Kolmogorov's martingale inequality

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Theorem (Kolmogorov's martingale inequality)**.** *Let $X(t)$, for $0 \le t \le T$, be a submartingale with continuous sample paths. Then for any constant $\alpha > 0$,*

$$\mathbb{P}\Big(\max_{0\le t\le T} X(t) \ge \alpha\Big) \le \frac{\mathbb{E}[X(T)^+]}{\alpha}\,.$$

(The notation $X(T)^+$ means $\max(X(T), 0)$, the positive part of $X(T)$.)

Notice the analogy with Markov's inequality. Of course, the conclusion is much stronger than Markov's inequality, as the probabilistic bound applies to an uncountable number of random variables. The continuity and submartingale hypotheses are used to establish the stronger bound.

Proof. Let $\{t_i\}_{i=1}^n$ be a partition of the interval $[0, T]$. Let

$$B = \Big\{\max_{1\le i\le n} X(t_i) \ge \alpha\Big\}$$

and split B into disjoint parts B_i, defined by

$$B_i = \Big\{X(t_j) < \alpha \text{ for all } j < i \text{ but } X(t_i) \ge \alpha\Big\}\,.$$

Also let $\{\mathcal{F}_t\}$ be the filtration under which $X(t)$ is a submartingale.

Then

$$
\begin{aligned}
\mathbb{P}(B) &= \sum_{i=1}^{n} \mathbb{E}\big[1(B_i)\big] \\
&\le \sum_{i=1}^{n} \mathbb{E}\left[\frac{X(t_i)}{\alpha}\,\mathbf{1}(B_i)\right] && \text{definition of } B_i \\
&\le \frac{1}{\alpha}\sum_{i=1}^{n} \mathbb{E}\Big[\mathbb{E}\big[X(T) \mid \mathcal{F}_{t_i}\big]\,\mathbf{1}(B_i)\Big] && X(t) \text{ is submartingale} \\
&= \frac{1}{\alpha}\sum_{i=1}^{n} \mathbb{E}\Big[\mathbb{E}\big[X(T)\,\mathbf{1}(B_i) \mid \mathcal{F}_{t_i}\big]\Big] && B_i \text{ is } \mathcal{F}_{t_i}\text{-measurable} \\
&= \frac{1}{\alpha}\sum_{i=1}^{n} \mathbb{E}\big[X(T)\,\mathbf{1}(B_i)\big] && \text{iterated expectation} \\
&= \frac{1}{\alpha}\mathbb{E}\big[X(T)\,\mathbf{1}(B)\big] \\
&\le \frac{1}{\alpha}\mathbb{E}\big[X(T)^+\,\mathbf{1}(B)\big] \\
&\le \frac{1}{\alpha}\mathbb{E}\big[X(T)^+\big] && \text{monotonicity.}
\end{aligned}
$$

Since the sample paths are continuous by hypothesis, the event

$$
A = \Big\{\max_{0\le t\le T} X(t) \ge \alpha\Big\}
$$

can be expressed as an countably infinite intersection of events of the form B with finer and finer partitions $\{t_i\}$ of the time interval $[0,T]$. By taking limits, it follows $\mathbb{P}(A)$ has the same bound as the probabilities $\mathbb{P}(B)$. □

Corollary. *Let $X(t)$, for $0 \le t \le T$, be a square-integrable martingale possessing continuous sample paths, whose unconditional mean is $m = \mathbb{E}[X(0)]$. For any constant $\alpha > 0$,*

$$
\mathbb{P}\Big(\max_{0\le t\le T}|X(t) - m| \ge \alpha\Big) \le \frac{\operatorname{Var}[X(T)]}{\alpha^2}\,.
$$

Proof. Apply Kolmogorov's martingale inequality to $(X(t)-m)^2$, which is a submartingale by Jensen's inequality. □