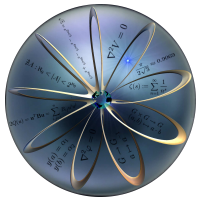

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criterion for almost-sure convergence

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Let $X_1, X_2, \ldots$ and X be random variables. If, for every $\epsilon > 0$, the sum $\sum_{n=1}^{\infty} \mathbb{P}(|X_n - X| > \epsilon)$ is finite, then X_n converge to X almost surely.

Proof. By the Borel-Cantelli lemma, we have $\mathbb{P}(\limsup_n \{|X_n - X| > \epsilon\}) = 0$. But $\limsup_n \{|X_n - X| > \epsilon\}$ is the same as the event $\{\limsup_n |X_n - X| > \epsilon\}$. (The latter event involves the `http://planetmath.org/LimitSuperior`limit superior of *numbers*; the former involves the `http://planetmath.org/InfinitelyOften`limit superior of *sets*.) So taking the limit $\epsilon \searrow 0$, we have $\mathbb{P}(\limsup_n |X_n - X| > 0) = 0$, or equivalently $\mathbb{P}(\limsup_n |X_n - X| = 0) = 1$. □