σ-algebra generated by a random variable

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Given the probability space $(\Omega, \mathcal{F}, P)$, any random variable $X\colon \Omega \to \mathbb{R}$ is $\mathcal{F}$- http://planetmath.org/MeasurableFunctionsmeasurable, in the following sense:

$$X^{-1}(U) = \{\omega \in \Omega\colon X(\omega) \in U\} \in \mathcal{F}$$

for any open sets $U \subseteq \mathbb{R}$, or equivalently any Borel sets $U \subset \mathbb{R}$.

We now define $\mathcal{F}_X$ as follows:

$$\mathcal{F}_X = X^{-1}(\mathcal{B}) := \{X^{-1}(B)\colon B \in \mathcal{B}\},$$

where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$. $\mathcal{F}_X$ is sometimes denoted as $\sigma(X)$. $\mathcal{F}_X$ is a sigma algebra since it satisfies the following:

- $\varnothing = X^{-1}(\varnothing) \in \mathcal{F}_X$,
- $\Omega - X^{-1}(B) = X^{-1}(\mathbb{R} - B) \in \mathcal{F}_X$, and
- $\bigcup X^{-1}(B_i) = X^{-1}(\bigcup B_i) \in \mathcal{F}_X$.

It is also clear that $\mathcal{F}_X$ is the smallest σ-algebra containing all sets of the form $X^{-1}(B)$, $B \in \mathcal{B}$. $\mathcal{F}_X$ as defined above is called the *σ-algebra* X.