conditional expectation

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Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X\colon \Omega \to \mathbb{R}$ a real random variable with $E[|X|] < \infty$.

Conditional Expectation Given an Event

Given an event $B \in \mathcal{F}$ such that $P(B) > 0$, then we define the *conditional expectation of X given B*, denoted by $E[X|B]$ to be

$$E[X|B] := \frac{1}{P(B)} \int_B X dP.$$

When $P(B) = 0$, $E[X|B]$ is sometimes defaulted to 0.

If X is discrete, then we can write $X = \sum_{i=1}^{\infty} w_i 1_{B_i}$, where 1_{B_i} are the indicator functions, $B_i = X^{-1}(\{w_i\})$ and $w_i \in \mathbb{R}$, then conditional expectation of X given B becomes

$$\begin{aligned} E[X|B] &= \frac{1}{P(B)} \int_B \Big(\sum_{i=1}^{\infty} w_i 1_{B_i} \Big) dP = \frac{1}{P(B)} \Big(\sum_{i=1}^{\infty} w_i \int_B 1_{B_i} dP \Big) \\ &= \frac{1}{P(B)} \Big(\sum_{i=1}^{\infty} w_i P(B_i \cap B) \Big) = \sum_{i=1}^{\infty} w_i P(B_i|B), \end{aligned}$$

where $P(B_i|B)$ is the conditional probability of B_i given B.

Conditional Expectation Given a Sigma Algebra

If $\mathcal{D} \subset \mathcal{F}$ is a sub σ-algebra, then the *conditional expectation of X given $\mathcal{D}$*, denoted by $E[X|\mathcal{D}]$ is defined as follows:

Definition $E[X|\mathcal{D}]$ is the function from Ω to $\mathbb{R}$ satisfying :

1. $E[X|\mathcal{D}]$ is $\mathcal{D}$-measurable
2. $\displaystyle\int_A E[X|\mathcal{D}] dP = \int_A X dP$, for all $A \in \mathcal{D}$.

It can be shown, via Radon-Nikodym Theorem, that $E[X|\mathcal{D}]$ always exists and is unique almost everywhere: any two $\mathcal{D}$-measurable random variables Y, Z with

$$\int_A Y dP = \int_A Z dP = \int_A X dP$$

differ by a null event in $\mathcal{D}$. We can in fact set up an equivalence relation on the set of all integrable $\mathcal{D}$-measurable functions satisfying condition 2 above. In this sense, $E[X|\mathcal{D}]$ is an equivalence class of random variables, and any two members in $E[X|\mathcal{D}]$ may qualify as conditional expectations of X given $\mathcal{D}$ (they are often called *versions* of the conditional expectation). In practice, however, we often think of $E[X|\mathcal{D}]$ as a function rather than a set of functions. As long as we realize that any two such functions are equal almost surely, we may blur such differences and abuse the language.

Suppose $Y : \Omega \to \mathbb{R}$ is another random variable with $E[|Y|] < \infty$ and let $\alpha, \beta \in \mathbb{R}$. Then

1. $E[\alpha X + \beta Y|\mathcal{D}] = \alpha E[X|\mathcal{D}] + \beta E[X|\mathcal{D}]$
2. $E[E[X|\mathcal{D}]] = E[X]$
3. $E[X|\mathcal{D}] = X$ if X is $\mathcal{D}$-measurable
4. $E[X|\mathcal{D}] = E[X]$ if X is `http://planetmath.org/IndependentSigmaAlgebras`independent of $\mathcal{D}$
5. $E[YX|\mathcal{D}] = YE[X|\mathcal{D}]$ if Y is $\mathcal{D}$-measurable

Conditional Expectation Given a Random Variable

Given any real random variable $Y : \Omega \to \mathbb{R}$, we define the *conditional expectation of* X *given* Y to be the conditional expectation of X given $\mathcal{F}_Y$, the `http://planetmath.org/MathcalFMeasurableFunction`sigma algebra generated by Y.