Álgebra Conmutativa (examen 2)

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Part I

Preliminaries. The following exercises are taken from [AM69].

Exercise 5.10. A ring homomorphism $\phi : A \to B$ is said to have the *going-up property* (resp. the *going-down property*) if the conclusion of the going-up theorem (resp. the going-down theorem) holds for B and its subring $f(A)$.

Let $\varphi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the induced morphism of affine schemes.

1. Consider the following statements:

   (a) φ is a closed map.

   (b) ϕ has the going-up property.

   (c) $\varphi : \mathrm{Spec}(B/\mathfrak{q}) \to \mathrm{Spec}(A/\mathfrak{p})$ is surjective for any $\mathfrak{q} \in \mathrm{Spec}(B)$ and $\mathfrak{p} = \mathfrak{q}^c$.

   Show that (a) $\implies$ (b) $\iff$ (c).

2. Consider the following statements:

   (a') φ is an open map.

   (b') ϕ has the going-down property.

   (c') $\varphi : \mathrm{Spec}(B_\mathfrak{q}) \to \mathrm{Spec}(A_\mathfrak{p})$ is surjective for any $\mathfrak{q} \in \mathrm{Spec}(B)$ and $\mathfrak{p} = \mathfrak{q}^c$.

   Show that (a') $\implies$ (b') $\iff$ (c').

Solution.

1. Let $\mathfrak{q} \in \mathrm{Spec}(B)$. Recall that

   - $V(\mathfrak{q})$ is the canonical image of $\mathrm{Spec}(B/\mathfrak{q})$ in $\mathrm{Spec}(B)$.

   Let $\mathfrak{p} = \varphi(\mathfrak{q})$. The following statements are equivalent:

   - $\varphi \circ V(\mathfrak{q})$ is a closed subset of $\mathrm{Spec}(A)$.
   - $\varphi \circ V(\mathfrak{q})$ contains every point of $V(\mathfrak{p})$.
   - $\varphi : \mathrm{Spec}(B/\mathfrak{q}) \to \mathrm{Spec}(A/\mathfrak{p})$ is surjective.

   Notice that

   - (a) is stronger than "the first statement holds for every $\mathfrak{q} \in \mathrm{Spec}(B)$".
   - (b) is equivalent to "the second statement holds for every $\mathfrak{q} \in \mathrm{Spec}(B)$".
   - (c) is equivalent to "the third statement holds for every $\mathfrak{q} \in \mathrm{Spec}(B)$".

   Therefore (a) $\implies$ (b) $\iff$ (c).

2. Let $\mathfrak{q} \in \operatorname{Spec}(B)$. Recall that

   - $B_\mathfrak{q} = \varinjlim B_f$, where f ranges over $B - \mathfrak{q}$.
   - $L(\mathfrak{q}) = \bigcap_{f \notin \mathfrak{q}} D_f$ is the canonical image of $\operatorname{Spec}(B_\mathfrak{q})$ in $\operatorname{Spec}(B)$.

   Let $\mathfrak{p} \in \varphi(\mathfrak{q})$ and $\mathfrak{p}' \in L(\mathfrak{p})$. Since the tensor product is a left adjoint functor,

   - $k(\mathfrak{p}') \otimes_A B_\mathfrak{q} = \varinjlim \left(k(\mathfrak{p}') \otimes_A B_f \right)$, where f ranges over $B - \mathfrak{q}$.
   - $k(\mathfrak{p}') \otimes_A B_\mathfrak{q} \neq 0$ if and only if $k(\mathfrak{p}') \otimes_A B_f \neq 0$ for every $f \notin \mathfrak{q}$.
   - $\mathfrak{p}' \in \varphi \circ L(\mathfrak{q})$ if and only if $\mathfrak{p}' \in \varphi(D_f)$ for every $f \notin \mathfrak{q}$.
   - $\varphi \circ L(\mathfrak{q}) = \bigcap_{f \notin \mathfrak{q}} \varphi(D_f)$

   Thus, the following statements are equivalent:

   - $\varphi(D_f)$ contains every point of $L(\mathfrak{p})$.
   - $\varphi \circ L(\mathfrak{q})$ contains every point of $L(\mathfrak{p})$.
   - $\varphi : \operatorname{Spec}(B_\mathfrak{q}) \to \operatorname{Spec}(A_\mathfrak{p})$ is surjective.

   Notice that

   - (a') is stronger than "the first statement holds for every $\mathfrak{q} \in \operatorname{Spec}(B)$".
   - (b') is equivalent to "the second statement holds for every $\mathfrak{q} \in \operatorname{Spec}(B)$".
   - (c') is equivalent to "the third statement holds for every $\mathfrak{q} \in \operatorname{Spec}(B)$".

   Therefore (a') $\implies$ (b') $\iff$ (c').

