proof of the début theorem

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Let $(\mathcal{F})_{t\in\mathbb{T}}$ be a right-continuous http://planetmath.org/FiltrationOfSigmaAlgebrasfiltrat on the measurable space $(\Omega, \mathcal{F})$, It is assumed that $\mathbb{T}$ is a closed subset of $\mathbb{R}$ and that $\mathcal{F}_t$ is universally complete for each $t \in \mathbb{T}$.

If $A \subseteq \mathbb{T} \times \Omega$ is a progressively measurable set, then we show that its début

$$D(A) = \inf \{t \in \mathbb{T} : (t, \omega) \in A\}$$

is a stopping time.

As A is progressively measurable, the set $A \cap ((-\infty, t) \times \Omega)$ is $\mathcal{B}(\mathbb{T}) \times \mathcal{F}_t$-measurable. By the measurable projection theorem it follows that

$$\{D(A) < t\} = \{\omega \in \Omega : (s, \omega) \in A \cap ((-\infty, t) \times \Omega) \text{ for some } s \in \mathbb{T}\}$$

is in $\mathcal{F}_t$. If there exists a sequence $t_n \in \mathbb{T}$ with $t_n > t$ and $t_n \to t$, then

$$\{D(A) \leq t\} = \bigcap_n \{D(A) < t_n\} \in \bigcap_n \mathcal{F}_{t_n} = \mathcal{F}_{t+} = \mathcal{F}_t.$$

On the other hand, if t is not a right limit point of $\mathbb{T}$ then

$$\{D(A) \leq t\} = \{D(A) < t\} \cup \{\omega \in \Omega : (t, \omega) \in A\} \in \mathcal{F}_t.$$

In either case, $\{D(A) \leq t\}$ is in $\mathcal{F}_t$, so $D(A)$ is a stopping time.