IPG Politécnico da Guarda — Escola Superior de Tecnologia e Gestão	**ENUNCIADO DE AVALIAÇÃO**	**MODELO** PED.018.01

Curso	**Eng. Informática**					*Ano letivo*	2019/2020
Unidade curricular	**Álgebra e Geometria Analítica**						
Ano curricular	1º	*Semestre*	1º	*Data*	14/02/2020	*Duração*	2h 00m

Exame

(Cotação)

1. Considere os complexos $z, w \in \mathbb{C}$.

(2.0) **a)** Calcule os complexos $z \in \mathbb{C}$ que verificam a equação, $z^4 = -\sqrt{2} + \sqrt{6}\, i$.

(2.0) **b)** Determine a imagem do conjunto $A = \{z \in \mathbb{C} : |z-4| > \sqrt{3}\,|z+4|\}$. por meio da transformação,

$$w = \frac{-1}{z+1}.$$

(3.0) **2.** Usando teoria dos determinantes classifique e, se possível, resolva o sistema,

$$\begin{cases} y + 2z + u = 0 \\ x + 2y + 2z = 0 \\ -2x + 4y + 2z + 3u = 1 \end{cases}.$$

3. Seja,

$$A = \begin{bmatrix} 1 & -2 & 1 & 2 \\ 2 & 1 & 0 & 3 \\ 2 & 2 & 1 & 0 \end{bmatrix},$$

a matriz que representa a aplicação linear $L : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ nas bases canónicas de $\mathbb{R}^4$ e $\mathbb{R}^3$.

(2.0) **a)** Calcule $L(1,0,0,-1)$.

(2.0) **b)** Determine a aplicação linear de $L(x,y,z,u)$.

(3.0) **c)** Calcule a dimensão e uma base do $\ker(L)$.

4. Seja o ponto $A = (2,0,0)$ e $\vec{u} = (0,k,-9)$, $\vec{v} = (0,1,k)$ e $\vec{w} = (1,-2,1)$, vetores de $\mathbb{R}^3$, com $k \in \mathbb{R}$.

(2.0) **a)** Determine o volume do paralelepípedo definido pelos vetores $\vec{u}, \vec{v}, \vec{w}$.

(2.0) **b)** Mostre que para $k = 0$ o ponto $(3,-1,1)$ pertence ao plano $\alpha : P = A + s\,\vec{v} + t\,\vec{w}$.

(2.0) **c)** Considerando ainda $k = 0$, determine o ângulo da reta $r : (x,y,z) = (2,0,3) + t\,(0,0,-9)$ com o plano α.

Resolução

1.

(2.0) **a)** $z=-\sqrt{2}+\sqrt{6}i \Longrightarrow \rho=\sqrt{2+6}=\sqrt{8}, \quad \theta=\operatorname{arctg}(-\sqrt{3})=\frac{2\pi}{3} \Longrightarrow z=\sqrt{8}\operatorname{cis}(\frac{2\pi}{3})$

$\sqrt[4]{z}=\sqrt[4]{\sqrt{8}\operatorname{cis}(\frac{2\pi}{3})}=\sqrt[8]{8}\operatorname{cis}\left(\frac{2\pi/3+2k\pi}{4}\right)=\sqrt[8]{8}\operatorname{cis}\left(\frac{(2+6k)\pi}{12}\right),\ k=0,1,2,3.$

$2^{3/8}cis(\frac{\pi}{6}) \quad \vee \quad 2^{3/8}cis(\frac{2\pi}{3}) \quad \vee \quad 2^{3/8}cis(\frac{7\pi}{6}) \quad \vee \quad 2^{3/8}cis(\frac{5\pi}{3})$

$2^{3/8}\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right) \quad \vee \quad 2^{3/8}\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right) \quad \vee \quad 2^{3/8}\left(-\frac{\sqrt{3}}{2}-\frac{i}{2}\right) \quad \vee \quad 2^{3/8}\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)$

(2.0) **b)** $w=\dfrac{-1}{z+1} \Longleftrightarrow z=\dfrac{-w-1}{w} \qquad A=\{z\in\mathbb{C} : |z-4|>\sqrt{3}\,|z+4|\}.$

$|z-4|^2>3|z+4|^2 \Longleftrightarrow \left|\frac{-w-1}{w}-4\right|^2>3\left|\frac{-w-1}{w}+4\right|^2 \Longleftrightarrow |-5w-1|^2>3|3w-1|^2 \Longleftrightarrow$

