mean hitting time

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Let $(X_n)_{n\geq 0}$ be a Markov chain with transition probabilities p_{ij} where i, j are states in an indexing set I. Let H^A be the hitting time of $(X_n)_{n\geq 0}$ for a subset $A \subseteq I$. That is, H^A is the random variable of the time it takes for the to first reach a in A.

Define the *mean hitting time* of A given the starts in state i to be

$$k_i^A := E(H^A | X_0 = i).$$

Proposition 1. *The mean hitting times are the minimal non negative solution to:*

$$k_i^A = \begin{cases} 0 & i \in A \\ 1 + \displaystyle\sum_{j\in I} p_{ij} k_j^A & i \notin A \end{cases}$$

Remark. In this case, a solution is minimal if for any non negative solution $\{y_i | i \in I\}$ we have $y_i \geq k_i^A$ for all $i \in I$.

Proof. If $i \in A$, then $H^A = \inf\{n \geq 0 \mid X_n \in A\} \equiv 0$, which means $k_i^A = 0$ (the is certain to be in a state in A at step $n = 0$).

If $i \notin A$ we condition on the first step:

$$\begin{aligned} k_i^A &= E(H^A \mid X_0 = i) \\ &= \sum_{j\in I} P(X_1 = j | X_0 = i) E(H^A | X_0 = i, X_1 = j) \\ &= \sum_{j\in I} p_{ij} E(H^A | X_1 = j) \quad \text{(by the Markov property)} \\ &= \sum_{j\in I} p_{ij}(1 + k_j^A) \\ &= 1 + \sum_{j\in I} p_{ij} k_j^A \end{aligned}$$

So the k_i^A satisfy the given equations.

Now suppose that $\{y_i | y \in I\}$ is any non-negative solution to the equations. Then for $i \in A$ we have $k_i^A = y_i = 0$. If $i \notin A$, then

$$
\begin{aligned}
y_i &= 1 + \sum_{j\in I} p_{ij} y_j \\
&= 1 + \sum_{j\notin A} p_{ij}(1 + \sum_{k\notin A} p_{jk} y_k) \\
&= 1 + \sum_{j\notin A} p_{ij} + \sum_{j\notin A}\sum_{k\notin A} p_{ij} p_{jk} y_k \\
&= 1 + q_1 + q_2 + \cdots + q_n + \sum \cdots \sum p_{ij} \cdots p_{uv} y_v,
\end{aligned}
$$

where $q_n = P(X_1 \notin A, X_1 \notin A, \cdots, X_n \notin A | X_0 = i)$ is the probability that the chain X does not enter A in the first n steps after the initial state i.

y_i is non negative by assumption, therefore so is the final term, and so

$$y_i \geq 1 + q_1 + q_2 + \cdots + q_n.$$

Since n is arbitrary, by taking the limit $n \to \infty$, we have that

$$y_i \geq \lim_{n\to\infty} (1 + q_1 + q_2 + \cdots + q_n) \geq k_i^A.$$

So $y_i \geq k_i^A$ for all $i \in I$ and therefore $\{k_i^A | i \in I\}$ is the minimal solution. $\square$