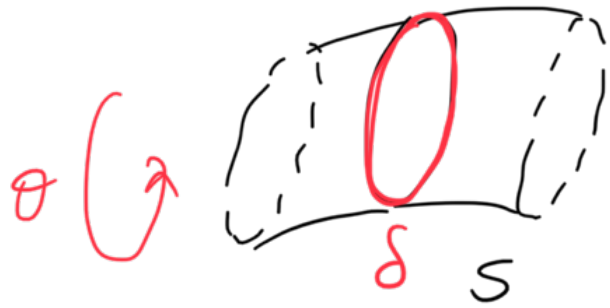
Otros ejemplos:

③ $S = S^1 \times (-1,1)$. cilindro.



$\delta = S^1 \times \{0\}$.

Calcular $H^1_{DR}(S)$

Consideramos

$$\mathcal{U} = (S^1 \setminus N) \times (-1,1) \approx \mathbb{R}^2$$

$$V = (S^1 \setminus S_{sur}) \times (-1,1) \approx \mathbb{R}^2$$

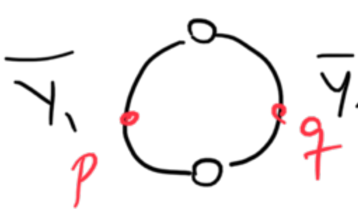
Notar

$\mathcal{U} \cap V$ no es conexa!

$\mathcal{U} \cap V$ Tiene dos componentes conexas:

$$\mathcal{U} \cap V = (S^1 \setminus N, S) \times (-1,1).$$

$$\gamma_1 \cup \gamma_2$$



(proy. a S^1).

Afirmación: $H^1 (S) \xrightarrow{\varphi} \mathbb{R}$

$H^1_{DR}(S) \longrightarrow \mathbb{R}$ isomorfismo.

$$\alpha \longmapsto \int_\delta \alpha$$

✓ Sobrey. $\alpha = d\theta$

· inyectiva:

sea α cerrada. $(d\alpha = 0)$

y $\varphi(\alpha) = 0 \overset{?}{\Longrightarrow} \alpha = 0.$

$$\int_\delta \alpha = 0 \;(*) \overset{?}{\Longrightarrow} \alpha = 0$$

α exacta.

Si $(*)$ se cumple

¿cómo construir $f \in \Omega^\infty(S)$ tq.

$$df = \alpha \;?$$

Como en el caso de $S = S^2$,

· $\alpha|_U = df_U \qquad f_U : U \to \mathbb{R}$

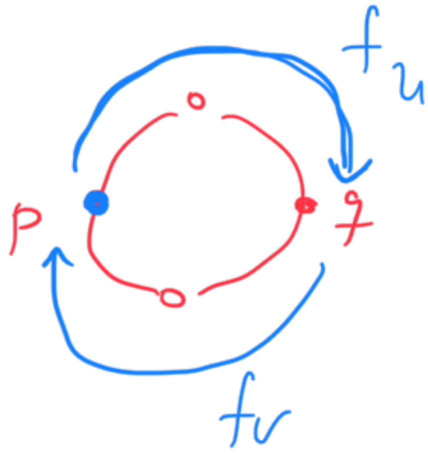
· $\alpha|_V = df_V \qquad f_V : V \to \mathbb{R}.$

$f_U - f_V$ definida en $U \cap V = Y_1 \cup Y_2$

(dos comp. conexas)

