Pontificia Universidad Católica del Perú Escuela de Posgrado
Doctorado en Matemáticas

Variedades Complejas
TAREA 1
2020-II

<u>Indicaciones Generales:</u> La TAREA 2, que provienen (textualmente) en su mayoría del libro texto de Huybrecht puede ser considerada un control de lectura. Ud. puede subir esta tarea a la plataforma Paideia o enviarla al correo electrónico jcuadros@pucp.edu.pe.

1. Show that $\mathbb{C}^n$ does not have any compact submanifolds of positive dimension. (This is in contrast to the real situation, where any manifold, compact or not, can be realized as a submanifold of some $\mathbb{R}^N$.)

2. Prove the **Implicit function theorem:** Let $U \subset \mathbb{C}^m$ be an open subset and let $f : U \rightarrow \mathbb{C}^n$ be a holomorphic map, where $m \geq n$. Suppose $z_0 \in U$ is a point such that

$$\det\left(\frac{\partial f_i}{\partial z_j}(z_0)\right)_{1 \leq i,j \leq n} \neq 0$$

   Then there exist open subsets $U_1 \subset \mathbb{C}^{m-n}, U_2 \subset \mathbb{C}^n$ and a holomorphic map $g : U_1 \rightarrow U_2$ such that $U_1 \times U_2 \subset U$ and $f(z) = f(z_0)$ if and only if $g(z_{n+1}, \ldots, z_m) = (z_1, \ldots, z_n)$.

3. On the projective space $\mathbb{P}^n$ a homogeneous polynomial $F(z_0, \ldots, z_n)$ of degree k is not a function, since its value at a point $[a_0 : \ldots : a_n]$ is not unique. However, the zero set in $\mathbb{P}^n$ of a homogeneous polynomial $F(z_0, \ldots, z_n)$ is well defined, since $F(a_0, \ldots, a_n) = 0$ if and only if

$$F(ta_0, \ldots, ta_n) = t^k F(a_0, \ldots, a_n) = 0 \quad \text{for all } t \in \mathbb{C}^\times := \mathbb{C} - \{0\}$$

   The zero set of finitely many homogeneous polynomials in $\mathbb{C}^n$ is called a **complex projective variety**. A projective variety defined by a single homogeneous polynomial of degree k is called a **hypersurface of degree** k. Show that the hypersurface $Z(F)$ defined by $F(z_0, z_1, z_2) = 0$ is a complex manifold if $\partial F/\partial z_0, \partial F/\partial z_1$, and $\partial F/\partial z_2$ are not simultaneously zero on $Z(F)$. (Hint: The standard coordinates on U_0, which is homeomorphic to $\mathbb{C}^2$, are $z = z_1/z_0, w = z_2/z_0$. In $U_0, F(z_0, z_1, z_2) = z_0^k F(1, z_1/z_0, z_2/z_0) = z_0^k F(1, z, w)$. Define $f(z, w) = F(1, z, w)$. Then f and F have the same zero set in U_0. It is enough to show that zero is a regular value of f, from there, implicit function theorem establishes the existence of charts with holomorphic transition functions.)

REMARK *In order to fully understand the statements of the next exercises, first you must understand the content of the following classical result (read Huybrechts text, pages 59-60):*

Proposition. *Let $G \times X \rightarrow X$ be the proper and free action of a complex Lie group G on a complex manifold X. Then the quotient X/G is a complex manifold in a natural way and the quotient map $\pi : X \rightarrow X/G$ is holomorphic.*

4. Let V be a complex vector space of dimension $n+1$. As a generalization of the projective space $\mathbb{P}(V)$, which is naturally identified with the set of all lines in V, one defines the **Grassmannian** $\mathrm{Gr}_k(V)$ for $k \leq n+1$ as the set of all k -dimensional subspaces of V, i.e.

$$\mathrm{Gr}_k(V) := \{W \subset V \mid \dim(W) = k\}$$

In particular, $\mathrm{Gr}_1(V) = \mathbb{P}(V)$ and $\mathrm{Gr}_n(V) = \mathbb{P}(V^*)$

In order to show that $\mathrm{Gr}_k(V)$ is a complex manifold, we may assume that $V = \mathbb{C}^{n+1}$. Any $W \in \mathrm{Gr}_k(V)$ is generated by the rows of a $(k, n+1)$ matrix A of rank k. Let us denote the set of these matrices by $M_{k,n+1}$, which is an open subset of the set of all $(k, n+1)$ -matrices. The latter space is a complex manifold canonically isomorphic to $\mathbb{C}^{k\cdot(n+1)}$. Thus, we obtain a natural surjection $\pi : M_{k,n+1} \to \mathrm{Gr}_k(\mathbb{C}^{n+1})$, which is the quotient by the natural action of $\mathrm{Gl}(k, \mathbb{C})$ on $M_{k,n+1}$

Let us fix an ordering $\{B_1, \ldots, B_m\}$ of all (k, k) -minors of matrices $A \in M_{k,n+1}$. Define an open covering $\mathrm{Gr}_k(\mathbb{C}^{n+1}) = \bigcup_{i=1}^m U_i$ where U_i is the open subset $\{\pi(A) \mid \det(B_i) \neq 0\}$. Note that if $\pi(A) = \pi(A')$, then $\det(B_i) \neq 0$ if and only if $\det(B_i') \neq 0$, i.e. the open subsets U_i are well-defined. After permuting the columns of $A \in \pi^{-1}(U_i)$ we may assume that A is of the form (B_i, C_i), where C_i is a $(k, n+1-k)$ -matrix. Then the map $\varphi_i : U_i \to \mathbb{C}^{k\cdot(n+1-k)}, \pi(A) \mapsto B_i^{-1}C_i$ is well-defined.

   a) Verify that $\{(U_i, \varphi_i)\}$ defines a holomorphic atlas of $\mathrm{Gr}_k(\mathbb{C}^{n+1})$, such that any $\mathbb{C}$ linear isomorphism of $\mathbb{C}^{n+1}$ induces a biholomorphic map of $\mathrm{Gr}_k(\mathbb{C}^{n+1})$.

   b) Determine the dimension of the Grassmannian manifolds.

