quadratic variation of Brownian motion

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Theorem. *Let $(W_t)_{t\in\mathbb{R}_+}$ be a standard Brownian motion. Then, its quadratic variation exists and is given by*

$$[W]_t = t.$$

As Brownian motion is a martingale and, in particular, is a semimartingale then http://planetmath.org/QuadraticVariationOfASemimartingaleits quadratic variation must exist. We just need to compute its value along a sequence of partitions.

If $P = \{0 = t_0 \le t_1 \le \cdots \le t_m = t\}$ is a http://planetmath.org/SubintervalPartitionparti of the interval $[0, t]$, then the quadratic variation on P is

$$[W]^P = \sum_{k=1}^{m} (W_{t_k} - W_{t_{k-1}})^2.$$

Using the property that the increments $W_{t_k} - W_{t_{k-1}}$ are independent normal random variables with mean zero and variance $t_k - t_{k-1}$, the mean and variance of $[W]^P$ are

$$\begin{aligned}
\mathbb{E}\left[[W]^P\right] &= \sum_{k=1}^{m} \mathbb{E}\left[(W_{t_k} - W_{t_{k-1}})^2\right] = \sum_{k=1}^{m} (t_k - t_{k-1}) = t,\\
\operatorname{Var}\left[[W]^P\right] &= \sum_{k=1}^{m} \operatorname{Var}\left[(W_{t_k} - W_{t_{k-1}})^2\right] = \sum_{k=1}^{m} 2(t_k - t_{k-1})^2\\
&\le 2|P| \sum_{k=1}^{m} (t_k - t_{k-1}) = 2|P|t.
\end{aligned}$$

Here, $|P| = \max_k (t_k - t_{k-1})$ is the mesh of the partition. If $(P_n)_{n=1,2,\ldots}$ is a sequence of partitions of $[0, t]$ with mesh going to zero as $n \to \infty$ then,

$$\mathbb{E}\left[([W]^{P_n} - t)^2\right] \le 2|P_n|t \to 0$$

as $n \to \infty$. This shows that $[W]^{P_n} \to t$ in the L^2 norm and, in particular, converges in probability. So, $[W]_t = t$.