Variedades Complejas (tarea 6)

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Preliminaries. Given a presheaf $\mathcal{F}$ on X, we will say that a family of local sections $\sigma_\alpha \in \mathcal{F}(U_\alpha)$ *overlaps correctly* if $\sigma_\alpha(p) = \sigma_\beta(p)$ for every $p \in U_\alpha \cap U_\beta$.

Exercise 1. Let X be a complex manifold. Show that

a) The presheaf of holomorphic functions $\mathcal{O}_X$ is a sheaf.

b) The presheaf of nowhere zero holomorphic functions $\mathcal{O}_X^\star$ is a sheaf.

c) The presheaf of holomorphic m-forms is a sheaf.

d) The presheaf of holomorphic vector fields is a sheaf.

Solution. Let $\pi : E \to X$ be a holomorphic fiber bundle. We are interested in the following cases:

a) E is the trivial bundle $X \times \mathbb{C}$.

b) E is the trivial bundle $X \times \mathbb{C}^\star$. (Note that this one is not a *vector* bundle.)

c) $E = \bigwedge^{m,0} M$ is the m-th exterior power of the holomorphic cotangent bundle.

d) $E = T^{1,0}M$ is the holomorphic tangent bundle.

By the gluing lemma, given a correctly overlapping family of continuous functions $\sigma_\alpha : U_\alpha \to E$, there is a unique continuous function $\sigma : \bigcup_\alpha U_\alpha \to E$ that restricts to σ_α on each U_α. Moreover,

- σ is a local section[1] $\iff$ each σ_α is a local section.
- σ is holomorphic $\iff$ each σ_α is holomorphic.

Therefore, the presheaf of holomorphic sections of E satisfies the gluing axiom, i.e., it is a sheaf.

Exercise 2. Let $Y \subset X$ be complex manifolds. Show that the ideal sheaf $\mathcal{I}_Y$ is a subsheaf of $\mathcal{O}_X$.

Solution. By definition, $\mathcal{I}_Y$ is the subpresheaf of $\mathcal{O}_X$ whose sections vanish on Y. That is,

$$\mathcal{I}_Y(U) = \{f \in \mathcal{O}_X(U) : f(p) = 0 \text{ for every } p \in Y \cap U\},$$

By the gluing lemma, given a correctly overlapping family of continuous functions $f_\alpha : U_\alpha \to \mathbb{C}$, there is a unique continuous function $f : \bigcup_\alpha U_\alpha \to \mathbb{C}$ that restricts to each f_α on each U_α. Moreover,

- f is holomorphic $\iff$ each f_α is holomorphic, as seen in the previous exercise.
- f vanishes on $Y \cap \bigcup_\alpha U_\alpha$ $\iff$ each f_α vanishes on $Y \cap U_\alpha$.

Therefore, $\mathcal{I}_Y$ is a subsheaf of $\mathcal{O}_X$. More generally, any subpresheaf of a sheaf $\mathcal{F}$ is itself a sheaf, hence a subsheaf of $\mathcal{F}$, by an argument using the stalkwise injectivity of the subpresheaf's embedding into $\mathcal{F}$.

[1] Recall that a local section of a fiber bundle $\pi : E \to X$ on an open subset $U \subset X$ is a continuous function $\sigma : U \to E$ such that $\pi \circ \sigma$ reproduces the embedding of U into X.

Exercise 3. Let X be a complex manifold. Show that

a) The sheaf of holomorphic vector fields is a locally free sheaf of $\mathcal{O}_X$-modules.

b) The sheaf of holomorphic m-forms is a locally free sheaf of $\mathcal{O}_X$-modules.

Solution. Let $\pi : E \to X$ be a holomorphic vector bundle. We are interested in the following cases:

a) $E = T^{1,0}M$ is the holomorphic tangent bundle.

b) $E = \bigwedge^{m,0} M$ is the m-th exterior power of the holomorphic cotangent bundle.

Let $f : U \to \mathbb{C}$ and $\sigma : V \to E$ be continuous functions defined on the opens $V \subset U \subset X$. Then,

- $f\sigma : V \to E$, $f\sigma(p) = f(p) \cdot \sigma(p)$ is a continuous function.
- σ is a local section $\implies$ $f\sigma$ is a local section.
- f, σ are holomorphic $\implies$ $f\sigma$ is holomorphic.

Let $\sigma : U \to E$ be a local holomorphic section and $\varphi_\alpha : E_\alpha \to U_\alpha \times \mathbb{C}^k$ the local trivializations of E on an open cover $\{U_\alpha\}$ of U. Then,

- Every $\sigma_\alpha = \sigma \mid U_\alpha$ is uniquely determined by k holomorphic functions $g_i : U_\alpha \to \mathbb{C}$, for $i = 1, \ldots, k$, such that $\sigma_\alpha = (g_1, \ldots, g_k) \circ \varphi_\alpha$.

Therefore, the sheaf $\mathcal{F}$ of sections of E satisfies the following conditions:

- $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, i.e., there is a ring homomorphism $\mathcal{O}_X(U) \to \mathrm{End} \circ \mathcal{F}(U)$.
- $\mathcal{F}(U) \to \mathcal{F}(V)$ is an $\mathcal{O}_X(U)$-module homomorphism, where $\mathcal{F}(V)$ has the $\mathcal{O}_X(U)$-module structure induced by the ring homomorphisms $\mathcal{O}_X(U) \to \mathcal{O}_X(V) \to \mathrm{End} \circ \mathcal{F}(V)$.
- There is always an open cover $\{U_\alpha\}$ of U such that each $\mathcal{F}(U_\alpha)$ is a free $\mathcal{O}(U_\alpha)$-module.

Hence, $\mathcal{F}$ is a locally free sheaf of $\mathcal{O}_X$-modules.