dominated convergence for stochastic integration

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Defines	locally bounded convergence theorem

The dominated convergence theorem for standard integration states that if a sequence of measurable functions converge to a limit, and are dominated by an integrable function, then their integrals converge to the integral of the limit. That is, the limit commutes with integration. A similar result holds for stochastic integration with respect to a semimartingale X, except the integrals are random variables, and the integrals converge in probability.

Theorem (Dominated convergence). *If ξ^n are predictable processes converging pointwise to ξ, and $|\xi^n| \le \alpha$ for every n and some X-integrable process α, then*

$$\int_0^t \xi^n \, dX \to \int_0^t \xi \, dX \tag{1}$$

in probability as $n \to \infty$. Furthermore, ucp convergence and semimartingale convergence hold.

Note that as ξ and ξ^n are bounded by an X-integrable process, they are guaranteed to also be X-integrable. Convergence in probability for each t was taken as part of the definition of the stochastic integral, but the dominated convergence theorem stated here says that the stronger ucp and semimartingale convergence also hold.

If α is a `http://planetmath.org/LocalPropertiesOfProcesses`locally bounded predictable process, then it is automatically X-integrable for any semimartingale X. It follows that if ξ^n are predictable processes converging to ξ and if $\sup_n |\xi^n|$ is locally bounded then the limit (**??**) holds. This result is sometimes known as the *locally bounded convergence theorem*.

To prove this result, it is enough to show that semimartingale convergence holds, as semimartingale convergence implies ucp convergence. So, let $|\alpha^n| \le 1$ be a sequence of simple predictable processes and set $Y^n = \int \xi^n \, dX$, $Y = \int \xi \, dX$. Associativity of stochastic integration gives

$$\int_0^t \alpha^n \, dY^n - \int_0^t \alpha^n \, dY = \int_0^t \alpha^n (\xi^n - \xi) \, dX$$

However, $|\alpha^n(\xi^n - \xi)| \le 2\alpha$, which is X-integrable. So, this converges to zero in probability by the definition of the stochastic integral, and $Y^n \to Y$ in the semimartingale topology.