periodicity of a Markov chain

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Entry type	Definition
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Defines	period of a state
Defines	aperiodic state
Defines	aperiodic Markov chain

Let $\{X_n\}$ be a http://planetmath.org/StationaryProcessstationary Markov chain with state space I. Let P_{ij}^n be the n-step transition probability that the process goes from state i at time 0 to state j at time n:

$$P_{ij}^n = P(X_n = j \mid X_0 = i).$$

Given any state $i \in I$, define the set

$$N(i) := \{n \geq 1 \mid P_{ii}^n > 0\}.$$

It is not hard to see that if $n, m \in N(i)$, then $n + m \in N(i)$. The *period* of i, denoted by $d(i)$, is defined as

$$d(i) := \begin{cases} 0 & \text{if } N(i) = \varnothing, \\ \gcd(N(i)) & \text{otherwise,} \end{cases}$$

where $\gcd(N(i))$ is the greatest common divisor of all positive integers in $N(i)$.

A state $i \in I$ is said to be *aperiodic* if $d(i) = 1$. A Markov chain is called *aperiodic* if every state is aperiodic.

Property. If states $i, j \in I$ http://planetmath.org/MarkovChainsClassStructurecommuni then $d(i) = d(j)$.

Proof. We will employ a common inequality involving the n-step transition probabilities:

$$P_{ij}^{m+n} \geq P_{ik}^m P_{kj}^n$$

for any $i, j, k \in I$ and non-negative integers m, n.

Suppose first that $d(i) = 0$. Since $i \leftrightarrow j$, $P_{ij}^n > 0$ and $P_{ji}^m > 0$ for some $n, m \geq 0$. This implies that $P_{ii}^{m+n} > 0$, which forces $m+n = 0$ or $m = n = 0$, and hence $j = i$.

Next, assume $d(i) > 0$, this means that $N(i) \neq \varnothing$. Since $i \leftrightarrow j$, there are $r, s \geq 0$ such that $P_{ji}^r > 0$ and $P_{ij}^s > 0$, and so $P_{jj}^{r+s} > 0$, showing $r + s \in N(j)$. If we pick any $n \in N$, we also have $P_{jj}^{r+n+s} \geq P_{ji}^r P_{ii}^n P_{ij}^s > 0$, or $r + s + n \in N(j)$. But this means $d(j)$ divides both $r + s$ and $r + s + n$, and so $d(j)$ divides their difference, which is n. Since n is arbitrarily picked, $d(j) \mid d(i)$. Similarly, $d(i) \mid d(j)$. Hence $d(i) = d(j)$. $\square$