semimartingale convergence implies ucp convergence

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Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{F}}, \mathbb{P})$ be a filtered probability space. On the space of cadlag adapted processes, the semimartingale topology is stronger than ucp convergence.

Theorem. *Let X^n be a sequence of cadlag adapted processes converging to X in the semimartingale topology. Then, X^n converges ucp to X.*

To show this, suppose that $X^n \to X$ in the semimartingale topology, and define the stopping times τ_n by

$$\tau_n = \inf\left\{t \geq 0 : |X^n_t - X_t| \geq \epsilon\right\} \tag{1}$$

(hitting times are stopping times). Then, letting ξ^n_t be the simple predictable process $1_{\{t\leq\tau_n\}}$,

$$X^n_{\tau_n\wedge t} - X_{\tau_n\wedge t} = X^n_0 - X_0 + \int_0^t \xi^n\, dX^n - \int_0^t \xi^n\, dX \to 0$$

in probability as $n \to \infty$. However, note that whenever $|X^n_s - X_s| > \epsilon$ for some $s < t$ then $\tau \leq s < t$ and $|X^n_{\tau_n} - X_{\tau_n}| \geq \epsilon$. So

$$\mathbb{P}\left(\sup_{s<t} |X^n_s - X_s| > \epsilon\right) \leq \mathbb{P}(\tau_n \leq t) \leq \mathbb{P}\left(|X^n_{\tau_n\wedge t} - X_{\tau_n\wedge t}| \geq \epsilon\right) \to 0$$

as $n \to \infty$, proving ucp convergence.

As a minor technical point, note that the result that the hitting times τ_n are stopping times requires the filtration to be at least universally complete. However, this condition is not needed. It is easily shown that semimartingale convergence is not affected by passing to the `http://planetmath.org/CompleteMeasure`completion of the filtered probability space or, alternatively, it is enough to define the stopping times in (**??**) by restricting τ_n to finite but suitably dense subsets of $[0, t]$ and using the right-continuity of the processes.