Álgebra Conmutativa (examen 1)

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Parte I

Ejercicio 1.

a) Muestre que, si m, n son relativamente primos, entonces $(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z})$ es trivial.

b) Sean A un anillo, $\mathfrak{a} \subset A$ un ideal y M un A-módulo. Pruebe que $(A/\mathfrak{a}) \otimes_A M$ es isomorfo a $M/\mathfrak{a}M$.

Solución.

a) Por conveniencia, abreviaremos $M_n = \mathbb{Z}/n\mathbb{Z}$ y $M = M_m \otimes M_n$. Por el algoritmo de Euclides, existen $a, b \in \mathbb{Z}$ tales que $am + bn = 1$. Entonces la transformación $\mathbb{Z}$-bilineal universal $\varphi : M_m \times M_n \to M$ es idénticamente cero, porque

$$\varphi(x, y) = (am + bn) \cdot \varphi(x, y) = a \cdot \varphi(mx, y) + b \cdot \varphi(x, ny) = a(0) + b(0) = 0$$

Por la propiedad universal de φ, el único $\mathbb{Z}$-homomorfismo $\tilde{\varphi} : M \to M$ tal que el diagrama

$$\begin{tikzcd} M_m \times M_n \arrow[r, "\varphi"] \arrow[dr, "\varphi"'] & M \arrow[d, dashed, "\tilde{\varphi}"] \\ & M \end{tikzcd}$$

conmuta es el homomorfismo identidad de M. Pero el homomorfismo cero también cumple con este requerimiento. Por lo tanto, M es un $\mathbb{Z}$-módulo cuyos homomorfismos cero e identidad coinciden, y esto exige que M sea el $\mathbb{Z}$-módulo trivial.

b) Sea N un A-módulo arbitrario. Por la propiedad universal del producto tensorial, cualquiera de los siguientes datos proporciona información canónicamente equivalente a los demás:

- Una aplicación A-lineal $f : (A/\mathfrak{a}) \otimes_A M \to N$.
- Una aplicación A-bilineal $f : A/\mathfrak{a} \times M \to N$.
- Una aplicación A-bilineal $f : A \times M \to N$ tal que $f(r, m) = 0$ para todo $r \in \mathfrak{a}$, $m \in M$.
- Una aplicación A-lineal $f : A \otimes_A M \to N$ tal que $f(r \otimes m) = 0$ para todo $r \in \mathfrak{a}$, $m \in M$.
- Una aplicación A-lineal $f : M \to N$ tal que $f(rm) = 0$ para todo $r \in \mathfrak{a}$, $m \in M$.
- Una aplicación A-lineal $f : M \to N$ tal que $\mathfrak{a}M \subset \ker f$.
- Una aplicación A-lineal $f : M/\mathfrak{a}M \to N$.

Al usar la misma letra f para datos ostensiblemente diferentes, enfatizamos que las diferencias entre estos datos son meramente cosméticas, ya que existe una manera canónica de “convertir el formato” sin pérdida o ganancia de información.

Entonces el producto tensorial $(A/\mathfrak{a}) \otimes_A M$ es canónicamente isomorfo a $M/\mathfrak{a}M$.

Ejercicio 2. Sean A un anillo local, M, N dos A-módulos finitamente generados y $P = M \otimes_A N$. Muestre que P es trivial si y sólo si alguno de los factores M, N es trivial.

Solución. Sean $\mathfrak{m} \subset A$ el ideal maximal y $k = A/\mathfrak{m}$ el cuerpo residual. Dado un A-módulo M, denotemos por M' el k-espacio vectorial $M' = M/\mathfrak{m}M$. Por el ejercicio anterior,

$$P' \cong k \otimes_A P \cong (k \otimes_A M) \otimes_k (k \otimes_A N) \cong M' \otimes_k N'$$

En la categoría de k-espacios vectoriales, se debe cumplir $\dim_k P' = \dim_k M' \cdot \dim_k N'$.

Veamos que P es un A-módulo finitamente generado. De hecho, si $X \subset M$ e $Y \subset N$ son subconjuntos finitos que generan a M, N, respectivamente, entonces $\{x \otimes y : x \in X, y \in Y\}$ es un subconjunto finito que genera a P. Entonces, por el lema de Nakayama, P es trivial si y sólo si P' es trivial, si y sólo si alguno de M', N' es trivial, si y sólo si alguno de M, N es trivial.

