proof of Doob's inequalities

Canonical name	ProofOfDoobsInequalities
Date of creation	2013-03-22 18:39:55
Last modified on	2013-03-22 18:39:55
Owner	gel (22282)
Last modified by	gel (22282)
Numerical id	5
Author	gel (22282)
Entry type	Proof
Classification	msc 60G46
Classification	msc 60G44
Classification	msc 60G42

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$ be a filtered probability space with countable index set $\mathbb{T}$. If $(X_t)_{t\in\mathbb{T}}$ is a submartingale, we show that

$$\mathbb{P}\left(\sup_{s\le t} X_s \ge K\right) \le K^{-1}\mathbb{E}[(X_t)_+] \tag{1}$$

and if X is a martingale or nonnegative submartingale then,

$$\mathbb{P}(X_t^* \ge K) \le K^{-1}\mathbb{E}[|X_t|], \tag{2}$$

$$\|X_t^*\|_p \le \frac{p}{p-1}\|X_t\|_p. \tag{3}$$

for every $K > 0$ and $p > 1$.

First, let us consider the case where $\mathbb{T}$ is finite. The first time at which $X_t \ge K$,

$$\tau = \inf\{t \in \mathbb{T} : X_t \ge K\}$$

is a stopping time (as hitting times are stopping times). By Doob's optional sampling theorem for submartingales $X_{\tau\wedge t} \le \mathbb{E}[X_t \mid \mathcal{F}_{\tau\wedge t}]$ and therefore,

$$K\mathbb{P}(\tau \le t) \le \mathbb{E}[1_{\{\tau\le t\}}X_{\tau\wedge t}] \le \mathbb{E}[1_{\{\tau\le t\}}X_t]$$

However, $\tau \le t$ if and only if $\sup_{s\le t} X_s \ge K$ giving,

$$\mathbb{P}\left(\sup_{s\le t} X_s \ge K\right) \le K^{-1}\mathbb{E}[1_{\{\sup_{s\le t} X_s \ge K\}}X_t], \tag{4}$$

where the supremum is understood to be over $s \in \mathbb{T}$. Now suppose that $\mathbb{T}$ is countable. Then choose finite subsets $\mathbb{T}_n \subseteq \mathbb{T}$ which increase to $\mathbb{T}$ as n goes to infinity. Replacing $\mathbb{T}$ by $\mathbb{T}_n$ in inequality (**??**) and using the monotone convergence theorem to take the limit $n \to \infty$ extends (**??**) to arbitrary uncountable index sets. Then, inequality (**??**) follows immediately from (**??**).

Now, suppose that X is a martingale. Jensen's inequality gives

$$\mathbb{E}[|X_t| \mid \mathcal{F}_s] \ge |\mathbb{E}[X_t \mid \mathcal{F}_s]| = |X_s|$$

for any $s < t$, so $|X|$ is a nonnegative submartingale. Therefore, it is enough to prove inequalities (**??**) and (**??**) for X a nonnegative submartingale, and the martingale case follows by replacing X by $|X|$.

So, we take X to be a nonnegative submartingale in the following. In this case, (**??**) just reduces to (**??**) and it only remains to prove inequality (**??**).

For $p > 1$, multiply (**??**) by K^{p-1} and integrate up to some limit $L > 0$,

$$\int_0^L K^{p-1}\mathbb{P}(X_t^* \geq K)\,dK \leq \int_0^L K^{p-2}\mathbb{E}[1_{\{X_t^* \geq K\}} X_t]\,dK. \tag{5}$$

The left hand side of this inequality can be computed by commuting the order of integration with respect to $\mathbb{P}$ and dK (Fubini's theorem),

$$\begin{aligned}\int_0^L K^{p-1}\mathbb{P}(X_t^* \geq K)\,dK &= \mathbb{E}\left[\int_0^L K^{p-1} 1_{\{X_t^* \geq K\}}\,dK\right] \\ &= \frac{1}{p}\mathbb{E}[(L \wedge X^*)^p].\end{aligned}$$

The right hand side of (**??**) can be computed similarly,

$$\begin{aligned}\int_0^L K^{p-2}\mathbb{E}[1_{\{X_t^* \geq K\}} X_t]\,dK &= \mathbb{E}\left[X_t \int_0^L K^{p-2} 1_{\{X_t^* \geq K\}}\,dK\right] \\ &= \frac{1}{p-1}\mathbb{E}[X_t(L \wedge X_t^*)^{p-1}].\end{aligned}$$

Putting these back into (**??**),

$$\|L \wedge X_t^*\|_p^p \leq \frac{p}{p-1}\mathbb{E}[X_t(L \wedge X_t^*)^{p-1}]. \tag{6}$$

Now let $q = p/(p-1)$, so that p, q are `http://planetmath.org/ConjugateIndex`conjugate and the Hölder inequality gives

$$\mathbb{E}[X_t(L \wedge X_t^*)^{p-1}] \leq \|X_t\|_p \|(L \wedge X_t^*)^{p-1}\|_q = \|X_t\|_p \|L \wedge X_t^*\|_p^{p-1}.$$

Substituting into (**??**), the finite term $\|L \wedge X_t^*\|_p^{p-1}$ cancels to get

$$\|L \wedge X_t^*\|_p \leq \frac{p}{p-1}\|X_t\|_p,$$

and the result follows by letting L increase to infinity.