semimartingale topology

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Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}), \mathbb{P})$ be a filtered probability space and (X_t^n), (X_t) be cadlag adapted processes. Then, X^n is said to converge to X in the *semimartingale topology* if $X_0^n \to X_0$ in probability and

$$\int_0^t \xi^n \, dX^n - \int_0^t \xi^n \, dX \to 0$$

in probability as $n \to \infty$, for every $t > 0$ and sequence of simple predictable processes $|\xi^n| \leq 1$.

This topology occurs with stochastic calculus where, according to the http://planetmath.org/DominatedConvergenceForStochasticIntegrationdominated convergence theorem, stochastic integrals converge in the semimartingale topology. Furthermore, stochastic integration with respect to any http://planetmath.org/LocalP bounded predictable process ξ is continuous under the semimartingale topology. That is, if X^n are semimartingales converging to X then $\int \xi \, dX^n$ converges to $\int \xi \, dX$, a fact which does not hold under weaker topologies such as ucp convergence.

Also, for cadlag martingales, L^1 convergence implies semimartingale convergence.

It can be shown that semimartingale convergence implies ucp convergence. Consequently, X^n converges to X in the semimartingale topology if and only if

$$X_0^n - X_0 + \int \xi^n \, dX^n - \int \xi^n \, dX \xrightarrow{\text{ucp}} 0$$

for all sequences of simple predictable processes $|\xi^n| \leq 1$.

The topology is described by a metric as follows. First, let $D^{\text{ucp}}(X - Y)$ be a metric defining the ucp topology. For example,

$$D^{\text{ucp}}(X) = \sum_{n=1}^{\infty} 2^{-n} \mathbb{E}\left[\min\left(1, \sup_{t<n} |X_t|\right)\right].$$

Then, a metric $D^{\text{s}}(X - Y)$ for semimartingale convergence is given by

$$D^{\text{s}}(X) = \sup\left\{D^{\text{ucp}}(X_0 + \xi \cdot X) : |\xi| \leq 1 \text{ is simple previsible}\right\}$$

($\xi \cdot X$ denotes the integral $\int \xi \, dX$). This is a proper metric under identification of processes with almost surely equivalent sample paths, otherwise it is a pseudometric.

If $\lambda_n \neq 0$ is a sequence of real numbers converging to zero and X is a cadlag adapted process then $\lambda_n X \to 0$ in the semimartingale topology if and only if

$$\lambda_n \int_0^t \xi^n \, dX \to 0$$

in probability, for every $t > 0$ and simple predictable processes $|\xi^n| \leq 1$. By the `http://planetmath.org/SequentialCharacterizationOfBoundedness`sequential characterization of boundedness, this is equivalent to the statement that

$$\left\{ \int_0^t \xi \, dX : |\xi| \leq 1 \text{ is simple predictable} \right\}$$

is bounded in probability for every $t > 0$. So, $\lambda_n X \to 0$ in the semimartingale topology if and only if X is a semimartingale. It follows that semimartingale convergence only becomes a `http://planetmath.org/TopologicalVectorSpace`vector topology when restricted to the space of semimartingales. Then, it can be shown that `http://planetmath.org/CompletenessOfSemimartingaleConvergence`the set of semimartingales is a complete topological vector space.