Pontificia Universidad Católica del Perú Escuela de Posgrado
Doctorado en Matemáticas

Variedades Complejas
TAREA 4
2020-II

Indicaciones Generales:

- La TAREA 4 puede ser subida a la plataforma Paideia o enviada al correo electrónico jcuadros@pucp.edu.pe.

1. Show that an almost complex structure on a real 2-dimensional manifold is always integrable.

2. Which spheres admit almost-complex structures? It is not difficult to show that if S^n admits an almost-complex structure, then the tangent bundle TS^{n+1} is a trivial bundle.

   i Think of an argument to show this last assertion.

   (Results from algebraic topology imply that $n = 2$ or $n = 6$ (Bott, Milnor, Kervaire). (For the interesting reader, see also `https://www.ocf.berkeley.edu/~rohanjoshi/2019/10/24/parallelizability-of-spheres/` .)

   Conversely, the spheres S^2 and S^6 actually admit almost-complex structures. For S^2, rotate each tangent plane through an angle $\pi/2$, in a fashion consistent with a choice of orientation: consider the usual embedding $S^2 \subset \mathbf{R}^3$; a point $x \in S^2$ may be regarded as a unit vector in $\mathbf{R}^3$, and via the cross product induces a linear transformation $J_x(y) = x \times y$ on $\mathbf{R}^3$. If $y \perp x$ (i.e. if $y \in T_xS^2$) , then $(x \times y) \perp x$ as well, so this linear transformation is an endomorphism J_x of the tangent space T_xS^2.

   ii Verify that $J_x^2 = -I$, so that this endomorphism is an almost-complex structure on S^2.

   iii Extend this idea to hypersurfaces $\Sigma \subset \mathbb{R}^3$: show that Σ carries an almost complex structure inherited from the cross product

   $$\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 : (u, v) \mapsto u \times v.$$

   (Hint: use the Gauss map $\nu : \Sigma \to S^2$ which maps every point $x \in \Sigma$ the outward unit normal vector $\nu(x) \perp T_x\Sigma$.)

   (In Introduction to Symplectic Topology by Salamon and McDuff (third edition), page 155 you find an extension of this construction to exhbit an almost complex structure on S^6 using the Cayley numbers: this almos complex structure has non-zero *torsion tensor*, that is it is not integrable. It is not known at present whether or not S^6 admits a holomorphic atlas, though the answer is believed to be negative.)

3. Show that the Nijenhuis tensor is a tensor indeed. That is, show that for all $f, g \in C^\infty(M)$ and for all $X, Y \in \Gamma^\infty(M)$ the following is satisfied

$$N_J(fX, gY) = f.gN_J(X, Y).$$

4. Show that $N_J(X, JX) = 0$.

5. A vector field on a complex manifold is a vector field Z of type (1,0) such that $Z(f)$ is holomorphic for every local holomorphic function f. Show that In local coordinates, Z can be written as follows

$$Z = \sum_{j=1}^{n} \xi^\alpha \frac{\partial}{\partial z^\alpha}$$

with ξ^α local holomorphic functions.

6. A form $\alpha \in \Omega^{(p,0)}(M)$ is holomorphic if $\bar{\partial}\alpha = 0$. Show that in local coordinates, α can be written as follows

$$\alpha = \sum_{|I|=p} \alpha_I dz^I$$

where the α_I are holomorphic functions. Here, we are using the multi-index notation, so $I = \{i_1, \dots i_p\}$ for instance.

7. Recall that a real vector field X on a complex manifold is called *real holomorphic* or *automorphic* if its $(1,0)$ component $X - iJX$ is a holomorphic vector field. In class we showed that X is real analytic if an only if $[X, JY] = J[X, Y]$ for all Y. Show that if X is real holomorphic, then JX is also real holomorphic.

8. Consider the complex projective space $\mathbf{P}^n$ and let M be the underlying real $2n$ dimensional smooth manifold.

   a) Prove that complex conjugation on $\mathbf{C}^{n+1}$ induces a diffeomorphism $F : M \to M$ such that $J \circ F_* = -J$, where F_* denotes the Jacobian of F.

   b) Determine the fixed points of F.

   c) Would you say that if $\overline{\mathbf{P}}^n$ denotes the complex manifold whose charts are complex conjugates of the standard charts, then $\mathbf{P}^n$ and $\overline{\mathbf{P}}^n$ are biholomorphic?

   The diffeomorphism F preserves orientation if $n = 2$; in fact, it is known that there is no holomorphic atlas on the smooth manifold underlying $\mathbf{P}^2$ which induces the orientation opposite to the standard orientation, that is "$-\mathbf{P}^2$ is not a complex manifold."