Pontificia Universidad Católica del Perú Escuela de Posgrado
Doctorado en Matemáticas

Variedades Complejas
TAREA 3
2020-II

Indicaciones Generales:

- La TAREA 3 puede ser subida a la plataforma Paideia o enviada al correo electrónico jcuadros@pucp.edu.pe.

1. **Toros de dimensión 1.**

   Let $\Lambda \subset \mathbb{C}$ be a lattice, and let $X := \mathbb{C}/\Lambda$ be the associated complex torus.

   a) Show that X is diffeomorphic to $S^1 \times S^1$.

   b) Let $\varphi : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ be a biholomorphic map such that $\varphi(0) = 0$. Show that there exists a unique $\alpha \in \mathbb{C}^* = \mathbb{C} - \{0\}$ such that $\alpha\Lambda = \Lambda'$ and such that the diagram

   $$\begin{array}{ccc} \mathbb{C} & \stackrel{z \mapsto \alpha z}{\longrightarrow} & \mathbb{C} \\ \pi \downarrow & & \pi' \downarrow \\ \mathbb{C}/\Lambda & \stackrel{\varphi}{\longrightarrow} & \mathbb{C}/\Lambda' \end{array}$$

   commutes. (Hint: recall that the group of biholomorphic automorphisms of $\mathbb{C}$ is $\mathrm{Aut}(\mathbb{C}) = \{z \mapsto \alpha z + \beta \mid \alpha \in \mathbb{C}^*, \beta \in \mathbb{C}\}$.)

   c) Show that X is biholomorphic to a torus of the form $X(\tau) := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ where $\tau \in \mathbb{C}$ such that $\mathrm{Im}(\tau) > 0$

   d) Let $\mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ be the Poincaré upper half plane. We define a group action

   $$\mathrm{SL}(2, \mathbb{Z}) \times \mathbb{H} \to \mathbb{H}, \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau \right) \mapsto \frac{a\tau + b}{c\tau + d}$$

   Show that the biholomorphic equivalence classes of complex tori of dimension 1 have a natural bijection with $\mathbb{H}/\mathrm{SL}(2, \mathbb{Z})$.

   (The interesting aspect of the set $\mathbb{H}/\mathrm{SL}(2, \mathbb{Z})$ is that it has a natural complex structure and that $\mathbb{H}/\mathrm{SL}(2, \mathbb{Z})$ is biholomorphic to $\mathbb{C}$.)

2. **Teorema de Chow.**

   Recall **Chow's theorem:** Let X be a compact complex manifold that admits an embedding $X \hookrightarrow \mathbb{P}^N$ into some projective space. Then X is algebraic, i.e. defined by a finite number of homogeneous polynomials.

   From the previous homework, recall that for V be a complex vector space of dimension n. For an integer $0 < r < n$, the **Grassmannian** $G_r(V) := \{S \subset V \text{ subspace of dimension } r\}$ is a complex manifold. We define a map

   $$\psi : G_k(V) \to \mathbb{P}(\bigwedge^k V)$$

   as follows: let $U \subset V$ be a subspace of dimension k and let $u_1, \ldots, u_k$ be a basis of U. The multivector

   $$u_1 \wedge \ldots \wedge u_k$$

   gives a point in $\mathbb{P}\left(\bigwedge^k V\right)$.

   a) Show that this map is well-defined.

   b) Show that defines an embedding, the so-called Plücker embedding and conclude that $G_k(V)$ is a projective manifold.

   c) From Chow' Theorem, the Grassmannian $G_k(V)$ is algebraic. Exhibit this explicitly for $G_2(V)$ with $V := \mathbb{C}^4$, with canonical basis $e_1, \ldots, e_4$: since every 2 -vector $w \in \Lambda^2\mathbb{C}^4$ has a unique decomposition

   $$w = X_0 e_1 \wedge e_2 + X_1 e_1 \wedge e_3 + X_2 e_1 \wedge e_4 + X_3 e_2 \wedge e_3 + X_4 e_2 \wedge e_4 + X_5 e_3 \wedge e_4,$$

   show that for the homogeneous coordinates $[X_0 : \ldots : X_5]$ on $\mathbb{P}\left(\Lambda^2\mathbb{C}^4\right)$, the Plücker embedding of $G_2\left(\mathbb{C}^4\right)$ in $\mathbb{P}\left(\Lambda^2\mathbb{C}^4\right) \simeq \mathbb{P}^5$ has the equation

   $$X_0X_5 - X_1X_4 + X_2X_3 = 0.$$

   For more on Plücker embeddings, check `http://www.math.uchicago.edu/~may/VIGRE/VIGRE2007/REUPapers/FINALFULL/Hudec.pdf` for instance.