stochastic differential equation

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Consider the ordinary differential equation, for example, the population growth model

$$\frac{dX(t)}{dt} = a(t)X(t),\ X(0) = X_0\,,$$

where $a(t)$ is the relative rate of growth at time t, and $X(t)$ is the solution-trajectory of the system.

But we may want to take into account, in our model, the randomness or the uncertainty of our knowledge of the data. In this case we may introduce the data $a(t)$ as:

$$a(t) = r(t) + N(t)\,,$$

where $N(t)$ is a noise term, represented by a random variable with some postulated probability distribution.

In general, stochastic differential equations can be posed in the case that the infinitesimal increment $dX(t)$ is a Gaussian random variable. (Other types of random variables are also possible, but require extensions of the basic theory.) A *stochastic differential equation* (SDE) is an equation of the form:

$$dX(t;\omega) = \mu(t;\omega)\,dt + \sigma(t;\omega)\,dW(t;\omega)$$

where ω lives in some probability space, and $W(t)$ is a Wiener process on that probability space. The real-valued functions μ and σ are to satisfy certain measurability requirements, and are usually assumed to be known, with the process $X(t)$ being sought.

The argument ω is usually suppressed in the notation:

$$dX(t) = \mu(t)\,dt + \sigma(t)\,dW(t)\,, \tag{1}$$

with the understanding that $X(t)$, $W(t)$, $\mu(t)$ and $\sigma(t)$ denote random variables for each time t.

The interpretation of the stochastic differential equation (**??**) is that a process $X(t)$ satisfies it if and only if it satisfies this relation amongst integrals:

$$X(t_1) - X(t_0) = \int_{t_0}^{t_1} \mu(t)\,dt + \int_{t_0}^{t_1} \sigma(t)\,dW(t) \tag{2}$$

for all times t_0 and t_1. The last integral is an Itô integral.

In many cases, the coefficients μ and σ depend on $X(t)$ itself:

$$dX(t) = \mu(t, X(t))\,dt + \sigma(t, X(t))\,dW(t)\,.$$

In this case, equation (**??**) does not give an explicit solution for the stochastic differential equation. Nevertheless, there are theorems analogous to those of ordinary differential equations, that guarantee existence of solutions given certain bounds on the growth of the coefficients $\mu(t,x)$ and $\sigma(t,x)$.

In simpler cases, stochastic differential equations that involve unknowns on the right-hand side may still be solved explicitly using changes of variables (often called Itô's formula in this context). For example,

$$X(t) = X_0\, e^{-\kappa t} + \theta\,(1 - e^{-\kappa t}) + \sigma \int_0^t e^{-\kappa(t-s)}\, dW(s)$$

(for any initial condition X_0) provides a solution to:

$$dX(t) = \kappa\,(\theta - X(t))\, dt + \sigma\, dW(t)\,.$$

References

[1] Bernt Øksendal. , *An Introduction with Applications.* 5th ed. Springer 1998.

[2] Lawrence Evans. . Department of Mathematics, U.C. Berkeley.