type of a distribution function

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Entry type	Definition
Classification	msc 60E05
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Synonym	centering factor
Synonym	scale parameter
Synonym	location parameter
Defines	type
Defines	scale factor
Defines	location factor
Defines	standard distribution function
Defines	location family
Defines	scale family

Two distribution functions $F, G : \mathbb{R} \to [0,1]$ are said to of the same *type* if there exist $a, b \in \mathbb{R}$ such that $G(x) = F(ax + b)$. a is called the *scale parameter*, and b the *location parameter* or *centering parameter*. Let's write $F \stackrel{t}{=} G$ to denote that F and G are of the same type.

Remarks.

- Necessarily $a > 0$, for otherwise at least one of $G(-\infty) = 0$ or $G(\infty) = 1$ would be violated.

- If $G(x) = F(x+b)$, then the graph of G is *shifted* to the right from the graph of F by b units, if $b > 0$ and to the left if $b < 0$.

- If $G(x) = F(ax)$, then the graph of G is *stretched* from the graph of F by a units if $a > 1$, and *compressed* if $a < 1$.

- If X and Y are random variables whose distribution functions are of the same type, say, F and G respectively, and related by $G(x) = F(ax+b)$, then X and $aY + b$ are identically distributed, since

  $$P(X \leq z) = F(z) = G(\frac{z-b}{a}) = P(Y \leq \frac{z-b}{a}) = P(aY + b \leq z).$$

  When X and $aY + b$ are identically distributed, we write $X \stackrel{t}{=} Y$.

- Again, suppose X and Y correspond to F and G, two distribution functions of the same type related by $G(x) = F(ax+b)$. Then it is easy to see that $E[X] < \infty$ iff $E[Y] < \infty$. In fact, if the expectation exists for one, then $E[X] = aE[Y] + b$. Furthermore, $Var[X]$ is finite iff $Var[Y]$ is. And in this case, $Var[X] = a^2 Var[Y]$. In general, convergence of moments is a "typical" property.

- We can partition the set of distribution functions into disjoint subsets of functions belonging to the same types, since the binary relation $\stackrel{t}{=}$ is an equivalence relation.

- By the same token, we can classify all real random variables defined on a fixed probability space according to their distribution functions, so that if X and Y are of the same type τ iff their corresponding distribution functions F and G are of type τ.

- Given an equivalence class of distribution functions belonging to a certain type τ, such that a random variable Y of type τ exists with finite expectation and variance, then there is one distribution function F of type τ corresponding to a random variable X such that $E[X] = 0$ and $Var[X] = 1$. F is called the *standard distribution function* for type τ. For example, the standard (cumulative) normal distribution is the standard distribution function for the type consisting of all normal distribution functions.

- Within each type τ, we can further classify the distribution functions: if $G(x) = F(x + b)$, then we say that G and F belong to the same *location family* under τ; and if $G(x) = F(ax)$, then we say that G and F belong to the same *scale family* (under τ).