proof of martingale convergence theorem

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Let $(X_n)_{n\in\mathbb{N}}$ be a supermartingale such that $\mathbb{E}|X_n| \leq M$, and let $a < b$. We define a random variable counting how many times the process crosses the stripe between a and b:

$U_n := \max\{0, r \in \{1,\ldots,n\} \mid \exists 0 \leq s_1 < t_1 < s_2 < t_2 < \ldots < s_r < t_r \leq n \, \forall i \in \{1,\ldots,r\} \colon X_{s_i} \leq a \wedge$

Obviously $U_{n+1} \geq U_n$ therefore $U_\infty = \lim_{n\to\infty} U_n$ exists almost surely. Next we will construct a new process that mirrors the movement of X_n but only if the original process is underway of going from below a to over b, and is constant otherwise. To do this let $C_1 := \chi[X_0 < a]$, $C_k := \chi[C_{k-1} = 0 \wedge X_{k-1} < a] + \chi[C_{k-1} = 1 \wedge X_{k-1} \leq b]$ for $k \geq 2$, and define $Y_0 := 0$, $Y_n := \sum_{k=1}^{n} C_k(X_k - X_{k-1})$. Then Y_n is also a supermartingale, and the inequality $Y_n \geq (b-a)U_n - |X_n - a|$ holds, which gives $0 \geq \mathbb{E}(Y_n) \geq (b-a)\mathbb{E}(U_n) - \mathbb{E}|X_n - a|$. After rearrangement we get

$$\mathbb{E}(U_n) \leq \frac{\mathbb{E}|X_n - a|}{b-a} \leq \frac{\mathbb{E}|X_n| + |a|}{b-a} \leq \frac{M + |a|}{b-a}.$$

Therefore by the monotone convergence theorem

$$\mathbb{E}(U_\infty) = \lim_{n\to\infty} \mathbb{E}(U_n) \leq \frac{M+|a|}{b-a} < \infty,$$

which means $\mathbb{P}(U_\infty = \infty) = 0$. Since a and b were arbitrary $X = \lim_{n\to\infty} X_n$ exists almost surely. Now the Fatou lemma gives

$$\mathbb{E}|X| = \mathbb{E}(\lim_{n\to\infty} |X_n|) \leq \liminf_{n\to\infty} \mathbb{E}|X_n| \leq M < \infty.$$

Thus X_n is in fact convergent almost surely, and $X \in L^1$. $\square$