Multidimensional Chebyshev's inequality

Canonical name	MultidimensionalChebyshevsInequality
Date of creation	2013-03-22 18:17:55
Last modified on	2013-03-22 18:17:55
Owner	daniWk (21206)
Last modified by	daniWk (21206)
Numerical id	5
Author	daniWk (21206)
Entry type	Theorem
Classification	msc 60A99

Let X be an N-dimensional random variable with mean $\mu = \mathbb{E}[X]$ and covariance matrix $V = \mathbb{E}\left[(X-\mu)\,(X-\mu)^T\right]$.

If V is invertible (i.e., strictly positive), for any $t > 0$:

$$\Pr\left(\sqrt{(X-\mu)^T\; V^{-1}\; (X-\mu)} > t\right) \leq \frac{N}{t^2}$$

Proof: V is positive, so V^{-1} is. Define the random variable

$$y = (X-\mu)^T\; V^{-1}\; (X-\mu)$$

y is positive, then Markov's inequality holds:

$$\Pr\left(\sqrt{(X-\mu)^T\; V^{-1}\; (X-\mu)} > t\right) = \Pr\left(\sqrt{y} > t\right) = \Pr\left(y > t^2\right) \leq \frac{\mathbb{E}[y]}{t^2}$$

Since V is symmetric, a rotation R (i.e., $R\,R^T = R^T\,R = I$) and a diagonal matrix D (i.e., $i \neq j \Rightarrow D_{i,j} = 0$) exist such that

$$V = R^T\,D\,R$$

Since V is positive $D_{ii} > 0$. Besides

$$V^{-1} = R^{-1}\,D^{-1}\,(R^T)^{-1} = R^T\,D^{-1}\,R$$

clearly $[D^{-1}]_{ii} = \frac{1}{D_{ii}}$.

Define $Z = R\,(X-\mu)$.

The following identities hold:

$$\mathbb{E}\left[Z\,Z^T\right] = R\,\mathbb{E}\left[(X-\mu)\,(X-\mu)^T\right]\,R^T = R\,R^T\,D\,R\,R^T = D \quad \Rightarrow \quad \forall i \quad \mathbb{E}\left[Z_i^2\right] = D_{ii}$$

and

$$y = Z^T\,R\,V^{-1}\,R^T\,Z = Z^T\,D^{-1}\,Z = \sum_{i=1}^{N} \frac{Z_i^2}{D_{ii}}$$

then

$$\mathbb{E}[y] = \sum_{i=1}^{N} \frac{\mathbb{E}\left[Z_i^2\right]}{D_{ii}} = N$$