Espaços Vetoriais

Exercício 1

a) $(\lambda+\mu)(x,y) = (x(\lambda+\mu),1) \neq (x(\lambda+\mu),2) = \lambda(x,y)+\mu(x,y)$

b) $(\lambda+\mu)(x,y) = \{x(\lambda+\mu), y(\lambda+\mu)\} \neq \{x(\lambda+\mu),0\} = \lambda(x,y)+\mu(x,y)$

c) $E.V.$

Exercício 2

Sejam $f,g \in F(A,E)$, $\lambda \in \mathbb{K}$.
$(f+g)(x) = f(x) +_E g(x) \in E,\ \forall x \in A$
$(f+g)(x) = f(x) +_E g(x) = g(x) +_E f(x) = (g+f)(x)$
$[(f+g)+h](x) = (f(x) +_E g(x)) +_E h(x) = f(x) +_E (g(x) +_E h(x)) = [f+(g+h)](x)$
$f(x) \equiv 0_E$
$(-f)(x) = -f(x)$
$[(\lambda f)](x) = \lambda \times_E f(x) \in E,\ \forall x \in A$
$[\mu(\lambda f)](x) = \mu(\lambda \times_E f(x)) = (\mu\lambda) \times_E f(x) = [(\mu\lambda)f](x)$
$f(x) \equiv 1_E$
$[(\mu+\lambda)f](x) = (\mu+\lambda) \times_E f(x) = \mu \times_E f(x) + \lambda \times_E f(x) = [\mu f + \lambda f](x)$
$[\lambda(f+g)](x) = \lambda \times_E (f(x) +_E g(x)) = \lambda \times_E f(x) + \lambda \times_E g(x) = [\lambda f + \lambda g](x)$

Exercício 3

a) $0 \in A \neq \emptyset$
$(-y_1,y_1,z_1)+(-y_2,y_2,z_2) = (-y_1-y_2, y_1+y_2, z_1+z_2) = \big(-(y_1+y_2), y_1+y_2, z_1+z2\big) \in A$
$\lambda(-y,y,z) = \big(-(\lambda y), \lambda y, \lambda z\big) \in A$

b) $0 \in A \neq \emptyset$
$2x_1+y_1-z_1=0 \wedge 2x_2+y_2-z_2=0 \implies 2(x_1+x_2)+y_1+y_2-(z_1+z_2)=0,$
$2\lambda x+\lambda y-\lambda z=\lambda(2x+y-z)=0$

c) $0 \in A \neq \emptyset$
$x_1+5y_1=0 \wedge x_2+5y_2=0 \implies (x_1+x_2)+5(y_1+y_2)=0 \quad \wedge \quad \lambda x+\lambda 5y=\lambda(x+5y)=0$

d) $0 \in A \neq \emptyset$
$x_1+y_1<1 \wedge x_2+y_2<1 \implies (x_1+x_2)+(y_1+y_2)<2 \neq 1$. *Por ex.* $(0.5,0)+(0.5,0)=(1,0) \notin A$

e) $0 \in A \neq \emptyset$
$$\begin{cases} x_1+y_1+z_1=0 \wedge x_2+y_2+z_2=0 \implies x_1+x_2+y_1+y_2+z_1+z_2=0 \wedge \lambda x+\lambda y+\lambda z=\lambda(x+y+z)=0 \\ x_1-y_1+z_1=0 \wedge x_2-y_2+z_2=0 \implies x_1+x_2-(y_1+y_2)+z_1+z_2=0 \wedge \lambda x-\lambda y+\lambda z=\lambda(x-y+z)=0 \end{cases}$$

f) $0 \in A \neq \emptyset$
$(1,1,1) \in A, \quad 1^2+1^2+1^2<4 \implies (x_1+x_2)^2+(y_1+y_2)^2+(z_1+z_2)^2=2^2+2^2+2^2=13>4 \neq 1.$ *Logo* $(1,1,1)+(1,1,1)=(2,2,2) \notin A$

Exercício 4

a) $\sqrt{2}w \notin \mathbb{Z}$

b) $0 \in S \neq \emptyset$
$z_1=0 \wedge z_2=0 \implies z_1+z_2=0,$
$\lambda z=0$

Exercício 5

$0 \in (S \cap T) \neq \emptyset$
$$\begin{cases} x_1+2y_1+z_1=0 \wedge x_2+2y_2+z_2=0 \implies x_1+x_2+2(y_1+y_2)+z_1+z_2=0 \wedge \lambda x+\lambda 2y+\lambda z=\lambda(x+2y+z)=0 \\ -3x_1+z_1=0 \wedge -3x_2+z_2=0 \implies -3(x_1+x_2)+z_1+z_2=0 \wedge -\lambda 3x+\lambda z=\lambda(-3x+z)=0 \end{cases}$$