Para probar que $f_u - f_v$ es cte.
sean $p, q \in U \cap Y$ $p \in Y_1$, $q \in Y_2$

$$(f_u - f_v)(p) - (f_u - f_v)(q)$$
$$= \pm \int_\delta \alpha$$



$0.$

$$f_u(q) - f_u(p) + f_v(p) - f_v(q)$$

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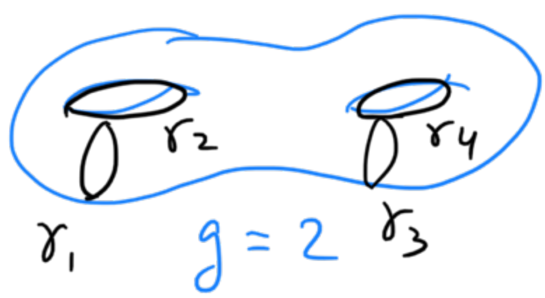
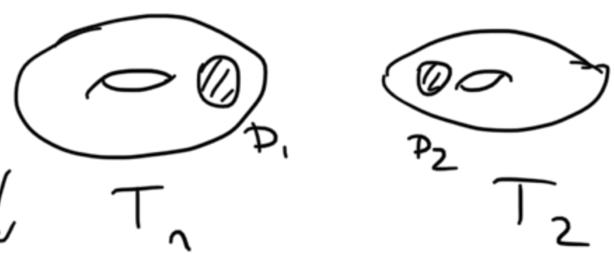
(4) S = bitoro

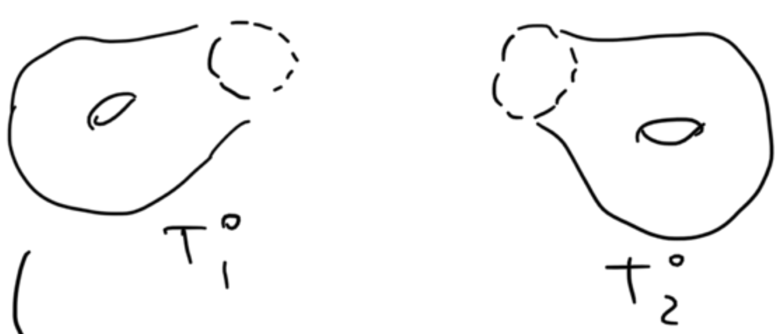
$= S_g$, $g = 2$



$= T_1 \# T_2$

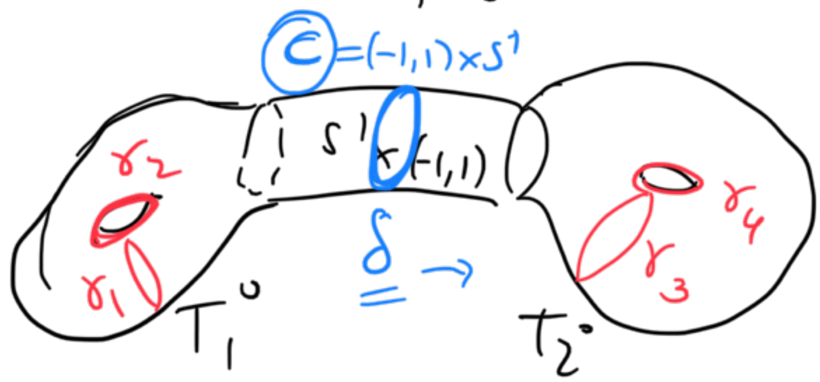


removemos los discos



usamos $S^1 \times (-1,1)$

para "pegarlos"



$$S_2 = T_1 \# T_2$$

Afirmación:

$$H^1_{DR}(S_2) = \mathbb{R}^4$$

(en general

$$H^1_{DR}(S_g) = \mathbb{R}^{2g} \;)$$

a través del isomorfismo

$$\mathcal{P} : H^1_{DR}(S_2) \longrightarrow \mathbb{R}^4$$

$$\alpha \longmapsto \left(\int_{\gamma_1} \alpha, \int_{\gamma_2} \alpha, \int_{\gamma_3} \alpha, \int_{\gamma_4} \alpha \right)$$

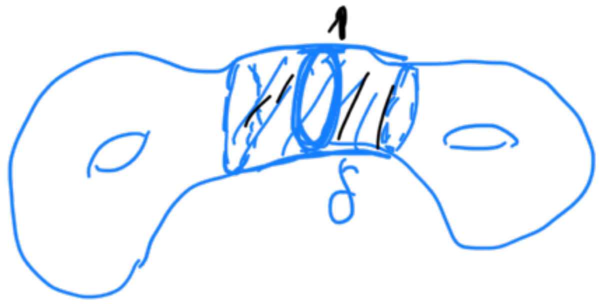
$\mathcal{P}$ inyectiva:

supongamos $\alpha \in H^1_{DR}(S_2)$ tal que $(d\alpha = 0)$

$$\mathcal{P}(\alpha) = 0 .$$

Así $\boxed{\int_{\gamma_i} \alpha = 0 \quad \forall i = 1, \dots, 4.}$ (**)

Debemos mostrar $\alpha = 0$ (α exacta).



