# existence of the conditional expectation

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Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X$ be a random variable. For any $\sigma$-algebra $\mathcal{G} \subseteq \mathcal{F}$, we show the existence of the conditional expectation $\mathbb{E}[X \mid \mathcal{G}]$. Although it is possible to do this using the Radon-Nikodym theorem, a different approach is used here which relies on the completeness of the vector space $L^2$. The defining property of the conditional expectation $Y = \mathbb{E}[X \mid \mathcal{G}]$ is

$$\mathbb{E}[1_G Y] = \mathbb{E}[1_G X] \tag{1}$$

for sets $G \in \mathcal{G}$. We shall prove the existence of the conditional expectation for all nonnegative random variables and, more generally, whenever $\mathbb{E}[|X| \mid \mathcal{G}]$ is almost surely finite.

First, the conditional expectation of every square-integrable random variable exists.

**Theorem 1.** *Suppose that $\mathbb{E}[X^2] < \infty$. Then there is a $\mathcal{G}$-measurable random variable $Y$ satisfying $\mathbb{E}[Y^2] < \infty$ and equation (**??**) is satisfied for all $G \in \mathcal{G}$.*

*Proof.* Consider the norm $\|Y\|_2 \equiv \mathbb{E}[Y^2]^{1/2}$ on the vector space $V = L^2(\Omega, \mathcal{F}, \mathbb{P})$ of real valued random variables $Y$ satisfying $\mathbb{E}[Y^2] < \infty$ (up to $\mathbb{P}$ almost everywhere equivalence). This is given by the following inner product

$$\langle Y_1, Y_2 \rangle \equiv \mathbb{E}[Y_1 Y_2].$$

As $L^p$-spaces are complete, this makes $V$ into a Hilbert space (see also, `http://planetmath.org/L2SpacesAreHilbertSpaces`$L^2$-spaces are Hilbert spaces). Then, $U \equiv L^2(\Omega, \mathcal{G}, \mathbb{P})$ is a complete, and hence closed, subspace of $V$.

By the `http://planetmath.org/ProjectionsAndClosedSubspaces`existence of orthogonal projections onto closed subspaces of Hilbert spaces, there is an orthogonal projection $\pi \colon V \to U$. In particular, $\langle \pi Y - Y, Z \rangle = 0$ for all $Y \in V$ and $Z \in U$. Setting $Y = \pi X$ gives

$$\mathbb{E}[1_G Y] - \mathbb{E}[1_G X] = \langle 1_G, \pi X - X \rangle = 0$$

as required. □

We can now prove the existence of conditional expectations of nonnegative random variables. Note that here there are no integrability conditions on $X$.

**Theorem 2.** *Let $X$ be a nonnegative random variable taking values in $\mathbb{R} \cup \{\infty\}$. Then, there exists a nonnegative $\mathcal{G}$-measurable random variable $Y$ taking values in $\mathbb{R} \cup \{\infty\}$ and satisfying (??) for all $G \in \mathcal{G}$. Furthermore, $Y$ is uniquely defined $\mathbb{P}$-`http://planetmath.org/AlmostSurely`almost everywhere.*

*Proof.* First, let $X_n = \min(n, X)$. As this is bounded, theorem **??** says that the conditional expectations $Y_n = \mathbb{E}[Y_n \mid \mathcal{G}]$ exist. Furthermore, as $X_0 = 0$, we may take $Y_0 = 0$. For any $n$, setting $G = \{Y_{n+1} < Y_n\} \in \mathcal{G}$ gives

$$\mathbb{E}[1_G(Y_n - Y_{n+1})] = \mathbb{E}[1_G(X_n - X_{n+1})] \leq 0.$$

So $1_G(Y_n - Y_{n+1})$ is a nonnegative random variable with nonpositive expectation, hence is almost surely equal to zero. Therefore, $Y_{n+1} \geq Y_n$ (almost surely) and, by replacing $Y_n$ with the maximum of $Y_1, \ldots Y_n$ we may suppose that $(Y_n)$ is an increasing sequence of random variables. Setting $Y = \sup_n Y_n$, the monotone convergence theorem gives

$$\mathbb{E}[1_G Y] = \lim_{n\to\infty} \mathbb{E}[1_G Y_n] = \lim_{n\to\infty} \mathbb{E}[1_G X_n] = \mathbb{E}[1_G X]$$

as required.

