# derivation of Black-Scholes formula in martingale form

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# Contents

This entry derives the Black-Scholes formula in martingale form.

The portfolio process $V_t$ representing a stock option will be shown to satisfy:

$$V_t = e^{-r(T-t)}\, \mathbb{E}^{\mathbb{Q}}\big[V_T \mid \mathcal{F}_t\big]\,. \tag{1}$$

(The quantities appearing here are defined precisely, in the section on "Assumptions" below.)

Equation (**??**) can be used in practice to calculate $V_t$ for all times $t$, because from the specification of a financial contract, the value of the portfolio at time $T$, or in other words, its pay-off at time $T$, will be a known function. Mathematically speaking, $V_T$ gives the *terminal condition* for the solution of a stochastic differential equation.

## 0.1 Assumptions

### 0.1.1 Asset price

The asset or stock price $X_t$ is to be modelled by the stochastic differential equation:

$$dX_t = \mu X_t\, dt + \sigma X_t\, dW_t\,, \tag{2}$$

where $\mu$ and $\sigma > 0$ are constants.

The stochastic process $W_t$ is a standard Brownian motion adapted to the filtration $\{\mathcal{F}_t\}$.

See the main article on the `http://planetmath.org/BlackScholesFormula`Black-Scholes formula for an explanation and justification of this modelling assumption.

### 0.1.2 Money-market account

The money-market account accumulates interest compounded continuously at a rate of $r$. It satisfies the stochastic differential equation:

$$dM_t = rM_t\, dt\,. \tag{3}$$

This happens to take the same form as an ordinary differential equation, for the process $M_t$ has no randomness in it at all, under the assumption of a fixed interest rate $r$.

The solution is to equation (**??**) with initial condition $M_0$ is $M_t = M_0\, e^{rt}$.

### 0.1.3 Portfolio process

The price of the option is derived by following a *replicating portfolio* consisting of $\Delta_t$ units of the stock $X_t$ and $\Theta_t$ units of the money-market account. If $V_t$ denotes the value of this portfolio at time $t$, then

$$V_t = \Delta_t\, X_t + \Theta_t\, M_t\,. \tag{4}$$

A certain "self-financing condition" on the portfolio requires that $V_t$ also satisfy the stochastic differential equation:

$$dV_t = \Delta_t\, dX_t + \Theta_t\, dM_t\,. \tag{5}$$

This condition essentially says that we cannot input extra amounts of money out of thin air into our portfolio; we must start with what we have.

Equation (**??**) is not a mathematically proven statement, but another modelling assumption, justified by an analogous equation governing trading in discretized time periods.

## 0.2 Derivation

We first manipulate the stochastic differential equation (**??**) for the portfolio process $V_t$, to express it in terms of the Brownian motion $W_t$.

$$\begin{aligned}
dV_t &= \Delta_t\, dX_t + \Theta_t\, r M_t\, dt && \text{from eq. (??) and (??)} \\
&= \Delta_t\, dX_t + r\big(V_t - \Delta_t X_t\big)\, dt && \text{from eq. (??)} \\
&= \Delta_t\, \big(\mu X_t\, dt + \sigma X_t\, dW_t\big) && \\
&\qquad + r\big(V_t - \Delta_t X_t\big)\, dt && \text{from eq. (??)} \\
&= rV_t\, dt + \Delta_t X_t\, \big((\mu - r)\, dt + \sigma\, dW_t\big) && \text{rearrangement}
\end{aligned}$$

### 0.2.1 Change of probability measure

Define the Brownian motion with drift $\lambda$:

$$\widetilde{W}_t = \lambda t + W_t\,, \quad \lambda = \frac{\mu - r}{\sigma}\,; \tag{6}$$

so that $d\widetilde{W}_t = \lambda\,dt + dW_t$, and

$$dV_t = rV_t\,dt + \sigma\,\Delta_t X_t\,d\widetilde{W}_t\,. \tag{7}$$

The introduction of the process $\widetilde{W}_t$ is not merely for notational convenience but is mathematically meaningful. If the probability space we are working in is $(\Omega, \mathcal{F}_T, \mathbb{P})$, and $W_t$, for $0 \leq t \leq T$, is a standard Wiener process on $(\Omega, \mathcal{F}_T, \mathbb{P})$, then $\widetilde{W}_t$ will not be a standard Wiener process on $(\Omega, \mathcal{F}_T, \mathbb{P})$, but it *will* be a standard Wiener process under $(\Omega, \mathcal{F}_T, \mathbb{Q})$ *with a different probability measure* $\mathbb{Q}$.

The probability measure $\mathbb{Q}$ is obtained by Girsanov's theorem. The exact form for $\mathbb{Q}$ can be calculated, but it will not be needed in this derivation.

In finance, $\mathbb{Q}$ is known as the *risk-neutral measure*, and the quantity $\lambda$ is the *market price of risk*.

### 0.2.2 Discounted portfolio process is a martingale

From equation (**??**), we see that the value of the portfolio grows at the risk-free interest rate of $r$, apart from the randomness associated due to the stochastic differential $d\widetilde{W}_t$.

