proof of Martingale criterion

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Let $(\tau_k)_{k\geq 1}$ be a localizing sequence of stopping times for X. Then:

$$\Lambda_{\{\tau_k \geq n\}} \uparrow \Lambda_\Omega \text{ a.s. } k \to \infty, \forall n \in \mathbb{N}$$

since $\{\tau_k \geq n\} \uparrow_k \bigcup_1^\infty \{\tau_k \geq n\} \forall\ n \in \mathbb{N}$.

$$\bigcup_{k=1}^{\infty} \{\tau_k \geq n\} = \Omega \text{ a.s., since } \tau_k \to \infty, \text{ a.s.}$$

Now assume $EX_n^- < \infty, \forall n \geq n_0$ (the case $EX_n^+ < \infty$ being analogous).
1) We have $EX_n^- < \infty\ \forall n \in \mathbb{N}$.
We proceed by (backward) induction. For $n = n_0$ the statement holds.
$n \mapsto n-1$:

$$(X^{\tau_k})^- = (X^-_{\tau_k \wedge n})_{n \in \mathbb{N}} \text{ submartingale}$$

We have:

$$\begin{aligned}
\int_{\{\tau_k \geq n\}} X^-_{n-1}\, dP &= \int_{\{\tau_k \geq n\}} X^-_{\tau_k \wedge (n-1)}\, dP \\
&\leq \int_{\{\tau_k \geq n\}} X^-_{\tau_k \wedge n}\, dP = \int_{\{\tau_k \geq n\}} X^-_n\, dP \\
&\leq \int X^-_n\, dP < \infty
\end{aligned}$$

Where the first to second line is the submartingale property and the last line follows by induction hypothesis.
Using Fatou we get:

$$\begin{aligned}
\int X^-_{n-1}\, dP &= \int \lim_{k\to\infty} X^-_{n-1} \Lambda_{\{\tau_k \geq n\}}\, dP \\
&\leq \liminf_{k\to\infty} \int X^-_{n-1} \Lambda_{\{\tau_k \geq n\}}\, dP \\
&\leq \int X^-_n\, dP < \infty
\end{aligned}$$

2) We have $X_n \in \mathscr{L}^1 (n \in \mathbb{N})$.

We have $X^+_{\tau_k \wedge n} \to X^+_n$ a.s., $k \to \infty, \forall n \in \mathbb{N}$. With Fatou we get:

$$\begin{aligned}
EX_n^+ &\leq \liminf_{k\to\infty} EX^+_{\tau_k \wedge n} \\
&= EX_0 + \liminf_{k\to\infty} EX^-_{\tau_k \wedge n} \\
&= EX_0 + \liminf_{k\to\infty} E\left(\sum_{j=0}^{n-1} X_j^- \Lambda_{\{\tau_k = j\}} + X_n^- \Lambda_{\{\tau_k \geq n\}}\right) \\
&\leq EX_0 + \sum_{j=0}^{n} EX_j^- < \infty
\end{aligned}$$

With *1)* $X_n \in \mathscr{L}^1$ follows.
3)
X is a martingale, because $X_n^{\tau_k} \to X_n$ a.s. $k \to \infty$ and:

$$|X_n^{\tau_k}| \leq \sum_{j=0}^{n} |X_j| \in \mathscr{L}^1 \text{ (} \mathscr{L}^1 \text{ -bound)}$$

Thus $X_n^{\tau_k} \xrightarrow{L^1} X_n, k \to \infty \ \forall n \in \mathbb{N}$ by bounded convergence theorem. Hence X must be martingale and we are done. $\square$