proof of Martingale criterion (continuous time)

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Proof. *1.* Let X be a martingale. By the optional sampling theorem we have $E(X_c|\mathcal{F}_\tau) = X_{c\wedge\tau} = X_\tau \forall \tau \leq c$. Since conditional expectations are uniformly integrable the first direction follows.

2. Let $(\tau_k)_{k\geq 1}$ be a local sequence of stopping times (i.e. $\tau_k \uparrow \infty$ a.s. and X^{τ_k} martingale $\forall k \in \mathbb{N}$). For each $t \in \mathbb{R}_+$ we have $X_{\tau_k \wedge t} \to X_t, k \to \infty$ almost surely. The set

$$\{X_{\tau_k\wedge t} : k \in \mathbb{N}\} \subset \{X_\tau : \tau \text{ stopping time}, \tau \leq c\}$$

is uniformly integrable (take $c = t$). It follows that $X_t^{\tau_k} \xrightarrow{\mathscr{L}^1} X_t, k \to \infty$. Since the martingale property is stable under $\mathscr{L}^1$ convergence, X is a martingale. $\square$