proof of Borel-Cantelli 1

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Let B_k be the event $\cup_{i=k}^{\infty} A_i$ for $k = 1, 2, \ldots,$. If x is in the event A_i's i.o., then $x \in B_k$ for all k. So $x \in \cap_{k=1}^{\infty} B_k$.

Conversely, if $x \in B_k$ for all k, then we can show that x is in A_i's i.o. Indeed, $x \in B_1 = \cup_{i=1}^{\infty} A_i$ means that $x \in A_{j(1)}$ for some $j(1)$. However $x \in B_{j(1)+1}$ implies that $x \in A_{j(2)}$ for some $j(2)$ that is strictly larger than $j(1)$. Thus we can produce an infinite sequence of integer $j(1) < j(2) < j(3) < \ldots$ such that $x \in A_{j(i)}$ for all i.

Let E be the event $\{x : x \in A_i \text{ i.o.}\}$. We have

$$E = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} A_i.$$

From $E \subseteq B_k$ for all k, it follows that $P(E) \leq P(B_k)$ for all k. By union bound, we know that $P(B_k) \leq \sum_{i=k}^{\infty} P(A_i)$. So $P(B_k) \to 0$, by the hypothesis that $\sum_{i=1}^{\infty} P(A_i)$ is finite. Therefore, $P(E) = 0$.