analytic solution of Black-Scholes PDE

Canonical name	AnalyticSolutionOfBlackScholesPDE
Date of creation	2013-03-22 16:31:34
Last modified on	2013-03-22 16:31:34
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Last modified by	stevecheng (10074)
Numerical id	6
Author	stevecheng (10074)
Entry type	Derivation
Classification	msc 60H10
Classification	msc 91B28
Related topic	ExampleOfSolvingTheHeatEquation
Related topic	BlackScholesPDE
Related topic	BlackScholesFormula

Here we present an analytical solution for the *Black-Scholes partial differential equation*,

$$rf = \frac{\partial f}{\partial t} + rx\,\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 x^2\,\frac{\partial^2 f}{\partial x^2}\,, \quad f = f(t,x)\,, \tag{1}$$

over the domain $0 < x < \infty$, $0 \leq t \leq T$, with terminal condition $f(T,x) = \psi(x)$, by reducing this parabolic PDE to the heat equation of physics.

We begin by making the substitution:

$$u = e^{-rt}\,f\,,$$

which is motivated by the fact that it is the portfolio value *discounted by the interest rate* r (see the derivation of the Black-Scholes formula) that is a martingale. Using the product rule on $f = e^{rt}\,u$, we derive the PDE that the function u must satisfy:

$$rf = re^{rt}\,u = re^{rt}\,u + e^{rt}\frac{\partial u}{\partial t} + rxe^{rt}\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2 x^2 e^{rt}\frac{\partial^2 u}{\partial x^2}\,;$$

or simply,

$$0 = \frac{\partial u}{\partial t} + rx\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 u}{\partial x^2}\,. \tag{2}$$

Next, we make the substitutions:

$$y = \log x\,, \quad s = T - t\,.$$

These changes of variables can be motivated by observing that:

- The underlying process described by the variable x is a geometric Brownian motion (as explained in the derivation of the Black-Scholes formula itself), so that $\log x$ describes a Brownian motion, possibly with a drift. Then $\log x$ should satisfy some sort of diffusion equation (well-known in physics).

- The evolution of the system is backwards from the terminal state of the system. Indeed, the boundary condition is given as a terminal state, and the coefficient of $\partial u/\partial t$ is positive in equation (**??**). (Compare with the standard heat equation, $0 = -\partial u/\partial t + \partial u/\partial x$, which describes a temperature evolving forwards in time.) So to get to the heat equation, we have to use a substitution to reverse time.

Since
$$\frac{\partial u}{\partial s} = -\frac{\partial u}{\partial t}\,, \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}\,\frac{dy}{dx} = \frac{1}{x}\,\frac{\partial u}{\partial y}\,,$$
and
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial u}{\partial y}\right) = -\frac{1}{x^2}\,\frac{\partial u}{\partial y} + \frac{1}{x^2}\frac{\partial^2 u}{\partial y^2}\,,$$
substituting in equation (**??**), we find:

$$0 = -\frac{\partial u}{\partial s} + (r - \tfrac{1}{2}\sigma^2)\frac{\partial u}{\partial y} + \frac{1}{2}\sigma^2\,\frac{\partial^2 u}{\partial y^2}\,. \tag{3}$$

The first partial derivative with respect to y does not cancel (unless $r = \frac{1}{2}\sigma^2$) because we have not take into account the drift of the Brownian motion. To cancel the drift (which is linear in time), we make the substitutions:

$$z = y + (r - \tfrac{1}{2}\sigma^2)\tau\,, \quad \tau = s\,.$$

Under the new coordinate system (z, τ), we have the relations amongst vector fields:
$$\frac{\partial}{\partial z} = \frac{\partial}{\partial y}\,, \quad \frac{\partial}{\partial \tau} = -(r - \tfrac{1}{2}\sigma^2)\frac{\partial}{\partial y} + \frac{\partial}{\partial s}\,,$$
leading to the following of equation (**??**):

$$0 = -\frac{\partial u}{\partial \tau} - (r - \tfrac{1}{2}\sigma^2)\frac{\partial u}{\partial z} + (r - \tfrac{1}{2}\sigma^2)\frac{\partial u}{\partial z} + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial z^2}\,;$$

or:

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial z^2}\,, \quad u = u(\tau, z)\,, \tag{4}$$

which is one form of the diffusion equation. The domain is on $-\infty < z < \infty$ and $0 \le \tau \le T$; the initial condition is to be:

$$u(0, z) = e^{-rT}\,\psi(e^z) := u_0(z)\,.$$

The original function f can be recovered by

$$f(t, x) = e^{rt}\,u\Big(T - t,\ \log x + (r - \tfrac{1}{2}\sigma^2)\tau\Big)\,.$$

The fundamental solution of the PDE (**??**) is known to be:

$$G_\tau(z) = \frac{1}{\sqrt{2\pi\sigma^2\tau}} \exp\Bigl(-\frac{z}{2\sigma^2\tau}\Bigr)$$

(derived using the Fourier transform); and the solution u with initial condition u_0 is given by the convolution:

$$u(\tau, z) = u_0 * G_\tau(z) = \frac{e^{-rT}}{\sqrt{2\pi\sigma^2\tau}} \int_{-\infty}^{\infty} \psi(e^\zeta)\, \exp\Bigl(-\frac{(z-\zeta)^2}{2\sigma^2\tau}\Bigr)\, d\zeta\,.$$

In terms of the original function f:

$$f(t, x) = \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}} \int_{-\infty}^{\infty} \psi(e^\zeta)\, \exp\Bigl(-\frac{\bigl(\log x + (r - \frac{1}{2}\sigma^2)\tau - \zeta\bigr)^2}{2\sigma^2\tau}\Bigr)\, d\zeta\,,$$

($\tau = T - t$) which agrees with the `http://planetmath.org/BlackScholesFormula`result derived using probabilistic methods.