Exercício 6

a) $0+0i \notin G$

b) $0+0i \notin G$

Exercício 7

a) $0 \in A \neq \emptyset$ $p(0)=0 \wedge q(0)=0 \Longrightarrow p(0)+q(0)=0 \Longrightarrow p+q \in A$
$p(0)=0 \Longrightarrow \lambda p(0)=\lambda 0=0 \Longrightarrow \lambda p \in A$

b) $0 \in A \neq \emptyset$ $p(-x)=p(x) \wedge q(-x)=q(x) \Longrightarrow p(-x)+q(-x)=p(x)+q(x) \Longrightarrow p+q \in A$
$p(-x)=p(x) \Longrightarrow \lambda p(-x)=\lambda p(x) \Longrightarrow \lambda p \in A$

Exercício 8

a) $\alpha = 2 \wedge \beta = 1$

b) $\alpha = 0 \wedge \beta = -2$

Exercício 9

$x = z - y \wedge \alpha = \frac{z}{3} \wedge \beta = \frac{2z}{3} - y$

Exercício 10

a) $k = \frac{12}{5} \wedge \alpha = \frac{1}{5} \wedge \beta = \frac{6}{5}$

b) $a = \frac{7b}{5} + c$

Exercício 11

a) $\alpha = \frac{x}{2} + \frac{y}{2} \wedge \beta = \frac{x}{4} - \frac{y}{4}$

b) $\alpha = \frac{4x}{3} - y \wedge \beta = \frac{x}{3}$

c) $\alpha = \frac{x}{2} + \frac{y}{2} \wedge \beta = -x$

d) $x = \frac{2y}{3} \wedge \beta = \alpha - \frac{y}{3}$, $\langle (2,3), (-2,-3) \rangle \neq \mathbb{R}^2$

Exercício 12

$x = z - y \wedge \alpha = \frac{z}{2} - \frac{3y}{2} \wedge \beta = z - y$

Exercício 13

$w = -x + y + z \wedge \alpha = -\frac{x}{4} + \frac{y}{4} + \frac{z}{4} \wedge \beta = -\frac{x}{2} - \frac{y}{2} + \frac{z}{2} \wedge \gamma = \frac{7x}{4} + \frac{y}{4} - \frac{3z}{4}$

Exercício 14

a) $\alpha = 0 \wedge \beta = 0$

b) $\beta = \alpha$

c) $\beta = -\frac{\alpha}{2}$

d) $\alpha = 0 \wedge \beta = 0 \wedge \gamma = 0$

e) $\alpha = 0 \wedge \beta = 0 \wedge \gamma = 0 \wedge \delta = 0$

Exercício 15

$\alpha = 0 \wedge \beta = 0 \wedge \gamma = 0$

Exercício 16

a) $\alpha = 0 \wedge \beta = 0 \wedge \gamma = 0$

b) $\alpha = 13 \wedge \beta = -9 \wedge \gamma = 2$

Exercício 17

a) $\alpha = 0 \wedge \beta = 0 \wedge \gamma = 0.$
3 vetores linearmente independentes geram $\mathbb{R}^3$*, e.v. dim 3.*

b) $\alpha = 4 \wedge \beta = -7 \wedge \gamma = -3$

Exercício 18

a) $\alpha = 0 \wedge \beta = 0$

b) $x = z \wedge \alpha = y \wedge \beta = z - y.$ $\quad A = \{(x, y, z) \in \mathbb{R}^3 : z = x\}$

c) $\dim(A) = 2$

Exercício 19

a) $x = -2y - 3z \wedge \beta = -\alpha + y + z \wedge \gamma = \alpha - y.$
Vetores linearmente dependentes, $\beta = -\alpha \wedge \gamma = \alpha$*, logo não formam uma base.*

b) $B = \{(x, y, z) \in \mathbb{R}^3 : x = -2y - 3z\}$, $(-2y - 3z, y, z) = y(-2, 1, 0) + z(-3, 0, 1)$.
$B = \langle(-2, 1, 0), (-3, 0, 1)\rangle$, $\dim(B) = 2$.

