Itô integral

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1 The need for a stochastic integral

Suppose that we wish to model a system which is continuously subject to random shocks. Then, we may want to consider a differential equation of the form

$$\frac{dX_t}{dt} = b(X_t) + a(X_t)\xi_t. \tag{1}$$

Here, X_t is the stochastic process describing the state of the system at each time $t \geq 0$, b describes the behavior in the absence of any random shocks and a is the sensitivity to the the random noise ξ_t. The random variables ξ_t are assumed to be independent and identically distributed with mean zero for each $t \geq 0$, and defined on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The first problem is that it is not clear how to define ξ continuously in time such that (**??**) can be solved. Let W be the integral

$$W_t = \int_0^t \xi_s\, ds.$$

Then, W_t should be continuous in t and, for each $h > 0$, the sequence of increments $(W_{(n+1)h} - W_{nh})_{n=0,1,\ldots}$ should be independent and identically distributed random variables. A possible candidate for W is Brownian motion, and it can be shown that, up to a constant scaling factor, this is the only possibility. However, Brownian motion is nowhere differentiable and $\xi_t = dW_t/dt$ cannot exist. We therefore re-express (**??**) in terms of W by integrating it

$$X_t = X_0 + \int_0^t b(X_s)\, ds + \int_0^t a(X_s)\, dW_s. \tag{2}$$

The idea of Itô integration is to give meaning to the final integral on the right hand side of (**??**). Defining the integral with respect to piecewise constant functions of the form

$$\alpha_t = \sum_{k=1}^n c_k 1_{\{t_{k-1} < t \leq t_k\}} \tag{3}$$

is easy. Here, $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$ is an increasing sequence of times and c_k are random variables. Then, the integral of α with respect to W is

$$\int_0^t \alpha_s\, dW_s = \sum_{k=1}^n c_k \left(W_{t_k \wedge t} - W_{t_{k-1} \wedge t}\right). \tag{4}$$

We could attempt to define the integral with respect to W on the right hand side of (**??**) by approximating $a(X_t)$ by such piecewise constant functions. However, the sample paths of Brownian motion are of infinite total variation over all intervals. This means that if we let α^n be

$$\alpha_t^n = \sum_{k=1}^n \operatorname{sign}(W_{k/n} - W_{(k-1)/n}) 1_{\{(k-1)/n < t \leq k/n\}}$$

then the integrals

$$\int_0^1 \alpha^n \, dW = \sum_{k=1}^n \left| W_{k/n} - W_{(k-1)/n} \right|$$

will tend to infinity as $n \to \infty$, despite the fact that α^n are all bounded by one. This means that even if we approximate $a(X_t)$ as closely as we like by piecewise constant functions, the integral can diverge to infinity.

2 The Itô integral

Itô's solution was to first note that if we can define the integral in a way such that (**??**) has a unique solution, then X_t should only depend on the values of W_s for $s \leq t$. So, define $\mathcal{F}_t$ to be the collection of events in the probability space observable up until time t. This is the smallest σ-algebra with respect to which W_s is measurable for $s \leq t$,

$$\mathcal{F}_t = \sigma \left(W_s : s \leq t \right).$$

Then, $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is a `http://planetmath.org/FiltrationOfSigmaAlgebras`filtration of σ-algebras to which W is adapted. If we assume that the piecewise constant process α_t is adapted, then the variables c_k in (**??**) will be $\mathcal{F}_{t_{k-1}}$-measurable. Such processes are called *elementary predictable*, and the following identity is obtained

$$\mathbb{E}\left[\left(\int_0^t \alpha \, dW \right)^2 \right] = \mathbb{E}\left[\int_0^t \alpha_s^2 \, ds \right]. \tag{5}$$

This is known as the *Itô isometry.* It ensures that, for bounded and adapted integrands α, the integral with respect to W cannot become too large, on average. Furthermore, the right hand side is defined for all jointly measurable processes α, and allows us extend the integral with respect to W to integrands which can be approximated by piecewise constant and adapted processes. Such processes are called *predictable* and include all continuous and adapted processes, so it gives meaning to (**??**). If we define

$$\|\alpha\|_{2,t} \equiv \mathbb{E}\left[\int_0^t \alpha_s^2 \, ds\right]^{\frac{1}{2}}$$

then this is a norm on the set of predictable processes — in fact, it is the $L^2(\mathbb{P} \times \lambda)$-norm, where λ is the Lebesgue measure on $[0,t]$. Equation (**??**) shows that the map taking the adapted and piecewise constant function α to $\int_0^t \alpha \, dW$ is an *isometry* with respect to the norms $\|\cdot\|_{2,t}$ and the $L^2(\mathbb{P})$-norm $\|\cdot\|_2$. Furthermore, it can be shown, by using the functional monotone class theorem, that every predictable process α with finite $\|\cdot\|_{2,t}$ norm can be approximated in this norm by a sequence elementary predictable processes α^n. That is, the elementary predictable processes are `http://planetmath.org/Dense`dense under the $\|\cdot\|_{2,t}$ norm, and the Itô integral is defined to be the unique `http://planetmath.org/BoundedLinearExtension`continuous extension from the elementary predictable processes. So, if $\|\alpha - \alpha^n\|_{2,t} \to 0$ then,

$$\int_0^t \alpha \, dW = \lim_{n\to\infty} \int_0^t \alpha^n \, dW$$

where convergence is in the $L^2(\mathbb{P})$ norm.

3 Extension of the integral

The definition above gives a construction of the Itô integral which is useful in many applications. However, the restriction that $\|\alpha\|_{2,t}$ must be finite is too restrictive for some situations. For example, it is possible to construct solutions to the stochastic differential equation

$$dX = X^c \, dW$$

for any constant $c > 0$ and any given initial value $X_0 > 0$. When $c > 1$ then it is known that X_t^{2c} does not have finite expectation for $t > 0$ and, therefore, X^c does not have finite $\|\cdot\|_{2,t}$ norm.

The Itô integral can be extended to all predictable processes α such that

$$\int_0^t \alpha_s^2 \, ds < \infty$$

with probability one, for each $t > 0$. Such processes are said to be W-integrable. Given any such α, the functional monotone class theorem can be used to show that there exists a sequence of elementary predictable processes α^n such that

$$\int_0^t (\alpha_s^n - \alpha_s)^2 \, ds \to 0$$

http://planetmath.org/ConvergenceInProbabilityin probability for every $t > 0$. Then, using the Itô isometry it can be shown that the limit

$$\int_0^t \alpha \, dW \equiv \lim_{n\to\infty} \int_0^t \alpha^n \, dW$$

exists, where convergence is in probability. Finally, we note that that this only defines the integral $\mathbb{P}$-almost everywhere, at each time. However, as probability measures only satisfy countable additivity, this only simultaneously defines the values of the integral at different times t on countable subsets of $\mathbb{R}_+$. The additional constraint is added that the sample paths $t \mapsto \int_0^t \alpha \, dW$ are continuous, and it can be shown that it is possible to take such continuous versions of the integral. This defines the integral simultaneously at all times up to a zero probability set.

We have presented a definition of the Itô integral with respect to a Brownian motion W. Then, an important result for manipulating such integrals is http://planetmath.org/ItosFormulaItô's lemma. Furthermore, it is sometimes necessary to define integration for more general processes. See http://planetmath.org/StochasticIntegrationstochastic integral for a definition of the integral with respect to general semimartingales, which includes discontinuous processes, such as Lévy processes.

References

[1] Bernt Øksendal. *, An Introduction with Applications.* 5th ed. Springer, 1998.