filtered probability space

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Defines	stochastic basis
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A filtered probability space, or *stochastic basis*, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in T}, \mathbb{P})$ consists of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a http://planetmath.org/FiltrationOfSigmaAlgebrasfiltra $(\mathcal{F}_t)_{t \in T}$ contained in $\mathcal{F}$. Here, T is the time index set, and is an ordered set — usually a subset of the real numbers — such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s < t$ in T.

Filtered probability spaces form the setting for defining and studying stochastic processes. A process X_t with time index t ranging over T is said to be adapted if X_t is an $\mathcal{F}_t$-measurable random variable for every t.

When the index set T is an http://planetmath.org/Intervalinterval of the real numbers (i.e., continuous-time), it is often convenient to impose further conditions. In this case, the filtered probability space is said to satisfy the *usual conditions* or *usual hypotheses* if the following conditions are met.

- The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is http://planetmath.org/CompleteMeasurecomplete.
- The σ-algebras $\mathcal{F}_t$ contain all the sets in $\mathcal{F}$ of zero probability.
- The filtration $\mathcal{F}_t$ is right-continuous. That is, for every non-maximal $t \in T$, the σ-algebra $\mathcal{F}_{t+} \equiv \bigcap_{s>t} \mathcal{F}_s$ is equal to $\mathcal{F}_t$.

Given any filtered probability space, it can always be enlarged by passing to the completion of the probability space, adding zero probability sets to $\mathcal{F}_t$, and by replacing $\mathcal{F}_t$ by $\mathcal{F}_{t+}$. This will then satisfy the usual conditions. In fact, for many types of processes defined on a complete probability space, their natural filtration will already be right-continuous and the usual conditions met. However, the process of completing the probability space depends on the specific probability measure $\mathbb{P}$ and in many situations, such as the study of Markov processes, it is necessary to study many different measures on the same space. A much weaker condition which can be used is that the σ-algebras $\mathcal{F}_t$ are universally complete, which is still strong enough to apply much of the 'heavy machinery' of stochastic processes, such as the Doob-Meyer decomposition, section theorems, etc.