proof of hitting times are stopping times for right-continuous processes

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Author	gel (22282)
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Let $(\mathcal{F})_{t\in\mathbb{T}}$ be a `http://planetmath.org/FiltrationOfSigmaAlgebras`filtration on the measurable space $(\Omega, \mathcal{F})$, It is assumed that $\mathbb{T}$ is a closed subset of $\mathbb{R}$ and that $\mathcal{F}_t$ is universally complete for each $t \in \mathbb{T}$.

Let X be a right-continuous and adapted process taking values in a metric space E and $S \subseteq E$ closed. We show that

$$\tau = \inf \{t \in \mathbb{T} : X_t \in S\}$$

is a stopping time. Assuming S is nonempty and defining the continuous function $d_S(x) \equiv \inf\{d(x,y) \colon y \in S\}$, then τ is the first time at which the right-continuous process $Y_t = d_S(X_t)$ hits 0.

Let us start by supposing that $\mathbb{T}$ has a minimum element t_0.

If $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$ and $\mathcal{F}_t^{\mathbb{P}}$ represents the `http://planetmath.org/Complet` of the σ-algebra $\mathcal{F}_t$ with respect to $\mathbb{P}$, then it is enough to show that τ is an $(\mathcal{F}_t^{\mathbb{P}})$-stopping time. By the universal completeness of $\mathcal{F}_t$ it would then follow that

$$\{\tau \leq t\} \in \bigcap_{\mathbb{P}} \mathcal{F}_t^{\mathbb{P}} = \mathcal{F}_t$$

for every $t \in \mathbb{T}$ and, therefore, that τ is a stopping time. So, by replacing $\mathcal{F}_t$ by $\mathcal{F}_t^{\mathbb{P}}$ if necessary, we may assume without loss of generality that $\mathcal{F}_t$ is complete with respect to the probability measure $\mathbb{P}$ for each t.

Let $\mathcal{T}$ consist of the set of measurable times $\sigma \colon \Omega \to \mathbb{T} \cup \{\infty\}$ such that $\{\sigma < t\} \in \mathcal{F}_t$ for every t and that $\sigma \leq \tau$. Then let σ^* be the essential supremum of $\mathcal{T}$. That is, σ^* is the smallest (up to sets of zero probability) random variable taking values in $\mathbb{R} \cup \{\pm\infty\}$ such that $\sigma^* \geq \sigma$ (almost surely) for all $\sigma \in \mathcal{T}$.

Then, by the properties of the essential supremum, there is a countable sequence $\sigma_n \in \mathcal{T}$ such that $\sigma^* = \sup_n \sigma_n$. It follows that $\sigma^* \in \mathcal{T}$.

For any $n = 1, 2, \ldots$ set

$$\sigma_1 = \inf \{t \in \mathbb{T} : t \geq \sigma^*, Y_t < 1/n\}.$$

Clearly, $\sigma_1 \leq \tau$ and, choosing any countable dense subset A of $\mathbb{T}$, the right-continuity of Y gives

$$\{\sigma_1 < t\} = \bigcup_{\substack{s<t,\\ s\in A}} \{\sigma^* < s, Y_s < 1/n\} \in \mathcal{F}_t.$$

So, $\sigma_1 \in \mathcal{T}$, which implies that $\sigma_1 \leq \sigma^*$ with probability one. However, by the right-continuity of Y, $\sigma_1 > \sigma^*$ whenever σ^* is finite and $Y_{\sigma^*} > 1/n$, so

$$\mathbb{P}(\sigma^* < \infty, Y_{\sigma^*} > 0) \leq \sum_n \mathbb{P}(\sigma^* < \infty, Y_{\sigma^*} > 1/n) = 0.$$

This shows that $Y_{\sigma^*} = 0$ and therefore $\sigma^* \geq \tau$ whenever $\sigma^* < \infty$. So, $\sigma^* = \tau$ almost surely and $\tau \in \mathcal{T}$ giving,

$$\{\tau \leq t\} = \{\tau < t\} \cup \{Y_t = 0\} \in \mathcal{F}_t.$$

So, τ is a stopping time.

Finally, suppose that $\mathbb{T}$ does not have a minimum element. Choosing a sequence $t_n \to -\infty$ in $\mathbb{T}$ then the above argument shows that

$$\tau_n = \inf \{t \in \mathbb{T} : t \geq t_n, Y_t = 0\}$$

are stopping times so,

$$\{\tau \leq t\} = \bigcup_n \{\tau_n \leq t\} \in \mathcal{F}_t$$

as required.