Aeroacoustics

Course Notes

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Contents

List of Figures

Acknowledgements

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Supporting Material

Other than these lecture notes, we recommend the following books:

- Sjoerd W Rienstra and Avraham Hirschberg (2004). “An Introduction to Acoustics”. In: *Eindhoven University of Technology* 18, p. 19
- Avraham Hirschberg and Sjoerd W Rienstra (2004). “An Introduction to Aeroacoustics”. In: *Eindhoven university of technology*

List of Symbols

$\vec{\cdot}$	Vector notation
$\overset{\leftrightarrow}{\cdot}$	Tensor notation
$\vec{u} = (u, v, w)$	Velocity vector and its components
ρ	Density
T	Temperature
$\vec{u}'$	Velocity vector associated with acoustic waves
ρ'	Density fluctuations associated with acoustic waves
T'	Temperature fluctuations associated with acoustic waves
$\overline{\vec{u}}$	Mean flow velocity vector
$\overline{\rho}$	Mean flow density
$\overline{T}$	Mean flow Temperature
$\vec{f}$	Force vector
$\vec{\nabla} f$	Gradient of a scalar field f
$\vec{\nabla} \cdot \vec{f}$	Divergent of a vector field $\vec{f}$
$\vec{\nabla}^2 f$	Laplacian of a scalar field f
$\langle f \rangle$	Statistical mean of f.

1
Introduction

1.1 Motivation

Aeroacoustics has two main elements. The fundamental science behind it is acoustics, a branch of fluid mechanics. The "aero" part is due to the application: the study of acoustic phenomena in aerodynamical systems.

In this "aeroacoustic" course, we assume no prior knowledge of acoustics. As "acoustics" is a prerequisite to the study of "aeroacoustics", we need to cover both. We will try to motivate the theorical acoustics concepts with concrete aeroacoustics problems, which will naturally need to be simplified.

Figure 1.1 illustrates two key industries where aeroacoustocs play important roles: aerospace and wind energy. The figures show empiral maps of the noise sources around an aircraft during take-off, and of a wind turbine. Three main sources are the jet, the trailing edge of the wing, and the landing gears, wind turbine blade tip and tower. To make a point, we will focus on the jet noise.

Figure 1.2 illustrates the workings of a typical jet engine, as well as a high-fidelity simula-

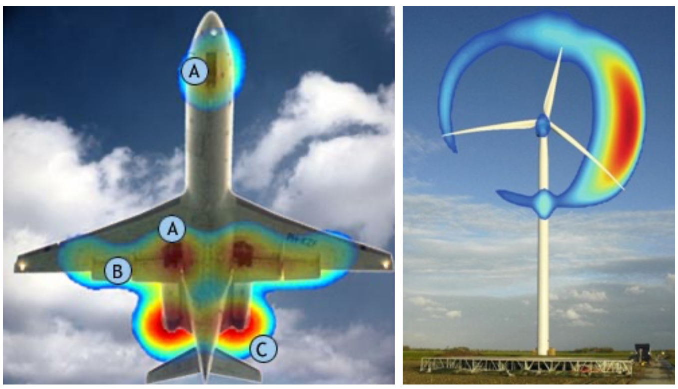
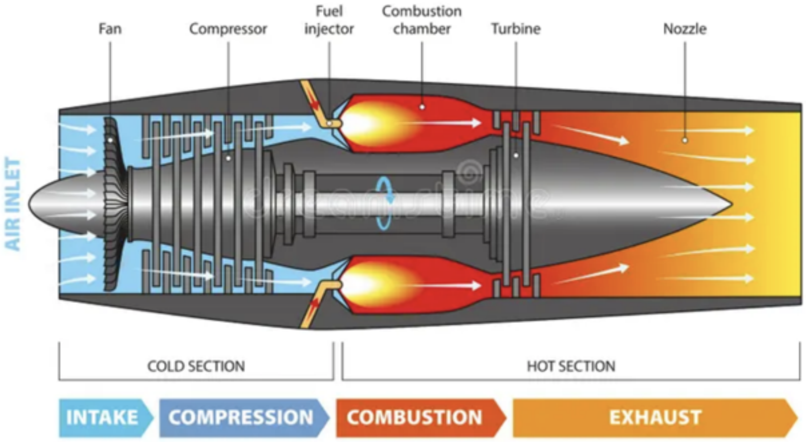
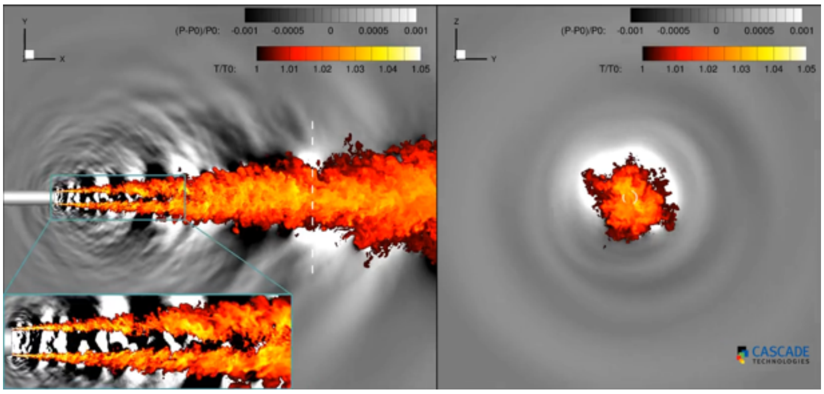


