proof of Kolmogorov's inequality

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For $k = 1, 2, \ldots, n$, let A_k be the event that $|S_k| \geq \lambda$ but $|S_i| < \lambda$ for all $i = 1, 2, \ldots, k-1$. Note that the events $A_1, A_2, \ldots, A_n$ are disjoint, and

$$\bigcup_{k=1}^{n} A_k = \Big\{ \max_{1 \leq k \leq n} |S_k| \geq \lambda \Big\}.$$

Let I_A be the indicator function of event A. Since $A_1, A_2, \ldots, A_n$ are disjoint, we have

$$0 \leq \sum_{k=1}^{n} I_{A_k} \leq 1.$$

Hence, we obtain

$$\sum_{k=1}^{n} \mathrm{Var}[X_k] = E[S_n^2] \geq \sum_{k=1}^{n} E[S_n^2 I_{A_k}].$$

After replacing S_n^2 by $S_k^2 + 2S_k(S_n - S_k) + (S_n - S_k)^2$, we get

$$\begin{aligned}
\sum_{k=1}^{n} \mathrm{Var}[X_k] &\geq \sum_{k=1}^{n} E[(S_k^2 + 2S_k(S_n - S_k) + (S_n - S_k)^2) I_{A_k}] \\
&\geq \sum_{k=1}^{n} E[(S_k^2 + 2S_k(S_n - S_k)) I_{A_k}] \\
&= \sum_{k=1}^{n} E[S_k^2 I_{A_k}] + 2 \sum_{k=1}^{n} E[S_n - S_k] E[S_k I_{A_k}] \\
&= \sum_{k=1}^{n} E[S_k^2 I_{A_k}] \\
&\geq \lambda^2 \sum_{k=1}^{n} E[I_{A_k}] \\
&= \lambda^2 \sum_{k=1}^{n} \Pr(A_k) \\
&= \lambda^2 \Pr\left(\bigcup_{k=1}^{n} A_k \right) \\
&= \lambda^2 \Pr\left(\max_{1 \leq k \leq n} |S_k| \geq \lambda \right),
\end{aligned}$$

where in the third line, we have used the assumption that $S_n - S_k$ is independent of $S_k I_{A_k}$.