convergence in probability is preserved under continuous transformations

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Theorem 1. *Let $g\colon \mathbb{R}^k \to \mathbb{R}^l$ be a continuous function. If $\{X_n\}$ are $\mathbb{R}^k$-valued random variables converging to X in probability, then $\{g(X_n)\}$ converge in probability to $g(X)$ also.*

Proof. Suppose first that g is uniformly continuous. Given $\epsilon > 0$, there is $\delta > 0$ such that $\|g(X_n) - g(X)\| < \epsilon$ whenever $\|X_n - X\| < \delta$. Therefore,

$$\mathbb{P}\bigl(\|g(X_n) - g(X)\| \geq \epsilon\bigr) \leq \mathbb{P}\bigl(\|X_n - X\| \geq \delta\bigr) \to 0$$

as $n \to \infty$.

Now suppose g is not necessarily uniformly continuous on $\mathbb{R}^k$. But it will be uniformly continuous on any compact set $\{x \in \mathbb{R}^k \colon \|x\| \leq m\}$ for $m \geq 0$. Consequently, if X_n and X are bounded (by m), then the proof just given is applicable. Thus we attempt to reduce the general case to the case that X_n and X are bounded.

Let

$$f_m(x) = \begin{cases} x\,, & \|x\| \leq m \\ mx/\|x\|\,, & \|x\| \geq m \end{cases}$$

Clearly, $f_m\colon \mathbb{R}^k \to \mathbb{R}^k$ is continuous; in fact, it can be verified that f_m is uniformly continuous on $\mathbb{R}^k$. (This is geometrically obvious in the one-dimensional case.)

Set $X_n^m = f_m(X_n)$ and $X^m = f_m(X)$, so that X_n^m converge to X^m in probability for each $m \geq 0$.

We now show that $g(X_n)$ converge to $g(X)$ in probability by a four-step estimate. Let $\epsilon > 0$ and $\delta > 0$ be given. For any $m \geq 0$ (which we will later),

$$\mathbb{P}\bigl(\|g(X_n) - g(X)\| \geq \delta\bigr) \leq \mathbb{P}\bigl(\|g(X_n^m) - g(X^m)\| \geq \delta\bigr) + \mathbb{P}\bigl(\|X_n\| \geq m\bigr) + \mathbb{P}\bigl(\|X\| \geq m\bigr)\,.$$

Choose M such that for $m \geq M$,

$$\mathbb{P}\bigl(\|X\| \geq m\bigr) \leq \mathbb{P}\bigl(\|X\| \geq M\bigr) < \frac{\epsilon}{4}\,.$$

(This is possible since $\lim_{m\to\infty} \mathbb{P}\bigl(\|X\| \geq m\bigr) = \mathbb{P}\bigl(\bigcap_{m=0}^{\infty}\{\|X\| \geq m\}\bigr) = \mathbb{P}(\emptyset) = 0$.)

In particular, let $m = M + 1$. Since X_n^m converge in probability to X^m and X_n^m, X^m are bounded, $g(X_n^m)$ converge in probability to $g(X^m)$. That means for n large enough,

$$\mathbb{P}\bigl(\|g(X_n^m) - g(X^m)\| \geq \delta\bigr) < \frac{\epsilon}{4}\,.$$

Finally, since $\|X_n\| \leq \|X_n - X\| + \|X\|$, and X_n converge to X in probability, we have

$$\mathbb{P}\big(\|X_n\| \geq m = M + 1\big) \leq \mathbb{P}\big(\|X_n - X\| \geq 1\big) + \mathbb{P}\big(\|X\| \geq M\big) < \frac{\epsilon}{4} + \frac{\epsilon}{4}$$

for large enough n.

Collecting the previous inequalities together, we have

$$\mathbb{P}\big(\|g(X_n) - g(X)\| \geq \delta\big) < \epsilon$$

for large enough n. $\square$