Figure 1.1: Measured noise sources on an airplane during takeoff (left) and a wind turbine.



(a) Jet engine illustration



(b) Formation of noise by turbulet jets (Guillaume A. Brès et al. 2018). Vorticity and pressure are shown in red and gray scales.

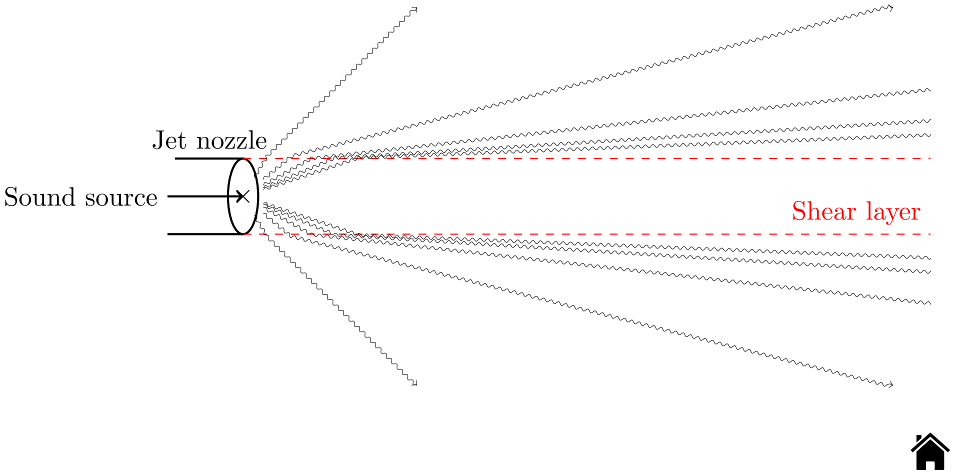
Figure 1.2: Illustration of jet engine and jet noise.

tion of a jet. The main sources of sound are the turbine blades (not shown here), mecanical vibrations, and the flow turbulence. While the blade noise is typically tonal, due to periodic motion of the blades, turbulence noise is typically broadband, due to the random nature of turbulence. Also, noise is only important as long as someone hears it. So, not only understanding how noise is created is important, but also how it propagates to the observer. Focuging on the noise created by the turbine blades, it has to travel within the turbine, interacting with the walls and the flow within, then leave the turbine, interact with the mixing layer, on which there are velocity an temperature gradients, and finally propagate to the observer.

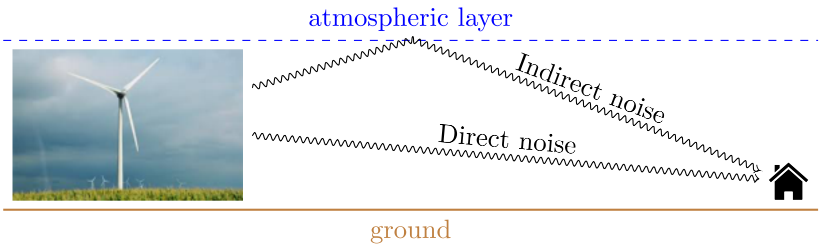
For wind turbines, where the noise is typically generated by the blade tips, the noise propagation is likewise important. Acoustic waves can propagate directly to the observer, or can be reflected, for example, at atmospheric stratification layers (sharp gradients in temperature or wind speed). An illustration of the paths that acoustic waves can take between a source and observers is illustrated in figure 1.3.

All of these stages are important in aeroacoustics. Selecting some key points, we have

1. Noise generation
   (a) By moving bodies (blades, structural vibrations),
   (b) By turbulent structures,



(a)



(b)

Figure 1.3: Illustration sound waves paths.