Notar: $\int_{\delta} \alpha = ?$

Por Stokes:

$$\int_{\delta} \alpha = \int_{S'} d\alpha = \underline{0}$$

Luego: por el ejm. anterior, existe $g \in C^{\infty}(C)$ tal que

$$\alpha = dg \quad \text{en } C.$$

- Sea P función suave en S con soporte en $C = (-1,1) \times S^1 \subseteq S$ tal que $P|_{\delta} = 1$

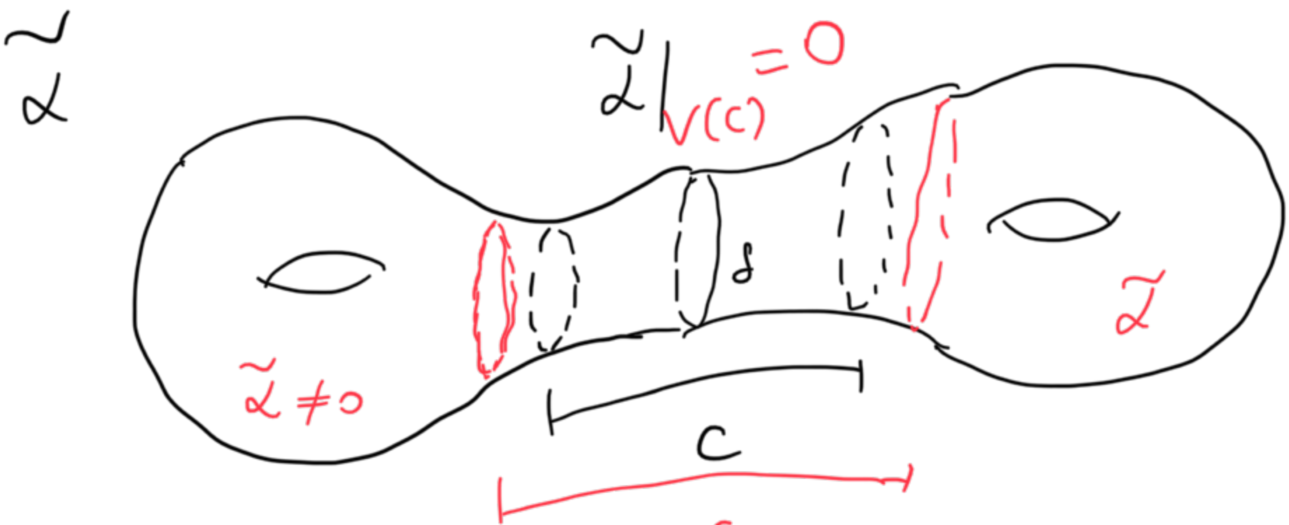
- Consideramos

$\tilde{\ }$ ... $(P\alpha)$

$\alpha = \alpha - d(\ g)$

Notar $\tilde{\alpha} = \alpha$ en $H^1_{DR}(S)$

Además
$\tilde{\alpha}$ se anula en una vecindad de C.



