Chernoff-Cramer bound

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The Chernoff-Cramèr inequality is a very general and powerful way of bounding random variables. Compared with the famous Chebyshev bound, which implies inverse polynomial decay inequalities, the Chernoff-Cramèr method yields exponential decay inequalities, at the cost of needing a few more hypotheses on random variables' .

Theorem: (Chernoff-Cramèr inequality)

Let $\{X_i\}_{i=1}^n$ be a collection of independent random variables such that $E[\exp(tX_i)] < +\infty \;\; \forall i$ in a right neighborhood of $t = 0$, i.e. for any $t \in (0, c)$ (Cramèr condition).

Then a zero-valued for $x = 0$, positive, strictly increasing, strictly convex function $\Psi(x) : [0, \infty) \mapsto R^+$ exists such that:

$$\Pr\left\{\sum_{i=1}^{n}(X_i - E[X_i]) > \varepsilon\right\} \leq \exp\left(-\Psi(\varepsilon)\right) \;\; \forall \varepsilon \geq 0$$

Namely, one has:

$$\begin{aligned}\Psi(x) &= \sup_{0<t<c}(tx - \psi(t)) \\ \psi(t) &= \sum_{i=1}^{n}\ln E\left[e^{t(X_i - EX_i)}\right] = \sum_{i=1}^{n}\left(\ln E\left[e^{tX_i}\right] - tE\left[X_i\right]\right)\end{aligned}$$

that is, $\Psi(x)$ is the Legendre of the cumulant generating function of the $\sum_{i=1}^{n}(X_i - E[X_i])$ random variable.

Remarks:

1) Besides its importance for theoretical questions, the Chernoff-Cramér bound is also the starting point to derive many deviation or concentration inequalities, among which http://planetmath.org/BernsteinInequalityBernstein, Kolmogorov, http://planetmath.org/ProhorovInequalityProhorov, http://planetmath.org/ Hoeffding and Chernoff ones are worth mentioning. All of these inequalities are obtained imposing various further conditions on X_i random variables, which turn out to affect the general form of the cumulant generating function $\psi(t)$.

2) Sometimes, instead of bounding the sum of n independent random variables, one needs to estimate they " i.e. the quantity $\frac{1}{n}\sum_{i=1}^{n}X_i$; in to reuse Chernoff-Cramér bound, it's enough to note that

$$\Pr\left\{\frac{1}{n}\sum_{i=1}^{n}(X_i - E[X_i]) > \varepsilon'\right\} = \Pr\left\{\sum_{i=1}^{n}(X_i - E[X_i]) > n\varepsilon'\right\}$$

so that one has only to replace, in the above stated inequality, ε with $n\varepsilon'$.
3) It turns out that the Chernoff-Cramer bound is asymptotically sharp, as Cramr theorem shows.