   (c) By the interaction of turbulence and boundaries,

2. Noise propagation in bounded domains,

3. Noise propagation in flow discontinuities,

4. Noise propagation to the observer.

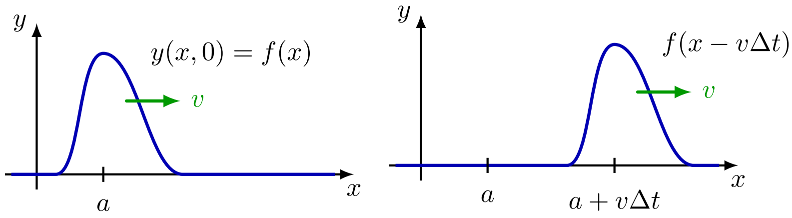
Simplified configurations that explore these phenomena are propagation of acoustic waves in free medias, in ducts, and in the presence of temperature and velocity gradients, noise sources due to moving boundaries and turbulent structures, and the interaction between turbulence and geometrical discontinuities. These will be the main topics of this course, which are explored in the following chapters.

But before that, let's take a broad view of "sound" in general.

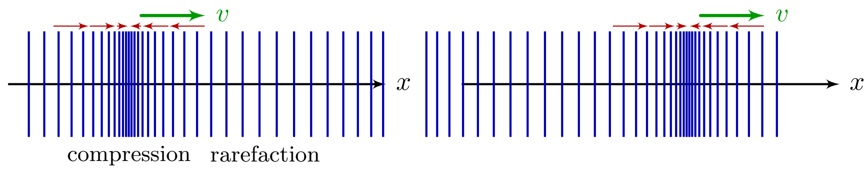
1.2 General aspects of sound

1.2.1 What is sound

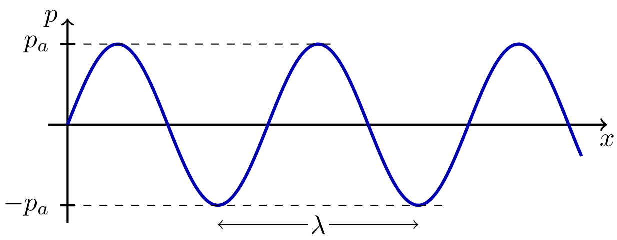
In this discussion of what is sound, we won't bother with formal definitions. Instead, we will just assume common knowledge: sound is the propagation of pressure waves in a medium, usually air.



(a) General non-dispersive wave



(b) Illustration of acoustic waves



(c) Harmonic wave

Figure 1.4: Wave illustration.

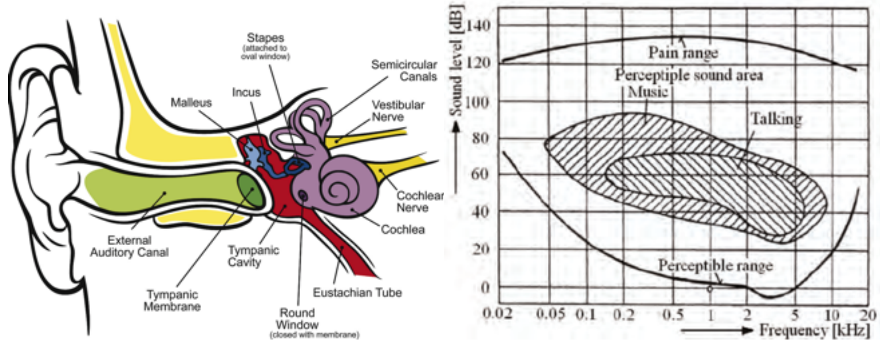
Waves can be described as a perturbation on a medium (air, in this case) that propagates, i.e, travells from one location to the other. If the "shape" os the perturbation is not altered while it travells, the wave is said to be *non-dispersive*, as illustrated in figure 1.4a. If the shape is altered, the wave is said to be *dispersive*.

A dynamic description of sound waves is the following. A region of more compressed air (higher pressure) pushes the air in front of it away. This air thus gains a velocity. As there is a small region where the pressure gradient is present, the now-moving air encounters a region of still air. There is now a new region with more air molecules per unit of volume (higher density), and thus with higher pressure, which slows down the moving air. Now there is a new region of higher pressure displaced a bit from the original one. The cycle thus restarts, and this exchange between pressure and velocity continues for a long time. This is illustrated in figure 1.4b.

1.2.2 Sound orders of magnitude

Sound is typically characterized by a few quantities:

- acoustic displacement: the molecular displacement amplitude
- acoustic velocity: the molecular velocity amplitude
- acoustic pressure: the pressure amplitude
- acoustic power/intensity: how much energy is carried by the wave
- (for "simple" waves) frequency and wavenumber: how fast the wave oscillates in space and time.



(a) Human ear (b) Noise sensitivity

Figure 1.5: Structure of the human ear (left) and its audition and pain limits (right).