Finally, suppose that $\tilde{Y}$ is also a nonnegative $\mathcal{G}$-measurable random variable satisfying (**??**). For any $x \in \mathbb{R}$, setting $G = \{\tilde{Y} > Y, x > Y\}$ then $1_G Y$ is bounded and,

$$\mathbb{E}[1_G(\tilde{Y} - Y)] = \mathbb{E}[1_G X] - \mathbb{E}[1_G X] = 0$$

showing that $\mathbb{P}(G) = 0$. Letting $x$ increase to infinity gives $\tilde{Y} \leq Y$ (almost surely) and, similarly, $Y \leq \tilde{Y}$ so that $Y = \tilde{Y}$ almost surely. □

Finally, we show existence of the conditional expectation of every random variable $X$ satisfying $\mathbb{E}[|X| \mid \mathcal{G}] < \infty$ almost surely. Note, in particular, that this is satisfied whenever $X$ is integrable, as

$$\mathbb{E}[\mathbb{E}[|X| \mid \mathcal{G}]] = \mathbb{E}[|X|] < \infty.$$

**Theorem 3.** *Let $X$ be a random variable such that $\mathbb{E}[|X| \mid \mathcal{G}] < \infty$ almost surely. Then, there exists a $\mathcal{G}$-measurable random variable $Y$ such that $\mathbb{E}[1_G|Y|] < \infty$ and (??) is satisfied for every $G \in \mathcal{G}$ with $\mathbb{E}[1_G|X|] < \infty$.*

*Furthermore, $Y$ is uniquely defined up to $\mathbb{P}$-a.e. equivalence.*

*Proof.* The positive and negative parts $X_+, X_-$ of $X$ satisfy

$$\mathbb{E}[X_+ \mid \mathcal{G}] + \mathbb{E}[X_- \mid \mathcal{G}] = \mathbb{E}[|X| \mid \mathcal{G}] < \infty$$

almost surely. We can therefore set $Y_\pm \equiv \mathbb{E}[X_\pm \mid \mathcal{G}]$ and $Y = Y_+ - Y_-$.

If $G \in \mathcal{G}$ satisfies $\mathbb{E}[1_G|X|] < \infty$ then $\mathbb{E}[1_G Y_\pm] = \mathbb{E}[1_G X_\pm] < \infty$, so $\mathbb{E}[1_G|Y|] < \infty$ and,

$$\mathbb{E}[1_G Y] = \mathbb{E}[1_G Y_+] - \mathbb{E}[1_G Y_-] = \mathbb{E}[1_G X_+] - \mathbb{E}[1_G X_-] = \mathbb{E}[1_G X]$$

as required.

Finally, suppose that $\tilde{Y}$ satisfies the same conditions as $Y$. For any $x \geq 0$ set $G = \{Y_+ + Y_- \leq x, \tilde{Y} > Y\} \in \mathcal{G}$. Then,

$$\mathbb{E}[1_G|X|] = \mathbb{E}[1_G(Y_+ + Y_-)] \leq x < \infty.$$

So, $\mathbb{E}[1_G|Y|]$ and $\mathbb{E}[1_G|\tilde{Y}|]$ are finite, hence (**??**) gives

$$\mathbb{E}[1_G(\tilde{Y} - Y)] = \mathbb{E}[1_G X] - \mathbb{E}[1_G X] = 0.$$

So $\mathbb{P}(G) = 0$ and, letting $x$ increase to infinity, $\tilde{Y} \leq Y$ almost surely. Similarly, $Y \leq \tilde{Y}$ and therefore $\tilde{Y} = Y$ almost surely. $\square$