Levy process

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Let $(\Omega, \Psi, P, (\mathcal{F})_{0 \leq t < \infty})$ be a filtered probability space. A Lèvy process on that space is an stochastic process $L \colon [0, \infty) \times \Omega \to \Re^n$ that has the following properties:

1. L has increments independent of the past: for any $t \geq 0$ and for all $s \geq 0$, $L_{t+s} - L_t$ in independent of $\mathcal{F}_t$
2. L has stationary increments: if $t \geq s \geq 0$ then $L_t - L_s$ and L_{t-s} have the same distribution. This particulary implies that $L_{t+s} - L_t$ and L_s have the same distribution.
3. L is continous in probability: for any $t, s \in [0, \infty)$, $\lim_{t \to s} = X_s$, the limit taken in probability.

Some important properties of any Lèvy processes L are:

1. There exist a modification of L that has càdlàg paths a.s. (càdlàg paths means that the paths are continuous from the right and that the left limits exist for any $t \geq 0$).
2. L_t is an infinite divisible random variable for all $t \in [0, \infty)$
3. *Lèvy -Itô decomposition*: L can be written as the sum of a diffusion, a continuous Martingale and a pure jump process; i.e:

$$L_t = \alpha t + \sigma B_t + \int_{|x|<1} x \, d\tilde{N}_t(\cdot, dx) + \int_{|x| \geq 1} x \, dN_t(\cdot, dx) \quad \text{for all } t \geq 0$$

   where $\alpha \in \Re$, B_t is a standard brownian motion. N is defined to be the Poisson random measure of the Lèvy process (the process that counts the jumps): for any Borel A in $\Re^n$ such that $0 \notin cl(A)$ then $N_t(\cdot, A) \colon = \sum_{0 < s \leq t} 1_A(\Delta L_s)$, where $\Delta L_s \colon = L_s - L_{s-}$; and $\tilde{N}_t(\cdot, A) = N_t(\cdot, A) - tE[N_1(\cdot, A)]$ is the compensated jump process, which is a martingale.
4. *Lèvy -Khintchine formula*: from the previous property it can be shown that for any $t \geq 0$ one has that

$$E[e^{iuL_t}] = e^{-t\psi(u)}$$

where

$$\psi(u) = -i\alpha u + \frac{\sigma^2}{2} u^2 + \int_{|x|\geq 1} (1 - e^{iux})\, d\nu(x) + \int_{|x|<1} (1 - e^{iux} + iux)\, d\nu(x)$$

with $\alpha \in \Re$, $\sigma \in [0, \infty)$ and ν is a positive, borel, σ-finite measure called *Lèvy measure*. (Actually $\nu(\cdot) = E[N_1(\cdot, A)]$). The second formula is usually called the Lèvy exponent or Lèvy symbol of the process.

5. L is a semimartingale (in the classical sense of being a sum of a finite variation process and a local martingale), so it is a *good* integrator, in the stochastic sense.

Some important examples of Lèvy processes include: the Poisson Process, the Compound Poisson process, Brownian Motion, Stable Processes, Subordinators, etc.

Bibliography

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