For harmonic ("simple") waves (illustrated in figure 1.4c), acoustic displacement, velocity, and pressure are all linked (as will be seen later). For audible waves, frequencies vary between 20 and 20.000Hz, and wavelengths of 17mm to 17m. That's three orders of magnitude. In terms of amplitudes/energy, the scales are even wider. On specific frequencies, we can hear sounds from 20 to 110 decibels (dB). In terms of amplitudes, this corresponds to 4.5 orders of magnitude, and in terms of energy, 9 orders of magnitude. These huge ranges are mainly due to how formidable our hearing is, and they are the reasons why acoustics we typically measure acoustic quantities using logarithm scales.

When we deal with a complex sound, an intensity measure on fluctuations root-mean-square (rms) values is typically used. That is equivalent to the sum of the energies in all frequency bands.

1.3 Psychoacoustics

Figure 1.5 shows a cartography of what are the sensitivities of the human hearing at different frequencies. Our ears are optimally tuned for $\approx 2.000Hz$. As we age, we typically lose hearing at the higher frequency range.

Note that, even if we do not hear anything above 20.000Hz, these frequencies still contribute to the rms. Thus, if we define "loudness" by the rms. Therefore, we can have a very "loud" sound that nobody can hear. When the interest is to study how noise can affect people, we should consider the sensitivity of our ears in how we measure sound. The classical one is to weigh different frequencies differently. For example, a noise can have a large rms, but a zero weighed rms. Different weights can be used in different situations.

To complexify things, our interpretation of sound also varies a lot. One interesting phenomena is the missing fundamental. We evolved to hear harmonics, i.e., a typical sound with frequency f typically also has harmonic frequencies, $2f$,$3f$, $4f$, etc. If we are presented *only* with harmonics, we can still hear the non-existing fundamental note. You can check this yourself[1]. Different sounds, even with the same rms, weighed or not, can be felt very differently. Notions such as noise roughness, intermittency, among others, have been created to try to capture how a given noise affects us. This field of study is called *psychoacoustics*. It lies outside the scope of its course, but it is important to know that it exists.

[1] https://www.youtube.com/watch?v=t-iWKvh6Fbw

1.4 Sound measures

As previously mentioned, sound is typically measured in decibels (dBs). This is a logarithm scale, which is useful to capture the wide range of sound amplitudes. For a harmonic sound, the sound intensity is simply measured as

$$I_{p_a} = 20 \log \left(\frac{p_a}{p_{ref}} \right), \tag{1.1}$$

where I is the sound intensity in dBs, p_a is the pressure amplitude, and p_{ref} a reference pressure value (typically $20 \mu Pa$), as to make the argument of the logarithm non-dimensional.

Alternativelly, sound intensity can be measured in terms of acoustic power, as

$$I_{P_a} = 10 \log \left(\frac{P_a}{P_{ref}} \right), \tag{1.2}$$

where P_a represents the acoustic power amplitude, and P_{ref} a reference power value. As will be seen later, the acoustic power is related to p_a^2. The diffornt constant in front of the logarithm in (1.1) and (1.2) compensates this difference.

When dealing with more complex signals, as stochastic noises, amplitudes are typically not defined, or at least less representative of the signal. In these cases, the root-mean square (rms) value of the signal is typically used. The pressure rms value is defined as

$$p_{rms} = \sqrt{\frac{1}{T} \int_0^T p^2(t) dt}, \tag{1.3}$$

and the sound intensity rms value is defined as

$$I_{p_{rms}} = 10 \log \left(\frac{p_{rms}^2}{p_{ref}^2} \right). \tag{1.4}$$

1.4.1 Sound interference

When a given observer receives acoustic signals from two different sources, these signals may or may not interfere. Considering the case of two harmonic signals of the same amplitude, they can intefere constructively, if they are in phase, doubling the amplitude; destructively, if they are out of phase, completely destroying the signal; or in any intermediary state, if they are neither in phase nor out of phase. In general,

$$\underbrace{\cos(t)}_{\text{Reference signal}} + \underbrace{\cos(t+\phi)}_{\text{Phased signal}} = 2\cos\left(\frac{\phi}{2}\right)\cos\left(t+\frac{\phi}{2}\right) = \begin{cases} 2\cos(t), & \phi = 0 : \text{ constructive int.} \\ 0, & \phi = \pi : \text{ destructive int.} \end{cases} \tag{1.5}$$

A few cases are illustrated in figure 1.6.

