Problema 1: 28/08

With the same assumptions as in Problem 6.10 discuss the conservation of angular momentum. Show that the differential and integral forms of the conservation law are

$$\frac{\partial}{\partial t}(\mathscr{L}_{mech} + \mathscr{L}_{held}) + \nabla \cdot \overleftrightarrow{\mathbf{M}} = 0$$

and

$$\frac{d}{dt}\int_V d^3x \quad (\mathscr{L}_{mech} + \mathscr{L}_{held}) + \int_S da \quad \hat{\mathbf{n}} \cdot \overleftrightarrow{\mathbf{M}} = 0$$

where the field angular-momentum density is

$$\mathscr{L}_{field} = \mathbf{x} \times \mathbf{g} = \frac{\mu\epsilon}{4\pi c} x \times (\mathbf{E} \times \mathbf{H})$$

and the flux of angular momentum is described by the tensor

$$\overleftrightarrow{\mathbf{M}} = \overleftrightarrow{\mathbf{T}} \times x$$

Note: Here we have used the diadic notation for $\mathbf{M}_{ij}$ and $\mathbf{T}_{ij}$. The double-headed arrow conveys a fairly obvious meaning. For example, $\hat{n} \cdot \overleftrightarrow{M}$ is a vector whose jth component is $\Sigma_i n_i M_{ij}$. The second-rank $\overleftrightarrow{M}$ can be written as a third-rank tensor $M_{ij} = T_{ij}x_k - T_{ik}x_j$. But the indices j and k is antisymmetric and so has only three independent elements. Including the index i, $M_i j$ therefore has nine components and can be written as a pseudo tensor of a second rank, as above.

Vamos assumir a conservação de momento linear deduzida em classe e também na lista de exercícios 3:

$$\frac{d}{dt}(\mathbf{P}_m + \mathbf{P}_c)_k = \sum_{m=1}^{3} \oint_{S(V)} T_{km} n_m \tag{1.1}$$

Uma vez que temos o momento angular definido como o produto vetorial da posição com o momento das massas carregadas definido como

$$\mathbf{L} = \mathbf{r} \times \mathbf{P}_m \tag{1.2}$$

Assim como no caso do momento linear queremos analisar suas características de conservação. Para ser conservada temos então que visualizar sua variação no tempo dada por

$$\begin{aligned}\frac{d}{dt}\mathbf{L} &= \frac{d}{dt}(\mathbf{r} \times \mathbf{P}_m) \\ &= \frac{d}{dt}\mathbf{r} \times \mathbf{P}_m + \mathbf{r} \times \frac{d}{dt}\mathbf{P}_m \\ &= \mathbf{r} \times \frac{d}{dt}\mathbf{P}_m\end{aligned} \tag{1.3}$$

Podemos substituir 1.1 em 1.3 para trabalharmos explicitamente

$$\begin{aligned}\frac{d}{dt}\mathbf{L} &= \mathbf{r}\times\frac{d}{dt}\mathbf{P}_m \\ &= \mathbf{r}\times\left[\oint_{S(V)} T_{km}n_m - \frac{d}{dt}\int_V d^3r'\frac{1}{4\pi c}\mathbf{E}\times\mathbf{B}\right]\end{aligned} \tag{1.4}$$

uma vez que

$$\frac{d}{dt}\mathbf{P}_c = \frac{d}{dt}\int_{V_\infty} d^3r'\frac{1}{4\pi c}\mathbf{E}\times\mathbf{B} \tag{1.5}$$

Aplicando o teorema de Gauss na integral de linha em 1.4 considerando as propriedades vetoriais apontadas no enunciado do problema chegamos em

$$\begin{aligned}\frac{d}{dt}\mathbf{L} &= \mathbf{r}\times\left[\int_{V_\infty} d^3r'\nabla\cdot\overleftrightarrow{\mathbf{T}} - \frac{d}{dt}\int_V d^3r'\frac{1}{4\pi c}\mathbf{E}\times\mathbf{B}\right] \\ &= \int_{V_\infty} d^3r' \quad \mathbf{r}\times\left[\nabla\cdot\overleftrightarrow{\mathbf{T}} - \frac{d}{dt}\frac{1}{4\pi c}\mathbf{E}\times\mathbf{B}\right] \\ &= \int_{V_\infty} d^3r' \quad \left[-\nabla\cdot\left(\mathbf{r}\times\overleftrightarrow{\mathbf{T}}\right) - \frac{d}{dt}\left(\mathbf{r}\times\mathbf{g}\right)\right] \\ &= \int_{V_\infty} d^3r' \quad \left[-\nabla\cdot\overleftrightarrow{\mathbf{M}} - \frac{d}{dt}\left(\mathbf{r}\times\mathbf{g}\right)\right. \\ &= \int_{V_\infty} d^3r' \quad \left[-\nabla\cdot\overleftrightarrow{\mathbf{M}} - \frac{d}{dt}\mathscr{L}_{field}\right]\end{aligned} \tag{1.6}$$

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$$\overleftrightarrow{\mathbf{M}} = \overleftrightarrow{\mathbf{T}}\times\mathbf{r} \qquad \text{e} \qquad \mathbf{g} \equiv \frac{1}{4\pi c}\mathbf{E}\times\mathbf{B}.$$

Finalmente, podemos demonstrar as equações enunciadas anteriormente. Para isso, vamos propor uma densidade de momento angular no espaço $\mathscr{L}_{mech}$.

$$\mathbf{L} = \int_V d^3r'\mathscr{L}_{mech} \implies \frac{d}{dt}\mathbf{L} = \frac{d}{dt}\int_V d^3r'\mathscr{L}_{mech} \tag{1.7}$$

logo,

$$\begin{aligned}\frac{d}{dt}\int_V d^3r'\mathscr{L}_{mech} &= \int_{V_\infty} d^3r' \quad \left[-\nabla\cdot\overleftrightarrow{\mathbf{M}} - \frac{d}{dt}\mathscr{L}_{field}\right] \\ \frac{d}{dt}\int_V d^3r'\left(\mathscr{L}_{mech}+\mathscr{L}_{field}\right) &= -\int_{V_\infty} d^3r'\nabla\cdot\overleftrightarrow{\mathbf{M}}\end{aligned} \tag{1.8}$$

assim,

$$\frac{d}{dt}\oint_V d^3r'\left(\mathscr{L}_{mech}+\mathscr{L}_{field}\right) + \int_{S(V)} da \quad \hat{\mathbf{n}}\cdot\overleftrightarrow{\mathbf{M}} = 0 \tag{1.9}$$

$$\frac{\partial}{\partial t}\left(\mathscr{L}_{mech}+\mathscr{L}_{field}\right) + \nabla\cdot\overleftrightarrow{\mathbf{M}} = 0 \tag{1.10}$$