proof of Bernstein inequalities

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1) By http://planetmath.org/ChernoffCramerBoundChernoff-Cramèr bound, we have:

$$\Pr\left\{\sum_{i=1}^{n}\left(X_i - E[X_i]\right) > \varepsilon\right\} \leq \exp\left[-\sup_{t>0}\left(t\varepsilon - \psi(t)\right)\right]$$

where

$$\psi(t) = \sum_{i=1}^{n}\left(\ln E\left[e^{tX_i}\right] - tE\left[X_i\right]\right)$$

Since $\ln x \leq x - 1 \ \forall x \geq 0$,

$$\begin{aligned}
\psi(t) &= \sum_{i=1}^{n}\left(\ln E\left[e^{tX_i}\right] - tE\left[X_i\right]\right) \\
&\leq \sum_{i=1}^{n} E\left[e^{tX_i}\right] - tE\left[X_i\right] - 1 \\
&= \sum_{i=1}^{n} E\left[1 + tX_i + \frac{1}{2}t^2X_i^2 + \sum_{k=3}^{+\infty}\frac{t^kX_i^k}{k!}\right] - tE\left[X_i\right] - 1 \\
&= \sum_{i=1}^{n}\left(\frac{1}{2}t^2E\left[X_i^2\right] + \sum_{k=3}^{+\infty}\frac{t^kE\left[X_i^k\right]}{k!}\right) \\
&= \frac{1}{2}t^2\sum_{i=1}^{n}E\left[X_i^2\right] + \sum_{k=3}^{+\infty}\frac{t^k\sum_{i=1}^{n}E\left[X_i^k\right]}{k!} \\
&\leq \frac{1}{2}t^2\sum_{i=1}^{n}E\left[X_i^2\right] + \sum_{k=3}^{+\infty}\frac{t^k\sum_{i=1}^{n}E\left[\left|X_i\right|^k\right]}{k!},
\end{aligned}$$

and, keeping in mind hypotheses a) and b),

$$\psi(t) \leq \frac{1}{2}t^2v^2 + \sum_{k=3}^{+\infty}\frac{t^k}{2}v^2c^{k-2} = \frac{1}{2}t^2v^2 + \frac{1}{2}t^3v^2c\sum_{k=0}^{+\infty}(tc)^k$$

Now, if $tc < 1$, we obtain

$$\psi(t) \leq \frac{1}{2}t^2v^2\left(1 + \frac{tc}{1-tc}\right) = \frac{v^2t^2}{2\left(1-tc\right)}$$

whence

$$\sup_{t>0}\left(t\varepsilon-\psi(t)\right)\geq\sup_{0<t<\frac{1}{c}}\left(t\varepsilon-\frac{v^2t^2}{2\left(1-tc\right)}\right)$$

By elementary calculus, we obtain the value of t that maximizes the expression in brackets (out of the two roots of the second degree polynomial equation, we choose the one which is $<\frac{1}{c}$):

$$t_{opt}=\frac{v^2+2c\varepsilon-v^2\sqrt{1+\frac{2c\varepsilon}{v^2}}}{c\left(v^2+2c\varepsilon\right)}=\frac{1}{c}\left(1-\frac{1}{\sqrt{1+\frac{2c\varepsilon}{v^2}}}\right)$$

which, once plugged into the bounds, yields

$$\Pr\left\{\sum_{i=1}^{n}\left(X_i-E[X_i]\right)>\varepsilon\right\}\leq\exp\left[-\frac{v^2}{c^2}\left(1+\frac{c\varepsilon}{v^2}-\sqrt{1+2\frac{c\varepsilon}{v^2}}\right)\right]$$

Observing that $\sqrt{1+x}\leq 1+\frac{1}{2}x$, one gets:

$$t_{opt}=\frac{1}{c}\left(1-\frac{1}{\sqrt{1+\frac{2c\varepsilon}{v^2}}}\right)\leq\frac{1}{c}\left(1-\frac{1}{1+\frac{c\varepsilon}{v^2}}\right)=\frac{\varepsilon}{v^2+c\varepsilon}=t^{'}<\frac{1}{c}$$

Plugging t' in the bound expression, the sub-optimal yet more easily manageable formula is obtained:

$$\Pr\left\{\sum_{i=1}^{n}\left(X_i-E[X_i]\right)>\varepsilon\right\}\leq\exp\left(-\frac{\varepsilon^2}{2\left(v^2+c\varepsilon\right)}\right)$$

which is obviously a worse bound than the preceeding one, since $t'\neq t_{opt}$. One can also verify the consistency of this inequality directly proving that, for any $x\geq 0$,

$$1+x-\sqrt{1+2x}\geq\frac{x^2}{2\left(1+x\right)}$$

(see `http://planetmath.org/ASimpleMethodForComparingRealFunctions`here for an easy way, which can be used with $x_0=0$)

2) To prove this more specialized statement let's recall that the condition

$$\Pr\left\{|X_i|\leq M\right\}=1\ \ \forall i$$

implies that, for all i,

$$E[|X_i|^k] \leq M^k \quad \forall k \geq 0$$

(See `http://planetmath.org/RelationBetweenAlmostSurelyAbsolutelyBoundedRandomVariab`
for a proof.)

Now, it's enough to verify that the condition

$$E[|X_i|^k] \leq M^k$$

imply both conditions a) and b) in part 1).

Indeed, part a) is obvious, while for part b) one happens to have:

$$E[|X_i|^k] \leq E\left[X_i^2\right] M^{k-2}$$

(see `http://planetmath.org/AbsoluteMomentsBoundingNecessaryAndSufficientCondition`he
for a proof).

So

$$\sum_{i=1}^{n} E[|X_i|^k] \leq \sum_{i=1}^{n} E\left[X_i^2\right] M^{k-2} = v^2 M^{k-2}$$

Let's find a value for c such that $v^2 M^{k-2} \leq \frac{k!}{2} v^2 c^{k-2}$, thus satisfying part b) of the hypotheses.

After simplifying, we have to study the inequality

$$k! c^{k-2} \geq 2 \cdot M^{k-2}$$

for any $k \geq 3$. Let's proceed by induction. For $k = 3$, we have

$$6c \geq 2M$$

which suggests $c = \frac{M}{3}$. Let's now verify if this position is consistent with the inductive hypothesis:

$$(k+1)! = (k+1)\, k! \geq (k+1) \cdot 2 \cdot 3^{k-2} \geq 3 \cdot 2 \cdot 3^{k-2} = 2 \cdot 3^{(k+1)-2}$$

which confirms the validity of the choice $c = \frac{M}{3}$, which has to be plugged into the former bound to obtain the new one.

[to be continued...]