martingale proof of the Radon-Nikodym theorem

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We apply the martingale convergence theorem to prove the Radon-Nikodym theorem, which states that if μ and ν are σ-finite measures on a measurable space $(\Omega, \mathcal{F})$ and ν is absolutely continuous with respect to μ then there exists a non-negative and measurable $f\colon \Omega \to \mathbb{R}$ such that $\nu(A) = \int_A f\,d\mu$ for all measurable sets A.

As `http://planetmath.org/AnySigmaFiniteMeasureIsEquivalentToAProbabilityMeasure` σ-finite measure is equivalent to a probability measure, it is enough to prove the result in the case where μ and ν are probability measures. Furthermore, by the Jordan decomposition, the result generalizes to the case where ν is a signed measure. So, we just need to prove the following.

Theorem (Radon-Nikodym). *Let $\mathbb{P}$ and $\mathbb{Q}$ be probability measures on the measurable space $(\Omega, \mathcal{F})$, such that $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$. Then, there exists a non-negative random variable X such that $\mathbb{E}_{\mathbb{P}}[X] = 1$ and $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[1_A X]$ for every $A \in \mathcal{F}$.*

Here, X is called the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$.

More generally, for any sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$ we can restrict the measures $\mathbb{P}$ and $\mathbb{Q}$ to $\mathcal{G}$ and ask if the Radon-Nikodym derivative of $\mathbb{Q}|_{\mathcal{G}}$ with respect to $\mathbb{P}|_{\mathcal{G}}$ exists. If it does we shall denote it by $X_{\mathcal{G}}$, which by definition is a non-negative $\mathcal{G}$-measurable random variable satisfying $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[1_A X_{\mathcal{G}}]$ for all $A \in \mathcal{G}$.

We note that if $X_{\mathcal{G}}$ does exist, then it is uniquely defined ($\mathbb{P}$-almost everywhere). Suppose that $\tilde{X}_{\mathcal{G}}$ also satisfied the required properties, then

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[\max(X_{\mathcal{G}} - \tilde{X}_{\mathcal{G}}, 0)] &= \mathbb{E}_{\mathbb{P}}[X_{\mathcal{G}} 1_{\{X_{\mathcal{G}} > \tilde{X}_{\mathcal{G}}\}}] - \mathbb{E}_{\mathbb{P}}[\tilde{X}_{\mathcal{G}} 1_{\{X_{\mathcal{G}} > \tilde{X}_{\mathcal{G}}\}}] \\ &= \mathbb{Q}(X_{\mathcal{G}} > \tilde{X}_{\mathcal{G}}) - \mathbb{Q}(X_{\mathcal{G}} > \tilde{X}_{\mathcal{G}}) = 0\end{aligned}$$

so $X_{\mathcal{G}} \leq \tilde{X}_{\mathcal{G}}$ almost surely. Similarly, $\tilde{X}_{\mathcal{G}} \leq X_{\mathcal{G}}$ and therefore $\tilde{X}_{\mathcal{G}} = X_{\mathcal{G}}$ (almost surely).

First, the easy case. For a finite σ-algebra, the Radon-Nikodym derivative can be written out explicitly.

Lemma 1. *If $\mathcal{G}$ is a finite sub-σ-algebra of $\mathcal{F}$ then the Radon-Nikodym derivative $X_{\mathcal{G}}$ exists.*

Proof. Let $A_1, A_2, \ldots, A_n$ be the minimal non-empty elements of $\mathcal{G}$. These are pairwise disjoint subsets of Ω such that every set in $\mathcal{G}$ is a union of a subcollection of the A_k. Set

$$X_{\mathcal{G}} = \sum_{k=1}^{n} \frac{\mathbb{Q}(A_k)}{\mathbb{P}(A_k)} 1_{A_k}$$

Note that whenever $\mathbb{P}(A_k) = 0$ then $\mathbb{Q}(A_k) = 0$, and we adopt the convention that $\frac{0}{0} = 0$. Clearly, $X_{\mathcal{G}}$ is $\mathcal{G}$-measurable, and

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}[1_{A_k} X_{\mathcal{G}}] &= \frac{\mathbb{Q}(A_k)}{\mathbb{P}(A_k)} \mathbb{E}_{\mathbb{P}}[1_{A_k}] + \sum_{j \neq k} \frac{\mathbb{Q}(A_j)}{\mathbb{P}(A_j)} \mathbb{E}_{\mathbb{P}}[1_{A_k \cap A_j}] \\
&= \mathbb{Q}(A_k).
\end{aligned}$$

