Variedades Complejas (tarea 7)

Eduardo León (梁遠光)

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Part I

Preliminaries. Let $\pi : E \to M$ be a complex vector bundle. Let $g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(k, \mathbb{C})$ be the transition functions of E with respect to an open cover $\{U_\alpha\}$ of M.

Exercise 1. Find the transition functions of $E^\star, \bar{E}$ with respect to the same open cover of M.

Solution.

a) Let $\sigma_\alpha : U_\alpha \to \mathbb{C}^k$ and $\xi_\alpha : U_\alpha \to \mathbb{C}^k$ be the local representatives of two local sections σ, ξ of $E, E^\star$, respectively. Consider the bilinear pairing $f = \langle \xi, \sigma \rangle$, whose value is

$$f(x) = \xi_\alpha(x)^T \cdot \sigma_\alpha(x) = \xi_\beta(x)^T \cdot \sigma_\beta(x) = \xi_\beta(x)^T \cdot g_{\alpha\beta}(x) \cdot \sigma_\alpha(x)$$

This equation holds no matter what σ is. Then ξ_α, ξ_β are related by

$$\xi_\alpha(x)^T = \xi_\beta(x)^T \cdot g_{\alpha\beta}(x) \implies \xi_\beta(x) = \left[g_{\alpha\beta}(x)^{-1}\right]^T \cdot \xi_\alpha(x)$$

b) Recall that $E, \bar{E}$ are isomorphic as smooth vector bundles. The only difference between them is the complex structure on the fibers; however, this has no bearing whatsoever on whether a local smooth section is well defined. Therefore, we may reinterpret a local section σ of E as a local section $\bar{\sigma}$ of $\bar{E}$, and vice versa.

Let $\sigma_\alpha : U_\alpha \to \mathbb{C}^k$ be the local representatives of σ. Since the rescaling action on $\bar{E}$ is related to the original rescaling action on E by complex conjugation, the natural local representatives of $\bar{\sigma}$ are the componentwise complex conjugates $\bar{\sigma}_\alpha : U_\alpha \to \mathbb{C}^k$. Then $\bar{\sigma}_\alpha, \bar{\sigma}_\beta$ are related by

$$\bar{\sigma}_\beta(x) = \overline{g_{\alpha\beta}(x) \cdot \sigma_\alpha(x)} = \bar{g}_{\alpha\beta}(x) \cdot \bar{\sigma}_\alpha(x)$$

Hence, the transition functions of the $\bar{E}$ are the componentwise complex conjugates $\bar{g}_{\alpha\beta}$.

Exercise 2. Show that a Hermitian structure on E induces a canonical isomorphism $\bar{E} \cong E^\star$.

Solution. A Hermitian structure on E is a smoothly chosen complex inner product on each fiber E_x. Such complex inner products admit several equivalent equivalent presentations:

- As sesquilinear maps $H_x : E_x \times E_x \to \mathbb{C}$, by definition.
- As $\mathbb{C}$-bilinear maps $H_x : E_x \times \bar{E}_x \to \mathbb{C}$, by conjugating the antilinear argument.
- As $\mathbb{C}$-linear maps $H_x : E_x \otimes_{\mathbb{C}} \bar{E}_x \to \mathbb{C}$, by the universal property of $- \otimes_{\mathbb{C}} -$.
- As $\mathbb{C}$-linear maps $H_x : \bar{E}_x \to E_x^\star$, by the Hom-tensor adjunction.

In each case, the Hermitian matrix that represents H_x must be positive definite, hence invertible. Thus, in the last presentation, H_x is a $\mathbb{C}$-linear isomorphism. Thus, $H : \bar{E} \to E^\star$ is a bundle isomorphism.

Exercise 3. Suppose that each transition function $g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{U}(k)$ is unitary matrix-valued. Show that E has a canonically induced Hermitian structure.

Solution. Let $\sigma_\alpha, \tau_\alpha : U_\alpha \to \mathbb{C}^k$ be the local representatives of two local sections σ, τ of E. Then,

$$\begin{aligned} H(\sigma,\tau)_x &= \sigma_\alpha(x)^\dagger \cdot \tau_\alpha(x) \\ &= \sigma_\alpha(x)^\dagger \cdot \left[g_{\alpha\beta}(x)^\dagger \cdot g_{\alpha\beta}(x)\right] \cdot \tau_\alpha(x) \\ &= \left[g_{\alpha\beta}(x) \cdot \sigma_\alpha(x)\right]^\dagger \cdot \left[g_{\alpha\beta}(x) \cdot \tau_\alpha(x)\right] \\ &= \sigma_\beta(x) \cdot \tau_\beta(x), \end{aligned}$$

is a well-defined smoothly chosen complex inner product on E_x, regardless of the chosen local coordinate system. Therefore, H is a Hermitian structure on E.

