filtration of σ-algebras

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For an ordered set T, a filtration of http://planetmath.org/SigmaAlgebraσ-algebras $(\mathcal{F}_t)_{t\in T}$ is a collection of σ-algebras on an underlying set Ω, satisfying $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s < t$ in T. Here, t is understood as the time variable, taking values in the index set T, and $\mathcal{F}_t$ represents the collection of all events observable up until time t. The index set is usually a subset of the real numbers, with common examples being $T = \mathbb{Z}_+$ for discrete-time and $T = \mathbb{R}_+$ for continuous-time scenarios. The collection $(\mathcal{F}_t)_{t\in T}$ is a filtration on a measurable space $(\Omega, \mathcal{F})$ if $\mathcal{F}_t \subseteq \mathcal{F}$ for every t. If, furthermore, there is a probability measure defined on the underlying measurable space then this gives a filtered probability space. The alternative notation $(\mathcal{F}_t, t \in T)$ is often used for the filtration or, when the index set T is clear from the context, simply $(\mathcal{F}_t)$ or $\mathbf{F}$.

Filtrations are widely used for studying stochastic processes, where a process X_t with time ranging over the set T is said to be adapted to the filtration if X_t is an $\mathcal{F}_t$-measurable random variable for each time t.

Conversely, any stochastic process $(X_t)_{t\in T}$ generates a filtration. Let $\mathcal{F}_t$ be the smallest σ-algebra with respect to which X_s is measurable for all $s \leq t$,

$$\mathcal{F}_t = \sigma\left(X_s : s \leq t\right).$$

This defines the smallest filtration to which X is adapted, known as the *natural filtration* of X.

Given a filtration, there are various limiting σ-algebras which can be defined. The values at plus and minus infinity are

$$\mathcal{F}_\infty = \sigma\left(\bigcup_t \mathcal{F}_t\right),\ \mathcal{F}_{-\infty} = \bigcap_t \mathcal{F}_t,$$

which satisfy $\mathcal{F}_{-\infty} \subseteq \mathcal{F}_t \subseteq \mathcal{F}_\infty$. In continuous-time, when the index set is an interval of the real numbers, the left and right limits can be defined at any time. They are,

$$\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s,\ \mathcal{F}_{t-} = \sigma\left(\bigcup_{s<t} \mathcal{F}_s\right),$$

except if t is the maximum of T it is often convenient to set $\mathcal{F}_{t+} = \mathcal{F}_t$ or, if t is the minimum, $\mathcal{F}_{t-} = \mathcal{F}_t$. It is easily verified that $\mathcal{F}_s \subseteq \mathcal{F}_{s+} \subseteq \mathcal{F}_{t-} \subseteq \mathcal{F}_t$ for all times $s < t$. Furthermore, $(\mathcal{F}_{t+})$ and $(\mathcal{F}_{t-})$ are themselves filtrations.

A filtration is said to be right-continuous if $\mathcal{F}_t = \mathcal{F}_{t+}$ for every t so, in particular, $(\mathcal{F}_{t+})$ is always the smallest right-continuous filtration larger than $(\mathcal{F}_t)$.