Here, we have used $\mathbb{E}_{\mathbb{P}}[1_{A_k}] = \mathbb{P}(A_k)$ and $1_{A_k \cap A_j} = 0$. By linearity, this equality remains true if both sides are replaced by any union of the A_k, and therefore $X_{\mathcal{G}}$ is the required Radon-Nikodym derivative. □

Next, martingale convergence is used to prove the existence of the Radon-Nikodym derivative in the case where the σ-algebra $\mathcal{G}$ is separable. By separable, we mean that there is a countable sequence of sets $A_1, A_2, \ldots$ generating $\mathcal{G}$. Note that if we let $\mathcal{G}_n$ be the σ-algebra generated by $A_1, A_2, \ldots, A_n$, then $\mathcal{G}_n$ is an increasing sequence of finite sub-σ-algebras such that $\bigcup_n \mathcal{G}_n$ generates $\mathcal{G}$. The following result is general enough to apply in many useful cases, such as with the Boral σ-algebra on $\mathbb{R}^n$.

Lemma 2. *Let $\mathcal{G}$ be a separable sub-σ-algebra of $\mathcal{F}$. Then, the Radon-Nikodym derivative $X_{\mathcal{G}}$ exists. If furthermore, $\mathcal{G}_n$ is an increasing sequence of finite σ-algebras satisfying $\mathcal{G} = \sigma(\bigcup_n \mathcal{G}_n)$ then $\mathbb{E}_{\mathbb{P}}[|X_{\mathcal{G}} - X_{\mathcal{G}_n}|] \to 0$ as $n \to \infty$.*

Proof. Let us set $X_n \equiv X_{\mathcal{G}_n}$. If $m < n$ then the conditional expectation $\mathbb{E}_{\mathbb{P}}[X_n \mid \mathcal{G}_m]$ is $\mathcal{G}_m$-measurable, and for every $A \in \mathcal{G}_m$,

$$\mathbb{E}_{\mathbb{P}}\left[1_A \mathbb{E}_{\mathbb{P}}[X_n \mid \mathcal{G}_m]\right] = \mathbb{E}_{\mathbb{P}}\left[1_A X_n\right] = \mathbb{Q}(A).$$

This equality just uses the definition of the conditional expectation and then the definition of X_n as the Radon-Nikodym derivative restricted to $\mathcal{G}_n$. So, $\mathbb{E}_{\mathbb{P}}[X_n \mid \mathcal{G}_m]$ is the Radon-Nikodym derivative restricted to $\mathcal{G}_m$, and equals X_m (almost-surely).

Therefore, X_n is a martingale and the martingale convergence theorem implies that the limit

$$X_{\mathcal{G}} = \lim_{n\to\infty} X_n \tag{1}$$

exists almost surely. We now show that the sequence X_n is uniformly integrable. Choose any $\epsilon > 0$. As $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$, there exists a $\delta > 0$ such that $\mathbb{Q}(A) < \epsilon$ whenever $\mathbb{P}(A) < \delta$. Using

$$\mathbb{P}(X_n > K) = \mathbb{E}_{\mathbb{P}}[1_{\{X_n > K\}}] \leq \mathbb{E}_{\mathbb{P}}\left[\frac{X_n}{K}\right] = \frac{1}{K}$$

we see that $\mathbb{P}(X_n > K) < \delta$ whenever $K > \delta^{-1}$ and, therefore, $\mathbb{Q}(X_n > K) < \epsilon$. So

$$\mathbb{E}_{\mathbb{P}}[X_n 1_{\{X_n > K\}}] = \mathbb{Q}(X_n > K) < \epsilon$$

for every n, showing that X_n is a uniformly integrable sequence with respect to $\mathbb{P}$. Therefore, convergence in (**??**) is in L^1, and $\mathbb{E}_{\mathbb{P}}[|X_n - X_{\mathcal{G}}|] \to 0$ as $n \to \infty$. So, for any $A \in \bigcup_n \mathcal{G}_n$,

$$\mathbb{E}_{\mathbb{P}}[X_{\mathcal{G}} 1_A] = \lim_{m\to\infty} \mathbb{E}_{\mathbb{P}}[X_m 1_A] = \mathbb{Q}(A). \tag{2}$$

