Proof of Bonferroni Inequalities

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Definitions and Notation. *A* measure space *is a triple* (X, Σ, μ)*, where* X *is a set,* Σ *is a* σ*-algebra over* X*, and* $\mu\colon \Sigma \to [0, \infty]$ *is a measure, that is, a non-negative function that is countably additive. If* $A \in \Sigma$*, the* characteristic function *of* A *is the function* $\chi_A\colon X \to \mathbb{R}$ *defined by* $\chi_A(x) = 1$ *if* $x \in A$*,* $\chi_A(x) = 0$ *if* $x \notin A$*. A* unimodal sequence *is a sequence of real numbers* $a_0, a_1, \ldots, a_n$ *for which there is an index* k *such that* $a_i \leq a_{i+1}$ *for* $i < k$ *and* $a_i \geq a_{i+1}$ *for* $i \geq k$*.*

The proof of the following easy lemma is left to the reader:

Lemma 1. *If* $a_0 \leq a_1 \leq \ldots \leq a_k \geq a_{k+1} \geq a_{k+2} \geq \ldots \geq a_n$ *is a unimodal sequence of non-negative real numbers with* $\sum_{i=0}^{n}(-1)^i a_i = 0$*, then* $\sum_{i=0}^{j}(-1)^i a_i \geq 0$ *for even* j *and* ≤ 0 *for odd* j*.*

Since the binomial sequence $(\binom{a}{i})_{0 \leq i \leq n}$ with integer $a > 0$ and integer $n \geq a$ satisfies the hypothesis of Lemma **??**, we have:

Corollary 1. *If* a *is a positive integer,* $\sum_{i=0}^{j}(-1)^i\binom{a}{i} \geq 0$ *for even* j *and* ≤ 0 *for odd* j*.*

Lemma 2. *Let* $(A_i)_{1 \leq i \leq n}$ *be a sequence of sets and let* $X = \bigcup_{1 \leq i \leq n} A_i$*. For* $x \in X$*, let* $I(x)$ *be the set of indices* j *such that* $x \in A_j$*. If* $1 \leq k \leq n$*,*

$$\sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \chi_{A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}}(x) = \binom{|I(x)|}{k}$$

for all $x \in X$*.*

Proof. $\chi_{A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}}(x) = 1$ if $\{i_1, i_2, \ldots, i_k\} \subseteq I(x)$, and $= 0$ otherwise. Therefore the sum equals the number of k-subsets of $I(x)$, which is $\binom{|I(x)|}{k}$. □

Theorem 1. *Let* (X, Σ, μ) *be a measure space. If* $(A_i)_{1 \leq i \leq n}$ *is a finite sequence of measurable sets all having finite measure, and*

$$S_j = \mu(A_1 \cup A_2 \cup \ldots \cup A_n) + \sum_{k=1}^{j}(-1)^k \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \mu(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k})$$

then $S_j \geq 0$ *for even* j*, and* ≤ 0 *for odd* j*. Moreover,* $S_n = 0$ *(Principle of Inclusion-Exclusion).*

Proof. Let $Y = \bigcup_{1\leq i\leq n} A_i$.

$$\begin{aligned} S_j &= \int_Y d\mu + \sum_{k=1}^{j}(-1)^k \sum_{1\leq i_1<i_2<\ldots<i_k\leq n} \int_Y \chi_{A_{i_1}\cap A_{i_2}\cap\ldots\cap A_{i_k}} d\mu \\ &= \int_Y d\mu + \sum_{k=1}^{j}(-1)^k \int_Y \Big(\sum_{1\leq i_1<i_2<\ldots<i_k\leq n} \chi_{A_{i_1}\cap A_{i_2}\cap\ldots\cap A_{i_k}} \Big) d\mu \end{aligned}$$

By Lemma **??**,

$$\begin{aligned} S_j &= \int_Y d\mu + \sum_{k=1}^{j}(-1)^k \int_Y \binom{|I(x)|}{k} d\mu \\ &= \sum_{k=0}^{j}(-1)^k \int_Y \binom{|I(x)|}{k} d\mu \\ &= \int_Y \sum_{k=0}^{j}(-1)^k \binom{|I(x)|}{k} d\mu \end{aligned}$$

Since $|I(x)| > 0$ for $x \in Y$, it follows from Corollary **??** that, in the last integral, the integrand is ≥ 0 for even j and ≤ 0 for odd j. Therefore the same is true for the integral itself. In addition, the integrand is identically 0 for $j = n$, hence $S_n = 0$. □

This proof shows that at the heart of Bonferroni's inequalities lie similar inequalities governing the binomial coefficients.