Part II

Preliminaries. Let (M, J) be an almost complex manifold, and let g be a Hermitian metric on it. Recall that the fundamental 2-form of h is $\omega(X, Y) = g(JX, Y)$.

Exercise 1. Suppose that M is a surface. Show that g is a Kähler metric.

Solution. We must show that ω is closed and J is integrable.

- Since M is a surface, $\Omega^3(M) = 0$. Therefore, $d\omega = 0$, i.e., ω is a closed 2-form.
- Let X be a nowhere vanishing local real vector field. Since J has no real eigenvalues, X and JX are fiberwise linearly independent. Since M is a surface, X and JX form a local basis of TM. Since

  $$N_J(X, JX) = [X, JX] + J[JX, JX] + J[X, J^2X] - [JX, J^2X] = 0,$$

  the Nijenhuis tensor is identically zero. Therefore, J is integrable.

Exercise 2. Show that ω is a $(1,1)$-form.

Solution. The fundamental 2-form ω satisfies the following identity:

$$\omega(JX, JY) = g(J^2X, JY) = g(JX, Y) = \omega(X, Y)$$

However, if X, Y are both of type $(1,0)$, then

$$\omega(JX, JY) = \omega(iX, iY) = i^2\,\omega(X, Y) = -\omega(X, Y)$$

Similarly, if X, Y are both of type $(0,1)$, then

$$\omega(JX, JY) = \omega(-iX, -iY) = (-i)^2\,\omega(X, Y) = -\omega(X, Y)$$

Hence ω vanishes if its arguments are both of the same type. Therefore, ω is a $(1,1)$-form.

Exercise 3. Show that the representation of g in local holomorphic coordinates is a Hermitian matrix.

Remark. The problem statement implicitly asserts that J is a complex structure. Otherwise we would not have any local holomorphic coordinates at our disposal.

Solution. Let $z = (z_1, \ldots, z_n)$ be a local holomorphic chart on M. We have

- The induced real coordinates $x = \Re(z)$ and $y = \Im(z)$ satisfying $z = x + iy$.
- The induced antiholomorphic coordinates $\bar{z} = x - iy$.

These coordinate systems are related to each other by the real change of basis

$$2\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} I & I \\ iI & -iI \end{bmatrix} \begin{bmatrix} z \\ \bar{z} \end{bmatrix}$$

Since g is a Hermitian metric, its real representation has the form

$$G = 4\begin{bmatrix} A & -B \\ B & A \end{bmatrix}, \quad \text{where} \quad \begin{cases} A^T = A \\ B^T = -B \end{cases}$$

Hence its complexified representation has the form

$$G^{\mathbb{C}} = \begin{bmatrix} I & I \\ iI & -iI \end{bmatrix}^T \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} I & I \\ iI & -iI \end{bmatrix} = 2\begin{bmatrix} 0 & A+iB \\ A-iB & 0 \end{bmatrix}$$

It only remains to show that $A - iB$ is a Hermitian matrix:

$$(A - iB)^\dagger = A^T + iB^T = A - iB$$

Exercise 4. Show that the complexification of g is a symmetric bilinear tensor satisfying

a) $g(\bar{Z}, \bar{W}) = \overline{g(Z, W)}$

b) $g(Z, \bar{Z}) > 0$, for every nonzero vector $Z \in TM^{\mathbb{C}}$.

c) $g(Z, W) = 0$, whenever both vectors are of the same type.

Solution. Recall that the complexified representation of g has the form

$$G^{\mathbb{C}} = \begin{bmatrix} 0 & A+iB \\ A-iB & 0 \end{bmatrix},$$

where A is real symmetric and B is real antisymmetric. Then,

a) This follows from the fact that $G^{\mathbb{C}}$ is a Hermitian matrix.

b) Let $Z = X + iY$ be a nonzero vector. Then at least one of X, Y is nonzero. Hence,

$$g(Z, \bar{Z}) = g(X + iY, X - iY) = g(X, X) + g(Y, Y) > 0$$

c) This follows from the fact that the diagonal blocks in $G^{\mathbb{C}}$ are zero.