$(-5x-1)^2+25y^2>3\big((3x-1)^2+9y^2\big) \Longleftrightarrow 25x^2+10x+25y^2+1>27x^2-18x+27y^2+3 \Longleftrightarrow$

$x^2-14x+1+y^2<0 \Longleftrightarrow (x-7)^2+y^2-48<0$

(3.0) **2.**

$$\begin{cases} y+2z+u=0 \\ x+2y+2z=0 \\ -2x+4y+2z+3u=1 \end{cases}, \quad \Delta=\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 2 \\ -2 & 4 & 2 \end{vmatrix}=10 \quad \Longrightarrow \quad \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 2 \\ -2 & 4 & 2 \end{bmatrix}=\begin{bmatrix} -u \\ 0 \\ 1-3u \end{bmatrix}$$

$$x=\frac{\begin{vmatrix} -u & 1 & 2 \\ 0 & 2 & 2 \\ 1-3u & 4 & 2 \end{vmatrix}}{10}=\frac{1}{10}(10u-2), \quad y=\frac{\begin{vmatrix} 0 & -u & 2 \\ 1 & 0 & 2 \\ -2 & 1-3u & 2 \end{vmatrix}}{10}=\frac{1}{5}, \quad z=\frac{\begin{vmatrix} 0 & 1 & -u \\ 1 & 2 & 0 \\ -2 & 4 & 1-3u \end{vmatrix}}{10}=\frac{1}{10}(-5u-1)$$

3.

(2.0) **a)** $L(1,0,0,-1)=(-1,-1,2)$.

(2.0) **b)** A aplicação linear de $L(x,y,z,u)$ é definida por,

$L(x,y,z,u)=A[x\ \ y\ \ z\ \ u]^T=(x-2y+z+2u,2x+y+3u,2x+2y+z)$.

(3.0) **c)**

$$A = \begin{bmatrix} 1 & -2 & 1 & 2 \\ 2 & 1 & 0 & 3 \\ 2 & 2 & 1 & 0 \end{bmatrix} \rightsquigarrow \cdots \rightsquigarrow A' = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

$$A'x = 0 \Longleftrightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Longleftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} u \end{bmatrix}$$

$$\begin{aligned} \ker(L) &= \{(x,y,z,u) \in \mathbb{R}^4 : x = -2u \wedge y = u \wedge z = 2u\} \\ &= \{(-2u, u, 2u, u),\, u \in \mathbb{R}\} = \langle(-2,1,2,1)\rangle, \qquad \dim(\ker(L)) = 1. \end{aligned}$$

4.

(2.0) **a)** O volume do paralelepípedo definido pelos vetores $\vec{u}, \vec{v}, \vec{w}$, é dado por,

$$|\,\vec{u} \wedge \vec{v}\,|\,\vec{w}\,| = \begin{vmatrix} 1 & -2 & 1 \\ 0 & k & -9 \\ 0 & 1 & k \end{vmatrix} = k^2 + 9.$$

(2.0) **b)** Para $k = 0$ o ponto $(3,-1,1)$ pertence ao plano α, definido por,

$(x,y,z) = (2,0,0) + s\,(0,1,0) + t\,(1,-2,1) = (2+t, s-2t, t)$ se e só se,

$$\begin{cases} 3 = 2 + t \\ -1 = s - 2t \\ 1 = t \end{cases} \Longleftrightarrow \begin{cases} 3 = 3 \quad \checkmark \\ s = -1 + 2 = 1 \\ t = 1 \end{cases}.$$

Portanto $(3,-1,1) = (2,0,0) + 1\,(0,1,0) + 1\,(1,-2,1)$.

(2.0) **c)** O ângulo da reta $r: (x,y,z) = (2,0,3) + t\,(0,0,-9)$ com o plano α, onde $\vec{n}_\alpha = (1,0,-1)$, é dado por,

$$\measuredangle(r,\alpha) = \tfrac{\pi}{2} - \arccos\left(\tfrac{|\,\vec{u}\,|\,\vec{n}_\alpha\,|}{\|u\|\,\|\vec{n}_\alpha\|}\right) = \tfrac{\pi}{2} - \arccos\left(\tfrac{|9|}{9\sqrt{2}}\right) = \tfrac{\pi}{2} - \arccos\left(\tfrac{1}{\sqrt{2}}\right) = \tfrac{\pi}{2} - \arccos\left(\tfrac{\sqrt{2}}{2}\right) = \tfrac{\pi}{4}$$