Süli, Cap. 12 "Problemas de valor inicial para EDOs"

12.1 Introducción

Theorem 12.1 (Picard's Theorem[1]) *Suppose that the real-valued function* $(x, y) \mapsto f(x, y)$ *is continuous in the rectangular region* D *defined by* $x_0 \le x \le X_M$, $y_0 - C \le y \le y_0 + C$*; that* $|f(x, y_0)| \le K$ *when* $x_0 \le x \le X_M$*; and that* f *satisfies the Lipschitz condition: there exists* $L > 0$ *such that*

$$|f(x,u) - f(x,v)| \le L|u - v| \quad \textit{for all} \ (x,u) \in D\,, \ \ (x,v) \in D\,.$$

Assume further that

$$C \ge \frac{K}{L}\left(\mathrm{e}^{L(X_M - x_0)} - 1\right)\,. \tag{12.3}$$

Then, there exists a unique function $y \in \mathrm{C}^1[x_0, X_M]$ *such that* $y(x_0) = y_0$ *and* $y' = f(x, y)$ *for* $x \in [x_0, X_M]$*; moreover,*

$$|y(x) - y_0| \le C\,, \quad x_0 \le x \le X_M\,.$$

12.2 Métodos de un paso

Ejemplos:

a) Euler (o forward Euler)

$$\boxed{u_{n+1} = u_n + hf_n, \qquad n = 0, \ldots, N_h - 1}$$

Programa "feuler" de Quarteroni

12.3 Consistencia y Convergencia

12.4 Un método implicíto de un paso

a) Trapecio (o Crank-Nicolson)

$$\boxed{u_{n+1} = u_n + \frac{h}{2}[f_n + f_{n+1}], \quad n = 0, \ldots, N_h - 1}$$

Programa "cranknic" de Quarteroni

b) Euler implícito (o backward Euler)

$$\boxed{u_{n+1} = u_n + hf_{n+1}, \qquad n = 0, \ldots, N_h - 1}$$

Programa "beuler" de Quarteroni

12.5 Métodos Runge-Kutta

a) Euler modificado (modified Euler)

$$y_{n+1} = y_n + h\,f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n)\right)$$

b) Euler mejorado (improved Euler o método de Heun)

$$\boxed{\begin{aligned} u^*_{n+1} &= u_n + hf_n, \\ u_{n+1} &= u_n + \frac{h}{2}\left[f_n + f(t_{n+1}, u^*_{n+1})\right] \end{aligned}}$$

Programa "predcor" en combinación con "feonestep" y "cnonestep" de Quarteroni

c) RK44 clásico

$$\boxed{u_{n+1} = u_n + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)}$$

where

$$\begin{aligned} K_1 &= f_n, \\ K_2 &= f(t_n + \tfrac{h}{2}, u_n + \tfrac{h}{2}K_1), \\ K_3 &= f(t_n + \tfrac{h}{2}, u_n + \tfrac{h}{2}K_2), \\ K_4 &= f(t_{n+1}, u_n + hK_3), \end{aligned}$$

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$