Bichteler-Dellacherie theorem

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The Bichteler-Dellacherie theorem is an important result in stochastic calculus, and states the equivalence of two very different definitions of semimartingales. The result also goes under other names, such as the *Dellacherie-Meyer-Mokobodzky theorem*. Prior to its discovery, a theory of stochastic integration had been developed for local martingales. As standard Lebesgue-Stieltjes integration can be applied to finite variation processes, this allowed an integral to be defined with respect to sums of local martingales and finite variation processes, known as a semimartingales. The Bichteler-Dellacherie theorem then shows that, as long as we require stochastic integration to satisfy bounded convergence, then semimartingales are actually the most general objects which can be used.

We consider a real valued stochastic process X adapted to a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$. Then, the integral $\int_0^t \xi\, dX$ can be written out explicitly for any simple predictable process ξ.

Theorem (Bichteler-Dellacherie)**.** *Let X be a cadlag adapted stochastic process. Then, the following are equivalent.*

1. *For every $t > 0$, the set*

$$\left\{ \int_0^t \xi\, dX : |\xi| \leq 1 \textit{ is simple predictable} \right\}$$

   is bounded in probability.

2. *A decomposition $X = M + V$ exists, where M is a local martingale and V is a finite variation process.*

3. *A decomposition $X = M + V$ exists, where M is locally a uniformly bounded martingale and V is a finite variation process.*

Condition **??** is equivalent to stating that if ξ^n is a sequence of simple predictable processes converging uniformly to zero, then the integrals $\int_0^t \xi^n\, dX$ tend to zero in probability as $n \to \infty$, which is a weak form of bounded convergence for stochastic integration.

Conditions **??** and **??** are the two definitions often used for the process X to be a semimartingale. However, condition **??** gives a stronger decomposition which is often more useful in practise. The property that M is locally a uniformly bounded martingale means that there exists a sequence of stopping times τ_n, almost surely increasing to infinity, such that the stopped processes M^{τ_n} are uniformly bounded martingales.

References

[1] Philip E. Protter, *Stochastic integration and differential equations.* Second edition. Applications of Mathematics, 21. Stochastic Modelling and Applied Probability. Springer-Verlag, 2004.