multinomial distribution

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Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random vector such that

1. $X_i \geq 0$ and $X_i \in \mathbb{Z}$
2. $X_1 + \cdots + X_n = N$, where N is a fixed integer

Then $\mathbf{X}$ has a *multinomial distribution* if there exists a parameter vector $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$ such that

1. $\pi_i \geq 0$ and $\pi_i \in \mathbb{R}$
2. $\pi_1 + \cdots + \pi_n = 1$
3. $\mathbf{X}$ has a discrete probability distribution function $f_{\mathbf{X}}(\boldsymbol{x})$ in the form:

$$f_{\mathbf{X}}(\boldsymbol{x}) = \frac{N!}{x_1! \cdots x_n!} \prod_{i=1}^{n} \pi_i^{x_i}$$

Remarks

- $\mathrm{E}[\mathbf{X}] = N\boldsymbol{\pi}$
- $\mathrm{Var}[\mathbf{X}] = (v_{ij})$, where

$$v_{ij} = \begin{cases} N\pi_i(1-\pi_i) & \text{if } i = j; \\ -N\pi_i\pi_j & \text{if } i \neq j. \end{cases}$$

- When $n = 2$, the multinomial distribution is the same as the binomial distribution
- If $X_1, \ldots, X_n$ are mutually independent Poisson random variables with parameters $\lambda_1, \ldots, \lambda_n$ respectively, then the conditional joint distribution of $X_1, \ldots, X_n$ given that $X_1 + \cdots + X_n = N$ is multinomial with parameters λ_i/λ, where $\lambda = \sum \lambda_i$.

  Sketch of proof. Each X_i is distributed as:

$$f_{X_i}(x_i) = \frac{e^{-\lambda_i}\lambda_i^{x_i}}{x_i!}$$

The mutual independence of the X_i's shows that the joint probability distribution of the X_i's is given by

$$f_{\mathbf{X}}(\boldsymbol{x}) = \prod_{i=1}^{n} \frac{e^{-\lambda_i}\lambda_i^{x_i}}{x_i!} = e^{-\lambda} \prod_{i=1}^{n} \frac{\lambda_i^{x_i}}{x_i!},$$

where $\mathbf{X} = (X_1, \ldots, X_n)$, $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\lambda = \lambda_1 + \cdots + \lambda_n$. Next, let $X = X_1 + \cdots + X_n$. Then X is Poisson distributed with parameter λ (which can be shown by using induction and the mutual independence of the X_i's):

$$f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}.$$

The conditional probability distribution of $\mathbf{X}$ given that $X = N$ is thus given by:

$$f_{\mathbf{X}}(\boldsymbol{x} \mid X = N) = \frac{f_{\mathbf{X}}(\boldsymbol{x})}{f_X(N)} = (e^{-\lambda} \prod_{i=1}^{n} \frac{\lambda_i^{x_i}}{x_i!})/(\frac{e^{-\lambda}\lambda^N}{N!}) = \frac{N!}{x_1! \cdots x_n!} \prod_{i=1}^{n} (\frac{\lambda_i}{\lambda})^{x_i},$$

where $\sum x_i = N$ and that $\sum \lambda_i / \lambda = 1$.