properties of expected value

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1) *(normalization)* Let X be almost surely constant random variable, i.e. $\Pr\{X = c\} = 1$; then $E[X] = c$.

2) *(linearity)* Let X, Y be random variables such that $E[|X|] < \infty$ and $E[|Y|] < \infty$ and let a, b be real numbers; then $E[|aX + bY|] < \infty$ and $E[aX + bY] = aE[X] + bE[Y]$.

3) *(monotonicity)* Let X, Y be random variables such that $\Pr\{X \leq Y\} = 1$ and $E[|X|] < \infty$, $E[|Y|] < \infty$; then $E[X] \leq E[Y]$.

Proof. 1) Let's define

$$F = \{\omega \in \Omega : X(\omega) = c\};$$

Then by hypothesis

$$\Pr\{\Omega \backslash F\} = 0$$

and

$$\Pr\{F\} = 1.$$

We have:

$$\begin{aligned} E[X] &= \int_{\Omega} X(\omega)\, dP \\ &= \int_{\Omega \backslash F} X(\omega)\, dP + \int_{F} X(\omega)\, dP \\ &= \int_{F} X(\omega)\, dP \\ &= \int_{F} c\, dP \\ &= c \Pr\{F\} = c. \end{aligned}$$

2) *[to be done]*.

3) Let's define

$$F = \{\omega \in \Omega : X(\omega) \leq Y(\omega)\};$$

Then by hypothesis

$$\Pr\{\Omega \backslash F\} = 0$$

and

$$\Pr\{F\} = 1.$$

We have, keeping in mind property 2),

$$
\begin{aligned}
E[Y]-E[X] &= E[Y-X] \\
&= \int_{\Omega} [Y(\omega)-X(\omega)]\,dP \\
&= \int_{\Omega\setminus F} [Y(\omega)-X(\omega)]\,dP + \int_{F} [Y(\omega)-X(\omega)]\,dP \\
&= \int_{F} [Y(\omega)-X(\omega)]\,dP \geq 0.
\end{aligned}
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$\square$