distribution function

Canonical name	DistributionFunction
Date of creation	2013-03-22 13:02:51
Last modified on	2013-03-22 13:02:51
Owner	Mathprof (13753)
Last modified by	Mathprof (13753)
Numerical id	16
Author	Mathprof (13753)
Entry type	Definition
Classification	msc 60E05
Synonym	cumulative distribution function
Synonym	distribution
Related topic	DensityFunction
Related topic	CumulativeDistributionFunction
Related topic	RandomVariable
Related topic	Distribution
Related topic	GeometricDistribution2
Defines	law of a random variable

[this entry is currently being revised, so hold off on corrections until this line is removed]

Let $F : \mathbb{R} \to \mathbb{R}$. Then F is a *distribution function* if

1. F is nondecreasing,
2. F is continuous from the right,
3. $\lim_{x\to-\infty} F(x) = 0$, and $\lim_{x\to\infty} F(x) = 1$.

As an example, suppose that $\Omega = \mathbb{R}$ and that $\mathcal{B}$ is the σ-algebra of Borel subsets of $\mathbb{R}$. Let P be a probability measure on $(\Omega, \mathcal{B})$. Define F by

$$F(x) = P((-\infty, x]).$$

This particular F is called the *distribution function* of P. It is easy to verify that 1,2, and 3 hold for this F.

In fact, every distribution function is the distribution function of some probability measure on the Borel subsets of $\mathbb{R}$. To see this, suppose that F is a distribution function. We can define P on a single half-open interval by

$$P((a, b]) = F(b) - F(a)$$

and extend P to unions of disjoint intervals by

$$P(\cup_{i=1}^{\infty}(a_i, b_i]) = \sum_{i=1}^{\infty} P((a_i, b_i]).$$

and then further extend P to all the Borel subsets of $\mathbb{R}$. It is clear that the distribution function of P is F.

0.1 Random Variables

Suppose that $(\Omega, \mathcal{B}, P)$ is a probability space and $X : \Omega \to \mathbb{R}$ is a random variable. Then there is an *induced* probability measure P_X on $\mathbb{R}$ defined as follows:

$$P_X(E) = P(X^{-1}(E))$$

for every Borel subset E of $\mathbb{R}$. P_X is called the *distribution* of X. The *distribution function* of X is

$$F_X(x) = P(\omega|X(\omega) \leq x).$$

The distribution function of X is also known as the law of X. Claim: $F_X =$ the distribution function of P_X.

$$\begin{aligned} F_X(x) &= P(\omega|X(\omega) \leq x) \\ &= P(X^{-1}((-\infty, x]) \\ &= P_X((-\infty, x]) \\ &= F(x). \end{aligned}$$

0.2 Density Functions

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a nonnegative function such that

$$\int_{-\infty}^{\infty} f(t)dt = 1.$$

Then one can define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \int_{-\infty}^{x} f(t)dt.$$

Then it is clear that F satisfies the conditions 1,2,and 3 so F is a distribution function. The function f is called a density function for the distribution F.

If X is a discrete random variable with density function f and distribution function F then

$$F(x) = \sum_{x_j \leq x} f(x_j).$$