stochastic integration by parts

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The stochastic integral satisfies a version of the classical integration by parts formula, which is just the integral version of the product rule. The only difference here is the existence of a quadratic covariation term.

Theorem. *Let X, Y be semimartingales. Then,*

$$X_t Y_t = X_0 Y_0 + \int_0^t X_{s-}\, dY_s + \int_0^t Y_{s-}\, dX_s + [X,Y]_t. \tag{1}$$

Alternatively, in differential notation, this reads

$$d(X_t Y_t) = X_{t-} dY_t + Y_{t-} dX_t + d[X,Y]_t.$$

The existence of the quadratic covariation term $[X,Y]$ in the integration by parts formula, and also in Itô's lemma, is an important difference between standard calculus and stochastic calculus. To see the need for this term, consider the following. Choosing any $h > 0$, write the increment of a process over a time step of size h as $\delta X_t \equiv X_{t+h} - X_t$. The increment of a product of processes satisfies the following simple identity,

$$\delta(XY)_t = X_t \delta Y_t + Y_t \delta X_t + \delta X_t \delta Y_t. \tag{2}$$

As we let h tend to zero, for differentiable processes the final term of (**??**) is of `http://planetmath.org/LandauNotation`order $O(h^2)$, so can be neglected in the limit. However, when X and Y are semimartingales, such as Brownian motion, the final term will be of order h, and needs to be retained even in the limit.

The proof of equation (**??**) is given by the proof of the `http://planetmath.org/QuadraticVari` of the quadratic variation of semimartingales and, in particular, is just a rearrangement of the formula given for the quadratic covariation of semimartingales. Whenever either of X or Y is a continuous finite variation process, the quadratic covariation term $[X,Y]$ is zero, so (**??**) becomes the standard integration by parts formula. More generally, for noncontinuous processes we have the following.

Corollary. *Let X be a semimartingale and Y be an adapted finite variation process. Then,*

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s\, dY_s + \int_0^t Y_{s-}\, dX_s. \tag{3}$$

As Y is a finite variation process, the first integral on the right hand side of (**??**) makes sense as a Lebesgue-Stieltjes integral. Equation (**??**) follows from the integration by parts formula by first substituting the following formula for the covariation whenever Y has finite variation into (**??**)

$$[X, Y]_t = \sum_{s \le t} \Delta X_s \Delta Y_s$$

and then using the following identity

$$\int_0^t X_s \, dY_s - \int_0^t X_{s-} \, dY_s = \int_0^t \Delta X_s \, dY_s = \int_0^t \sum_u \Delta X_u 1_{\{u=s\}} \, dY_s$$
$$= \sum_{u \le t} \Delta X_u \Delta Y_u.$$