relative entropy

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Let p and q be probability distributions with supports $\mathcal{X}$ and $\mathcal{Y}$ respectively, where $\mathcal{X} \subset \mathcal{Y}$. The *relative entropy* or *Kullback-Leibler* distance between two probability distributions p and q is defined as

$$D(p||q) := \sum_{x \in \mathcal{X}} p(x) \log \frac{p(x)}{q(x)}. \tag{1}$$

While $D(p||q)$ is often called a distance, it is not a true metric because it is not symmetric and does not satisfy the triangle inequality. However, we do have $D(p||q) \geq 0$ with equality iff $p = q$.

$$\begin{aligned}
-D(p||q) &= -\sum_{x \in \mathcal{X}} p(x) \log \frac{p(x)}{q(x)} && (2)\\
&= \sum_{x \in \mathcal{X}} p(x) \log \frac{q(x)}{p(x)} && (3)\\
&\leq \log \left(\sum_{x \in \mathcal{X}} p(x) \frac{q(x)}{p(x)} \right) && (4)\\
&= \log \left(\sum_{x \in \mathcal{X}} q(x) \right) && (5)\\
&\leq \log \left(\sum_{x \in \mathcal{Y}} q(x) \right) && (6)\\
&= 0 && (7)
\end{aligned}$$

where the first inequality follows from the concavity of $\log(x)$ and the second from expanding the sum over the support of q rather than p.

Relative entropy also comes in a continuous version which looks just as one might expect. For continuous distributions f and g, $\mathcal{S}$ the support of f, we have

$$D(f||g) := \int_{\mathcal{S}} f \log \frac{f}{g}. \tag{8}$$