regular conditional probability

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Introduction

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $B \in \mathcal{F}$ be an event with $P(B) > 0$. It is easy to see that $P_B : \mathcal{F} \to [0,1]$ defined by

$$P_B(A) := P(A|B),$$

the conditional probability of event A given B, is a probability measure defined on $\mathcal{F}$, since:

1. P_B is clearly non-negative;
2. $P_B(\Omega) = \dfrac{P(\Omega \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$;
3. P_B is countably additive: for if $\{A_1, A_2, \ldots\}$ is a countable collection of pairwise disjoint events in $\mathcal{F}$, then

$$P_B(\bigcup_{i=1}^{\infty} A_i) = \frac{P\big(B \cap (\bigcup A_i)\big)}{P(B)} = \frac{P\big(\bigcup (B \cap A_i)\big)}{P(B)} = \frac{\sum P(B \cap A_i)}{P(B)} = \sum_{i=1}^{\infty} P_B(A_i),$$

as $\{B \cap A_1, B \cap A_2, \ldots\}$ is a collection of pairwise disjoint events also.

Regular Conditional Probability

Can we extend the definition above to $P_{\mathcal{G}}$, where $\mathcal{G}$ is a sub sigma algebra of $\mathcal{F}$ instead of an event? First, we need to be careful what we mean by $P_{\mathcal{G}}$, since, given any event $A \in \mathcal{F}$, $P(A|\mathcal{G})$ is not a real number. And strictly speaking, it is not even a random variable, but an equivalence class of random variables (each pair differing by a null event in $\mathcal{G}$).

With this in mind, we start with a probability measure P defined on $\mathcal{F}$ and a sub sigma algebra $\mathcal{G}$ of $\mathcal{F}$. A function $P_{\mathcal{G}} : \mathcal{G} \times \Omega \to [0,1]$ is a called a *regular conditional probability* if it has the following properties:

1. for each event $A \in \mathcal{G}$, $P_{\mathcal{G}}(A, \cdot) : \Omega \to [0,1]$ is a `http://planetmath.org/ProbabilityConditi` probability (as a random variable) of A given $\mathcal{G}$; that is,
   (a) $P_{\mathcal{G}}(A, \cdot)$ is `http://planetmath.org/MathcalFMeasurableFunction`$\mathcal{G}$-measurable and
   (b) for every $B \in \mathcal{G}$, we have $\displaystyle\int_B P_{\mathcal{G}}(A, \cdot) dP = P(A \cap B)$.

2. for every outcome $\omega \in \Omega$, $P_{\mathcal{G}}(\cdot, \omega) : \mathcal{G} \to [0, 1]$ is a probability measure.

There are probability spaces where no regular conditional probabilities can be defined. However, when a regular conditional probability function does exist on a space Ω, then by condition 2 of the definition, we can define a "conditional" probability measure on Ω for each outcome in the sense of the first two paragraphs.