proof of Prohorov inequality

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Starting from the basic inequality $\exp(-x) \geq 1 - x$, it's easy to derive by elementary algebraic manipulations the two inequalities

$$\begin{aligned} \exp(x) - x - 1 &\leq 2(\cosh(x) - 1) \\ 2(\cosh(x) - 1) &\leq x \sinh(x) \end{aligned}$$

By the `http://planetmath.org/ChernoffCramerBound`Chernoff-Cramèr bound, we have:

$$\Pr\left\{\sum_{i=1}^{n}(X_i - E[X_i]) > \varepsilon\right\} \leq \exp\left[-\sup_{t>0}(t\varepsilon - \psi(t))\right]$$

where

$$\psi(t) = \sum_{i=1}^{n}\left(\ln E\left[e^{tX_i}\right] - tE\left[X_i\right]\right)$$

Keeping in mind that the condition

$$\Pr\{|X_i| \leq M\} = 1 \quad \forall i$$

implies that, for all i,

$$E[|X_i|^k] \leq M^k \quad \forall k \geq 0$$

(see `http://planetmath.org/RelationBetweenAlmostSurelyAbsolutelyBoundedRandomVariab` for a proof) and since $\ln x \leq x - 1$ $\forall x > 0$, and

$$E[|X|^k] \leq M^k \implies E\left[|X|^k\right] \leq E\left[X^2\right]M^{k-2} \qquad \forall k \geq 2, k \in N$$

(see `http://planetmath.org/AbsoluteMomentsBoundingNecessaryAndSufficientConditionhe` for a proof), one has:

$$\begin{aligned}
\psi(t) &= \sum_{i=1}^{n}\left(\ln E\left[e^{tX_i}\right] - tE\left[X_i\right]\right) \\
&\leq \sum_{i=1}^{n} E\left[e^{tX_i}\right] - tE\left[X_i\right] - 1 \\
&= \sum_{i=1}^{n} E\left[e^{tX_i} - tX_i - 1\right] \\
&\leq \sum_{i=1}^{n} 2E\left[\cosh\left(tX_i\right) - 1\right] \\
&\leq \sum_{i=1}^{n} E\left[tX_i \sinh\left(tX_i\right)\right] \\
&\leq \sum_{i=1}^{n} E\left[\left|tX_i \sinh\left(tX_i\right)\right|\right] \\
&= \sum_{i=1}^{n} tE\left[\left|X_i\right| \sinh\left(t\left|X_i\right|\right)\right] \\
&= \sum_{i=1}^{n} tE\left[\sum_{k=0}^{\infty} \frac{t^{2k+1}\left|X_i\right|^{2k+2}}{(2k+1)!}\right] \\
&= \sum_{i=1}^{n} t\sum_{k=0}^{\infty} \frac{t^{2k+1}E\left[\left|X_i\right|^{2k+2}\right]}{(2k+1)!} \\
&\leq \sum_{i=1}^{n} t\sum_{k=0}^{\infty} \frac{t^{2k+1}E\left[X_i^2\right]M^{2k}}{(2k+1)!} \\
&= \frac{t}{M}\sum_{k=0}^{\infty} \frac{t^{2k+1}M^{2k+1}\sum_{i=1}^{n} E\left[X_i^2\right]}{(2k+1)!} \\
&= \frac{tv^2}{M}\sum_{k=0}^{\infty} \frac{(tM)^{2k+1}}{(2k+1)!} \\
&= \frac{tv^2}{M}\sinh\left(tM\right).
\end{aligned}$$

One can now write

$$\sup_{t>0} (t\varepsilon - \psi(t)) \geq \sup_{t>0} \left(t\varepsilon - \frac{tv^2}{M} \sinh(tM) \right) = \sup_{t>0} \left[\frac{v^2}{M^2} \left(\frac{M^2 \varepsilon}{v^2} t - tM \sinh(tM) \right) \right]$$

Optimizing this expression with respect to t would lead to solving the transcendental equation:

$$\frac{M\varepsilon}{v^2} = Mt_{opt} \cosh(Mt_{opt}) + \sinh(Mt_{opt})$$

which is analytically infeasible. So, one can choose the sup-optimal yet manageable solution

$$\widetilde{t} = \frac{1}{M} \operatorname{arsinh} \left(\frac{M\varepsilon}{2v^2} \right)$$

which, once plugged into the bound, yields

$$\Pr \left\{ \sum_{i=1}^{n} (X_i - E[X_i]) > \varepsilon \right\} \leq \exp \left[-\frac{v^2}{M^2} \left(\frac{M\varepsilon}{2v^2} \operatorname{arsinh} \left(\frac{M\varepsilon}{2v^2} \right) \right) \right]$$