generalized Ito formula

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The generalized Itô formula, or *generalized Itô's lemma*, is an extension of http://planetmath.org/ItosLemma2Itô's lemma that applies also to discontinuous processes. For a cadlag process X, we write $\Delta X_t \equiv X_t - X_{t-}$ for its jump at time t.

Theorem. *Suppose that $X = (X^1, \dots, X^n)$ is a semimartingale taking values in an open subset U of $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}$ is twice continuously differentiable. Then,*

$$\begin{aligned} df(X_s) = &\sum_{i=1}^n f_{,i}(X_{t-})\,dX_t^i + \frac{1}{2}\sum_{i,j=1}^n f_{,ij}(X_{t-})\,d[X^i,X^j]_t^c \\ &+ \left(\Delta f(X_t) - \sum_{i=1}^n f_{,i}(X_{t-})\,\Delta X_t^i\right). \end{aligned} \tag{1}$$

Here, $[X^i, X^j]^c$ represents the continuous part of the quadratic covariation,

$$[X^i, X^j]_t^c = [X^i, X^j]_t - \sum_{s\le t} \Delta X_s^i \Delta X_s^j$$

which is a continuous finite variation process. The final term on the right hand side of (**??**) involving the jumps of X represents the differential dZ of the process

$$Z_t = \sum_{s\le t}\left(\Delta f(X_s) - \sum_{i=1}^n f_{,i}(X_{s-})\,\Delta X_s^i\right).$$

This is indeed a well defined finite variation process, as the sum of the absolute values

$$\sum_{s\le t}\left|\Delta f(X_s) - \sum_{i=1}^n f_{,i}(X_{s-})\,\Delta X_s^i\right| \le K\sum_{s\le t}\|\Delta X_s\|^2 \le K\sum_{i=1}^n [X^i]_t$$

is finite. Here, K is a finite random variable, and this bound follows from expanding f as a Taylor series to second order.

The reason for using differential notation and writing the formula in terms of the continuous part of the quadratic covariation should be clear when it is considered that writing out the expression in full gives the following rather messy formula.

$$f(X_t) = f(X_0) + \sum_{i=1}^n \int_0^t f_{,i}(X_{s-})\,dX^i_s + \frac{1}{2}\sum_{i,j=1}^n \int_0^t f_{,ij}(X_{s-})\,d[X^i,X^j]_s$$
$$+\sum_{s\le t}\left(\Delta f(X_s) - \sum_{i=1}^n f_{,i}(X_{s-})\,\Delta X^i_s - \frac{1}{2}\sum_{i,j=1}^n \int_0^t f_{,ij}(X_{s-})\,\Delta X^i_s \Delta X^j_s\right).$$

This formula may be understood as `http://planetmath.org/ItosLemma2`Itô's formula for continuous processes together with an additional term to ensure that the jumps of the right hand side are equal to $\Delta f(X_t)$. The need for this adjustment term comes from the fact that Itô's formula for continuous processes is essentially a Taylor expansion to second order, which only applies when the increments $\delta X_t = X_{t+\delta t} - X_t$ vanish in the limit of small δt. This needs adjusting whenever the process jumps.

The first term on the right hand side of (**??**) is a stochastic integral and, hence, is a semimartingale. As the remaining terms are finite variation processes, the following consequence is obtained.

Corollary. *Suppose that $X = (X^1,\ldots,X^n)$ is a semimartingale taking values in an open subset U of $\mathbb{R}^n$ and $f\colon U\to\mathbb{R}$ is twice continuously differentiable. Then $f(X)$ is a semimartingale.*