Ejercicio 3. Sean A un anillo, M un A-módulo finitamente generado y $\phi : M \to A^n$ un A-homomorfismo sobreyectivo. Pruebe que $K = \ker(\phi)$ también es un A-módulo finitamente generado.

Solución. Puesto que ϕ es sobreyectivo, existen elementos $x_1, \dots, x_n \in M$ tales que cada $e_i = \phi(x_i)$ es el i-ésimo elemento de la base canónica de A^n. Definamos el A-homomorfismo $\psi : A^n \to M$ por $\psi(e_i) = m_i$ y denotemos por $M' = \psi(A^n)$ su imagen. Entonces ψ rompe la sucesión exacta corta

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & M \arrow[r, "\phi"'] & A^n \arrow[r] \arrow[l, bend right, dashed, "\psi"'] & 0 \end{tikzcd}$$

Tomemos un elemento arbitrario $x \in M$ y pongamos $x' = \psi \circ \phi(x)$. Por construcción, $x' \in M'$, mientras que $x - x' \in K$. Entonces $M = M' + K$. Además, tenemos $M' \cap K = 0$, porque la restricción $\psi \mid M'$ es un isomorfismo de A-módulos. Entonces $M = M' \oplus K$. Finalmente, como M es finitamente generado, el cociente $K \cong M/M'$ con mayor razón aún es finitamente generado.

Ejercicio 4. Sean A un anillo local, $\mathfrak{m} \subset A$ su ideal maximal y $k = A/\mathfrak{m}$ su cuerpo residual.

a) Pruebe que todo A-módulo proyectivo finitamente generado es libre.

b) Sea $\varphi : M \to N$ un A-homomorfismo entre dos A-módulos finitamente generados. Denotando por un apóstrofe la aplicación del funtor $k \otimes_A -$, suponga que $\varphi' : M' \to N'$ es un isomorfismo de k-espacios vectoriales. Muestre que, si N es un A-módulo libre, entonces M también es un A-módulo libre y φ es un A-isomorfismo.

c) Suponga, además, que A es un dominio local con cuerpo de fracciones K. Denotando por una viñeta la aplicación del funtor $K \otimes_A -$, muestre que un A-módulo finitamente generado es libre si y solamente si $\dim_k M' = \dim_K M^\bullet$.

Solución. Sea $e_1, \dots, e_n$ la base estándar de A^n. Dados un A-módulo M y un elemento $x \in M$, denotemos por x' la imagen de x en M' y denotemos por $x^\bullet$ la imagen de x en $M^\bullet$.

a) Sea M un módulo proyectivo finitamente generado y sean $x_1, \dots, x_n \in M$ una cantidad minimal de generadores de M. Definamos el homomorfismo $\pi : A^n \to M$ por $\phi(e_i) = x_i$. Por construcción, π es sobreyectivo, así que existe una solución $\psi : M \to A^n$ al problema de levantamiento

$$\begin{tikzcd} & A^n \arrow[d, two heads, "\pi"] \\ M \arrow[ur, dashed, "\psi"] \arrow[r, "\mathrm{id}"] & M \end{tikzcd}$$

Pongamos $K = \ker \pi$. Entonces ψ rompe la sucesión exacta

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & A^n \arrow[r, "\pi"'] & M \arrow[r] \arrow[l, bend right, dashed, "\psi"'] & 0 \end{tikzcd}$$

Entonces $A^n \cong K \oplus M$. Por ende, $k^n \cong K' \oplus M'$. Las clases $x'_1, \dots, x'_n$ forman una base de M', así que $M' \cong k^n$. Entonces $K' = 0$. Por ende, $K = 0$. Por ende, $M \cong A^n$ es un A-módulo libre.

b) Sean $x_1, \dots, x_n$ una cantidad minimal de generadores de M y sea $y_i = \varphi(x_i)$. Entonces,

- Por el lema de Nakayama, $x_1', \dots, x_n'$ es una base de M'.
- Puesto que φ' es un isomorfismo, $y_1', \dots, y_n'$ es una base de N'.
- Por el lema de Nakayama, $y_1, \dots, y_n$ es un conjunto minimal de generadores de N.
- Puesto que N es libre, $y_1, \dots, y_n$ es una base de N.