A similar dynamics can occur with stochastic signals. But they can only interfere constructively or destructively if they are correlated. If they are uncorrelated, they will combine, but the amplitude is not doubled. The combined signal mean square reads

$$\left\langle (p_1+p_2)^2 \right\rangle = \underbrace{\left\langle p_1^2 \right\rangle}_{\text{rms } p_1} + \underbrace{\left\langle p_2^2 \right\rangle}_{\text{rms } p_2} + 2 \underbrace{\left\langle p_1 p_2 \right\rangle}_{p_1\text{-}p_2 \text{ correlation}}. \tag{1.6}$$

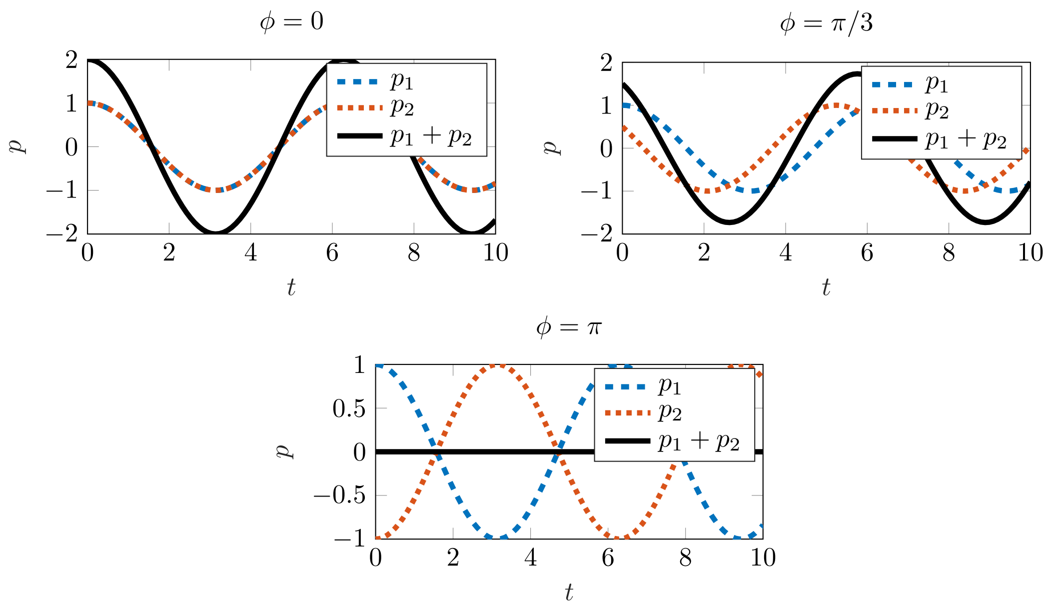


Figure 1.6: Difference types of interference of harmonic signals.

If the signals are idential, $\langle p_1^2 \rangle = \langle p_2^2 \rangle = \langle p_1 p_2 \rangle$, and thus the combined mean square value is multiplied by 4, i.e., the root mean square value is doubled. If the signals are correlated, but oposite ($p_1 = -p_2$), the combined signal is thus zero. Finally, if the signals are uncorrelated, the rms value of the combined signal is $\sqrt{2}$ times greater than the original signals. This are illustrated in figure 1.7.

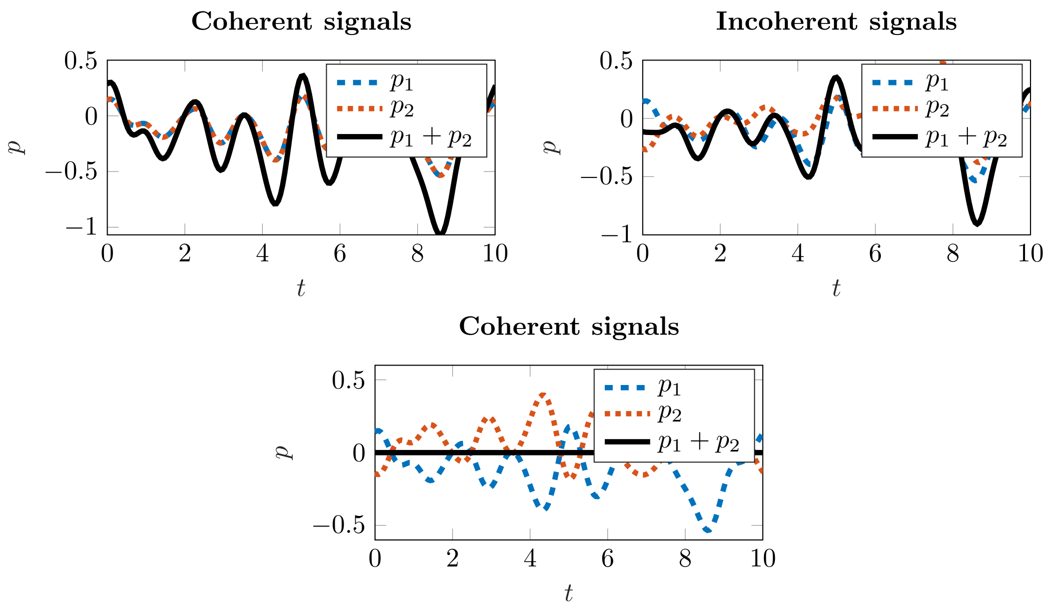


Figure 1.7: Difference types of interference of stochastic signals.

