martingale

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Related topic	DoobsOptionalSamplingTheorem
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Related topic	MartingaleConvergenceTheorem
Defines	martingale
Defines	supermartingale
Defines	submartingale
Defines	reverse submartingale
Defines	reverse supermartingale

Definition. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$ be a filtered probability space and (X_t) be a stochastic process such that X_t is http://planetmath.org/Integral2integrable for all $t \in \mathbb{T}$. Then, $X = (X_t, \mathcal{F}_t)$ is called a *submartingale* if

$$\mathbb{E}^{\mathbb{P}}[X_t|\mathcal{F}_s] \geq X_s, \text{ for every } s < t, \text{ a.e.}[\mathbb{P}],$$

and a *supermartigale* if

$$\mathbb{E}^{\mathbb{P}}[X_t|\mathcal{F}_s] \leq X_s, \text{ for every } s < t, \text{ a.e.}[\mathbb{P}].$$

A submartingale that is also a supermartingale is called a *martingale*, i.e., a martingale satisfies

$$\mathbb{E}^{\mathbb{P}}[X_t|\mathcal{F}_s] = X_s, \text{ for every } s < t, \text{ a.e.}[\mathbb{P}].$$

Similarly, if the $\{\mathcal{F}_t\}$ form a decreasing collection of σ-subalgebras of $\mathcal{F}$, then X is called a *reverse submartingale* if

$$\mathbb{E}^{\mathbb{P}}[X_s|\mathcal{F}_t] \geq X_t, \text{ for every } s < t, \text{ a.e.}[\mathbb{P}],$$

and a *reverse supermartingale* if

$$\mathbb{E}^{\mathbb{P}}[X_s|\mathcal{F}_t] \leq X_t, \text{ for every } s < t, \text{ a.e.}[\mathbb{P}].$$

Remarks

- The martingale property captures the idea of a fair bet, where the expected future value is equal to the current value.
- The submartingale property is equivalent to

$$\int_A X_t \, d\mathbb{P} \geq \int_A X_s \, d\mathbb{P} \text{ for every } A \in \mathcal{F}_s \text{ and } s < t$$

  and similarly for the other definitions. This is immediate from the definition of conditional expectation.