multidimensional Gaussian integral

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Let $\mathbf{x} = [x_1\ x_2\ \ldots\ x_n]^T$ and $d^n\mathbf{x} \equiv \prod_{i=1}^n dx_i$.

Theorem 1 () *Let K be a symmetric* `http://planetmath.org/PositiveDefinite`*positive definite matrix and $f : R^n \to R$, where $f(x) = \exp\left(-\frac{1}{2}\mathbf{x}^T\mathbf{K}^{-1}\mathbf{x}\right)$. Then*

$$\int e^{-\frac{1}{2}\mathbf{x}^T\mathbf{K}^{-1}\mathbf{x}} d^n\mathbf{x} = \left((2\pi)^n|\mathbf{K}|\right)^{\frac{1}{2}} \tag{1}$$

where $|\mathbf{K}| = \det\mathbf{K}$.

Proof. $\mathbf{K}^{-1}$ is real and symmetric (since $(\mathbf{K}^{-1})^{\mathrm{T}} = (\mathbf{K}^{\mathrm{T}})^{-1} = \mathbf{K}^{-1}$). For convenience, let $\mathbf{A} = \mathbf{K}^{-1}$. We can decompose $\mathbf{A}$ into $\mathbf{A} = \mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1}$, where $\mathbf{T}$ is an orthonormal ($\mathbf{T}^{\mathrm{T}}\mathbf{T} = \mathbf{I}$) matrix of the eigenvectors of $\mathbf{A}$ and $\boldsymbol{\Lambda}$ is a diagonal matrix of the eigenvalues of $\mathbf{A}$. Then

$$\int e^{-\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}} d^n\mathbf{x} = \int e^{-\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1}\mathbf{x}} d^n\mathbf{x}. \tag{2}$$

Because $\mathbf{T}$ is orthonormal, we have $\mathbf{T}^{-1} = \mathbf{T}^{\mathrm{T}}$. Now define a new vector variable $\mathbf{y} \equiv \mathbf{T}^{\mathrm{T}}\mathbf{x}$, and substitute:

$$\int e^{-\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1}\mathbf{x}} d^n\mathbf{x} = \int e^{-\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{\mathrm{T}}\mathbf{x}} d^n\mathbf{x} \tag{3}$$

$$= \int e^{-\frac{1}{2}\mathbf{y}^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{y}} |\mathbf{J}| d^n\mathbf{y} \tag{4}$$

$$\tag{5}$$

where $|\mathbf{J}|$ is the determinant of the Jacobian matrix $J_{mn} = \frac{\partial x_m}{\partial y_n}$. In this case, $\mathbf{J} = \mathbf{T}$ and thus $|\mathbf{J}| = 1$.

Since $\boldsymbol{\Lambda}$ is diagonal, the integral may be separated into the product of n independent Gaussian distributions, each of which we can integrate separately using the well-known formula

$$\int e^{-\frac{1}{2}at^2} dt = \left(\frac{2\pi}{a}\right)^{\frac{1}{2}}. \tag{6}$$

Carrying out this program, we get

$$
\begin{aligned}
\int e^{-\frac{1}{2}\mathbf{y}^{\mathrm{T}}\mathbf{\Lambda}\mathbf{y}} d^n\mathbf{y} &= \prod_{k=1}^{n} \int e^{-\frac{1}{2}\lambda_k y_k^2} dy_k & (7) \\
&= \prod_{k=1}^{n} \left(\frac{2\pi}{\lambda_k}\right)^{\frac{1}{2}} & (8) \\
&= \left(\frac{(2\pi)^n}{\prod_{k=1}^{n}\lambda_k}\right)^{\frac{1}{2}} & (9) \\
&= \left(\frac{(2\pi)^n}{|\mathbf{\Lambda}|}\right)^{\frac{1}{2}}. & (10) \\
& & (11)
\end{aligned}
$$

Now, we have $|\mathbf{A}| = |\mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1}| = |\mathbf{T}||\mathbf{\Lambda}||\mathbf{T}^{-1}| = |\mathbf{\Lambda}|$, so this becomes

$$
\int e^{-\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}} d^n\mathbf{x} = \left(\frac{(2\pi)^n}{|\mathbf{A}|}\right)^{\frac{1}{2}}. \tag{12}
$$

Substituting back in for $\mathbf{K}^{-1}$, we get

$$
\int e^{-\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{x}} d^n\mathbf{x} = \left(\frac{(2\pi)^n}{|\mathbf{K}^{-1}|}\right)^{\frac{1}{2}} = \left((2\pi)^n|\mathbf{K}|\right)^{\frac{1}{2}}, \tag{13}
$$

as promised.