2

Waves on a free media at rest

We will start our study of acoustics investigating waves on a media at rest, i.e., in still air, away from any walls, temperature gradients, etc. This the simplest case we can study in acoustics, but undersanding it will allow us to better interpret what happens in more complex cases in the next chapters.

As stated in chapter 1, the sound intensity has a huge range. You have probably studied the high-intensity limit before: shock waves. These can be viewed as sound waves which exhibit non-linear behavior due to their large amplitudes. In this course, we are interested in the other limit: "small" amplitudes.

By "small" here, we mean sound waves whose amplitudes are small enough such that their dynamics are linear. Do not mistake "small" as "silent". The noise created by the noisiest commercial aircraft is still "small" by this definition[1]

2.1 The wave equation

Acoustics is fundamentally a fluid mechanics problem. We must thus start with the Euler equations,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0 \tag{2.1}$$

$$\rho\Big(\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u}\Big) = -\vec{\nabla} p \tag{2.2}$$

$$\frac{\partial s}{\partial t} + s\vec{\nabla} \cdot \vec{u} = 0, \tag{2.3}$$

written here in terms of density (ρ), velocity ($\vec{u}$), pressure (p), and entropy (s). We must also add an equation of state,

$$s = s(p, \rho) \;\text{ or }\; p = p(\rho, s) \tag{2.4}$$

[1] Although this will likely change with the re-emergence of supersonic flight.

We know perform the the following decomposition,

$$\rho = \overline{\rho} + \rho', \qquad \vec{u} = \overline{\vec{u}} + \vec{u}', \tag{2.5}$$

$$s = \overline{s} + s', \qquad p = \overline{p} + p'. \tag{2.6}$$

where the $\overline{\cdot}$ variables are assumed to be constant in time and space, and the primed variables are assumed small[2].

We will first study a flow at rest ($\overline{\vec{u}} = 0$) and with uniform density and temperature. Re-arranging these equations, we obtain

$$\frac{\partial \rho'}{\partial t} + \overline{\rho}\vec{\nabla} \cdot \vec{u}' = 0, \tag{2.7}$$

$$\overline{\rho}\frac{\partial \vec{u}'}{\partial t} = -\vec{\nabla} p', \tag{2.8}$$

$$\frac{\partial s'}{\partial t} = 0. \tag{2.9}$$

Equation (2.9) implies that the entropy is constant, and thus, assuming that s is also constant in space in the initial state, that the equation of state for an ideal gas can be simplified to

$$p' = c_0^2 \rho', \tag{2.10}$$

where

$$c_0 = \frac{\beta}{\overline{\rho}} = \sqrt{\gamma R \overline{T}}, \tag{2.11}$$

is the sound speed. The bulk modulus $\beta = \partial V/\partial p = \partial(1/\rho)/\partial p$ measures the "stiffness of air", i.e., how hard it is to (adiabatically) compress it.

The equations are then re-written as

$$\frac{1}{c_0^2}\frac{\partial p'}{\partial t} + \overline{\rho}\vec{\nabla} \cdot \vec{u}' = 0, \tag{2.12}$$

$$\overline{\rho}\frac{\partial \vec{u}'}{\partial t} = -\vec{\nabla} p'. \tag{2.13}$$

Our final equation is obtained by taking the time derivative of the first equation, the divergence of the second, and combining them to remove the dependency on $\vec{u}'$. This gives

$$\frac{\partial^2 p'}{\partial t^2} - c_0^2 \vec{\nabla}^2 p' = 0, \tag{2.14}$$

which is known as the *wave equation*. This allows us to solve for the pressure only, which makes it much more convinent, and easy, to work with than the original set of equations. Nevertheless, the physical system has oscilating velocities, temperatures, etc, can be recovered from the pressure using (2.8) and (2.3).

Note 1: The analysis above makes it seems that (2.14) is equivalent to (2.7)-(2.9). It is not. Hidden in our analysis there were some passages where we "potentially multiplied the equations by 0". For example, when we took the divergenceof the momentum equations, any rotational component of the velocity was lost. Likewise, when we used the entropy equation to write (2.3), we neglected the possibility of having entropy perturbations in the flow.

