proof of completeness under ucp convergence

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Owner	gel (22282)
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Author	gel (22282)
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Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$ be a filtered probability space, $\mathcal{M}$ be a sub-σ-algebra of $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}$, and S be a set of real valued functions on $\mathbb{R}_+$ which is `http://planetmath.org/Closed`closed under uniform convergence on compacts. We show that both the set of $\mathcal{M}$-measurable processes and the set of jointly measurable processes with sample paths almost surely in S are `http://planetmath.org/Complete`complete under ucp convergence. The method used will be to show that we can pass to a subsequence which almost surely converges uniformly on compacts.

We start by writing out the metric generating the topology of uniform convergence on compacts (compact-open topology) for functions $\mathbb{R}_+ \to \mathbb{R}$. This is the same as uniform convergence on each of the bounded intervals $[0, n)$ for positive integers n,

$$d(X) \equiv \sum_{n=1}^{\infty} 2^{-n} \min\left(1, \sup_{t<n} |X_t|\right).$$

Then, the metric is $(X, Y) \mapsto d(X-Y)$. Convergence under the ucp topology is given by

$$D^{\rm ucp}(X) = \mathbb{E}[d(X)]$$

for any jointly measurable stochastic process X, with the (pseudo)metric being $(X, Y) \mapsto D^{\rm ucp}(X - Y)$.

Now, suppose that X^n is a sequence of jointly measurable processes such that $X^n - X^m \xrightarrow{\rm ucp} 0$ as $m, n \to \infty$. Then, $D^{\rm ucp}(X^n - X^m) \to 0$ and we may pass to a subsequence X^{n_k} satisfying $D^{\rm ucp}(X^{n_j} - X^{n_k}) \le 2^{-j}$ whenever $k > j$. So,

$$\mathbb{E}\left[\sum_k d(X^{n_k} - X^{n_{k+1}})\right] = \sum_k D^{\rm ucp}(X^{n_k} - X^{n_{k+1}}) \le \sum_k 2^{-k} = 1.$$

In particular, this shows that $\sum_k d(X^{n_k} - X^{n_{k+1}})$ is almost surely finite and, therefore,

$$d(X^{n_j} - X^{n_k}) \le \sum_{i=j}^{k-1} d(X^{n_i} - X^{n_{i+1}}) \to 0$$

as $k > j \to \infty$, with probability one.

So, the sequence X^{n_k} is almost surely `http://planetmath.org/CauchySequence`Cauchy, under the topology of uniform convergence on compacts. We set

$$X_t(\omega) \equiv \begin{cases} \lim_{k\to\infty} X^{n_k}_t(\omega), & \text{if the limit exists,} \\ 0, & \text{otherwise.} \end{cases}$$

As measurability of real valued functions is preserved under pointwise convergence, it follows that if X^n are $\mathcal{M}$-measurable, then so is X. In particular, X is a jointly measurable process. Furthermore, since convergence is almost surely uniform on compacts, if X^n have sample paths in S with probability one then so does X.

It only remains to show that $X^n \xrightarrow{\text{ucp}} X$. However, we have already shown that $d(X^{n_k} - X) \to 0$ with probability one, hence $D^{\text{ucp}}(X^{n_k} - X) \to 0$.

$$D^{\text{ucp}}(X^n - X) \le D^{\text{ucp}}(X^n - X^{n_k}) + D^{\text{ucp}}(X^{n_k} - X).$$

Letting k go to infinity, this is bounded by $\sup_{m>n} D^{\text{ucp}}(X^n - X^m)$, which goes to zero as $n \to \infty$.