Calculando o Campo Indução Magnética e o Campo Elétrico

Vamos agora calcular os campos $\mathbf{B}$ e $\mathbf{E}$. Comecemos com o cálculo de $\mathbf{B}$:

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

ou, em termos de componentes,

$$B_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} \partial_j A_k,$$

onde ε_{ijk} é o símbolo de Levi-Civita, que é dado por

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{se } (i,j,k) \text{ for uma permutação par de } (1,2,3)\,, \\ 0, & \text{se pelo menos dois dos índices } i,j,k \text{ forem iguais e} \\ -1, & \text{se } (i,j,k) \text{ for uma permutação ímpar de } (1,2,3)\,. \end{cases}$$

Também utilizemos a notação

$$\partial_j = \frac{\partial}{\partial x_j},$$

para $j = 1, 2, 3$. A convenção de Einstein para somas permite que escrevamos

$$B_i = \varepsilon_{ijk} \partial_j A_k,$$

onde subentendemos que os índices j e k estão somados de 1 a 3, porque aparecem repetidos no mesmo termo. Temos, assim,

$$\begin{aligned} \partial_j A_k &= \frac{q}{c} \partial_j \left(\frac{v_k}{R - \mathbf{R} \cdot \boldsymbol{\beta}} \right) \\ &= \frac{q}{c} \left[\frac{\partial_j v_k}{R - \mathbf{R} \cdot \boldsymbol{\beta}} + v_k \partial_j \left(\frac{1}{R - \mathbf{R} \cdot \boldsymbol{\beta}} \right) \right] \\ &= \frac{q}{c} \left[\frac{\partial_j v_k}{R - \mathbf{R} \cdot \boldsymbol{\beta}} - \frac{v_k}{(R - \mathbf{R} \cdot \boldsymbol{\beta})^2} \left(\partial_j R - \boldsymbol{\beta} \cdot \partial_j \mathbf{R} - \mathbf{R} \cdot \partial_j \boldsymbol{\beta} \right) \right]. \end{aligned}$$

Também,

$$\begin{aligned} \partial_j v_k &= \frac{d v_k}{d t_R} \partial_j t_R \\ &= a_k \partial_j t_R, \end{aligned}$$

onde

$$\begin{aligned} \mathbf{a} &\equiv \frac{d\mathbf{v}}{dt_R} \\ &= \frac{d\mathbf{v}(t_R)}{dt_R} \end{aligned}$$

e, portanto,

$$a_k = \hat{\mathbf{x}}_k \cdot \mathbf{a}.$$

Façamos agora o cálculo de $\partial_j t_R$:

$$\begin{aligned}
\partial_j t_R &= \partial_j \left(t - \frac{R}{c}\right) \\
&= -\frac{1}{c}\partial_j R.
\end{aligned}$$

Mas,

$$\begin{aligned}
\partial_j R &= \frac{1}{2R}\partial_j R^2 \\
&= \frac{1}{2R}\partial_j \left(\mathbf{R} \cdot \mathbf{R}\right) \\
&= \frac{\mathbf{R}}{R} \cdot \partial_j \mathbf{R}
\end{aligned}$$

e, portanto,

$$\begin{aligned}
\partial_j t_R &= -\frac{1}{c}\frac{\mathbf{R}}{R} \cdot \partial_j \mathbf{R} \\
&= -\frac{1}{c}\hat{\mathbf{R}} \cdot \partial_j \mathbf{R}
\end{aligned}$$

Como

$$\mathbf{R} = \mathbf{r} - \mathbf{r}_0\left(t_R\right),$$

segue que

$$\begin{aligned}
\partial_j \mathbf{R} &= \hat{\mathbf{x}}_j - \frac{d\mathbf{r}_0\left(t_R\right)}{dt_R}\partial_j t_R \\
&= \hat{\mathbf{x}}_j - \mathbf{v}\partial_j t_R.
\end{aligned}$$

Sendo assim,

$$\begin{aligned}
\partial_j t_R &= -\frac{1}{c}\hat{\mathbf{R}} \cdot \partial_j \mathbf{R} \\
&= -\frac{1}{c}\hat{\mathbf{R}} \cdot \left(\hat{\mathbf{x}}_j - \mathbf{v}\partial_j t_R\right) \\
&= -\frac{1}{c}\hat{\mathbf{R}} \cdot \hat{\mathbf{x}}_j + \hat{\mathbf{R}} \cdot \boldsymbol{\beta}\partial_j t_R,
\end{aligned}$$

ou seja,

$$\left(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta}\right)\partial_j t_R = -\frac{1}{c}\hat{\mathbf{R}} \cdot \hat{\mathbf{x}}_j,$$

resultando em

$$\partial_j t_R = -\frac{1}{c}\frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)}.$$