$\tilde{\alpha} = 0.$ ←

$\tilde{\alpha}$ genera dos formas en T_1 y T_2

$V(C)$

$\tilde{\alpha} = 0.$ ←

$\tilde{\alpha}$ genera dos formas en T_1 y T_2

También

$\int_{\gamma_1} \beta = \int_{\gamma_2} \beta = 0$ | $\int_{\gamma_3} \beta' = \int_{\gamma_4} \beta' = 0$

β es 1-forma exacta en T_1 (ejm. anterior)

$\beta = df$

$f \in C^\infty(T_1)$

β' es 1-forma exacta en T_2 (ejm. anterior)

$\beta' = df'$

$f' \in C^\infty(T_2)$

$d(f - f') = 0$

$f - f'$ def. en $V(C)$

$f = f'$ en $V(C) \approx \tilde{T}_1 \cap \tilde{T}_2$

vecindad de T_1

vecindad de T_2

$\Rightarrow$ f y f' se pueden pegar y obtenemos

así $\exists\ \tilde{f} = \begin{cases} f & \text{en } \tilde{T}_1 \\ f' & \text{— } \tilde{T}_2 \end{cases}$

de modo que

$\alpha = df$.

$$\Rightarrow \alpha = 0 \text{ en } H^1(S_2).$$

Obs:

En general

$$H^1_{DR}(S_g) \simeq \mathbb{R}^{2g} \simeq \underbrace{\mathbb{R}^g}_{H^{1,0}} \oplus \underbrace{\mathbb{R}^g}_{H^{0,1}}$$

Sea X una superficie de Riemann.

$p \in X$

$$T^*X_p = \mathrm{Hom}_{\mathbb{R}}(TX, \mathbb{R})$$

Complexificamos:

$$T^*X_p^{\mathbb{C}} = \mathrm{Hom}_{\mathbb{R}}(TX, \mathbb{C})$$

$$\simeq T^*X_p \otimes_{\mathbb{R}} \mathbb{C}.$$

Notar:

$$T^*X_p \simeq \mathbb{R}\{dx, dy\}.$$

$$T^*X_p \otimes \mathbb{C} \simeq \mathbb{C}\{dx, dy\}.$$

$$\simeq \mathbb{C}\{dz, d\bar{z}\}$$

donde $\begin{cases} dz = dx + i\,dy \\ d\bar{z} = dx - i\,dy \end{cases}$

Así:

- $T^*X_p \otimes \mathbb{C} = \underbrace{T^*X_p'}_{\text{parte holomorfa}} \oplus \underbrace{T^*X_p''}_{\text{parte antiholomorfa}}$

$$\mathbb{C}\{dz\} \oplus \mathbb{C}\{d\bar{z}\}$$

de ese modo $f : X \to \mathbb{C}$ función holomorfa, entonces

$$df \in T^*X_p'$$

$$d\bar{f} \in T^*X_p''.$$

Asimismo

T^*X

fibrado cotang.

$$\begin{tikzcd} T^*X \arrow[d, "\pi"'] \\ X \arrow[u, bend right, "s"'] \end{tikzcd}$$

s : 1-formas dif. son secciones globales del fibrado cotangente.

$$\Omega^1_{X,\mathbb{C}} = \underbrace{\Gamma\left(X, T^*X_p{}^{\mathbb{C}}\right)}_{\text{secciones globales}}$$

$$= \underbrace{\Omega^{1,0}_X}_{\substack{\text{rango}\\ \text{en } T^*X_p{}'}} \oplus \underbrace{\Omega^{0,1}_X}_{\substack{\text{rango en}\\ T^*X_p{}''}}$$

Recordar:

$$0 \to \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \to 0$$

similarmente, se tiene

$$0 \to \Omega^0_{\mathbb{C}}(X) \xrightarrow{d} \underbrace{\Omega^1_{\mathbb{C}}(X)}_{\Omega^{1,0} \oplus \Omega^{0,1}} \xrightarrow{d} \Omega^2_{\mathbb{C}}(X) \to 0$$

¿cómo se descompone d en este contexto?

