We appreciate the referee's careful review of our manuscript and the insightful feedback provided. Below, we provide detailed responses to each point and explain the corresponding changes made to the manuscript where applicable:

1. (a) To obtain the matter Lagrangian for this model, we use the Schutz formalism, following Ref. [1]. In this framework, the Lagrangian corresponds to the fluid pressure, which depends on the specific entropy and enthalpy. A detailed discussion of this approach is provided in the cited reference. For clarity, we rewrote the description of the Schutz formalism in Section 2.2 and ensured that the references are correctly cited.

   (b) Indeed, if the sound speed were to approach zero, we would encounter a divergence problem. This is why we limit ourselves to small but non-zero equations of state for dust, which is in agreement with the literature. Regarding the ζ field, there are always field configurations that would lead to non-perturbative behavior. However, in the present context, given the initial conditions in equation 3.32, we are considering the adiabatic vacuum for which $\zeta \propto 1/\sqrt{z^2 c_s} \propto \sqrt{c_s}$. Therefore, terms of the form $z^2 c_s^2 \zeta^2$ are $\propto c_s$ while $\zeta^3 \propto \sqrt{c_s}^3$ This means that the perturbative expansion is valid for small sound speeds, as discussed in Ref. [2].

   We have reorganized Sections 4.3 and 4.4 and considered a very small, but non-zero, equation of state for dark matter to analyze the pressureless case.

2. (a) We improved Sec. 2.2 by adding a more detailed explanation of the Schutz formalism and the fluid Lagrangian. We also included a discussion of the lapse function and how it can be defined as a posteriori for simplicity.

   (b) In the improved Sec. 2.2, we have added a brief discussion of the meaning of the effective energy density and pressure in the context of the quantum corrections. In Ref. [1], the authors show that the energy density and pressure appearing in the perturbed equations are only the classical ones computed with the Bohmian trajectory. Therefore, the equation-of-state parameter is always $w = p_d/\rho_d$.

   (c) In the revised Sec. 2.2, we have included the effective equation corresponding to the second Friedmann equation, now presented as Eq. 2.17: $-2\dot{\bar{H}} = \bar{\rho}(1 + w) - 2\rho_q$. Notably, the right-hand side of this equation approaches zero around the bounce. However, as discussed previously in Ref. [1] it has been shown that the energy density and pressure in the second-order Lagrangian for the perturbations are only the classical components evaluated along the Bohmian trajectory. Consequently, this term is never zero, ensuring that the model is free from ghost instability issues.

   (d) In Ref. 33 the author chooses a particular form for the wave function at the bounce. This is indeed not an initial condition, but a choice made to simplify the analysis. The same wave-function propagated to $T \to -\infty$ would have an extra phase factor proportional to the q^2, exactly like a wave function of a free

Gaussian package in 1d. One can always start with a different wave function and define the bounce as the time where the phase factor proportional to q^2 is zero. Again, this is equivalent to a free Gaussian package in 1d. Thus, in short, it has to start at a large but finite negative time, with a Gaussian wave-function containing a phase factor proportional to q^2. Then, the bounce is defined as the time when this phase factor is zero. We have included a brief discussion of this point in the revised manuscript.

(e) In our approach, we use a perturbative method: the background is solved first, and the Bohmian trajectory is derived from the background wave function. Perturbations are then computed conditioned on this Bohmian trajectory. In practice, during the contraction phase, the significant part of the power spectrum is generated in the classical regime, effectively following a semi-classical approach. Quantum effects on the background become relevant only near the bounce, where the relevant modes are on super-Hubble scales. If the background is in a superposition of Gaussian states, the perturbations would naturally reflect this superposition. However, this aspect lies outside the scope of our work, as we are focused here on black hole formation during the contraction phase.

3. In general, when both the background and the perturbations are quantized it is necessary to develop the second order Lagrangian for the perturbations without assuming the background is classical. Once this is done, one can show that for particular choices of the wave function for the whole system (background plus perturbations), the perturbations can be treated as quantum fields evolving around the Bohmian trajectory. We included a brief discussion of this point in the revised manuscript, Sec. 3.3.

4. In Ref. [2], it is shown that for a Friedmann background with scalar perturbations, an appropriate gauge choice ensures the perturbative series remains valid in the bounce model. Nevertheless, this only shows that vacuum initial conditions around an initial Friedmann background are consistent with the perturbative expansion. The referee is correct in pointing out that the perturbative expansion may not be valid for more general initial conditions. In our study, we focus on this simpler model, leaving the exploration of a homogeneous and anisotropic background, specifically Bianchi-I models, for future work. A brief discussion of this possibility has been included in the conclusion.

5. To simplify the presentation, we set the speed of light $c = 1$ in the main text. We have now included a brief discussion of this choice in the revised manuscript. We also clarified the definition of energy density and perturbation in the main text.

We hope that these changes address the referee's concerns and improve the clarity of the manuscript.

References

[1] S. D. P. Vitenti, F. T. Falciano, and N. Pinto-Neto. Quantum cosmological perturbations of generic fluids in quantum universes. *Physical Review D*, 87(10), may 2013.

[2] Sandro Dias Pinto Vitenti and Nelson Pinto-Neto. Large adiabatic scalar perturbations in a regular bouncing universe. *Physical Review D*, 85(2):023524, 2012.