_{\parallel} v}{c^2}\right)^2}\left[\left(1 + \frac{u'_{\parallel} v}{c^2}\right)\frac{du'_{\parallel}}{dt} - (u'_{\parallel} + v)\frac{v}{c^2}\frac{du'_{\parallel}}{dt}\right]$$

$$= \frac{1}{\left(1 + \frac{u_{\parallel} v}{c^2}\right)^2}\left[1 + \frac{u'_{\parallel} v}{c^2} - \frac{u'_{\parallel} v}{c^2} - \frac{v^2}{c^2}\right]\left[\frac{dt'}{dt}\right]\frac{du'_{\parallel}}{dt'}$$

$$= \frac{1}{\left(1 + \frac{u_{\parallel} v}{c^2}\right)^2}\frac{1}{\gamma^2}\left[\frac{dt'}{dt}\right]a'_{\parallel} = \frac{a'_{\parallel}}{\gamma^2\left(1 + \frac{u'_{\parallel} v}{c^2}\right)^2}\left[\frac{d}{dt}\left(\gamma t - \frac{\beta\gamma x_{\parallel}(t)}{c}\right)\right]$$

$$= \frac{a'_{\parallel}}{\gamma\left(1 + \frac{u'_{\parallel} v}{c^2}\right)^2}\left[1 - \frac{v}{c^2}\frac{dx_{\parallel}}{dt}\right] = \frac{a'_{\parallel}}{\gamma\left(1 + \frac{u'_{\parallel} v}{c^2}\right)^2}\left[1 - \frac{v}{c^2}u_{\parallel}\right]$$

$$= \frac{a'_{\parallel}}{\gamma\left(1 + \frac{u'_{\parallel} v}{c^2}\right)^2}\left[1 - \frac{v}{c^2}u_{\parallel}\right] = \frac{a'_{\parallel}}{\gamma\left(1 + \frac{u'_{\parallel} v}{c^2}\right)^2}\left[1 - \frac{v}{c^2}\left(\frac{u_{\parallel} + v}{1 + \frac{u' v}{c^2}}\right)\right]$$

$$= \frac{a'_{\parallel}}{\gamma\left(1+\frac{u'_{\parallel} v}{c^2}\right)^3}\left[1+\frac{u'_{\parallel} v}{c^2}-\frac{v\left(u'_{\parallel}+v\right)}{c^2}\right]$$

$$= \frac{a'_{\parallel}}{\gamma^3\left(1+\frac{u'_{\parallel} \cdot v}{c^2}\right)^3}$$

temos então,
$$\boxed{\vec{a}_{\parallel}(t) = \frac{\vec{a}'_{\parallel}}{\gamma^3\left(1+\frac{\vec{u}\cdot\vec{v}}{c^2}\right)^3}}$$

Para as acelerações transversais:

$$\vec{a}_{\perp} = \frac{1}{\gamma}\frac{d}{dt}\left(\frac{\vec{u}'_{\perp}}{1+\frac{u'_{\parallel} v}{c^2}}\right)$$

$$= \frac{1}{\gamma\left(1+\frac{u'_{\parallel} v}{c^2}\right)^2}\left[\left(1+\frac{u'_{\parallel} v}{c^2}\right)\frac{d}{dt}\vec{u}'_{\perp} - \vec{u}'_{\perp}\frac{d}{dt}\left(1+\frac{u'_{\parallel} v}{c^2}\right)\right]$$

$$= \frac{1}{\gamma\left(1+\frac{u'_{\parallel} v}{c^2}\right)^2}\left[\frac{dt'}{dt}\right]\left[\left(1+\frac{u'_{\parallel} v}{c^2}\right)\frac{d}{dt}\vec{u}'_{\perp} - \vec{u}'_{\perp}\frac{v}{c^2}\frac{d}{dt'}u'_{\parallel}\right]$$

$$= \frac{1}{\gamma^2\left(1+\frac{u'_{\parallel} v}{c^2}\right)^3}\left[\left(1+\frac{u'_{\parallel} v}{c^2}\right)\frac{d}{dt}\vec{u}'_{\perp} - \vec{u}'_{\perp}\frac{v}{c^2}\frac{d}{dt'}u'_{\parallel}\right]$$

$$= \frac{1}{\gamma^2\left(1+\frac{u'_{\parallel} v}{c^2}\right)^3}\left[\left(1+\frac{u'_{\parallel} v}{c^2}\right)\vec{a}'_{\perp} - \vec{u}'_{1}\frac{v}{c^2}a'_{\parallel}\right]$$

$$= \frac{1}{\gamma^2\left(1+\frac{u'_{\parallel} v}{c^2}\right)^3}\left[\vec{a}'_{\perp} + \vec{a}'_{\perp}\frac{u'_{\parallel} v}{c^2} - \vec{u}'_{\perp}\frac{v}{c^2}a'_{\parallel}\right]$$

$$= \frac{1}{\gamma^2\left(1+\frac{u'_{\parallel} v}{c^2}\right)^3}\left[\vec{a}'_{\perp} + \vec{a}'\frac{u'_{\parallel} v}{c^2} + \frac{v a'_{\parallel}}{c^2}\left(-\vec{u}'_{\perp} - \hat{v}\, u'_{\parallel}\right)\right]$$

$$= \frac{1}{\gamma^2\left(1+\frac{u'_{\parallel} v}{c^2}\right)^3}\left[\vec{a}_{\perp} + \frac{1}{c^2}\left(\vec{a}'\, u'_{\parallel} v - v a'_{\parallel}\, \vec{u}'\right)\right]$$

$$= \frac{1}{\gamma^2\left(1+\frac{v'_{\parallel} v}{c^2}\right)^3}\left[\vec{a}'_{\perp} + \frac{1}{c^2}\left(\vec{a}'(\vec{u}'\cdot\vec{v}) - (\vec{a}'\cdot\vec{v})\,\vec{u}'\right)\right]$$

finalmente:

$$\vec{a}_{\perp} = \frac{1}{\gamma^2\left(1+\frac{\vec{u}'\cdot\vec{v}}{c^2}\right)^3}\left[\vec{a}'_{\perp} + \frac{1}{c^2}\vec{v}\times\left(\vec{a}'\times\vec{u}'\right)\right]$$