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Doob's optional sampling theorem

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Author	skubeedooo (5401)
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Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$, a process $(X_t)_{t\in\mathbb{T}}$ is a martingale if it satisfies the equality

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s$$

for all $s < t$ in the index set $\mathbb{T}$. Doob's optional sampling theorem says that this equality still holds if the times s, t are replaced by bounded stopping times S, T. In this case, the σ-algebra $\mathcal{F}_s$ is replaced by the collection of `http://planetmath.org/SigmaAlgebraAtAStoppingTime`events observable at the random time S,

$$\mathcal{F}_S = \{A \in \mathcal{F} : A \cap \{S \leq t\} \in \mathcal{F}_t \text{ for all } t \in \mathbb{T}\}.$$

In discrete-time, when the index set $\mathbb{T}$ is countable, the result is as follows.

Doob's Optional Sampling Theorem. *Suppose that the index set $\mathbb{T}$ is countable and that $S \leq T$ are stopping times bounded above by some constant $c \in \mathbb{T}$. If (X_t) is a martingale then X_T is an integrable random variable and*

$$\mathbb{E}[X_T|\mathcal{F}_S] = X_S,\ \mathbb{P} \text{ almost surely.} \tag{1}$$

Similarly, if X is a submartingale then X_T is integrable and

$$\mathbb{E}[X_T|\mathcal{F}_S] \geq X_S,\ \mathbb{P} \text{ almost surely.} \tag{2}$$

If X is a supermartingale then X_T is integrable and

$$\mathbb{E}[X_T|\mathcal{F}_S] \leq X_S,\ \mathbb{P} \text{ almost surely.} \tag{3}$$

This theorem shows, amongst other things, that in the case of a fair casino, where your return is a martingale, betting strategies involving 'knowing when to quit' do not enhance your expected return.

In continuous-time, when the index set $\mathbb{T}$ an interval of the real numbers, then the stopping times S, T can have a continuous distribution and X_S, X_T need not be measurable quantities. Then, it is necessary to place conditions on the sample paths of the process X. In particular, Doob's optional sampling theorem holds in continuous-time if X is assumed to be right-continuous.