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σ-algebra at a stopping time

Canonical name	sigmaalgebraAtAStoppingTime
Date of creation	2013-03-22 18:38:56
Last modified on	2013-03-22 18:38:56
Owner	gel (22282)
Last modified by	gel (22282)
Numerical id	6
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Entry type	Definition
Classification	msc 60G40
Related topic	DoobsOptionalSamplingTheorem

Let $(\mathcal{F}_t)_{t\in\mathbb{T}}$ be a `http://planetmath.org/FiltrationOfSigmaAlgebras`filtration on a measurable space $(\Omega,\mathcal{F})$. For every $t\in\mathbb{T}$, the σ-algebra $\mathcal{F}_t$ represents the collection of events which are observable up until time t. This concept can be generalized to any stopping time $\tau\colon\Omega\to\mathbb{T}\cup\{\infty\}$.

For a stopping time τ, the collection of events observable up until time τ is denoted by $\mathcal{F}_\tau$ and is generated by sampling progressively measurable processes

$$\mathcal{F}_\tau=\sigma\left(\{X_{\tau\wedge t}:X\text{ is progressive, }t\in\mathbb{T}\}\right).$$

The reason for sampling X at time $\tau\wedge t$ rather than at τ is to include the possibility that $\tau=\infty$, in which case X_τ is not defined.

A random variable V is $\mathcal{F}_\tau$-measurable if and only if it is $\mathcal{F}_\infty$-measurable and the process $X_t\equiv 1_{\{\tau\le t\}}V$ is adapted.

This can be shown as follows. If X is a progressively measurable process, then the stopped process $X^{\tau\wedge s}$ is also progressive. In particular, $V\equiv X_{\tau\wedge s}=X^{\tau\wedge s}_s$ is $\mathcal{F}_\infty$-measurable and $1_{\{\tau\le t\}}V=1_{\{\tau\le t\}}X^{\tau\wedge s}_t$ is $\mathcal{F}_t$-measurable. Conversely, if V is $\mathcal{F}_t$-measurable then $X_s\equiv 1_{\{s>t\}}V$ is a progressive process and $1_{\{\tau>t\}}V=X_{\tau\wedge t}$ is $\mathcal{F}_\tau$-measurable. By letting t increase to infinity, it follows that $1_{\{\tau=\infty\}}V$ is $\mathcal{F}_\tau$-measurable for every $\mathcal{F}_\infty$-measurable random variable V. Now suppose also that $X_t\equiv 1_{\{\tau\le t\}}V$ is adapted, and hence progressive. Then, $1_{\{\tau\le t\}}V=X_{\tau\wedge t}$ is $\mathcal{F}_\tau$-measurable. Letting t increase to infinity shows that $V=1_{\{\tau<\infty\}}V+1_{\{\tau=\infty\}}V$ is $\mathcal{F}_\tau$-measurable.

As a set A is $\mathcal{F}_\tau$-measurable if and only if 1_A is an $\mathcal{F}_\tau$-measurable random variable, this gives the following alternative definition,

$$\mathcal{F}_\tau=\left\{A\in\mathcal{F}_\infty:A\cap\{\tau\le t\}\in\mathcal{F}_t\text{ for all }t\in\mathbb{T}\right\}.$$

From this, it is not difficult to show that the following properties are satisfied

1. Any stopping time τ is $\mathcal{F}_\tau$-measurable.

2. If $\tau(\omega)=t\in\mathbb{T}\cup\{\infty\}$ for all $\omega\in\Omega$ then $\mathcal{F}_\tau=\mathcal{F}_t$.

3. If σ,τ are stopping times and $A\in\mathcal{F}_\sigma$ then $A\cap\{\sigma\le\tau\}\in\mathcal{F}_\tau$. In particular, if $\sigma\le\tau$ then $\mathcal{F}_\sigma\subseteq\mathcal{F}_\tau$.

4. If σ,τ are stopping times and $A\in\mathcal{F}_\sigma$ then $A\cap\{\sigma=\tau\}\in\mathcal{F}_\tau$.

5. if the filtration $(\mathcal{F}_t)$ is right-continuous and $\tau_n \geq \tau$ are stopping times with $\tau_n \to \tau$ then $\mathcal{F}_\tau = \bigcap_n \mathcal{F}_{\tau_n}$. More generally, if $\tau_n = \tau$ eventually then this is true irrespective of whether the filtration is right-continuous.

6. If τ_n are stopping times with $\tau_n = \tau$ eventually then $\mathcal{F}_{\tau_n} \to \mathcal{F}_\tau$. That is,

$$\mathcal{F}_\tau = \bigcap_n \sigma\left(\bigcup_{m\geq n} \mathcal{F}_{\tau_m}\right).$$

In continuous-time, for any stopping time τ the σ-algebra $\mathcal{F}_{\tau+}$ is the set of events observable up until time t with respect to the right-continuous filtration $(\mathcal{F}_{t+})$. That is,

$$\begin{aligned}\mathcal{F}_{\tau+} &= \{A \in \mathcal{F}_\infty : A \cap \{\tau \leq t\} \in \mathcal{F}_{t+} \text{ for every } t \in \mathbb{T}\}\\ &= \{A \in \mathcal{F}_\infty : A \cap \{\tau < t\} \in \mathcal{F}_t \text{ for every } t \in \mathbb{T}\}.\end{aligned}$$

If $\tau_n \geq \tau$ are stopping times with $\tau_n > \tau$ whenever $\tau < \infty$ is not a maximal element of $\mathbb{T}$, and $\tau_n \to \tau$ then,

$$\mathcal{F}_{\tau+} = \bigcap_n \mathcal{F}_{\tau_n} = \bigcap_n \mathcal{F}_{\tau_n+}.$$

The σ-algebra of events observable up until just before time τ is denoted by $\mathcal{F}_{\tau-}$ and is generated by sampling predictable processes

$$\mathcal{F}_{\tau-} = \sigma\left(\{X_{\tau\wedge t} : X \text{ is predictable, } t \in \mathbb{T}\}\right).$$

Suppose that the index set $\mathbb{T} \subseteq \mathbb{R}$ has minimal element t_0. As the predictable σ-algebra is generated by sets of the form $(s,\infty) \times A$ for $s \in \mathbb{T}$ and $A \in \mathcal{F}_s$, and $\{t_0\} \times A$ for $A \in \mathcal{F}_{t_0}$, the definition above can be rewritten as,

$$\mathcal{F}_{\tau-} = \sigma\left(\{A \cap \{\tau > s\} : s \in \mathbb{T}, A \in \mathcal{F}_s\} \cup \mathcal{F}_{t_0}\right).$$

Clearly, $\mathcal{F}_{\tau-} \subseteq \mathcal{F}_\tau \subseteq \mathcal{F}_{\tau+}$. Furthermore, for any stopping times σ, τ then $\mathcal{F}_{\sigma+} \subseteq \mathcal{F}_{\tau-}$ when restricted to the set $\{\sigma < \tau\}$.

If τ_n is a sequence of stopping times `http://planetmath.org/PredictableStoppingTime`annou
τ, so that τ is predictable, then

$$\mathcal{F}_{\tau-} = \sigma\left(\bigcup_n \mathcal{F}_{\tau_n}\right).$$