Sean $a_1, \dots a_n \in A$ tales que $a_1 x_1 + \dots + a_n x_n = 0$. Por A-linealidad, tenemos

$$\varphi(a_1 x_1 + \dots + a_n x_n) = a_1 y_1 + \dots + a_n y_n = 0$$

Como $y_1, \dots, y_n$ son linealmente independientes, tenemos $a_i = 0$ para todo $i = 1, \dots, n$. Entonces los generadores $x_1, \dots, x_n$ son una base de M. Por lo tanto, M es libre y φ es un A-isomorfismo.

c) Sean $x_1, \dots, x_n$ una cantidad minimal de generadores de M. Puesto que A no tiene divisores de cero, toda combinación K-lineal no trivial de la forma

$$\frac{p_1}{q_1} x_1^\bullet + \dots + \frac{p_n}{q_n} x_n^\bullet$$

se puede reescalar para obtener una combinación A-lineal no trivial

$$a_1 x_1 + \dots + a_n x_n$$

Entonces las siguientes proposiciones son equivalentes:

- M es un A-módulo libre.
- $x_1, \dots, x_n$ son A-linealmente independientes.
- $x_1^\bullet, \dots, x_n^\bullet$ son K-linealmente independientes.
- $\dim_K M^\bullet = n$

Por el lema de Nakayama, siempre se cumple $\dim_k M' = n$, sea M libre o no. Por lo tanto, M es un A-módulo libre si y sólo si $\dim_k M' = \dim_K M^\bullet$.

Ejercicio 5. Sean A un anillo conmutativo y M un A-módulo noetheriano. Recuerde que un A-módulo M se dice *noetheriano* si toda cadena ascendente de A-submódulos de M eventualmente se estabiliza. Muestre que todo A-endomorfismo sobreyectivo $\varphi : M \to M$ es un A-isomorfismo.

Solución. Sea $M_n = \ker(\varphi^n)$. Existe un instante $n \in \mathbb{N}$ en el cual la cadena ascendente

$$0 = M_0 \subset M_1 \subset M_2 \subset M_3 \subset \dots$$

se estabiliza. Restringiendo φ^n al A-submódulo $M_{2n} = M_n$, obtenemos un endomorfismo $\varphi^n : M_n \to M_n$ que es simultáneamente cero y sobreyectivo, lo cual sólo es posible si $M_n = 0$. Como la cadena se estabiliza en cero, en particular, tenemos $\ker \varphi = M_1 = 0$. Así pues, φ es un isomorfismo.

Parte II

Exercise 1. Let f be a nonconstant polynomial over a field k. By Gauss' lemma, f admits an essentially unique prime factorization $f = f_1^{r_1} f_2^{r_2} f_3^{r_3} \cdots f_n^{r_n}$. Consider $V(f)$ as a subset of a suitable affine space $\mathbb{A}^m$, regarded as the spectrum of the polynomial ring $R = k[X_1, \dots, X_m]$.

a) Show that $\operatorname{rad}(f) = (f_1 f_2 f_3 \cdots f_n)$.

b) Show that the irreducible components of $V(f)$ are precisely $V(f_i)$ for every $i = 1, \dots, n$.

Remark. It is *absolutely* not necessary to require k to be algebraically closed.

Solution.

a) Given a polynomial $g \in R$, we have

$$g \in \operatorname{rad}(f) \iff \exists n \in \mathbb{N} : g^k \in (f) \iff \forall i : g \in (f_i) \iff g \in (f_1 \cdots f_n)$$

Thus, $\operatorname{rad}(f)$ and $(f_1 \cdots f_n)$ are one and the same ideal of R.

b) Given a point $\mathfrak{p} \in \mathbb{A}^m$, i.e., a prime ideal $\mathfrak{p} \subset R$, we have

$$\mathfrak{p} \notin V(f) \iff f \notin \mathfrak{p} \iff \forall i : f_i \notin \mathfrak{p} \iff \forall i : \mathfrak{p} \notin V(f_i)$$

Thus, $V(f)$ is the union of the subvarieties $V(f_i)$. Each $V(f_i)$ is irreducible, because it has a generic point, namely, the prime ideal (f_i). Moreover, this generic point is not contained in any of the other $V(f_j)$, hence the subvarieties $V(f_i)$ are the irreducible components of $V(f)$.

