quadratic variation of a semimartingale

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Given any semimartingale X, its quadratic variation $[X]$ exists and, for any two semimartingales X, Y, their quadratic covariation $[X, Y]$ exists. This is a consequence of the existence of the stochastic integral, and the covariation can be expressed by the integration by parts formula

$$[X, Y]_t = X_t Y_t - X_0 Y_0 - \int_0^t X_{s-}\,dY_s - \int_0^t Y_{s-}\,dX_s.$$

Furthermore, suppose that P_n is a sequence of http://planetmath.org/Partition3partitions of $\mathbb{R}_+$,

$$P_n = \{0 = \tau_0^n \leq \tau_1^n \leq \cdots \uparrow \infty\}$$

where, τ_k^n can, in general, be stopping times. Suppose that the mesh $|P_n^t| = \sup_k(\tau_k^n \wedge t - \tau_{k-1}^n \wedge t)$ tends to zero in probability as $n \to \infty$, for each time $t > 0$. Then, the approximations $[X, Y]^{P_n}$ to the quadratic covariation http://planetmath.org/UcpConvergenceconverge ucp to $[X, Y]$ and, convergence also holds in the semimartingale topology.

A consequence of ucp convergence is that the jumps of the quadratic variation and covariation satisfy

$$\Delta[X] = (\Delta X)^2,\ \Delta[X, Y] = \Delta X \Delta Y$$

at all times. In particular, $[X, Y]$ is continuous whenever X or Y is continuous. As quadratic variations are increasing processes, this shows that the sum of the squares of the jumps of a semimartingale is finite over any bounded interval

$$\sum_{s \leq t} (\Delta X_s)^2 \leq [X]_t < \infty.$$

Given any two semimartingales X,Y, the polarization identity $[X, Y] = ([X + Y] - [X - Y])/4$ expresses the covariation as a difference of increasing processes and, therefore is of http://planetmath.org/FiniteVariationProcessfinite variation, So, the continuous part of the covariation

$$[X, Y]_t^c \equiv [X, Y]_t - \sum_{s \leq t} \Delta X_s \Delta Y_s$$

is well defined and continuous.