proof of measurability of stopped processes

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Owner	gel (22282)
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Author	gel (22282)
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Let $(\mathcal{F}_t)_{t\in\mathbb{T}}$ be a http://planetmath.org/FiltrationOfSigmaAlgebrasfiltration on the measurable space $(\Omega,\mathcal{F})$, τ be a stopping time, and X be a stochastic process. We prove the following measurability properties of the stopped process X^τ.

If X is jointly measurable then so is X^τ.

Suppose first that X is a process of the form $X_t = 1_A 1_{\{t\geq s\}}$ for some $A\in\mathcal{F}$ and $t\in\mathbb{T}$. Then, $X^\tau_t = 1_{A\cap\{\tau\geq s\}}1_{\{t\geq s\}}$ is $\mathcal{B}(\mathbb{T})\otimes\mathcal{F}$-measurable. By the functional monotone class theorem, it follows that X^τ is a bounded $\mathcal{B}(\mathbb{T})\otimes\mathcal{F}$-measurable process whenever X is. By taking limits of bounded processes, this generalizes to all jointly measurable processes.

If X is progressively measurable then so is X^τ.

For any given $t\in\mathbb{T}$, let Y be the $\mathcal{B}(\mathbb{T})\otimes\mathcal{F}_t$-measurable process $Y_s = X^t_s = X_{s\wedge t}$. As $\tau\wedge t$ is $\mathcal{F}_t$-measurable, the result proven above says that $(X^\tau)^t = Y^{\tau\wedge t}$ will also be $\mathcal{B}(\mathbb{T})\otimes\mathcal{F}_t$-measurable, so X^τ is progressive.

If X is optional then so is X^τ.

As the optional processes are generated by the right-continuous and adapted processes then it is enough to prove this result when X is right-continuous and adapted. Clearly, X^τ will be right-continuous. Also, X will be progressive (see measurability of stochastic processes) and, by the result proven above, it follows that X^τ is progressive and, in particular, is adapted.

If X is predictable then so is X^τ.

By the definition of predictable processes, it is enough to prove the result in the cases where $X_t = 1_A 1_{\{t>s\}}$ for some $s\in\mathbb{T}$ and $A\in\mathcal{F}_s$, and $X_t = 1_A 1_{\{t=t_0\}}$ where t_0 is the minimal element of $\mathbb{T}$ and $A\in\mathcal{F}_{t_0}$.

In the first case, $X^\tau_t = 1_{A\cap\{\tau>s\}}1_{\{t>s\}}$ is predictable and, in the second case, $X^\tau_t = 1_A 1_{\{t=t_0\}} + 1_{A\cap\{\tau=t_0\}}1_{\{t>t_0\}}$ is predictable.