a simple method for comparing real functions

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Theorem:

Let $f(x)$ and $g(x)$ be real-valued, twice differentiable functions on $[a, b]$, and let $x_0 \in [a, b]$.

If $f(x_0) = g(x_0)$, $f'(x_0) = g'(x_0)$, $f''(x) \leq g''(x)$ for all x in $[a, b]$, then $f(x) \leq g(x)$ for all x in $[a, b]$.

Proof. Let $h(x) = g(x) - f(x)$; by our hypotheses, $h(x)$ is a twice differentiable function on $[a, b]$, and by the Taylor formula with `http://planetmath.org/RemainderVariou` form remainder one has for any $x \in [a, b]$:

$$h(x) = h(x_0) + h'(x_0)(x - x_0) + \frac{1}{2}h''(\xi)(x - x_0)^2$$

where $\xi = \xi(x) \in [x, x_0]$.

Then by hypotheses,

$$\begin{aligned} h(x_0) &= g(x_0) - f(x_0) = 0 \\ h'(x_0) &= g'(x_0) - f'(x_0) = 0 \\ h''(\xi) &= g''(\xi) - f''(\xi) \geq 0 \end{aligned}$$

so that

$$h(x) = \frac{1}{2}h''(\xi)(x - x_0)^2 \geq 0$$

whence the thesis. □