conditional expectation under change of measure

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Let $\mathbb{P}$ be a given probability measure on some σ-algebra $\mathcal{F}$. Suppose a new probability measure $\mathbb{Q}$ is defined by $d\mathbb{Q} = Z\,d\mathbb{P}$, using some $\mathcal{F}$-measurable random variable Z as the Radon-Nikodym derivative. (Necessarily we must have $Z \geq 0$ almost surely, and $\mathbb{E}Z = 1$.)

We denote with $\mathbb{E}$ the expectation with respect to the measure $\mathbb{P}$, and with $\mathbb{E}^{\mathbb{Q}}$ the expectation with respect to the measure $\mathbb{Q}$.

Theorem 1. *If $\mathbb{Q}$ is restricted to a sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, then the restriction has the conditional expectation $\mathbb{E}[Z \mid \mathcal{G}]$ as its Radon-Nikodym derivative: $d\mathbb{Q}_{|\mathcal{G}} = \mathbb{E}[Z \mid \mathcal{G}]\,d\mathbb{P}_{|\mathcal{G}}$.*

In other words,

$$\frac{d\mathbb{Q}_{|\mathcal{G}}}{d\mathbb{P}_{|\mathcal{G}}} = \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_{|\mathcal{G}}.$$

Proof. It is required to prove that, for all $B \in \mathcal{G}$,

$$\mathbb{Q}(B) = \mathbb{E}\big[\mathbb{E}[Z \mid \mathcal{G}]\,1_B\big].$$

But this follows at once from the law of iterated conditional expectations:

$$\mathbb{E}\big[\mathbb{E}[Z \mid \mathcal{G}]\,1_B\big] = \mathbb{E}\big[\mathbb{E}[Z1_B \mid \mathcal{G}]\big] = \mathbb{E}[Z1_B] = \mathbb{Q}(B). \qquad \square$$

Theorem 2. *Let $\mathcal{G} \subseteq \mathcal{F}$ be any sub-σ-algebra. For any $\mathcal{F}$-measurable random variable X,*

$$\mathbb{E}[Z \mid \mathcal{G}]\,\mathbb{E}^{\mathbb{Q}}[X \mid \mathcal{G}] = \mathbb{E}[ZX \mid \mathcal{G}].$$

That is,

$$\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_{|\mathcal{G}} \mathbb{E}^{\mathbb{Q}}[X \mid \mathcal{G}] = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\,X \mid \mathcal{G}\right].$$

Proof. Let $Y = \mathbb{E}[Z \mid \mathcal{G}]$, and $B \in \mathcal{G}$. We find:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\big[1_B\,\mathbb{E}[ZX \mid \mathcal{G}]\big] &= \mathbb{E}\big[Y1_B\,\mathbb{E}[ZX \mid \mathcal{G}]\big] && (\text{since } d\mathbb{Q}_{|\mathcal{G}} = Y\,d\mathbb{P}_{|\mathcal{G}}) \\
&= \mathbb{E}\big[\mathbb{E}[Y1_B\,ZX \mid \mathcal{G}]\big] \\
&= \mathbb{E}[Y1_B\,ZX] \\
&= \mathbb{E}^{\mathbb{Q}}[Y1_B\,X] && (\text{since } d\mathbb{Q} = Z\,d\mathbb{P}) \\
&= \mathbb{E}^{\mathbb{Q}}\big[1_B\,\mathbb{E}^{\mathbb{Q}}[YX \mid \mathcal{G}]\big].
\end{aligned}$$

Since $B \in \mathcal{G}$ is arbitrary, we can equate the $\mathcal{G}$-measurable integrands:

$$\mathbb{E}[ZX \mid \mathcal{G}] = \mathbb{E}^{\mathbb{Q}}[YX \mid \mathcal{G}] = Y\mathbb{E}^{\mathbb{Q}}[X \mid \mathcal{G}]. \qquad \square$$

Observe that if $d\mathbb{Q}/d\mathbb{P} > 0$ almost surely, then

$$\mathbb{E}^{\mathbb{Q}}[X \mid \mathcal{G}] = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} X \mid \mathcal{G}\right] \Big/ \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_{|\mathcal{G}} .$$

Theorem 3. *If X_t is a martingale with respect to $\mathbb{Q}$ and some filtration $\{\mathcal{F}_t\}$, then $X_t Z_t$ is a martingale with respect to $\mathbb{P}$ and $\{\mathcal{F}_t\}$, where $Z_t = \mathbb{E}[Z \mid \mathcal{F}_t]$.*

Proof. First observe that $X_t Z_t$ is indeed $\mathcal{F}_t$-measurable. Then, we can apply Theorem 2, with X in the statement of that theorem replaced by X_t, Z replaced by Z_t, $\mathcal{F}$ replaced by $\mathcal{F}_t$, and $\mathcal{G}$ replaced by $\mathcal{F}_s$ $(s \leq t)$, to obtain:

$$\mathbb{E}[X_t Z_t \mid \mathcal{F}_s] = Z_s \, \mathbb{E}^{\mathbb{Q}}[X_t \mid \mathcal{F}_s] = Z_s X_s ,$$

thus proving that $X_t Z_t$ is a martingale under $\mathbb{P}$ and $\{\mathcal{F}_t\}$. $\square$

Sometimes the random variables Z_t in Theorem 3 are written as $\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_t$. (This is a Radon-Nikodym derivative *process*; note that Z_t defined as $Z_t = \mathbb{E}[Z \mid \mathcal{F}_t]$ is always a martingale under $\mathbb{P}$ and $\{\mathcal{F}_t\}$.)

Under the hypothesis $Z_t > 0$, there is an alternate restatement of Theorem 3 that may be more easily remembered:

Theorem 4. *Let $Z_t = (d\mathbb{Q}/d\mathbb{P})_t > 0$ almost surely. Then X_t is a martingale with respect to $\mathbb{P}$, if and only if X_t/Z_t is a martingale with respect to $\mathbb{Q}$.*