# absolute moments bounding (necessary and sufficient condition)


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Let $X$ be a random variable; then

$$E[|X|^k] \leq M^k \qquad \forall k \geq 1, k \in \mathbf{N}$$

if and only if,$\forall i \geq 0, i \in \mathbf{N}$

$$E\left[|X|^k\right] \leq E\left[|X|^i\right] M^{k-i} \qquad \forall k \geq i, k \in \mathbf{N}$$

*Proof.* a) $(E\left[|X|^k\right] \leq E\left[|X|^i\right] M^{k-i} \implies E[|X|^k] \leq M^k)$

It's enough to take $i = 0$ and the thesis follows easily.

b) $(E[|X|^k] \leq M^k \Longrightarrow E\left[|X|^k\right] \leq E\left[|X|^i\right] M^{k-i})$

Let $1 \leq i \leq k$ (the case $i = 0$ is trivial). Then, using Cauchy-Schwarz inequality $N$ times, one has:

$$\begin{aligned}
E[|X|^k] &= E\left[|X|^{\frac{i}{2}} |X|^{k-\frac{i}{2}}\right] \\
&\leq E\left[|X|^i\right]^{\frac{1}{2}} E\left[|X|^{2k-i}\right]^{\frac{1}{2}} \\
&= E\left[|X|^i\right]^{\frac{1}{2}} E\left[|X|^{\frac{i}{2}} |X|^{2k-\frac{3}{2}i}\right]^{\frac{1}{2}} \\
&\leq E\left[|X|^i\right]^{\left(\frac{1}{2}+\frac{1}{4}\right)} E\left[|X|^{4k-3i}\right]^{\frac{1}{4}} \\
&\leq E\left[|X|^i\right]^{\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right)} E\left[|X|^{(8k-7i)}\right]^{\frac{1}{8}} \\
&\quad \ldots \\
&\leq E\left[|X|^i\right]^{\left(\sum_{m=1}^{N} \frac{1}{2^m}\right)} E\left[|X|^{2^N k-\left(2^N-1\right)i}\right]^{\frac{1}{2^N}} \\
&= E\left[|X|^i\right]^{\left(1-\frac{1}{2^N}\right)} E\left[|X|^{2^N(k-i)+i}\right]^{\frac{1}{2^N}} \\
&\leq E\left[|X|^i\right]^{\left(1-\frac{1}{2^N}\right)} M^{(k-i)+\frac{i}{2^N}},
\end{aligned}$$

and since this must hold for any $N$, we obtain

$$E[|X|^k] \leq E\left[|X|^i\right] M^{k-i}$$

□