Exercício 24, alínea a)

$$w = \frac{1}{z} \Leftrightarrow z = \frac{1}{w}$$

$A = \{z \in \mathbb{C}: \quad |z| > |z - i| \quad \}$

$A' = \left\{w \in \mathbb{C}: \quad \left|\frac{1}{w}\right| > \left|\frac{1}{w} - i\right| \quad \right\}$

$$\left|\frac{1}{w}\right| > \left|\frac{1}{w} - i\right| \Leftrightarrow \left|\frac{1}{w}\right| > \left|\frac{1 - wi}{w}\right| \Leftrightarrow \frac{1}{|w|} > \frac{|1 - wi|}{|w|} \Leftrightarrow |1 - wi| < 1 \Leftrightarrow |1 - wi|^2 < 1$$
$$\Leftrightarrow |1 - (x + yi)i|^2 < 1 \Leftrightarrow |1 + y - xi|^2 < 1 \Leftrightarrow (y + 1)^2 + x^2 < 1$$

$A' = \{w = x + yi \in \mathbb{C}: \quad (y + 1)^2 + x^2 < 1 \quad \}$

Exercício 24, alínea b)

$$w = \frac{i + iz}{z + i} \Leftrightarrow wz + wi = i + iz \Leftrightarrow wz - iz = i - wi \Leftrightarrow z = \frac{i - wi}{w - i}$$

$B = \{z \in \mathbb{C}: \quad |z - i| \leq 1 \quad \}$

$B' = \left\{ w \in \mathbb{C}: \quad \left| \frac{i - wi}{w - i} - i \right| \leq 1 \quad \right\}$

$$\left| \frac{i - wi}{w - i} - i \right| \leq 1 \Leftrightarrow \left| \frac{i - wi}{w - i} - \frac{i(w - i)}{w - i} \right| \leq 1 \Leftrightarrow \left| \frac{i - wi - iw - 1}{w - i} \right| \leq 1 \Leftrightarrow \left| \frac{i - 1 - 2wi}{w - i} \right| \leq 1$$

$$\Leftrightarrow |i - 1 - 2wi| \leq |w - i| \Leftrightarrow |i - 1 - 2xi + 2y| \leq |x + yi - i|$$

$$\Leftrightarrow (2x + 1)^2 + (2y - 1)^2 \leq x^2 + (y - 1)^2$$

$$\Leftrightarrow 4x^2 + 4x + 1 + 4y^2 - 4y + 1 \leq x^2 + y^2 - 2y + 1$$

$$\Leftrightarrow 3x^2 + 4x + 3y^2 - 2y \leq -1$$

$$\Leftrightarrow 3\left(x^2 + \frac{4}{3}x + \frac{4}{9}\right) + 3\left(y^2 - \frac{2}{3}y + \frac{1}{9}\right) \leq -1 + \frac{4}{3} + \frac{1}{3}$$

$$\Leftrightarrow 3\left(x - \frac{2}{3}\right)^2 + 3\left(y - \frac{1}{3}\right)^2 \leq \frac{2}{3}$$

$$\Leftrightarrow \left(x - \frac{2}{3}\right)^2 + \left(y - \frac{1}{3}\right)^2 \leq \frac{2}{9}$$

$B' = \left\{ w = x + yi \in \mathbb{C}: \quad \left(x - \frac{2}{3}\right)^2 + \left(y - \frac{1}{3}\right)^2 \leq \frac{2}{9} \quad \right\}$

Exercício 24, alínea c)

$$w=\frac{i+z}{iz+1}\Leftrightarrow wiz+w=i+z\Leftrightarrow wiz-z=i-w\Leftrightarrow z=\frac{i-w}{wi-1}$$

$C=\{z\in\mathbb{C}:\quad z+\overline{z}=i(\overline{z}-z)\quad\}=\{z\in\mathbb{C}:\quad 2Re(z)=2Im(z)\quad\}$

$C'=\left\{w\in\mathbb{C}: Re\left(\frac{i-w}{wi-1}\right)\underbrace{=}_{(1)}Im\left(\frac{i-w}{wi-1}\right)\right\}$

$$\frac{i-w}{wi-1}=\frac{i-w}{wi-1}\times\frac{\overline{wi-1}}{\overline{wi-1}}=\frac{(i-w)\times\left(\overline{i}\ \overline{w}-\overline{1}\right)}{|wi-1|^2}$$

$$=\frac{(i-w)\times(-i\,\overline{w}-1)}{|wi-1|^2}$$

$$=\frac{-i^2\overline{w}-i+iw\overline{w}+w}{|wi-1|^2}$$

$$=\frac{(\overline{w}+w)+(w\overline{w}-1)i}{|wi-1|^2}$$

$$=\frac{2Re(w)+(|w|^2-1)i}{|wi-1|^2}$$

$$=\underbrace{\frac{2Re(w)}{|wi-1|^2}}_{=Re}+\underbrace{\frac{|w|^2-1}{|wi-1|^2}}_{=Im}i$$

$$Re\left(\frac{i-w}{wi-1}\right)\underbrace{=}_{(1)}Im\left(\frac{i-w}{wi-1}\right)\Leftrightarrow\frac{2Re(w)}{|wi-1|^2}=\frac{|w|^2-1}{|wi-1|^2}$$

$$\Leftrightarrow 2Re(w)=|w|^2-1\Leftrightarrow 2x=x^2+y^2-1$$

$$\Leftrightarrow x^2-2x+y^2-1=0\Leftrightarrow (x-1)^2+y^2=2$$

$C'=\{w=x+yi\in\mathbb{C}:(x-1)^2+y^2=2\}$

Exercício 24, alínea d)

$$w = \frac{1+z}{z-1} \Leftrightarrow wz - w = 1 + z \Leftrightarrow wz - z = 1 + w \Leftrightarrow z = \frac{1+w}{w-1}$$

$D = \{z \in \mathbb{C}: |z| < 1 \wedge x > 0\}$

$D' = \{w \in \mathbb{C}: \left|\frac{1+w}{w-1}\right| < 1 \wedge Re\left(\frac{1+w}{w-1}\right) > 0\}$

$\left|\frac{1+w}{w-1}\right| < 1 \Leftrightarrow |w+1| < |w-1| \Leftrightarrow (x+1)^2 + y^2 < (x-1)^2 + y^2 \Leftrightarrow 4x < 0 \Leftrightarrow x < 0$

$$\begin{aligned}\frac{1+w}{w-1} &= \frac{(w+1)(\overline{w}-1)}{|w-1|^2} \\ &= \frac{w\overline{w} - w + \overline{w} - 1}{|w-1|^2} \\ &= \frac{|w|^2 - 2Im(w)i - 1}{|w-1|^2} \\ &= \frac{x^2 + y^2 - 2yi - 1}{|w-1|^2} \\ &= \frac{x^2 + y^2 - 1}{|w-1|^2} - \frac{2yi}{|w-1|^2}\end{aligned}$$

$Re\left(\frac{1+w}{w-1}\right) > 0 \Leftrightarrow \frac{x^2 + y^2 - 1}{|w-1|^2} > 0 \Leftrightarrow x^2 + y^2 - 1 > 0$

$D' = \{w = x + yi \in \mathbb{C}: \quad x < 0 \wedge x^2 + y^2 - 1 > 0 \ \}$