# stochastic integration as a limit of Riemann sums

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As with the http://planetmath.org/RiemannIntegralRiemann and Riemann-Stieltjes integrals, the stochastic integral can be calculated as a limit of approximations computed on http://planetmath.org/Partition3partitions, called Riemann sums.

Let $P_n$ be a sequence of partitions of $\mathbb{R}_+$,

$$P_n = \{0 = \tau_0^n \leq \tau_1^n \leq \cdots \uparrow \infty\}$$

where, $\tau_k^n$ can, in general, be stopping times. Suppose that the mesh $|P_n^t| = \sup_k(\tau_k^n \wedge t - \tau_{k-1}^n \wedge t)$ tends to zero http://planetmath.org/ConvergenceInProbabilityin probability as $n \to \infty$, for each time $t > 0$.

The stochastic integral of a process $Y$ with respect to $X$ can be defined on each of the partitions,

$$I_t^n(Y, X) \equiv \sum_k Y_{\tau_{k-1}^n}(X_{\tau_k^n \wedge t} - X_{\tau_{k-1}^n \wedge t}).$$

Since the times $\tau_k^n$ tend to infinity as $k \to \infty$, all but finitely many terms are zero. Note that here, the process $Y$ is sampled at $\tau_{k-1}^n$, which are the left hand points of the intervals. For this reason, the stochastic integral is sometimes called the forward integral. Alternatively, the backward integral can be computed by sampling $Y$ at time $t_k$ and the Stratonovich integral takes the average of $Y_{t_{k-1}}$ and $Y_{t_k}$. However, these alternative integrals are less general and may not exist even when $Y$ is a continuous and adapted process.

For left-continuous integrands, the approximations do indeed converge to the stochastic integral.

**Theorem 1.** *Suppose that $X$ is a semimartingale and $Y$ is an adapted, left-continuous and locally bounded process. Then,*

$$I_t^n(Y, X) \to \int_0^t Y\, dX$$

*in probability as $n \to \infty$. Furthermore, this converges ucp and in the semimartingale topology.*

Similarly, convergence is also obtained for cadlag integrands. However, in this case, it is necessary to use the left limit $Y_{s-}$ in the integral. The integral of $Y$ does not even exist when it is a general cadlag adapted process, as it might not be predictable.

**Theorem 2.** *Suppose that $X$ is a semimartingale and $Y$ is a cadlag adapted process. Then,*

$$I_t^n(Y, X) \to \int_0^t Y_{s-} \, dX_s$$

*in probability as $n \to \infty$. Furthermore, this converges ucp and in the semimartingale topology.*