Lindeberg's central limit theorem

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Theorem (Lindeberg's central limit theorem)

Let $X_1, X_2, \dots$ be independent random variables with distribution functions $F_1, F_2, \dots$, respectively, such that $EX_n = \mu_n$ and $\operatorname{Var} X_n = \sigma_n^2 < \infty$, with at least one $\sigma_n > 0$. Let

$$S_n = X_1 + \cdots + X_n \text{ and } s_n = \sqrt{\operatorname{Var}(S_n)} = \sqrt{\sigma_1^2 + \cdots + \sigma_n^2}.$$

Then the normalized partial sums $\frac{S_n - ES_n}{s_n}$ converge `http://planetmath.org/ConvergenceInDi` distribution to a random variable with normal distribution $N(0,1)$ (i.e. the *normal convergence* holds,) if the following *Lindeberg condition* is satisfied:

$$\forall \varepsilon > 0, \lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^{n} \int_{|x-\mu_k|>\varepsilon s_n} (x-\mu_k)^2 dF_k(x) = 0.$$

Corollary 1 (Lyapunov's central limit theorem)

If the Lyapunov condition

$$\frac{1}{s_n^{2+\delta}} \sum_{k=1}^{n} E|X_k - \mu_k|^{2+\delta} \xrightarrow[n\to\infty]{} 0$$

is satisfied for some $\delta > 0$, the normal convergence holds.

Corollary 2

If $X_1, X_2, \dots$ are identically distributed random variables, $EX_n = \mu$ and $\operatorname{Var} S_n = \sigma^2$, with $0 < \sigma < \infty$, then the normal convergence holds; i.e. $\frac{S_n - n\mu}{\sigma\sqrt{n}}$ converges `http://planetmath.org/ConvergenceInDistribution`in distribution to a random variable with distribution $N(0,1)$.

Reciprocal (Feller)

The reciprocal of Lindeberg's central limit theorem holds under the following additional assumption:

$$\max_{1\le k\le n} \left(\frac{\sigma_k^2}{s_n^2} \right) \xrightarrow[n\to\infty]{} 0.$$

Historical remark