Exercise 2. Fix a perfect field k and consider the affine plane $\mathbb{A}^2$, whose standard coordinates functions are $T_1, T_2 \in k[\mathbb{A}^2]$. Identify $\mathbb{A}^1$ with the T_1 axis and let $f \in k[\mathbb{A}^1]$ be an arbitrary polynomial on just T_1.

a) Show that $X_1 = V(T_2 - f)$ is a plane curve isomorphic to $\mathbb{A}^1$.

b) Show that $X_2 = V(1 - fT_2)$ is a plane curve isomorphic to a cofinite open subset of $\mathbb{A}^1$.

c) Show that X_1, X_2 are *not* isomorphic.

Remark. The same absolute Galois group joke as in the previous exercise.

Solution. Let $K = k(T_1)$ be the field of fractions of the unique factorization domain $R = k[T_1]$.

a) Regard $T_2 - f$ as a polynomial on just T_2. By Gauss' lemma, $T_2 - f$ is irreducible over R, because it is primitive over R as well as irreducible over K. Therefore, X_1 is an irreducible plane curve, and its coordinate ring is $k[X_1] = k[\mathbb{A}^2]/(T_2 - f)$.

Define the k-algebra homomorphism $\varphi : k[\mathbb{A}^2] \to k[\mathbb{A}^1]$ by $\varphi(T_1) = T_1$ and $\varphi(T_2) = f$. Since φ is a surjection, it induces a closed embedding $\varphi^\# : \mathbb{A}^1 \to \mathbb{A}^2$. Moreover, $\ker(\varphi) = (T_2 - f)$, hence, by the first isomorphism theorem, φ descends to a k-algebra isomorphism $\tilde{\varphi} : k[X_1] \to k[\mathbb{A}^1]$ such that

$$\begin{tikzcd}
k[\mathbb{A}^2] \arrow[d, "\pi"'] \arrow[dr, "\varphi"] & \\
k[X_1] \arrow[r, dashed, "\tilde{\varphi}"'] & k[\mathbb{A}^1]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{A}^2 \\
\mathbb{A}^1 \arrow[ur, "\varphi^\#"] \arrow[r, dashed, "\tilde{\varphi}^\#"'] & X_1 \arrow[u, "\pi^\#"']
\end{tikzcd}$$

Thus the induced map $\tilde{\varphi}^\# : \mathbb{A}^1 \to X_1$ is the sought isomorphism.

b) Using the same argument as in the preceding item, we deduce that X_2 is an irreducible plane curve, and its coordinate ring is $k[X_2] = k[\mathbb{A}^2]/(1 - fT_2)$.

Define the k-algebra homomorphism $\psi : k[\mathbb{A}^2] \to k[\mathbb{A}^1]_f$ by $\varphi(T_1) = T_1$ and $\psi(T_2) = 1/f$. Since ψ is a surjection, it induces a closed embedding $\psi^\# : D_f \to \mathbb{A}^2$, where $D_f \subset \mathbb{A}^1$ is the open subset where f does not vanish. Moreover, $\ker(\psi) = (1 - fT_2)$, hence, by the first isomorphism theorem, φ descends to a k-algebra isomorphism $\tilde{\psi} : k[X_2] \to k[\mathbb{A}^1]_f$ such that

$$\begin{tikzcd} k[\mathbb{A}^2] \arrow[d, "\pi"'] \arrow[dr, "\psi"] & \\ k[X_2] \arrow[r, dashed, "\tilde{\psi}"'] & k[\mathbb{A}^1]_f \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{A}^2 \\ D_f \arrow[ur, "\psi^\#"] \arrow[r, dashed, "\tilde{\psi}^\#"'] & X_2 \arrow[u, "\pi^\#"'] \end{tikzcd}$$

Thus the induced map $\tilde{\psi}^\# : D_f \to X_1$ is the sought isomorphism.