So (2.14) cannot (and does not) models vorticity and entropy waves. All the waves in

[2] In some books, you may see this written as $\vec{u} = \overline{\vec{u}} + \epsilon\vec{u}'$, to construct a Taylor series in terms of ϵ. The result is the same as the one presented here, but we are keeping a simpler notation here.

medias at rest can be easily classified into one of these three classes of waves (acoustic, vorticity, and entropy). Equation (2.14) models only acoustic waves, but as this is the goal of this course, it is perfectly suitable.

Note however, that the classification is only unambiguous for uniforms medias. The presence of flow/velocity gradients can couple these waves, and their classification becomes less clear.

2.1.1 One dimensional waves

To understand why (2.14) is known as the wave equation, it is useful to look at the 1D problem, when (2.14) reduces to

$$\frac{\partial^2 p'}{\partial t^2} - c_0^2 \frac{\partial^2 p'}{\partial x^2} = 0. \tag{2.15}$$

The differential equation can be factored as

$$\left(\frac{\partial p'}{\partial t} + c_0 \frac{\partial p'}{\partial x}\right)\left(\frac{\partial p'}{\partial t} - c_0 \frac{\partial p'}{\partial x}\right) p' = 0. \tag{2.16}$$

Thus, any function p' which is a function of $x + t/c_0$ or $x - t/c_0$ will be a solution of (2.16). A general solution can thus be written as

$$p'(x,t) = h(x + c_0 t) + g(x - c_0 t). \tag{2.17}$$

The terms h and g can be interpred as left- and right-travelling waves. To see why, assume $h = 0$. Then $p(x, 0) = g(x)$, i.e.n g gives the pressure distribution. At a latter time t, the pressure can be written as $p(x,t) = h(x - x_t)$, where $x_t = c_0 t$. That is, the pressure has the same spatial support than it had at time $t = 0$, but translated to the right by x_t. The same resoaning shows that g travels to the left.

Similarly,

$$p'(\vec{x},t) = f(\vec{d} \cdot \vec{x} - t/c_0), \tag{2.18}$$

is a solution of (2.14), with $\vec{d}$ a unity vector which gives the propagation velocity of the wave.

The quantity $t - c_0 \vec{d} \cdot \Delta\vec{x}$ is called the *retarded time*. It considers the time it takes for a wave emitted at the point $\vec{x}$ to reach the point $x + \vec{\Delta} \vec{x}$. This is a concept which will be revisited in chapter 6.

2.2 Acoustic Energy

To understand how acoustic waves transport energy, we need to obtain an energy conservation equation. We can obtain it by multiplying (2.19) by $p'/\overline{\rho}$ and taking the scallar product of (2.20) with $\vec{u}'$,

$$\frac{p'}{\overline{\rho} c_0^2} \frac{\partial p'}{\partial t} + p' \vec{\nabla} \cdot \vec{u}' = 0, \tag{2.19}$$

$$\overline{\rho} \vec{u}' \cdot \frac{\partial \vec{u}'}{\partial t} = -\vec{u}' \cdot \vec{\nabla} p'. \tag{2.20}$$

Adding the two equations gives

$$\frac{1}{2\overline{\rho} c_0^2} \frac{\partial p'^2}{\partial t} + \frac{\overline{\rho}}{2} \frac{\partial |\vec{u}'|^2}{\partial t} + \vec{\nabla} \cdot (p' \vec{u}') = 0. \tag{2.21}$$

When compared to the general form of a conservation law,

$$\frac{\partial \mathcal{E}}{\partial t} + \vec{\nabla} \cdot \vec{\mathcal{J}} = \mathcal{D}, \tag{2.22}$$

where $\mathcal{E}$, $\vec{\mathcal{J}}$, and $\mathcal{D}$ correspond to the energy fluctuation, the energy flux, and source/sink terms, we can identify

$$\mathcal{E} = \frac{1}{2\overline{\rho}c_0^2}p'^2 + \frac{\overline{\rho}}{2}||\vec{u}'||^2 \tag{2.23}$$

$$\vec{\mathcal{J}} = p'\vec{u}' \tag{2.24}$$

$$\mathcal{D} = 0 \tag{2.25}$$

The last equation indicates that there is no source or sink (e.g., dissipation) term for acoustic waves. This is of course not true in general. These terms will be addressed later.

Note 2: The definition of acoustic energy is more subtle than this section suggests. While the analysis here is straight forward for a media at rest, when the meanflow is not homogenious, there are any different definitions of what *energy* means. One approach is to linearize the physical energy: this seems the most logical approach, but leads to some unexpected results, as negative-energy waves, i.e., waves with a negative energy associated to them. The results here will suffice for the pourposes of this course, but the interested reader is encouraged take a look on a paper by Myers 1986 for a more detailed discussion.

