Dirac measure

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Let X be a nonempty set. Let $\mathcal{P}(X)$ denote the power set of X. Then $(X, \mathcal{P}(X))$ is a measurable space.

Let $x \in X$. The *Dirac measure* concentrated at x is $\delta_x \colon \mathcal{P}(X) \to \{0, 1\}$ defined by

$$\delta_x(E) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$$

Note that the Dirac measure δ_x is indeed a measure:

1. Since $x \notin \emptyset$, we have $\delta_x(\emptyset) = 0$.

2. If $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint subsets of X, then one of the following must happen:

   - $x \notin \bigcup\limits_{n\in\mathbb{N}} A_n$, in which case $\delta_x\left(\bigcup\limits_{n\in\mathbb{N}} A_n\right) = 0$ and $\delta_x(A_n) = 0$ for every $n \in \mathbb{N}$;
   - $x \in \bigcup\limits_{n\in\mathbb{N}} A_n$, in which case $x \in A_{n_0}$ for exactly one $n_0 \in \mathbb{N}$, causing $\delta_x\left(\bigcup\limits_{n\in\mathbb{N}} A_n\right) = 1$, $\delta_x(A_{n_0}) = 1$, and $\delta_x(A_n) = 0$ for every $n \in \mathbb{N}$ with $n \neq n_0$.

Also note that $(X, \mathcal{P}(X), \delta_x)$ is a probability space.

Let $\overline{\mathbb{R}}$ denote the extended real numbers. Then for any function $f \colon X \to \overline{\mathbb{R}}$, the integral of f with respect to the Dirac measure δ_x is

$$\int\limits_X f \, d\delta_x = f(x).$$

In other words, integration with respect to the Dirac measure δ_x amounts to evaluating the function at x.

If $X = \mathbb{R}$, m denotes Lebesgue measure, A is a Lebesgue measurable subset of $\mathbb{R}$, and δ (no) denotes the Dirac delta function, then for any measurable function $f \colon \mathbb{R} \to \mathbb{R}$, we have

$$\int\limits_A \delta(t - x) f(t) \, dm(t) = \int\limits_A f \, d\delta_x = f(x)\delta_x(A).$$

Moreover, if f is defined so that $f(t) = 1$ for all $t \in A$, the above becomes

$$\int_A \delta(t - x)\, dm(t) = \int_A d\delta_x = \delta_x(A).$$

In other words, the function $\delta(t - x)$ (with $x \in \mathbb{R}$ fixed and t a real variable) behaves like a Radon-Nikodym derivative of δ_x with respect to m.

Note that, just as the Dirac delta function is a misnomer (it is not really a function), there is not really a Radon-Nikodym derivative of δ_x with respect to m, since δ_x is not absolutely continuous with respect to m.