Doob's inequalities

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Doob's inequalities place bounds on the maximum value attained by a martingale in terms of the terminal value. We consider a process $(X_t)_{t\in\mathbb{T}}$ defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in\mathbb{T}}, \mathbb{P})$. The associated maximum process (X^*_t) is

$$X^*_t \equiv \sup_{s\le t} |X_s|.$$

The notation $\|\cdot\|_p$ for the `http://planetmath.org/LpSpace`L^p-norm of a random variable will be used. In discrete-time or, more generally whenever the index set $\mathbb{T}$ is countable, then Doob's inequalities are as follows.

Theorem 1 (Doob)**.** *Let $(X_t)_{t\in\mathbb{T}}$ be a submartingale with countable index set $\mathbb{T}$. Then,*

$$\mathbb{P}\left(\sup_{s\le t} X_s \ge K\right) \le K^{-1}\mathbb{E}[(X_t)_+] \tag{1}$$

If X is either a martingale or nonnegative submartingale then,

$$\mathbb{P}(X^*_t \ge K) \le K^{-1}\mathbb{E}[|X_t|], \tag{2}$$

$$\|X^*_t\|_p \le \frac{p}{p-1}\|X_t\|_p. \tag{3}$$

for every $K > 0$ and $p > 1$.

In particular, (**??**) shows that the maximum of any L^p-bounded martingale is itself L^p-bounded and, martingales X^n converge to X in the L^p-norm if and only if $(X^n - X)^* \to 0$ in the L^p-norm. The special case where $p = 2$ gives

$$\mathbb{E}[(X^*_t)^2] \le 4\mathbb{E}[X^2_t]$$

which is known as *Doob's maximal quadratic inequality.*

Similarly, (**??**) shows that any L^1-bounded martingale is almost surely bounded and that convergence in the L^1-norm implies ucp convergence. Inequality (**??**) is also known as Kolmogorov's submartingale inequality.

Doob's inequalities are often applied to continuous-time processes, where $\mathbb{T} = \mathbb{R}_+$. In this case, $X^*_t = \sup_{s\le t}|X_s|$ is a supremum of uncountably many random variables, and need not be measurable. Instead, it is typically assumed that the processes are right-continuous, in which case, for any $t > 0$ the supremum may be restricted to the countable set

$$\mathbb{T}' = \{s \in \mathbb{R}_+ : s/t \in \mathbb{Q}\}.$$

Putting this into Theorem **??** gives the following continuous-time version of the inequalities.

Theorem 2 (Doob)**.** *Let* $(X_t)_{t \in \mathbb{R}_+}$ *be a right-continuous submartingale. Then,*

$$\mathbb{P}\left(\sup_{s \le t} X_s \ge K\right) \le K^{-1}\mathbb{E}[(X_t)]$$

for every $K > 0$*. If* X *is right-continuous and either a martingale or non-negative submartingale then,*

$$\mathbb{P}(X_t^* \ge K) \le K^{-1}\mathbb{E}[|X_t|],$$
$$\|X_t^*\|_p \le \frac{p}{p-1}\|X_t\|_p.$$

for every $K > 0$ *and* $p > 1$*.*