Finally, notice that D_f is a proper subset of $\mathbb{A}^1$, even if f has no roots in k. To see why, recall that the closed points of $\mathbb{A}^1$ are the absolute Galois orbits in the algebraic closure $\bar{k}$, and we do have such orbits for the roots of f in $\bar{k}$.

c) The k-algebras $k[X_1]$ and $k[X_2]$ cannot possibly be isomorphic, because the former has no invertible elements besides those of the ground field k, whereas the latter also has an inverse for f. Therefore, X_1, X_2 cannot possibly be isomorphic as schemes over $\operatorname{Spec} k$.

Exercise 3. Fix a ground field k and identify $\mathbb{A}^4$ with the space 2×2 matrices

$$A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

For convenience, define the following abbreviations:

$$p = x^2 + yz, \qquad q = w^2 + yz, \qquad d = xw - yz, \qquad t = x + w$$

Recall that A is nilpotent if the following equivalent conditions hold:

- A squares to zero, i.e., $A \in V(\mathfrak{a})$, where $\mathfrak{a} = (p, q, yt, zt)$
- A's characteristic polynomial is λ^2, i.e., $A \in V(\mathfrak{p})$, where $\mathfrak{p} = (d, t)$

Thus, we expect the ideals $\mathfrak{a}, \mathfrak{p} \subset k[\mathbb{A}^4]$ to have the same radical.

a) Show that $\mathfrak{p}$ is a prime ideal of $k[\mathbb{A}^4]$.

b) Show that $\sqrt{\mathfrak{a}} = \mathfrak{p}$.

c) Show that $V(\mathfrak{p})$ is an irreducible closed affine cone in $\mathbb{A}^4$.

Remark. The Galois group joke is getting tired, sorry. But it is still true.

Solution.

a) Regard $V(x + w)$ as a copy of $\mathbb{A}^3$ with standard coordinate functions x, y, z. Then $V(\mathfrak{p})$ is the quadric hypersurface $V(p)$ of $\mathbb{A}^3$. Thus, $\mathfrak{p}$ is a prime ideal of $k[\mathbb{A}^4]$ if and only if p is irreducible. Once again, we appeal to Gauss' lemma: p is primitive over $k[y, z]$ and irreducible over $k(y, z)$, therefore it has to be irreducible over $k[y, z]$.

b) To show that $\mathfrak{a} \subset \mathfrak{p}$, notice that

- $p = xt - d$ and $q = wt - d$ are elements of $\mathfrak{p}$.
- yt and zt are self-evidently elements of $\mathfrak{p}$.

To show that $\mathfrak{p} \subset \sqrt{\mathfrak{a}}$, notice that

- $d^2 = pq - yzt^2$, hence $d^2 \in \mathfrak{a}$, hence $d \in \sqrt{\mathfrak{a}}$.
- $t^2 = p + q + 2d$, hence $t^2 \in (\mathfrak{a}, d)$, hence $t \in \sqrt{\mathfrak{a}}$.

c) Since $\mathfrak{p}$ is a prime ideal of $k[\mathbb{A}^4]$, it follows that $V(\mathfrak{p})$ is an irreducible closed subscheme of $\mathbb{A}^4$. It only remains to show that $V(\mathfrak{p})$ is a cone, i.e., it is invariant under the group scheme action of G_m on $\mathbb{A}^4$ by homogeneously rescaling the standard coordinates.

Recall that, classically, a subset V of a G-space X is G-invariant if $G \cdot V = V$. (In general, $G \cdot V$ can be larger than V.) However, when working with schemes, it is usually not a good idea to work with the points of a space (since set-theoretic points often behave in rather strange ways), but rather with morphisms into or out of the space. Thus, we shall consider the following maps:

- $\varphi : G \times X \to X$, the group action itself,
- $\pi : G \times X \to X$, the projection onto the second factor,

and define V to be G-invariant if $\varphi \circ \pi^{-1}(V) = V$. In our particular situation, we have

- $G = G_m$ is the affine line $\mathbb{A}^1$ minus the origin, i.e., $\operatorname{Spec} k[\lambda, \lambda^{-1}]$,
- $X = \mathbb{A}^4$ and $V = V(\mathfrak{p})$,
- $\varphi : k[\mathbb{A}^4] \to k[\mathbb{A}^4] \otimes_k k[G_m]$, defined by $\varphi(x) = \lambda x$, $\varphi(y) = \lambda y$, $\varphi(z) = \lambda z$, $\varphi(w) = \lambda w$,
- $\pi : k[\mathbb{A}^4] \to k[\mathbb{A}^4] \otimes_k k[G_m]$, defined by $\pi(x) = x$, $\pi(y) = y$, $\pi(z) = z$, $\pi(w) = w$,

and we need to show that $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$, where $\mathfrak{q}$ is the ideal generated by $\pi(\mathfrak{p})$. Then,