Obj: descomponer d usando la descomp. de $\Omega^*_{\mathbb{C}}(X)$. $\quad (d = \partial + \bar{\partial})$

$$
\begin{tikzcd}
\Omega^{0,1} \arrow[r, "\partial"] & \Omega^{2} \\
\Omega^{0} \arrow[u, "\bar\partial"] \arrow[r, "\partial"'] & \Omega^{(1,0)} \arrow[u, "\bar\partial"']
\end{tikzcd}
$$

En coordenadas locales

$$z = x + iy.$$

$$dz = dx + i\,dy \quad \text{base de } T^{*}X_p'$$

$$d\bar z = dx - i\,dy \quad \text{———} \quad T^{*}X_p''$$

Escribir

$$\cdot\ dx = \frac{1}{2}(dz + d\bar z)$$

$$\cdot\ dy = \frac{1}{2i}(dz - d\bar z)$$

Así:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)d\bar{z}$$

$$\underline{\underline{\partial f}} \checkmark \quad + \quad \underline{\underline{\bar{\partial} f}} \checkmark$$

Definimos

- $$\partial f = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)dz$$

- $$\bar{\partial} f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)d\bar{z}$$

Además

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \checkmark$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \checkmark$$

Así

$$df = \frac{\partial f}{\partial z}\cdot dz + \frac{\partial f}{\partial \bar{z}}\cdot d\bar{z}.$$

Obs:

(*) $\bar{\partial} f = 0 \iff f$ holomorfa.

(ec. Cauchy Riemann (*))

$$f = u + iv$$

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial v} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$$\left\{ \begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial v} &= -\frac{\partial v}{\partial x} \end{aligned} \right\} (*)$$

Así si f es holomorfa

$$df = \partial f = \underbrace{f'(z)}\, dz.$$

Obs:

$$d^2 = 0 \Rightarrow \partial\partial = 0 \quad \checkmark$$

$$(\partial + \bar\partial)^2 = 0 \qquad \bar\partial\bar\partial = 0 \quad \checkmark$$

$$\partial\bar\partial = -\bar\partial\partial.$$

Ahora consideremos $\partial, \bar\partial$ en $\Omega^{0,1}, \Omega^{1,0}$

$$\odot\ \partial(A\, d\bar z) = \frac{\partial A}{\partial z} dz\, d\bar z = 2i \frac{\partial A}{\partial z} dx\, dy.$$

$\odot\ \bar\partial(B\,dz) = \dfrac{\partial B}{\partial \bar z}\, d\bar z\, dz = -2i\, \dfrac{\partial B}{\partial \bar z}\, dx\, dy.$

Def: una $(1,0)$-forma B es una 1-forma holomorfa si

$$\bar\partial B = 0.$$

(i.e. en cord. locales una 1-forma holo. se puede escribir como $\underline{\underline{B\,dz}}$ con B función holomorfa).

$S \subseteq X$

Sea S una superficie compacta con frontera, $S \subseteq X$.

Sea $\boxed{\alpha \text{ 1-forma holomorfa en una vecindad de } S.}$

$\Rightarrow$ α es cerrada. $\quad$ ($d\alpha =$

$(d\alpha = 0)$ $\quad$ $((\partial + \bar\partial)\alpha =$ α holo

(*)

$\partial\alpha = \frac{\partial f}{\partial \bar z} d\bar z\, dz$

$\alpha = \underline{f dz}$)

Por Stokes

$$\Rightarrow \boxed{\int_{\partial S} \alpha = 0}$$

(versión del T. de Cauchy para sup. de Riemann).

X sup. de Riemann.

$\Omega^*_{\mathbb{C}}(X)$

- $\Omega^2_{\mathbb{C}}(X)$ gen. por $\underline{dz\, d\bar z}$. $(\cong \Omega^{1,1}(X))$

- $\Omega^1_{\mathbb{C}}(X) = \Omega^{1,0}(X) \oplus \Omega^{0,1}(X)$

- $\Omega^0(X)$

X var. compleja

- $\Omega^{p,q}_x(X) = \Gamma\left(X, \Lambda^p T^*_x X' \wedge \Lambda^q T^*_x X''\right)$