By linearity and the monotone convergence theorem, the collection of sets A satisfying (**??**) is a Dynkin system containing the π-system $\bigcup_n \mathcal{G}_n$ so, by Dynkin's lemma, is satisfied for every $A \in \sigma(\bigcup_n \mathcal{G}_n) = \mathcal{G}$ and, by definition, $X_{\mathcal{G}}$ is the Radon-Nikodym derivative restricted to $\mathcal{G}$. □

Finally, by approximating by finite σ-algebras we can prove the Radon-Nikodym theorem for arbitrary inseparable σ-algebras $\mathcal{F}$.

Proof of the Radon-Nikodym theorem:

First, we use contradiction to show that for any $\epsilon > 0$ there exists a finite σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ satisfying $\mathbb{E}_{\mathbb{P}}[|X_{\mathcal{G}} - X_{\mathcal{H}}|] < \epsilon$ for every finite σ-algebra $\mathcal{H}$ with $\mathcal{G} \subseteq \mathcal{H} \subseteq F$. If this were not the case, then by induction we could find an increasing sequence of finite sub-σ-algebras of $\mathcal{F}$ satisfying $\mathbb{E}_{\mathbb{P}}[|X_{\mathcal{G}_n} - X_{\mathcal{G}_m}|] \geq \epsilon$. However, letting $\mathcal{G} = \sigma(\bigcup_n \mathcal{G}_n)$, Lemma **??** shows that $X_{\mathcal{G}}$ exists and

$$\epsilon \leq \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}[|X_{\mathcal{G}_n} - X_{\mathcal{G}_{n+1}}|] \leq \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}[|X_{\mathcal{G}_n} - X_{\mathcal{G}}|] + \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}[|X_{\mathcal{G}_{n+1}} - X_{\mathcal{G}}|] = 0$$

— a contradiction.

So, there exists a sequence of finite sub-σ-algebras $\mathcal{G}_n$ of $\mathcal{F}$ such that $\mathbb{E}_{\mathbb{P}}[|X_{\mathcal{G}_n} - X_{\mathcal{H}}|] < 2^{-n}$ for every finite sub-σ-algebra $\mathcal{H}$ of $\mathcal{F}$ containing $\mathcal{G}_n$. Let $\mathcal{G}$ be the (separable) σ-algebra generated by $\bigcup_n \mathcal{G}_n$, and set $\tilde{\mathcal{G}}_n = \sigma(\bigcup_{k=1}^n \mathcal{G}_k)$. By Lemma **??**, the Radon-Nikodym derivative restricted to $\mathcal{G}$, $X_{\mathcal{G}}$, exists, and we show that it is the required derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$.

Choose any set $A \in \mathcal{F}$ and let $\mathcal{H}_n$ be the (finite) σ-algebra generated by $\mathcal{G}_n \cup \{A\}$. Then, $X_{\mathcal{H}_n}$ exists and satisfies $\mathbb{E}_{\mathbb{P}}[X_{\mathcal{H}_n} 1_A] = \mathbb{Q}(A)$ and,

$$\begin{aligned}
|\mathbb{E}_{\mathbb{P}}[X_{\mathcal{G}} 1_A] - \mathbb{Q}(A)| &= \lim_{n\to\infty} \left|\mathbb{E}_{\mathbb{P}}[X_{\tilde{\mathcal{G}}_n} 1_A] - \mathbb{Q}(A)\right| \\
&= \lim_{n\to\infty} \left|\mathbb{E}_{\mathbb{P}}[X_{\tilde{\mathcal{G}}_n} 1_A] - \mathbb{E}_{\mathbb{P}}[X_{\mathcal{H}_n} 1_A]\right| \\
&\leq \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}[|X_{\tilde{\mathcal{G}}_n} - X_{\mathcal{G}_n}|] + \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}[|X_{\mathcal{H}_n} - X_{\mathcal{G}_n}|] \\
&\leq \lim_{n\to\infty} (2^{-n} + 2^{-n}) = 0.
\end{aligned}$$

So, $\mathbb{E}_{\mathbb{P}}[X_{\mathcal{G}} 1_A] = \mathbb{Q}(A)$ as required.

References

[1] David Williams, *Probability with martingales*, Cambridge Mathematical Textbooks, Cambridge University Press, 1991.