analytic solution to Ornstein-Uhlenbeck SDE

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This entry derives the analytical solution to the stochastic differential equation for the Ornstein-Uhlenbeck process:

$$dX_t = \kappa(\theta - X_t)\,dt + \sigma\,dW_t\,, \tag{1}$$

where W_t is a standard Brownian motion, and $\kappa > 0$, θ, and $\sigma > 0$ are constants.

Motivated by the observation that θ is supposed to be the long-term mean of the process X_t, we can simplify the SDE (**??**) by introducing the change of variable

$$Y_t = X_t - \theta$$

that subtracts off the mean. Then Y_t satisfies the SDE:

$$dY_t = dX_t = -\kappa Y_t\,dt + \sigma\,dW_t\,. \tag{2}$$

In SDE (**??**), the process Y_t is seen to have a drift towards the value zero, at an exponential rate κ. This motivates the change of variables

$$Y_t = e^{-\kappa t} Z_t \quad \Leftrightarrow \quad Z_t = e^{\kappa t} Y_t\,,$$

which should remove the drift. A calculation with the product rule for Itô integrals shows that this is so:

$$\begin{aligned} dZ_t &= \kappa e^{\kappa t} Y_t\,dt + e^{\kappa t}\,dY_t \\ &= \kappa e^{\kappa t} Y_t\,dt + e^{\kappa t}\big(-\kappa Y_t\,dt + \sigma\,dW_t\big) \\ &= 0\,dt + \sigma e^{\kappa t}\,dW_t\,. \end{aligned}$$

The solution for Z_t is immediately obtained by Itô-integrating both sides from s to t:

$$Z_t = Z_s + \sigma \int_s^t e^{\kappa u}\,dW_u\,.$$

Reversing the changes of variables, we have:

$$Y_t = e^{-\kappa t} Z_t = e^{-\kappa(t-s)} Y_s + \sigma e^{-\kappa t} \int_s^t e^{\kappa u}\,dW_u\,,$$

and

$$X_t = Y_t + \theta = \theta + e^{-\kappa(t-s)}(X_s - \theta) + \sigma \int_s^t e^{-\kappa(t-u)}\,dW_u\,.$$