martingale convergence theorem

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There are several convergence theorems for martingales, which follow from Doob's upcrossing lemma. The following says that any L^1-bounded martingale X_n in discrete time converges almost surely. Note that almost-sure convergence (i.e. convergence with probability one) is quite strong, implying the weaker property of convergence in probability. Here, a martingale $(X_n)_{n\in\mathbb{N}}$ is understood to be defined with respect to a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

Theorem (Doob's Forward Convergence Theorem)**.** *Let $(X_n)_{n\in\mathbb{N}}$ be a martingale (or submartingale, or supermartingale) such that $\mathbb{E}[|X_n|]$ is bounded over all $n \in \mathbb{N}$. Then, with probability one, the limit $X_\infty = \lim_{n\to\infty} X_n$ exists and is finite.*

The condition that X_n is L^1-bounded is automatically satisfied in many cases. In particular, if X is a non-negative supermartingale then $\mathbb{E}[|X_n|] = \mathbb{E}[X_n] \le \mathbb{E}[X_1]$ for all $n \ge 1$, so $\mathbb{E}[|X_n|]$ is bounded, giving the following corollary.

Corollary. *Let $(X_n)_{n\in\mathbb{N}}$ be a non-negative martingale (or supermartingale). Then, with probability one, the limit $X_\infty = \lim_{n\to\infty} X_n$ exists and is finite.*

As an example application of the martingale convergence theorem, it is easy to show that a standard random walk started started at 0 will visit every level with probability one.

Corollary. *Let $(X_n)_{n\in\mathbb{N}}$ be a standard random walk. That is, $X_1 = 0$ and*

$$\mathbb{P}(X_{n+1} = X_n + 1 \mid \mathcal{F}_n) = \mathbb{P}(X_{n+1} = X_n - 1 \mid \mathcal{F}_n) = 1/2.$$

Then, for every integer a, with probability one $X_n = a$ for some n.

Proof. Without loss of generality, suppose that $a \le 0$. Let $T : \Omega \to \mathbb{N} \cup \{\infty\}$ be the first time n for which $X_n = a$. It is easy to see that the stopped process X^T_n defined by $X^T_n = X_{\min(n,T)}$ is a martingale and $X^T - a$ is non-negative. Therefore, by the martingale convergence theorem, the limit $X^T_\infty = \lim_{n\to\infty} X^T_n$ exists and is finite (almost surely). In particular, $|X^T_{n+1} - X^T_n|$ converges to 0 and must be less than 1 for large n. However, $|X^T_{n+1} - X^T_n| = 1$ whenever $n < T$, so we have $T < \infty$ and therefore $X_n = a$ for some n. $\square$