proof of Borel-Cantelli 2

Canonical name	ProofOfBorelCantelli2
Date of creation	2013-03-22 14:29:35
Last modified on	2013-03-22 14:29:35
Owner	kshum (5987)
Last modified by	kshum (5987)
Numerical id	4
Author	kshum (5987)
Entry type	Proof
Classification	msc 60A99

Let E denote the set of samples that are in A_i infinitely often. We want to show that the complement of E has probability zero.

As in the proof of Borel-Cantelli 1, we know that

$$E^c = \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} A_i^c$$

where the superscript ^{c} means set complement. But for each k,

$$\begin{aligned} P(\cap_{i=k} A_i^c) &= \prod_{i=k}^{\infty} P(A_i^c) \\ &= \prod_{i=k}^{\infty} (1 - P(A_i)) \end{aligned}$$

Here we use the assumption that the event A_i's are independent. The inequality $1 - a \leq e^{-a}$ and the assumption that the sum of $P(A_i)$ diverges together imply that

$$P(\cap_{i=k} A_i^c) \leq \exp(-\sum_{i=k}^{\infty} P(A_i)) = 0$$

Therefore E^c is a union of countable number of events, each of them has probability zero. So $P(E^c) = 0$.