$\omega \in$

$$\omega = \sum_{I,J} f_{IJ}\, dz_I \wedge d\bar z_J$$

$|I| = p$

$|J| = q.$

· fijamos p:

$$\xrightarrow{\bar\partial} \Omega^{p,q}(X) \xrightarrow{\bar\partial} \Omega^{p,q+1}(X) \xrightarrow{\bar\partial} \cdots$$

complejo de cadenas, con $\bar\partial$ "derivada exterior". $(\bar\partial \circ \bar\partial = 0)$

Coh. de Dolbeault:

$$H^{p,q}(X) = \frac{\operatorname{Ker} \bar\partial^{p,q}}{\operatorname{Im} \bar\partial^{p,q-1}}$$

Para X sup. de Riemann:

· $H^2(X) \simeq H^{1,1}(X) \oplus \cancel{H^{2,0}(X)}^{0} \oplus \cancel{H^{0,2}(X)}^{0}$

· $H^1(X) \simeq H^{1,0}(X) \oplus H^{0,1}(X)$

· $H^0(X)$

$\dim X = n$ (X var. $\mathbb{C}x$)

$$H^k_{DR}(X, \mathbb{C}) = \bigoplus H^{p,q}(X)$$

$p+q=k$.

$$\overline{H^{p,q}(X)} = H^{q,p}(X).$$

Def: (1-forma meromorfa) (X: sup. de Riemann)

Decimos α 1-forma meromorfa en X

si: α holomorfa en $X \setminus D$, D subcjto. discreto de X, de modo que α se puede escribir localm. como

$$f(z)dz \quad , \quad f \text{ meromorfa.}$$

D: cjto de polos de f.

Sea p un polo de f.

Definimos:

$$\mathrm{Res}_p(f) = \frac{1}{2\pi i}\int_C \alpha$$

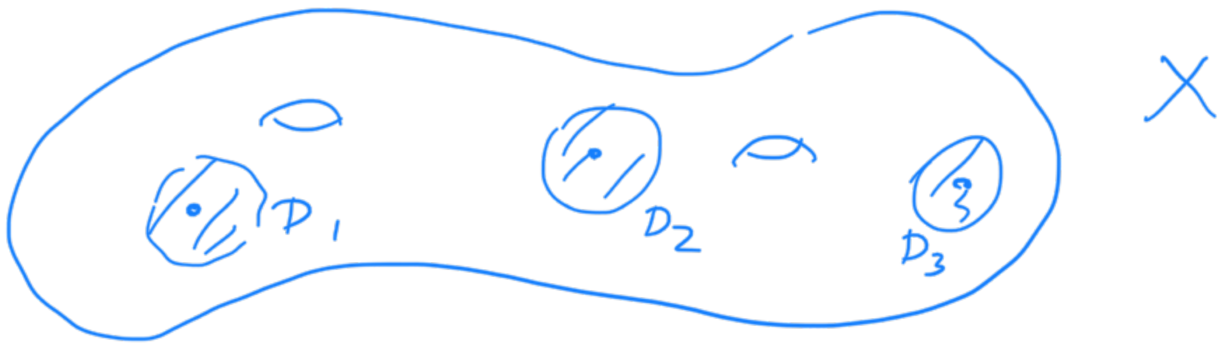
$\uparrow$ pequeño lazo alrededor de p.

Prop: α 1-forma meromorfa en X

... f en X.

$$\sum_{\substack{P_i \in \mathcal{P} \\ \mathcal{P} \text{ polos de } f}} \operatorname{Res}_{P_i}(f) = 0.$$

<u>prueba:</u>



D_i : discos alrededor de los polos.

S = superficie con frontera $\bigcup_{i=1}^{p} \partial D_i$

$\partial D_j = C_j$

Por Stokes

$$\int_{\partial S} \alpha = 0$$

$$\sum_j \int_{C_j} \alpha = 0$$

<u>Próxima clase:</u>

$$\Delta = 2i\,\bar{\partial}\partial$$

Laplaciano

φ armónica si $\Delta\varphi = 0$.

φ armónica $\iff$ $\varphi = \mathrm{Re}(f)$

f holomorfa.