- $\mathfrak{q} = (d, t)$, because $\mathfrak{p} = (d, t)$ and π is an extension of scalars map,
- $\mathfrak{q} = (\lambda^2 d, \lambda t)$, because λ is a unit of $k[G_m]$, hence a unit of $k[\mathbb{A}^4] \otimes_k k[G_m]$,
- $\varphi(d) = \lambda^2 d$, because d is homogeneous of degree 2,
- $\varphi(t) = \lambda t$, because t is homogeneous of degree 1,
- $\varphi(\mathfrak{p})$ generates $\mathfrak{q}$, because $\varphi(d)$ and $\varphi(t)$ generate $\mathfrak{q}$.
- $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$, because φ is injective.

Therefore $V(\mathfrak{p})$ is an affine cone.

Exercise 4. Let X be an affine scheme and let $p \in X$ be any point. Construct the 1-to-1 correspondence between the prime ideals of the local ring $\mathcal{O}_{X,p}$ and the closed irreducible subsets of X containing p.

Remark. This is essentially the proof of Atiyah and MacDonald's proposition 3.11, specialized to the case when we localize at a prime ideal. However, as countless graduate students have complained about on the Internet, the authors' writing style leans too much on the terse side for the reader's good, so I have chosen to explain the proof's key insights using *English words*.

Solution. Since X is an affine scheme, there exists a ring A whose prime spectrum is isomorphic to X as a locally ringed space. Under this identification, p is a prime ideal $\mathfrak{p} \subset A$, and $\mathcal{O}_{X,p}$ is the localization $A_\mathfrak{p}$ at this prime ideal.

Let $\varphi : A \to A_\mathfrak{p}$ be the localization map. Since φ maps every denominator $g \notin \mathfrak{p}$ to a unit of $A_\mathfrak{p}$, every fraction $f/g \in A_\mathfrak{p}$ is associate to the image of its own numerator, i.e., $\varphi(f) = f/1$. Thus, every ideal of $A_\mathfrak{p}$ contracts to, and is the extension of its own *ideal of numerators* in A. In particular, this means that each prime ideal of $A_\mathfrak{p}$ contracts to a distinct prime ideal of A, which then must be contained in $\mathfrak{p}$.

Conversely, we need to ask under what circumstances $\mathfrak{a} \subset A$ is an ideal of numerators. Since $\mathfrak{a}$ cannot possibly be the ideal of numerators of any ideal of $A_\mathfrak{p}$ other than its own extension $\mathfrak{a}_\mathfrak{p}$, we deduce that $\mathfrak{a}$ is an ideal of numerators if and only if $\mathfrak{a}$ contains every numerator of $\mathfrak{a}_\mathfrak{p}$, if and only if every denominator is cancellable (i.e., not a zero divisor) in the quotient ring $A/\mathfrak{a}$. In particular, a prime ideal $\mathfrak{q} \subset A$ is an ideal of numerators if and only if it is contained in $\mathfrak{p}$.

Reinterpreting our results geometrically, the morphism $\varphi^\# : \operatorname{Spec} A_\mathfrak{p} \to \operatorname{Spec} A$ is an embedding whose image consists of the generic points $\mathfrak{q} \in \operatorname{Spec} A$ of closed irreducible subsets $V(\mathfrak{q})$ containing $\mathfrak{p}$. Therefore, the prime ideals of $\mathcal{O}_{X,p}$ correspond to the closed irreducible subsets of X containing p.

Exercise 5. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X, and let $\mathcal{H}$ be its *presheaf* image.

a) Show that the presheaf inclusion $\iota : \mathcal{H} \to \mathcal{G}$ gives rise to a sheaf inclusion $\iota^+ : \mathcal{H}^+ \to \mathcal{G}$.

b) Show that φ is surjective if and only if every local section $s \in \mathcal{G}(U)$ arises from gluing the images of local sections $t_\alpha \in \mathcal{F}(U_\alpha)$ defined on an open cover $\{U_\alpha\}$ of U.

c) Show that, even if φ is surjective, $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ need not be surjective.