Portanto,

$$\partial_j v_k = -\frac{a_k}{c}\frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)}.$$

O próximo passo é calcularmos $\partial_j R$, que, como vimos logo acima,

$$\partial_j R = \frac{\mathbf{R}}{R}\cdot\partial_j\mathbf{R}$$

e, então, como já temos $\partial_j\mathbf{R}$, vem[1]:

$$\begin{aligned}
\partial_j R &= \hat{\mathbf{R}}\cdot\left(\hat{\mathbf{x}}_j-\mathbf{v}\partial_j t_R\right)\\
&= \hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j-\hat{\mathbf{R}}\cdot\mathbf{v}\partial_j t_R\\
&= \hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j+\frac{\hat{\mathbf{R}}\cdot\frac{\mathbf{v}}{c}}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)}\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j\\
&= \left(1+\frac{\hat{\mathbf{R}}\cdot\boldsymbol{\beta}}{1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}}\right)\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j\\
&= \frac{1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}+\hat{\mathbf{R}}\cdot\boldsymbol{\beta}}{1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}}\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j\\
&= \frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}}.
\end{aligned}$$

A seguir, o próximo termo no cálculo de $\partial_j A_k$ tem a quantidade $\partial_j\mathbf{R}$ que, já vimos que é dada por:

$$\partial_j\mathbf{R} = \hat{\mathbf{x}}_j-\mathbf{v}\partial_j t_R.$$

Logo, usando nosso resultado para $\partial_j t_R$, dá

$$\partial_j\mathbf{R} = \hat{\mathbf{x}}_j+\boldsymbol{\beta}\frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)}.$$

[1]Notemos que, só para esclarecer a notação, escrevemos, por exemplo,

$$\mathbf{r} = \hat{\mathbf{x}}_k x_k.$$

Logo,

$$\begin{aligned}
\mathbf{r}\cdot\hat{\mathbf{x}}_j &= \left(\hat{\mathbf{x}}_k\cdot\hat{\mathbf{x}}_j\right)x_k\\
&= \delta_{kj}x_k\\
&= x_j.
\end{aligned}$$

Então, segue que

$$
\begin{aligned}
\boldsymbol{\beta}\cdot\partial_j\mathbf{R} &= \boldsymbol{\beta}\cdot\hat{\mathbf{x}}_j+\boldsymbol{\beta}\cdot\boldsymbol{\beta}\frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)} \\
&= \boldsymbol{\beta}\cdot\hat{\mathbf{x}}_j+\beta^2\frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)}.
\end{aligned}
$$

Finalmente, só falta calcularmos o seguinte fator:

$$
\begin{aligned}
\partial_j\boldsymbol{\beta} &= \frac{1}{c}\partial_j\mathbf{v} \\
&= \frac{1}{c}\frac{d\mathbf{v}}{dt_R}\partial_j t_R \\
&= -\frac{\mathbf{a}}{c^2}\frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)}.
\end{aligned}
$$

Assim,

$$
\begin{aligned}
\partial_j A_k &= \frac{q}{c}\left[\frac{\partial_j v_k}{R-\mathbf{R}\cdot\boldsymbol{\beta}}-\frac{v_k}{\left(R-\mathbf{R}\cdot\boldsymbol{\beta}\right)^2}\left(\partial_j R-\boldsymbol{\beta}\cdot\partial_j\mathbf{R}-\mathbf{R}\cdot\partial_j\boldsymbol{\beta}\right)\right] \\
&= \frac{q}{c}\left[\frac{\partial_j v_k}{R\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)}-\frac{v_k\left(\partial_j R-\boldsymbol{\beta}\cdot\partial_j\mathbf{R}-R\hat{\mathbf{R}}\cdot\partial_j\boldsymbol{\beta}\right)}{R^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2}\right] \\
&= \frac{q}{c}\left[\frac{\partial_j v_k}{R\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)}+\frac{v_k\hat{\mathbf{R}}\cdot\partial_j\boldsymbol{\beta}}{R\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2}-\frac{v_k\left(\partial_j R-\boldsymbol{\beta}\cdot\partial_j\mathbf{R}\right)}{R^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2}\right].
\end{aligned}
$$