Ornstein-Uhlenbeck process

Canonical name	OrnsteinUhlenbeckProcess
Date of creation	2013-03-22 17:19:26
Last modified on	2013-03-22 17:19:26
Owner	stevecheng (10074)
Last modified by	stevecheng (10074)
Numerical id	4
Author	stevecheng (10074)
Entry type	Definition
Classification	msc 60H10
Classification	msc 60-00
Synonym	Ornstein-Uhlenbeck equation

Definition

The *Ornstein-Uhlenbeck* process is a stochastic process that satisfies the following stochastic differential equation:

$$dX_t = \kappa(\theta - X_t)\, dt + \sigma\, dW_t\,, \tag{1}$$

where W_t is a standard Brownian motion on $t \in [0, \infty)$.

The constant parameters are:

- $\kappa > 0$ is the rate of mean reversion;
- θ is the long-term mean of the process;
- $\sigma > 0$ is the volatility or average magnitude, per square-root time, of the random fluctuations that are modelled as Brownian motions.

Mean-reverting property

If we ignore the random fluctuations in the process due to dW_t, then we see that X_t has an overall drift towards a mean value θ. The process X_t reverts to this mean exponentially, at rate κ, with a magnitude in direct proportion to the distance between the current value of X_t and θ.

This can be seen by looking at the solution to the ordinary differential equation $dx_t = \kappa(\theta - x)dt$ which is

$$\frac{\theta - x_t}{\theta - x_0} = e^{-\kappa(t-t_0)}\,, \qquad \text{or } x_t = \theta + (x_0 - \theta)e^{-\kappa(t-t_0)}\,. \tag{2}$$

For this reason, the Ornstein-Uhlenbeck process is also called a *mean-reverting process*, although the latter name applies to other types of stochastic processes exhibiting the same property as well.

Solution

The solution to the stochastic differential equation (**??**) defining the Ornstein-Uhlenbeck process is, for any $0 \leq s \leq t$, is

$$X_t = \theta + (X_s - \theta)e^{-\kappa(t-s)} + \sigma \int_s^t e^{-\kappa(t-u)}\, dW_u\,.$$

where the integral on the right is the Itô integral.

For any fixed s and t, the random variable X_t, conditional upon X_s, is normally distributed with

$$\text{mean} = \theta + (X_s - \theta)e^{-\kappa(t-s)}\,, \quad \text{variance} = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa(t-s)})\,.$$

Observe that the mean of X_t is exactly the value derived heuristically in the solution (**??**) of the ODE.

The Ornstein-Uhlenbeck process is a time-homogeneous Itô diffusion.

Applications

The Ornstein-Uhlenbeck process is widely used for modelling biological processes such as neuronal response, and in mathematical finance, the modelling of the dynamics of interest rates and volatilities of asset prices.

References

[1] Martin Jacobsen. "Laplace and the Origin of the Ornstein-Uhlenbeck Process". Bernoulli, Vol. 2, No. 3. (Sept. 1996), pp. 271 – 286.

[2] Bernt Øksendal. *Stochastic Differential Equations, An Introduction with Applications*, 5th edition. Springer, 1998.

[3] Steven E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Springer, 2004.

[4] Sebastian Jaimungal. Lecture notes for *Pricing Theory*. University of Toronto.

[5] Dmitri Rubisov. Lecture notes for *Risk Management*. University of Toronto.