relation between almost surely absolutely bounded random variables and their absolute moments

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Let $\{\Omega, E, P\}$ a probability space and let X be a random variable; then, the following are equivalent:

1) $\Pr\{|X| \leq M\} = 1$ i.e. X is absolutely bounded almost surely;

2) $E[|X|^k] \leq M^k \qquad \forall k \geq 1, k \in N$

Proof. 1) $\Longrightarrow$ 2)

Let's define

$$F = \{\omega \in \Omega : |X(\omega)| > M\};$$

Then by hypothesis

$$\Pr\{\Omega \backslash F\} = 1$$

and

$$\Pr\{F\} = 0.$$

We have:

$$\begin{aligned}
E[|X|^k] &= \int_\Omega |X|^k \, dP \\
&= \int_{\Omega \backslash F} |X|^k \, dP + \int_F |X|^k \, dP \\
&= \int_{\Omega \backslash F} |X|^k \, dP \\
&\leq \int_{\Omega \backslash F} M^k dP \\
&= M^k \Pr\{\Omega \backslash F\} = M^k.
\end{aligned}$$

2) $\Longrightarrow$ 1)

Let's define

$$\begin{aligned}
F &= \{\omega \in \Omega : |X(\omega)| > M\} \\
F_n &= \left\{\omega \in \Omega : |X(\omega)| > M + \frac{1}{n}\right\} \quad \forall n \geq 1.
\end{aligned}$$

Then we have obviously $F_n \subseteq F_{n+1}$ (in fact, if $\omega \in F_n \Longrightarrow |X(\omega)| > M + \frac{1}{n} > M + \frac{1}{n+1} \Longrightarrow \omega \in F_{n+1}$) and $F = \bigcup_{n=1}^\infty F_n$ (in fact, let $\omega \in F$; let $N = \left\lceil \frac{1}{|X(\omega)| - M} \right\rceil$; then $|X(\omega)| > M + \frac{1}{N}$, that is $\omega \in F_N$); this means that

$$F = \lim_{n \to \infty} F_n$$

in the meaning of http://planetmath.org/SequenceOfSetsConvergencesets sequences convergence.

So http://planetmath.org/PropertiesForMeasurethe continuity from below property of probability can be applied:

$$\Pr\left\{F\right\} = \Pr\left\{\lim_{n\to\infty} F_n\right\} = \lim_{n\to\infty} \Pr\left\{F_n\right\}.$$

Now, for any $k \geq 1$,

$$\begin{aligned}
M^k &\geq E\left[|X|^k\right] \\
&= \int_\Omega |X(\omega)|^k \, dP \\
&= \int_{\Omega\setminus F_n} |X(\omega)|^k \, dP + \int_{F_n} |X(\omega)|^k \, dP \\
&\geq \int_{F_n} |X(\omega)|^k \, dP \\
&\geq \int_{F_n} \left(M + \frac{1}{n}\right)^k dP \\
&= \left(M + \frac{1}{n}\right)^k \Pr\left\{F_n\right\}.
\end{aligned}$$

that is

$$\Pr\left\{F_n\right\} \leq \left(\frac{M}{M+\frac{1}{n}}\right)^k \quad \text{for any } k \geq 1$$

so that the only acceptable value for $\Pr\left\{F_n\right\}$ is

$$\Pr\left\{F_n\right\} = 0$$

whence the thesis. □

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