Itô's formula

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0.1 Case of single space dimension

Let X_t be an Itô process satisfying the stochastic differential equation

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t \, ,$$

with μ_t and σ_t being adapted processes, adapted to the same filtration as the Brownian motion W_t. Let f be a function with continuous partial derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$.

Then $Y_t = f(X_t)$ is also an Itô process, and its stochastic differential equation is

$$\begin{aligned} dY_t &= \frac{\partial f}{\partial t} \, dt + \frac{\partial f}{\partial x} \, dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dX_t)(dX_t) \\ &= \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \mu_t + \frac{1}{2} \sigma_t^2 \right) dt + \frac{\partial f}{\partial x} \sigma_t \, dW_t \, , \end{aligned}$$

where all partial derivatives are to be taken at (t, X_t).

0.2 Case of multiple space dimensions

There is also an analogue for multiple space dimensions.

Let X_t be a $\mathbb{R}^n$-valued Itô process satisfying the stochastic differential equation

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t \, ,$$

with μ_t and σ_t being adapted processes, adapted to the same filtration as the m-dimensional Brownian motion W_t. μ_t is $\mathbb{R}^n$-valued and σ_t is $L(\mathbb{R}^m, \mathbb{R}^n)$-valued.

Let $f \colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a function with continuous partial derivatives.

Then $Y_t = f(X_t)$ is also an Itô process, and its stochastic differential equation is

$$\begin{aligned} dY_t &= \frac{\partial f}{\partial t} \, dt + (\mathrm{D}\, f) \, dX_t + \tfrac{1}{2} dX_t^* (\mathrm{D}^2 \, f) dX_t \\ &= \frac{\partial f}{\partial t} \, dt + (\mathrm{D}\, f) \mu_t \, dt + (\mathrm{D}\, f) \sigma_t \, dW_t + \tfrac{1}{2} dW_t^* \, \sigma_t^* (\mathrm{D}^2 \, f) \sigma_t \, dW_t \\ &= \frac{\partial f}{\partial t} \, dt + (\mathrm{D}\, f) \mu_t \, dt + (\mathrm{D}\, f) \sigma_t \, dW_t + \tfrac{1}{2} \operatorname{tr}\bigl(\sigma_t^* \, (\mathrm{D}^2 \, f) \, \sigma_t \bigr) \, dt \\ &= \left(\frac{\partial f}{\partial t} + (\mathrm{D}\, f) \mu_t + \tfrac{1}{2} \operatorname{tr}\bigl((\sigma_t \sigma_t^*) (\mathrm{D}^2 \, f) \bigr) \right) dt + (\mathrm{D}\, f) \sigma_t \, dW_t \, , \end{aligned}$$

where

- tr is the trace operation; * is the transpose
- $\mathrm{D}\, f$ is the derivative with respect to the space variables; its value is a linear transformation from $L(\mathbb{R}^n, \mathbb{R})$
- $\mathrm{D}^2 f$ is the second derivative with respect to space variables; represented as the Hessian matrix
- the third line follows because $dW_t^i\, dW_t^j = \delta_{ij}\, dt$.

The quadratic form $\mathrm{tr}(\sigma_t \sigma_t^* \,\mathrm{D}^2 f)\, dt$ represents the quadratic variation of the process. When σ_t is the identity transformation, this reduces to the Laplacian of f.

Itô's formula in multiple dimensions can also be written with the standard vector calculus operators. It is in the similar notation typically used for the related parabolic partial differential equation describing an Itô diffusion:

$$dY_t = \left(\frac{\partial f}{\partial t} + \mu_t \cdot \nabla f + \tfrac{1}{2} \big(\nabla \cdot (\sigma_t \sigma_t^*) \nabla \big) f \right) dt + (\sigma_t\, dW_t) \cdot \nabla f\,.$$

References

[1] Bernt Øksendal. , *An Introduction with Applications*. 5th ed., Springer 1998.

[2] Hui-Hsiung Kuo. *Introduction to Stochastic Integration*. Springer 2006.