Exercício 20

a) $\{a = 0, b = 0\} \Longrightarrow \{a, 0, b, 4a + 5b\} = \{0, 0, 0, 0\} \in C$
$\{a_1, 0, b_1, 4a_1 + 5b_1\} + \{a_2, 0, b_2, 4a_2 + 5b_2\} = \{a_1 + a_2, 0, b_1 + b_2, 4a_1 + 4a_2 + 5b_1 + 5b_2\} \in C$
$\lambda\{a, 0, b, 4a + 5b\} = \{a\lambda, 0, b\lambda, 4a\lambda + 5b\lambda\} \in C.$

b) $C = \{(a, 0, b, 4a + 5b) \in \mathbb{R}^4 : a, b \in \mathbb{R}\}$, $(a, 0, b, 4a + 5b) = a(1, 0, 0, 4) + b(0, 0, 1, 5)$.
$C = \langle(1, 0, 0, 4), (0, 0, 1, 5)\rangle$, $\dim(C) = 2$.

c) $\alpha = 2 \wedge \beta = -1$

Exercício 21

a) $T=\{(x,y,z,-y-z)\in\mathbb{R}^4 : x,y,z\in\mathbb{R}\}$, $(x,y,z,-y-z)=x(1,0,0,0)+y(0,1,0,-1)+z(0,0,1,-1)$.
$T=\langle(1,0,0,0),(0,1,0,-1),(0,0,1,-1)\rangle$, $\dim(T)=3$.

b) $x=2y-z \wedge \alpha=\frac{8z}{3}-3y \wedge \beta=2y-\frac{5z}{3}$
$T=\langle(4,5,6),(7,8,9)\rangle$, $\dim(T)=2$ *Ou* $T=\langle(2y-z,y,z)\rangle=\langle(2,1,0),(-1,0,1)\rangle$.

Exercício 22

a) $\beta=0 \wedge \gamma=0 \wedge \alpha=0$.
$P_2=\left\{a+bx+cx^2=\alpha+\beta(1+x)\gamma(1+x^2)\right\}$, *com* $\alpha=a-b-c, \beta=b, \gamma=c$.
$P_2=\langle 1,1+x,1+x^2\rangle$, $\dim(P_2)=3$.

b) $\alpha=0 \wedge \beta=-2 \wedge \gamma=3$

Exercício 23

a) $x=z-y \wedge \alpha=\frac{y}{5}+\frac{z}{5} \wedge \beta=\frac{2z}{5}-\frac{3y}{5}$.
$S=\langle(z-y,y,z)\rangle=\langle(-1,1,0),(1,0,1)\rangle$

b) $(1,0,1)=\frac{1}{5}(1,2,3)+\frac{2}{5}(2,-1,1)$, $(0,1,1)=\frac{2}{5}(1,2,3)-\frac{1}{5}(2,-1,1)$.

c) $\dim(S)=2$.

Exercício 24

Se $i,j,k\neq 0$, $\alpha=0 \wedge \beta=0 \wedge \gamma=0$ *e* $x=i(\alpha+\beta+\gamma)\wedge y=\alpha j \wedge z=k(\alpha+\beta)$.
$\mathbb{R}^3=\langle(1,1,1),(1,0,1),(1,0,0)\rangle$

Se um dos parâmetros i,j,k *for nulo os vetores são linearamente dependentes e não poderão formar uma base de* $\mathbb{R}^3$.

Exercício 25

a) $c=3a \wedge d=0$.
$0=3\times 0 \wedge 0=0 \Longrightarrow (0,0,0,0)\in G$
$c_1+c_2=3(a_1+a_2) \wedge d_1+d_2=0 \Longrightarrow (a_1,b_1,c_1,d_1)+(a_2,b_2,c_2,d_2)\in G$
$\lambda c=3\lambda a \wedge \lambda d=0 \Longrightarrow \lambda(a,b,c,d)\in G$

b) $G=\langle(a,b,3a,0)\rangle=\langle(1,0,3,0),(0,1,0,0)\rangle$
$\dim(S)=2$.

Exercício 26

$\alpha+\beta(x-1)+\gamma\left(x^2-1\right)+\delta\left(x^3-1\right)=0.$

$\beta=0 \wedge \gamma=0 \wedge \alpha=0 \wedge \delta=0$

$\alpha+\beta(x-1)+\gamma\left(x^2-1\right)+\delta\left(x^3-1\right)=ax^3+bx^2+cx+d.$

$\delta=a \wedge \gamma=b \wedge \beta=c \wedge \alpha=d+a+b+c$

Exercício 27

a) $2x+y-2z=k+2.$

$2\times 0+0-2\times 0=k+2, \qquad (0,0,0,0)\in V \Longleftrightarrow 0=k+2 \Longleftrightarrow k=-2$

$2(x_1+x_2)+y_1+y_2-2(z_1+z_2)=0 \Longrightarrow (x_1,y_1,z_1)+(x_2,y_2,z_2)\in V$

$2\lambda x+\lambda y-2\lambda z=0 \Longrightarrow \lambda(x,y,z)\in V$

b) $k=-2$

i) $V=\langle(x,2z-2x,z)\rangle=\langle(1,-2,0),(0,2,1)\rangle, \quad \dim(S)=2.$

ii) $V\cap S=\{x+y-z=0 \wedge 2x+y-2z=0\}=\{y=0\wedge z=x\}=\langle(x,0,x)\rangle, \quad x\in\mathbb{R}$