Exercise 5.13. Let G be a finite subgroup of the automorphisms of a ring B.

1. Show that B is integral over $A = B^G$, the subring of fixed points of G.
2. Show that each fiber of the canonical map $\varphi : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is a G-orbit, hence finite.

Solution.

1. Let $x \in B$ and $x_i = g_i(x)$, where $g_1, \dots, g_n$ are the elements of G. Then,

   $$0 = (x - x_1) \cdots (x - x_n) = x^n - a_1 x^{n-1} + a_2 x^{n-2} - a_3 x^{n-3} + \dots$$

   Then x is integral over A, because

   - $a_1, \dots, a_n$ are the elementary symmetric polynomials evaluated at $x_1, \dots, x_n$.
   - The action of G merely permutes $x_1, \dots, x_n$.
   - The action of G fixes $a_1, \dots, a_n \in A$.

2. Let $\mathfrak{p} \in \operatorname{Spec}(A)$ and $\mathfrak{q}, \mathfrak{q}' \in \varphi^{-1}(\mathfrak{p})$. Given $x \in \mathfrak{q}$, we deduce that

   - $\mathfrak{q}'$ contains the product $x_1 \cdots x_n$.
   - $\mathfrak{q}'$ contains at least one of the factors $x_1, \dots, x_n$.

   Let $O = G \cdot \mathfrak{p}$ be the G-orbit of $\mathfrak{p}$. Since x is arbitrary,

   - $\mathfrak{q}'$ is contained in $\bigcup O$.
   - $\mathfrak{q}'$ is contained in some $g_i(\mathfrak{q}) \in O$, by the prime avoidance lemma.
   - $\mathfrak{q}' = g_i(\mathfrak{q})$, because φ has the going-up property.

   Therefore, $\varphi^{-1}(\mathfrak{p}) = O$ is a G-orbit, hence finite.

Exercise 5.15. Let A be an integrally closed domain and K its field of fractions. Consider a finite Galois extension L/K with Galois group G. Let B be the integral closure of A in L.

1. Show that B is a G-invariant subset of L.
2. Show that $A = B^G$.
3. Show that the canonical morphism $\varphi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ has finite fibers.

Solution.

1. Let $x \in B$ and $\sigma \in G$. Then $\sigma(x) \in B$, because
   - There exists a monic polynomial $p(t) \in A[t]$ such that $p(x) = 0$.
   - $p(t)$ is G-invariant, because its coefficients are in $K = L^G$.
   - $p \circ \sigma(x) = \sigma \circ p(x) = 0$, hence $\sigma(x)$ is integral over A.
2. This follows from a two-step calculation:
   - Since L/K is Galois, $K = L^G$.
   - Since A is integrally closed, $A = B \cap K = B \cap L^G = B^G$.
3. This also follows from a two-step proof:
   - By the preceding two items, G is a finite group of automorphisms of B, and $A = B^G$.
   - By exercise 5.13, the canonical morphism φ has finite fibers.

Exercise 6.11. Let $\phi : A \to B$ be a ring homomorphism. Suppose further that $\mathrm{Spec}(B)$ is a *Noetherian* topological space, i.e., its closed subsets satisfy the descending chain condition.

1. Show that every closed subset of $\mathrm{Spec}(B)$ is a finite union of *irreducible* closed subsets.
2. Show that the induced morphism $\varphi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is closed if and only if ϕ has the going-up property.

Solution.

1. Let us call a closed subset of $\mathrm{Spec}(B)$ *unwieldy* if it is not a union of finitely many irreducible closed subsets. Suppose $Y \subset \mathrm{Spec}(B)$ is an unwieldy subset. Then,
   - By Noetherian induction, we may assume Y is a *minimal* unwieldy subset.
   - By definition, Y is not irreducible. (Irreducible closed subsets are trivially not unwieldy.)

   We may take two proper closed subsets Y_1, Y_2 whose union is Y. Then,
   - At least one of Y_1, Y_2 must be unwieldy as well.
   - Hence, Y is not a *minimal* unwieldy set, a contradiction.
   - Hence, $\mathrm{Spec}(B)$ has no unwieldy subsets.
2. By exercise 5.10, if φ is a closed map, then ϕ has the going-up property. Thus, we only need to show the reverse implication. Let $Y \subset \mathrm{Spec}(B)$ be a closed subset. Then,
   - Y is the union of finitely many irreducible closed subsets $Y_1, \dots, Y_n$.
   - Each $\varphi(Y_i)$ is closed, because φ has the going-up property.
   - Hence, $\varphi(Y) = \varphi(Y_1) \cup \cdots \cup \varphi(Y_n)$ is closed.
   - Hence, φ is a closed map.

Exercise 7.15. Let A be a Noetherian local ring, $\mathfrak{m}$ its maximal ideal and k its residue field. Show that the following statements about a finitely generated A-module M are equivalent:

(a) M is free.

