properties for measure

Canonical name	PropertiesForMeasure
Date of creation	2013-03-22 13:45:28
Last modified on	2013-03-22 13:45:28
Owner	matte (1858)
Last modified by	matte (1858)
Numerical id	8
Author	matte (1858)
Entry type	Theorem
Classification	msc 60A10
Classification	msc 28A10

Theorem [?, ?, ?, ?] Let $(E, \mathcal{B}, \mu)$ be a measure space, i.e., let E be a set, let $\mathcal{B}$ be a σ-algebra of sets in E, and let μ be a measure on $\mathcal{B}$. Then the following properties hold:

1. **Monotonicity:** If $A, B \in \mathcal{B}$, and $A \subset B$, then $\mu(A) \leq \mu(B)$.
2. If A, B in $\mathcal{B}$, $A \subset B$, and $\mu(A) < \infty$, then
$$\mu(B \setminus A) = \mu(B) - \mu(A).$$
3. For any A, B in $\mathcal{B}$, we have
$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B).$$
4. **Subadditivity:** If $\{A_i\}_{i=1}^{\infty}$ is a collection of sets from $\mathcal{B}$, then
$$\mu\big(\bigcup_{i=1}^{\infty} A_i\big) \leq \sum_{i=1}^{\infty} \mu(A_i).$$
5. **Continuity from below:** If $\{A_i\}_{i=1}^{\infty}$ is a collection of sets from $\mathcal{B}$ such that $A_i \subset A_{i+1}$ for all i, then
$$\mu\big(\bigcup_{i=1}^{\infty} A_i\big) = \lim_{i \to \infty} \mu(A_i).$$
6. **Continuity from above:** If $\{A_i\}_{i=1}^{\infty}$ is a collection of sets from $\mathcal{B}$ such that $\mu(A_1) < \infty$, and $A_i \supset A_{i+1}$ for all i, then
$$\mu\big(\bigcap_{i=1}^{\infty} A_i\big) = \lim_{i \to \infty} \mu(A_i).$$

Remarks In (2), the assumption $\mu(A) < \infty$ assures that the right hand side is always well defined, i.e., not of the form $\infty - \infty$. Without the assumption we can prove that $\mu(B) = \mu(A) + \mu(B \setminus A)$ (see below). In (3), it is tempting to move the term $\mu(A \cap B)$ to the other side for aesthetic reasons. However, this is only possible if the term is finite.

Proof. For (1), suppose $A \subset B$. We can then write B as the disjoint union $B = A \cup (B \setminus A)$, whence
$$\mu(B) = \mu(A \cup (B \setminus A)) = \mu(A) + \mu(B \setminus A).$$

Since $\mu(B \setminus A) \geq 0$, the claim follows. Property (2) follows from the above equation; since $\mu(A) < \infty$, we can subtract this quantity from both sides. For property (3), we can write $A \cup B = A \cup (B \setminus A)$, whence

$$\begin{aligned} \mu(A \cup B) &= \mu(A) + \mu(B \setminus A) \\ &\leq \mu(A) + \mu(B). \end{aligned}$$

If $\mu(A \cup B)$ is infinite, the last inequality must be equality, and either of $\mu(A)$ or $\mu(B)$ must be infinite. Together with (1), we obtain that if any of the quantities $\mu(A), \mu(B), \mu(A \cap B)$ or $\mu(A \cup B)$ is infinite, both sides in the equation are infinite and the claim holds. We can therefore without loss of generality assume that all quantities are finite. From $A \cup B = B \cup (A \setminus B)$, we have

$$\mu(A \cup B) = \mu(B) + \mu(A \setminus B)$$

and thus

$$2\mu(A \cup B) = \mu(A) + \mu(B) + \mu(A \setminus B) + \mu(B \setminus A).$$

For the last two terms we have

$$\begin{aligned} \mu(A \setminus B) + \mu(B \setminus A) &= \mu((A \setminus B) \cup (B \setminus A)) \\ &= \mu((A \cup B) \setminus (A \cap B)) \\ &= \mu(A \cup B) - \mu(A \cap B), \end{aligned}$$

where, in the second equality we have used properties for the `http://planetmath.org/SymmetricD` set difference, and the last equality follows from property (2). This completes the proof of property (3). For property (4), let us define the sequence $\{D_i\}_{i=1}^{\infty}$ as

$$D_1 = A_1, \qquad D_i = A_i \setminus \bigcup_{k=1}^{i-1} A_k.$$

Now $D_i \cap D_j = \emptyset$ for $i < j$, so $\{D_i\}$ is a sequence of disjoint sets. Since $\cup_{i=1}^{\infty} D_i = \cup_{i=1}^{\infty} A_i$, and since $D_i \subset A_i$, we have

$$\begin{aligned} \mu(\bigcup_{i=1}^{\infty} A_i) &= \mu(\bigcup_{i=1}^{\infty} D_i) \\ &= \sum_{i=1}^{\infty} \mu(D_i) \\ &\leq \sum_{i=1}^{\infty} \mu(A_i), \end{aligned}$$

and property (4) follows.

TODO: proofs for (5)-(6).

References

[1] G.B. Folland, *Real Analysis: Modern Techniques and Their Applications*, 2nd ed, John Wiley & Sons, Inc., 1999.

[2] A. Mukherjea, K. Pothoven, *Real and Functional analysis*, Plenum press, 1978.

[3] D.L. Cohn, *Measure Theory*, Birkhäuser, 1980.

[4] A. Friedman, *Foundations of Modern Analysis*, Dover publications, 1982.