Solution.

a) The existence and uniqueness of ι^+ follows from the universal property of $\mathcal{H}^+$. By construction, ι^+'s stalks are those of ι. We know that ι's stalks are injective, because they are direct limits of injective maps $\iota(U) : \mathcal{H}(U) \to \mathcal{G}(U)$. Therefore, ι^+ is stalkwise injective.

   Suppose $t, t' \in \mathcal{H}^+(U)$ have the same image in $\mathcal{G}(U)$. For each point $p \in U$, the stalks t_p, t'_p have the same image in $\mathcal{G}_p$, hence they are equal. In other words, t, t' agree on an open $U_p \subset U$ containing p. By the gluing axiom, t, t' agree on the union of the opens U_p, which is all of U, of course. Therefore, ι^+ is not just stalkwise injective, but also injective on each open $U \subset X$.

b) Given a local section $s \in \mathcal{G}(U)$, the following statements are equivalent:

   - For each point $p \in U$, there is a stalk $t_p \in \mathcal{F}_p$ such that $\varphi_p(t_p) = s_p$
   - For each point $p \in U$, there is a section $t \in \mathcal{F}(V)$, defined on a neighborhood $V \subset X$ of p, such that $\varphi(V)(t)$ agrees with s on a nested neighborhood $W \subset U \cap V$ of p.
   - For each point $p \in U$, there is a section $t \in \mathcal{F}(W)$, defined on a nested neighborhood $W \subset U$ of p, such that $\varphi(W)(t)$ agrees with the restriction $s \mid W$.
   - There exist sections $t_\alpha \in \mathcal{F}(U_\alpha)$, defined on an open cover $\{U_\alpha\}$ of U, such that each $\varphi(U_\alpha)(t_\alpha)$ agrees with the restriction $s \mid U_\alpha$.

   Hence the following statements are equivalent:

   - φ is surjective, i.e., $\mathcal{H}^+ = \mathcal{G}$.
   - φ is stalkwise surjective, i.e., $\mathcal{H}_p = \mathcal{G}_p$.
   - φ is locally surjective, i.e., every local section $s \in \mathcal{G}(U)$ is obtained by gluing the images of local sections $t_\alpha \in \mathcal{F}(U_\alpha)$ defined on an open covering $\{U_\alpha\}$ of U.

c) Let X be any complex manifold, $\mathcal{F} = \mathcal{O}_X$ its structure sheaf, $\mathcal{G} = \mathcal{O}_X^\star$ its group sheaf of units, and $\varphi : \mathcal{F} \to \mathcal{G}$ the morphism of sheaves induced by the covering map $\pi : \mathbb{C} \to \mathbb{C}^\star$, $\pi(z) = e^z$.

   A preimage of $s \in \mathcal{G}(U)$ under $\varphi(U)$ is a solution of the following lifting problem:

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\pi"] \\ U \arrow[r, "s"] \arrow[ur, dashed, "t"] & \mathbb{C}^\star \end{tikzcd}$$

   Since $\mathbb{C}$ is simply connected, the lift $t \in \mathcal{F}(U)$ exists if and only if $s_\star : \pi_1(U) \to \pi_1(\mathbb{C}^\star)$ is the trivial group homomorphism. In particular, when U is simply connected, t is guaranteed to exist no matter what s is. Even if U is not simply connected, we may cover it with simply connected opens $U_\alpha \subset U$, on each of which a lift $t_\alpha \in \mathcal{F}(U_\alpha)$ exists. Therefore, φ is a surjective morphism of sheaves.

   However, unless X has dimension zero, it also has a neighborhood $U \subset X$ biholomorphic to $D \times Y$, where $D \subset \mathbb{C}^\star$ is the punctured unit disc. Let $s \in \mathcal{G}(U)$ be the coordinate of D, and $\gamma : S^1 \to U$ any loop that projects to the circle of radius $1/2$ in D. Then $s_\star(\gamma)$ is not the identity element of $\pi_1(\mathbb{C}^\star)$, which implies that there is no lift $t \in \mathcal{F}(U)$. Hence $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is not surjective.