hitting times are stopping times

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Let $(\mathcal{F}_t)_{t\in\mathbb{T}}$ be a http://planetmath.org/FiltrationOfSigmaAlgebrasfiltration on a measurable space $(\Omega, \mathcal{F})$. If X is an adapted stochastic process taking values in a measurable space $(E, \mathcal{A})$ then the *hitting time* of a set $S \in \mathcal{A}$ is defined as

$$\begin{aligned}
&\tau\colon \Omega \to \mathbb{T} \cup \{\pm\infty\},\\
&\tau(\omega) = \inf\left\{t \in \mathbb{T} : X_t(\omega) \in S\right\}.
\end{aligned}$$

We suppose that $\mathbb{T}$ is a closed subset of $\mathbb{R}$, so the hitting time τ will indeed lie in $\mathbb{T}$ whenever it is finite. The main cases are discrete-time when $\mathbb{T} = \mathbb{Z}_+$ and continuous-time where $\mathbb{T} = \mathbb{R}_+$. An important property of hitting times is that they are stopping times, as stated below for the different cases.

Discrete-time processes

For discrete-time processes, hitting times are easily shown to be stopping times.

Theorem. *If the index set $\mathbb{T}$ is discrete, then the hitting time τ is a stopping time.*

Proof. For any $s \leq t \in \mathbb{T}$ then X_s will be $\mathcal{F}_t/\mathcal{A}$-measurable, as it is adapted. So, by the fact that the σ-algebra $\mathcal{F}_t$ is closed under taking countable unions,

$$\{\tau \leq t\} = \bigcup_{\substack{s\in\mathbb{T},\\ s\leq t}} X_s^{-1}(S) \in \mathcal{F}_t$$

as required. $\square$

Continuous processes

For continuous-time processes it is not necessarily true that a hitting time is even measurable, unless further conditions are imposed. Processes with continuous sample paths can be dealt with easily.

Theorem. *Suppose that X is a continuous and adapted process taking values in a metric space E. Then, the hitting time τ of any closed subset $S \subseteq E$ is a stopping time.*

Proof. We may suppose that S is nonempty, and define the continuous function $d_S(x) \equiv \inf\{d(x,y)\colon y \in S\}$ on E. Then, τ is the first time at which $Y_t \equiv d_S(X_t)$ hits 0. Letting U be any countable and dense subset of $\mathbb{T} \cap [0,t]$ then the continuity of the sample paths of Y gives,

$$\{\tau \le t\} = \left\{\inf_{u \in U} Y_u = 0\right\}.$$

As the infimum of a countable set of measurable functions is measurable, this shows that $\{\tau \le t\}$ is in $\mathcal{F}_t$. □

Right-continuous processes

Right-continuous processes are more difficult to handle than either the discrete-time and continuous sample path situations. The first time at which a right-continuous process hits a given value need not be measurable. However, it can be shown to be universally measurable, and the following result holds.

Theorem. *Suppose that X is a right-continuous and adapted process taking values in a metric space E, and that the filtration $(\mathcal{F}_t)$ is universally complete. Then, the hitting time τ of any closed subset $S \subseteq E$ is a stopping time.*

In particular, the hitting time of any closed set $S \subseteq \mathbb{R}$ for an adapted right-continuous and real-valued process is a stopping time.

The proof of this result is rather more involved than the case for continuous processes, and the condition that $\mathcal{F}_t$ is universally complete is necessary.

Progressively measurable processes

The début $D(A)$ of a set $A \subseteq \mathbb{T} \times \Omega$ is defined to be the hitting time of $\{1\}$ for the process 1_A,

$$D(A)(\omega) = \inf\{t \in \mathbb{T} : (t,\omega) \in A\}.$$

An important result for continuous-time stochastic processes is the début theorem.

Theorem (Début theorem)**.** *Suppose that the filtration $(\mathcal{F}_t)$ is right-continuous and universally complete. Then, the début $D(A)$ of a progressively measurable $A \subseteq \mathbb{T} \times \Omega$ is a stopping time.*

Proofs of this typically rely upon properties of analytic sets, and are therefore much more complicated than the result above for right-continuous processes.

A process X taking values in a measurable space $(E, \mathcal{A})$ is said to be progressive if the set $X^{-1}(S)$ is progressively measurable for every $S \in \mathcal{A}$. In particular, the hitting time of S is equal to the début of $X^{-1}(S)$ and the début theorem has the following immediate corollary.

Theorem. *Suppose that the filtration $(\mathcal{F}_t)$ is right-continuous and universally complete, and that X is a progressive process taking values in a measurable space $(E, \mathcal{A})$. Then, the hitting time τ of any set $S \in \mathcal{A}$ is a stopping time.*