(b) M is flat.

(c) The canonical map $\varphi : \mathfrak{m} \otimes_A M \to M$ is injective.

(d) $\mathrm{Tor}_1^A(k, M) = 0$

Solution.

- (a) $\implies$ (b): Every free module is automatically flat.
- (b) $\implies$ (c): Consider the short exact sequence

  $$0 \longrightarrow \mathfrak{m} \longrightarrow A \longrightarrow k \longrightarrow 0$$

  Since M is flat, after tensoring with M, we obtain

  $$0 \longrightarrow \mathfrak{m} \otimes_A M \xrightarrow{\varphi} M \longrightarrow k \otimes_A M \longrightarrow 0$$

  Therefore, φ is injective.
- (c) $\implies$ (d): Consider the following fragment of the long exact sequence

  $$\ldots \longrightarrow \mathrm{Tor}_1^A(k, A) \xrightarrow{\pi} \mathrm{Tor}_1^A(k, M) \xrightarrow{\zeta} \mathfrak{m} \otimes_A M \xrightarrow{\varphi} M \longrightarrow \ldots$$

  We then perform diagram chasing:

  - Since φ is injective, ζ is the zero map.
  - Since ζ is the zero map, π is surjective.
  - Since A is free, $\mathrm{Tor}_1^A(k, A) = 0$.
  - Therefore, $\mathrm{Tor}_1^A(k, M) = \rho \circ \mathrm{Tor}_1^A(k, A) = \rho(0) = 0$.
- (d) $\implies$ (a): Let $n \in \mathbb{N}$ be the minimal number of generators of M. We have

  $$0 \longrightarrow L \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

  Consider the following fragment of the long exact sequence

  $$\ldots \longrightarrow \mathrm{Tor}_1^A(k, M) \xrightarrow{\pi} k \otimes_A L \xrightarrow{\zeta} k^n \xrightarrow{\phi} k \otimes_A M \longrightarrow 0$$

  Let $x_1, \ldots, x_m$ be a k-basis of $k \otimes_A M$. Then,

  - By Nakayama's lemma, $x_1, \ldots, x_m$ lifts to a generating set of M. Hence $m = n$.
  - Since ϕ is a surjective, ϕ is an isomorphism.

  We then perform diagram chasing:

  - Since ϕ is an isomorphism, ζ is the zero map.
  - Since ζ is the zero map, π is surjective.
  - By (d), we have $k \otimes_A L = \pi \circ \mathrm{Tor}_1^A(k, M) = \pi(0) = 0$.
  - By Nakayama's lemma, $L = 0$.

Exercise 7.16. Let A be a Noetherian ring and M a finitely generated A-module. Show that the following statements are equivalent:

(a) M is a flat A-module.

(b) $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module, for each point $\mathfrak{p} \in \mathrm{Spec}(A)$.

(c) $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module, for each closed point $\mathfrak{m} \in \mathrm{Spec}(A)$.

Solution. Let $\mathfrak{p} \in \mathrm{Spec}(A)$. Notice that

- $A_{\mathfrak{p}}$ is a Noetherian ring:
  - Every ascending chain of ideals of $A_{\mathfrak{p}}$ contracts to an ascending chain in A.
  - This ascending chain in A is a fortiori stationary, since A is Noetherian.
  - This ascending chain in A extends back to the *original* chain in $A_{\mathfrak{p}}$.
  - Hence the original chain $A_{\mathfrak{p}}$ is also stationary.
- $M_{\mathfrak{p}}$ is a finitely generated $A_{\mathfrak{p}}$-module:
  - Any (finite) generating set of M extends to a (finite) generating set of $M_{\mathfrak{p}}$.
- $M_{\mathfrak{p}}$ is free $\iff$ $M_{\mathfrak{p}}$ is flat, by the previous exercise.

Consider an arbitrary A-monomorphism, i.e., an exact sequence of the form

$$0 \longrightarrow N \xrightarrow{\phi} P$$

After tensoring with M, we have a short exact sequence

$$0 \longrightarrow L \longrightarrow M \otimes_A N \longrightarrow M \otimes_A P$$

For each $\mathfrak{p} \in \mathrm{Spec}(A)$, we also have the short exact sequence

$$0 \longrightarrow L_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} P_{\mathfrak{p}}$$

Then we may rephrase the original statements as follows:

(a) $L = 0$, no matter what ϕ is.

(b) $L_{\mathfrak{p}} = 0$ for each point $\mathfrak{p} \in \mathrm{Spec}(A)$, no matter what ϕ is.

(c) $L_{\mathfrak{m}} = 0$ for each closed point $\mathfrak{m} \in \mathrm{Spec}(A)$, no matter what ϕ is.

These statements are now obviously equivalent, since being zero is a local property.