It is thus reasonable to expect that, if we normalize the portfolio value amount by the amount that cash grows due to accumulation of risk-free interest, the resulting process, $V_t/M_t$, should have a zero growth rate. That this is indeed the case can be verified by a computation with Itô's formula — more specifically, the  for Itô integrals:

$$\begin{aligned}
d\left(\frac{V_t}{M_t}\right) &= d\left(V_t \cdot \frac{1}{M_t}\right) \\
&= \big(dV_t\big)\,\frac{1}{M_t} + V_t\,d\left(\frac{1}{M_t}\right) \\
&= r\frac{V_t}{M_t}\,dt + \sigma\Delta_t\frac{X_t}{M_t}\,d\widetilde{W}_t + V_t\,d\left(\frac{1}{M_t}\right) && \text{from eq. (\textbf{??})} \\
&= r\frac{V_t}{M_t}\,dt + \sigma\Delta_t\frac{X_t}{M_t}\,d\widetilde{W}_t + \frac{V_t}{M_t}\,dt && \text{from } \frac{1}{M_t} = \frac{e^{-rt}}{M_0}.
\end{aligned}$$

Thus,

$$d\left(\frac{V_t}{M_t}\right) = \sigma\Delta_t\,\frac{X_t}{M_t}\,d\widetilde{W}_t\,.$$

Or, in integral form:

$$\frac{V_{t_1}}{M_{t_1}} = \frac{V_{t_0}}{M_{t_0}} + \int_{t_0}^{t_1} \sigma \Delta_t \, \frac{X_t}{M_t} \, d\widetilde{W}_t \,, \quad 0 \leq t_0 \leq t_1 \leq T \,. \tag{8}$$

Assuming $\Delta_t$ is a $\mathcal{F}_t$-adapted process — where $\{\mathcal{F}_t\}$ is the filtration generated by the Brownian motion $W_t$ (or equivalently $\widetilde{W}_t$) — the Itô integral in equation (**??**) is a martingle under the probability space $(\Omega, \mathcal{F}_T, \mathbb{Q})$.

### 0.2.3 Portfolio process as a conditional expectation

Then by the definition of a martingale, we have

$$\frac{V_{t_0}}{M_{t_0}} = \mathbb{E}^{\mathbb{Q}} \left[ \frac{V_{t_1}}{M_{t_1}} \mid \mathcal{F}_{t_0} \right] , \quad 0 \leq t_0 \leq t_1 \leq T \,,$$

where $\mathbb{E}^{\mathbb{Q}}[\cdot \mid \mathcal{F}_t]$ denotes the conditional expectation, of a random variable on the measurable space $(\Omega, \mathcal{F}_T)$, under the probability measure $\mathbb{Q}$.

In particular, setting $t_0 = t \leq T$ and $t_1 = T$, and rearranging the factors of $M_t = e^{rt}$, we obtain the desired result, equation (**??**).

## 0.3 Existence of solutions

So far, we have derived the form of the solution for the portfolio value process $V_t$ , *assuming that it exists.* Actually, if we were to take only equations (**??**) and (**??**) as the problem to solve mathematically, without any reference to the financial motivations, it is possible to work backwards and deduce the existence of the solution.

### 0.3.1 Proposed construction

Let $\mathbb{Q}$ be the risk-neutral probability measure, and let $U$ be any given $\mathbf{L}^1(\Omega, \mathcal{F}_T, \mathbb{Q})$ random variable, representing the *terminal condition.* Define the family of random variables dependent on time,

$$V_t = e^{-r(T-t)} \, \mathbb{E}^{\mathbb{Q}}[U \mid \mathcal{F}_t] \,, \quad 0 \leq t \leq T \,. \tag{9}$$

It is easy to verify that, for any $U$, the process $V_t \, e^{-rt}$ is a martingale with respect to $\mathcal{F}_t$, the filtration generated by the Wiener process $\widetilde{W}_t$ under the probability measure $\mathbb{Q}$.

### 0.3.2 Verification

We now invoke the martingale representation theorem for Itô processes: for any martingale $Z_t$, with respect to $\mathcal{F}_t$ under the probability measure $\mathbb{Q}$, there exists a $\mathcal{F}_t$-adapted process $G_t$ such that $Z_t$ has the representation:

$$Z_{t_1} - Z_{t_0} = \int_{t_0}^{t_1} G_t \, d\widetilde{W}_t \, .$$

Letting $Z_t = V_t \, e^{-rt}$ and comparing with equations (**??**) and (**??**), we are motivated to *define* the $\mathcal{F}_t$-adapted processes:

$$\Delta_t = \frac{G_t \, e^{rt}}{\sigma X_t} \, , \quad \Theta_t = \frac{V_t - \Delta_t X_t}{M_t} = \frac{Z_t - G_t/\sigma}{M_0} \, .$$

Then the process $V_t$ constructed by equation (**??**) trivially satisfies equation (**??**). And it is a simple matter to check that equation (**??**) holds as well:

$$\begin{aligned}
dV_t = d\big(Z_t \, e^{rt}\big) &= e^{rt} \, dZ_t + re^{rt} \, Z_t \, dt && \text{Itô's product rule} \\
&= e^{rt} \, G_t \, d\widetilde{W}_t + rV_t \, dt \\
&= \sigma \Delta_t X_t \, d\widetilde{W}_t + rV_t \, dt \\
&= \Delta_t \, X_t (r \, dt + \sigma d\widetilde{W}_t) && \text{add and subtract} \\
&\qquad + r\big(V_t - \Delta_t X_t\big) && \text{the } dt \text{ term} \\
&= \Delta_t \, dX_t + r \, \Theta_t \, M_t \, dt \, ,
\end{aligned}$$

where in the last equality we have used the SDE for $X_t$ in terms of $d\widetilde{W}_t$ in place of $dW_t$:

$$dX_t = rX_t \, dt + \sigma X_t \, d\widetilde{W}_t \, ,$$

obtained by substituting in equation (**??**), the differential of equation (**??**).

# References


[1] Bernt Øksendal. *Stochastic Differential Equations, An Introduction with Applications*, 5th edition. Springer, 1998.

[2] Steven E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Springer, 2004.