martingale proof of Kolmogorov's strong law for square integrable variables

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We apply the martingale convergence theorem to prove the following result.

Theorem. *Let $X_1, X_2, \dots$ be independent random variables such that $\sum_n \operatorname{Var}[X_n]/n^2 < \infty$. Then, setting*

$$S_n = \frac{1}{n}\sum_{k=1}^{n}(X_k - \mathbb{E}[X_k])$$

we have $S_n \to 0$ as $n \to \infty$, with probability one.

To prove this, we start by constructing a martingale,

$$M_n = \sum_{k=1}^{n}\frac{X_k - \mathbb{E}[X_k]}{k}.$$

If $\mathcal{F}_n$ is the `http://planetmath.org/SigmaAlgebra`σ-algebra generated by $X_1, \dots X_n$ then

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = M_n + \frac{\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] - \mathbb{E}[X_{n+1}]}{n+1} = M_n.$$

Here, the independence of X_{n+1} and $\mathcal{F}_n$ has been used to imply that $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[X_{n+1}]$. So, M is a martingale with respect to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

Also, by the independence of the X_n, the variance of M_n is

$$\operatorname{Var}[M_n] = \sum_{k=1}^{n}\operatorname{Var}[X_k/k] \leq \sum_{k=1}^{\infty}\frac{\operatorname{Var}[X_k]}{k^2} < \infty.$$

So, the inequality $\mathbb{E}[|M_n|] \leq \sqrt{\mathbb{E}[M_n^2]} = \sqrt{\operatorname{Var}[M_n]}$ shows that M is an L^1-bounded martingale, and the martingale convergence theorem says that the limit $M_\infty = \lim_{n\to\infty} M_n$ exists and is finite, with probability one.

The strong law now follows from Kronecker's lemma, which states that for sequences of real numbers $x_1, x_2, \dots$ and $0 < b_1, b_2, \dots$ such that b_n strictly increases to infinity and $\sum_n x_n/b_n$ converges to a finite limit, then $b_n^{-1}\sum_{k=1}^{n} x_k$ tends to 0 as $n \to \infty$. In our case, we take $x_n = X_n - \mathbb{E}[X_n]$ and $b_n = n$ to deduce that $n^{-1}\sum_{k=1}^{n}(X_k - \mathbb{E}[X_k])$ converges to zero with probability one.

References

[1] David Williams, *Probability with martingales*, Cambridge Mathematical Textbooks, Cambridge University Press, 1991.

[2] Olav Kallenberg, *Foundations of modern probability*, Second edition. Probability and its Applications. Springer-Verlag, 2002.