completeness under ucp convergence

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Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$ be a filtered probability space. Then, under the ucp topology, various classes of stochastic processes form http://planetmath.org/Completecompl vector spaces.

For a σ-algebra $\mathcal{M}$ on $\mathbb{R}_+ \times \Omega$, we can look at the processes which are $\mathcal{M}$-measurable when regarded as a map from $\mathbb{R}_+ \times \Omega$ to $\mathbb{R}$. As the ucp topology is only defined for jointly measurable processes, we restrict attention to sub-σ-algebras of the product $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$.

Theorem 1. *Let $\mathcal{M}$ be a sub-σ-algebra of $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$. Then, the set of $\mathcal{M}$-measurable processes is complete under ucp convergence.*

That is, if X^n is a sequence of $\mathcal{M}$-measurable processes such that $X^n - X^m \xrightarrow{\text{ucp}} 0$ as $m, n \to \infty$ then $X^n \xrightarrow{\text{ucp}} X$ for some $\mathcal{M}$-measurable process X.

In particular, the spaces of jointly measurable, progressive, optional and predictable processes are each complete under ucp convergence. We can also look at the properties of the sample paths of the processes.

Theorem 2. *Let S be any set of functions $\mathbb{R}_+ \to \mathbb{R}$ which is http://planetmath.org/Closedclose under uniform convergence on compacts (the compact-open topology). Then, the set of jointly measurable processes whose sample paths are almost surely in S is complete under ucp convergence.*

So, for example, the continuous, right-continuous, left-continuous and cadlag processes are each complete under the ucp topology. Furthermore, combining theorems **??** and **??**, and using the fact that a cadlag process is adapted if and only if it is jointly measurable (see http://planetmath.org/MeasurabilityOfStoch of stochastic processes), the following useful result is obtained.

Corollary. *The space of cadlag adapted processes is complete under ucp convergence.*