proof of Kolmogorov's strong law for IID random variables

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Kolmogorov's strong law for square integrable random variables states that if $X_1, X_2, \ldots$ is a sequence of independent random variables with $\sum_n \mathrm{Var}[X_n]/n^2 < \infty$ then $n^{-1}\sum_{k=1}^n (X_k - \mathbb{E}[X_k])$ converges to zero with probability one as $n \to \infty$ (see martingale proof of Kolmogorov's strong law for square integrable variables). We show that the following version of the strong law for IID random variables follows from this.

Theorem (Kolmogorov)**.** *Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with $\mathbb{E}[|X_n|] < \infty$. Then, $n^{-1}\sum_{k=1}^n (X_k - \mathbb{E}[X_k]) \to 0$ as $n \to \infty$, with probability one.*

Note that here, the random variables X_n are not necessarily square integrable. Let us set $\tilde{X}_n = X_n - \mathbb{E}[X_n]$, so that $\tilde{X}_n$ are IID random variables with $\mathbb{E}[\tilde{X}_n] = 0$. Then, set

$$Y_n = \begin{cases} \tilde{X}_n, & \text{if } \left|\tilde{X}_n\right| < n, \\ 0, & \text{otherwise.} \end{cases}$$

Using the fact that X_n has the same distribution as X_1 gives

$$\begin{aligned} \sum_n \mathbb{E}[Y_n^2]/n^2 &= \sum_n \mathbb{E}\left[1_{\{|\tilde{X}_n| < n\}} n^{-2} \tilde{X}_n^2\right] \\ &= \sum_n \mathbb{E}\left[1_{\{|\tilde{X}_1| < n\}} n^{-2} \tilde{X}_1^2\right] \\ &= \mathbb{E}\left[\sum_n 1_{\{|\tilde{X}_1| < n\}} n^{-2} \tilde{X}_1^2\right]. \end{aligned} \tag{1}$$

Letting N be the smallest integer greater than $|\tilde{X}_1|$,

$$\begin{aligned} \sum_n 1_{\{|\tilde{X}_1| < n\}} n^{-2} &\le \sum_{n=N}^{\infty} \frac{4}{4n^2 - 1} = \sum_{n=N}^{\infty} \left(\frac{2}{2n-1} - \frac{2}{2n+1}\right) \\ &= \frac{2}{2N-1} \le \frac{2}{N} \le \frac{2}{|\tilde{X}_1|}. \end{aligned}$$

So, putting this into equation (**??**),

$$\sum_n \mathrm{Var}[Y_n]/n^2 \le \sum_n \mathbb{E}[Y_n^2]/n^2 \le \mathbb{E}[2|\tilde{X}_1|] < \infty.$$

Therefore, Y_n satisfies the required properties to apply the strong law for square integrable random variables,

$$n^{-1}\sum_{k=1}^{n}(Y_k - \mathbb{E}[Y_k]) \to 0 \tag{2}$$

as $n \to \infty$, with probability one. Also,

$$\mathbb{E}[Y_n] = \mathbb{E}[Y_n - \tilde{X}_n] = -\mathbb{E}[1_{\{|\tilde{X}_n| \geq n\}}\tilde{X}_n] = -\mathbb{E}[1_{\{|\tilde{X}_1| \geq n\}}\tilde{X}_1]$$

converges to 0 as $n \to \infty$ (by the dominated convergence theorem). So, the $\mathbb{E}[Y_k]$ terms in (**??**) vanish in the limit, giving

$$n^{-1}\sum_{k=1}^{n} Y_k \to 0 \tag{3}$$

as $n \to \infty$ with probability one.

We finally note that

$$\mathbb{E}\left[\sum_n 1_{\{\tilde{X}_n \neq Y_n\}}\right] = \mathbb{E}\left[\sum_n 1_{\{|\tilde{X}_1| \geq n\}}\right] \leq \mathbb{E}[|\tilde{X}_1|] < \infty,$$

so $\sum_n 1_{\{\tilde{X}_n \neq Y_n\}} < \infty$, and $\tilde{X}_n = Y_n$ for large n (with probability one). So, Y_k can be replaced by $\tilde{X}_k$ in (**??**), giving the result.

References

[1] David Williams, *Probability with martingales*, Cambridge Mathematical Textbooks, Cambridge University Press, 1991.

[2] Olav Kallenberg, *Foundations of modern probability*, Second edition. Probability and its Applications. Springer-Verlag, 2002.