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probability distribution function

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1 Definition

Let $(\Omega, \mathfrak{B}, \mu)$ be a measure space. A *probability distribution function* on Ω is a function $f : \Omega \longrightarrow \mathbb{R}$ such that:

1. f is μ-measurable
2. f is nonnegative μ-almost everywhere.
3. f satisfies the equation
$$\int_\Omega f(x)\ d\mu = 1$$

The main feature of a probability distribution function is that it induces a probability measure P on the measure space $(\Omega, \mathfrak{B})$, given by
$$P(A) := \int_A f(x)\ d\mu = \int_\Omega 1_A f(x)\ d\mu,$$
for all $A \in \mathfrak{B}$. The measure P is called the *associated probability measure* of f. Note that P and μ are different measures, though they both share the same underlying measurable space $(\Omega, \mathfrak{B})$.

2 Examples

2.1 Discrete case

Let I be a countable set, and impose the counting measure on I ($\mu(A) := |A|$, the cardinality of A, for any subset $A \subset I$). A probability distribution function on I is then a nonnegative function $f : I \longrightarrow \mathbb{R}$ satisfying the equation
$$\sum_{i \in I} f(i) = 1.$$

One example is the Poisson distribution P_r on $\mathbb{N}$ (for any real number r), which is given by
$$P_r(i) := e^{-r}\frac{r^i}{i!}$$
for any $i \in \mathbb{N}$.

Given any probability space $(\Omega, \mathfrak{B}, \mu)$ and any random variable $X : \Omega \longrightarrow I$, we can form a distribution function on I by taking $f(i) := \mu(\{X = i\})$. The resulting function is called the distribution of X on I.

2.2 Continuous case

Suppose $(\Omega, \mathfrak{B}, \mu)$ equals $(\mathbb{R}, \mathfrak{B}_\lambda, \lambda)$, the real numbers equipped with Lebesgue measure. Then a probability distribution function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is simply a measurable, nonnegative almost everywhere function such that

$$\int_{-\infty}^{\infty} f(x)\ dx = 1.$$

The associated measure has `http://planetmath.org/RadonNikodymTheorem`Radon–Nikodym derivative with respect to λ equal to f:

$$\frac{dP}{d\lambda} = f.$$

One defines the *cumulative distribution function* F of f by the formula

$$F(x) := P(\{X \leq x\}) = \int_{-\infty}^{x} f(t)\ dt,$$

for all $x \in \mathbb{R}$. A well known example of a probability distribution function on $\mathbb{R}$ is the Gaussian distribution, or normal distribution

$$f(x) := \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2}.$$