Wiener measure

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Definition 1. The *Wiener space* $W(\mathbb{R})$ is just the set of all continuous paths $\omega : [0, \infty) \to \mathbb{R}$ satisfying $\omega(0) = 0$. It may be made into a measurable space by equipping it with the σ-algebra $\mathcal{F}$ generated by all projection maps $\omega \mapsto \omega(t)$ (or the completion of this under Wiener measure, see below).

Thus, an $\mathbb{R}$-valued continuous-time stochastic process X_t with continuous sample paths can be thought of as a random variable taking its values in $W(\mathbb{R})$.

Definition 2. In the case where $X_t = W_t$ is Brownian motion, the distribution measure P induced on $W(\mathbb{R})$ is called the *Wiener measure*. That is, P is the unique probability measure on $W(\mathbb{R})$ such that for any finite sequence of times $0 < t_1 < \ldots < t_n$ and Borel sets $A_1, \ldots, A_n \subset \mathbb{R}$

$$P(\{\omega : \omega(t_1) \in A_1, \ldots, \omega(t_n) \in A_n\}) = \int_{A_1} \cdots \int_{A_n} p(t_1, 0, x_1) p(t_2 - t_1, x_1, x_2) \tag{1}$$

$$\cdots p(t_n - t_{n-1}, x_{n-1}, x_n)\, dx_1 \cdots\, dx_n, \tag{2}$$

where $p(t, x, y) = \frac{1}{\sqrt{2\pi t}} \exp(-\frac{(x-y)^2}{2t})$ defined for any $x, y \in \mathbb{R}$ and $t > 0$.

This of course corresponds to the defining property of Brownian motion. The other properties carry over as well; for instance, the set of paths in $W(\mathbb{R})$ which are nowhere differentiable is of P-measure 1.

The Wiener space $W(\mathbb{R}^d)$ and corresponding Wiener measure are defined similarly, in which case P is the distribution of a d-dimensional Brownian motion.