



planetmath.org

Math for the people, by the people.

local martingale

Canonical name	LocalMartingale
Date of creation	2013-03-22 15:12:43
Last modified on	2013-03-22 15:12:43
Owner	skubeedooo (5401)
Last modified by	skubeedooo (5401)
Numerical id	8
Author	skubeedooo (5401)
Entry type	Definition
Classification	msc 60G07
Classification	msc 60G48
Related topic	Martingale
Related topic	LocalPropertiesOfProcesses

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{P})$  be a filtered probability space, where the time index set  $\mathbb{T} \subseteq \mathbb{R}$  has minimal element  $t_0$ . The most common cases are discrete-time, with  $\mathbb{T} = \mathbb{Z}_+$ , and continuous time where  $\mathbb{T} = \mathbb{R}_+$ , in which case  $t_0 = 0$ .

A process  $X$  is said to be a *local martingale* if it is <http://planetmath.org/LocalPropertiesOfMartingales> a right-continuous martingale. That is, if there is a sequence of stopping times  $\tau_n$  almost surely increasing to infinity and such that the stopped processes  $1_{\{\tau_n > t_0\}} X^{\tau_n}$  are martingales. Equivalently,  $1_{\{\tau_n > t_0\}} X_{\tau_n \wedge t}$  is integrable and

$$1_{\{\tau_n > t_0\}} X_{\tau_n \wedge s} = \mathbb{E}[1_{\{\tau_n > t_0\}} X_{\tau_n \wedge t} \mid \mathcal{F}_s]$$

for all  $s < t \in \mathbb{T}$ . In the discrete-time case where  $\mathbb{T} = \mathbb{Z}_+$  then it can be shown that a local martingale  $X$  is a martingale if and only if  $\mathbb{E}[|X_t|] < \infty$  for every  $t \in \mathbb{Z}_+$ . More generally, in continuous-time where  $\mathbb{T}$  is an interval of the real numbers, then the stronger property that

$$\{X_\tau : \tau \leq t \text{ is a stopping time}\}$$

is uniformly integrable for every  $t \in \mathbb{T}$  gives a necessary and sufficient condition for a local martingale to be a martingale.

Local martingales form a very important class of processes in the theory of stochastic calculus. This is because the local martingale property is preserved by the stochastic integral, but the martingale property is not. Examples of local martingales which are not proper martingales are given by solutions to the stochastic differential equation

$$dX = X^\alpha dW$$

where  $X$  is a nonnegative process,  $W$  is a Brownian motion and  $\alpha > 1$  is a fixed real number.

An alternative definition of local martingales which is sometimes used requires  $X^{\tau_n}$  to be a martingale for each  $n$ . This definition is slightly more restrictive, and is equivalent to the definition given above together with the condition that  $X_{t_0}$  must be integrable.