One can further generalize the notion of Grassmannians to so called **flag manifolds**. Let again V be a complex vector space of dimension $n+1$ and fix $0 \leq k_1 \leq k_2 \leq \ldots \leq k_\ell \leq n+1$. Then $\mathrm{Flag}(V, k_1, \ldots, k_\ell)$ is the manifold of all flags $W_1 \subset W_2 \subset \ldots \subset W_\ell \subset V$ with $\dim(W_i) = k_i$. E.g. $\mathrm{Flag}(V, k) = \mathrm{Gr}_k(V)$. Furthermore, $\mathrm{Flag}(V, 1, n)$ is the incidence variety $\{(\ell, H) \mid \ell \subset H\} \subset \mathbb{P}(V) \times \mathbb{P}(V^*)$ of all pairs (ℓ, H) consisting of a line $\ell \subset V$ contained in a hyperplane $H \subset V$. The fibre of the projections $\mathrm{Flag}(V, 1, n) \to \mathbb{P}(V)$ $(\ell, H) \mapsto \ell$ and $\mathrm{Flag}(V, 1, n) \to \mathbb{P}(V^*), (\ell, H) \mapsto H$ over ℓ respectively H are canonically isomorphic to $\mathbb{P}(V/\ell)$ and $\mathbb{P}(H)$, respectively.

5. **Hopf Manifolds** Let $\mathbb{Z}$ act on $\mathbb{C}^n \backslash \{0\}$ by $(z_1, \ldots, z_n) \mapsto (\lambda^k z_1, \ldots, \lambda^k z_n)$ for $k \in \mathbb{Z}$. Show that

   a) for $0 < \lambda < 1$ the action is free and discrete.

   b) Show that the quotient complex manifold $X = (\mathbb{C}^n \backslash \{0\})/\mathbb{Z}$ is diffeomorphic to $S^1 \times S^{2n-1}$. For $n = 1$ this manifold is isomorphic to a complex torus $\mathbb{C}/\Gamma$. For $n > 1$ Hopf manifolds are examples of complex manifolds that are not symplectic (for this last fact, use Küneth formula and conclude that $H^2(X, \mathbb{R}) = 0$, since X is compact, it follows that is not symplectic...hence non-Kähler...hence non-projective!)

   c) Show that any Hopf surface contains elliptic curves. (Hint: Observe that when $n = 2$, we obtain something diffeomorphic to $S^1 \times S^3$. Since S^3 is the total space of a Hopf fibration over S^2 with S^1 fibers, the Hopf surface is like an elliptic fibration over S^2 but it is not projective.)

d) Generalize the construction of the Hopf manifolds by considering the action of $\mathbb{Z}$ given by $(z_1, \ldots, z_n) \mapsto \left(\lambda_1^k z_1, \ldots, \lambda_n^k z_n\right)$, where $0 < \lambda_i < 1$. Show that the quotient $\left(\mathbb{C}^n \backslash \{0\}\right)/\mathbb{Z}$ is again diffeomorphic to $S^1 \times S^{2n-1}$.

6. **Iwasawa manifold.** Let G be the complex Lie group that consists of all matrices of the form $\begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(3, \mathbb{C})$. Clearly, G is biholomorphic (as a complex manifold not as a complex Lie group! because of the different group structure) to $\mathbb{C}^3$. The group G (and every subgroup of it) acts on G by multiplication in $\mathrm{GL}(3, \mathbb{C})$

   Consider the subgroup $\Gamma := G \cap GL(3, \mathbb{Z} + i\mathbb{Z})$. Then $(w_1, w_2, w_3) \in \Gamma$ acts on G by left matrix multiplication:

$$(z_1, z_2, z_3) \mapsto (z_1 + w_1, z_2 + w_1 z_3 + w_2, z_3 + w_3)\,.$$

   Show that this action is properly discontinuous. Thus, the quotient $X := G/\Gamma$ is a complex manifold of dimension three. (Note that the first and third coordinates give a holomorphic map $f : X \to \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}) \times \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ which fibers are isomorphic to $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$. Then locally, it looks like three copies of an elliptic curve (which is not T^6.)

7. Let ρ be a fifth root of unity. The group $G = \langle \rho \rangle \cong \mathbb{Z}/5\mathbb{Z}$ acts on $\mathbb{P}^3$ by

$$(z_0 : z_1 : z_2 : z_3) \mapsto \left(z_0 : \rho z_1 : \rho^2 z_2 : \rho^3 z_3\right)$$

   Describe all fix points of this action. Show that the surface Y defined by $\sum_{i=0}^{3} z_i^5 = 0$ is G -invariant and that the induced action is fix point free. (The quotient $X = Y/G$ is called **Godeaux surface** and historically was the first compact complex surface with $H^i(X, \mathcal{O}) = 0, i = 1, 2$, that is not rational.)