2.3 Harmonic waves

In acoustics, it is frequent to conduct analysis in the frequency domain. In fact, this is the framework we will use in most of the course. Some time domain analaysis will be presented in the last chapter.

Taking a Fourier transform (in time) of (2.14) (which, for the purposes here is equivalet to using the ansatz $p' = \hat{p}'(\vec{x})e^{-i\omega t}$), gives

$$\left(-\omega^2 - c_0^2\vec{\nabla}^2\right)\hat{p}' = 0. \tag{2.26}$$

A second Fourier transform, now in space, i.e. $\hat{p}' = \hat{\hat{p}}'e^{i\vec{k}\cdot\vec{x}}$, gives

$$\left(-\omega^2 + c_0^2||\vec{k}||^2\right)\hat{\hat{p}}' = 0. \tag{2.27}$$

This equation is satisfied if

$$-\frac{\omega^2}{c_0^2} + ||\vec{k}||^2 = 0. \tag{2.28}$$

This expression contains only the wave frequency and wavenumber. Such expressions are known as *disperion relations*. This function is illustrated in figure 2.1 for a fixed value of ω. The pressure field is given by

$$p'(\vec{x}, t) = \hat{\hat{p}}'e^{-i(\omega t - \vec{k}\cdot\vec{x})}. \tag{2.29}$$

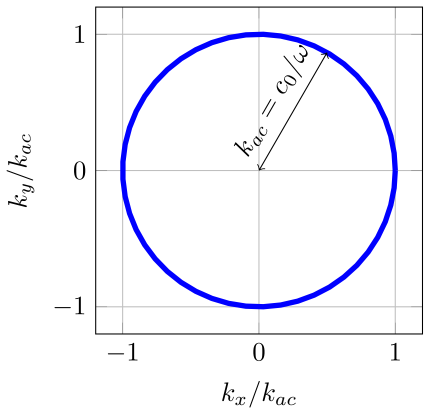


Figure 2.1: Visual representation of the dispersion relation for acoustic waves on a media at rest.

From acoustic velocities and density fluctuations can be obtained using (2.29) into and (2.8), from which

$$\rho'(\vec{x},t) = \frac{\hat{p}'}{c_0^2} e^{-i(\omega t - \vec{k}\cdot\vec{x})} \tag{2.30}$$

$$\vec{u}'(\vec{x},t) = \vec{k}\frac{\hat{p}'}{\omega\overline{\rho}} e^{-i(\omega t - \vec{k}\cdot\vec{x})} = \frac{\vec{k}}{||\vec{k}||}\frac{\hat{p}'}{c_0\overline{\rho}} e^{-i(\omega t - \vec{k}\cdot\vec{x})} \tag{2.31}$$

You can check by yourself that this solution does indeed correspond to plane waves traveling in a direction given by $\vec{k}$. This wave is called a *plane wave*. Understanding it is one of the key concepts behind the interpretation of acoustics phenomena.

It is thus worthy to explore some of its properties.

2.3.1 Acoustic wavenumber

Equation 2.29 can be put in the same form as in 2.18, with $\vec{d} = \frac{\vec{k}}{\omega/c_0}$. Note that (2.28) imposes that

$$||\vec{k}|| = \frac{\omega}{c_0} = k_{ac}, \tag{2.32}$$

for any valid solution, and thus $\vec{d}$ is a unit vector. The quantity k_{ac} is the acoustic wavenumber, which is a function of frequency and temperature (speed of sound).

2.3.2 Plane wave energy

Equations (2.23)-(2.25) can be used to analyze how plane waves propagate energy. But care must be taken before using (2.27) into these expressions. The energy equations are non-linear, so the complex form of (2.27) cannot be used. Instead, we need to reconstruct real pressure and velocity fields as

$$p' = \Re\left(\hat{p}' e^{-i(\omega t - \vec{k}\cdot\vec{x})}\right), \tag{2.33}$$

$$\vec{v}' = \Re\left(\frac{\vec{k}}{||\vec{k}||}\frac{\hat{p}'}{c_0\overline{\rho}} e^{-i(\omega t - \vec{k}\cdot\vec{x})}\right). \tag{2.34}$$

Without loss of generality, let's assume $\hat{p}'$ real. Then (2.23)-(2.24) become

$$\mathcal{E} = \frac{1}{\overline{\rho}c_0^2}\hat{p}'^2\cos^2\left(\omega t - \vec{k}\cdot\vec{x}\right) \tag{2.35}$$

$$\vec{\mathcal{J}} = \hat{p}'^2\frac{\vec{k}}{||\vec{k}||}\frac{1}{c_0\overline{\rho}}\cos^2\left(\omega t - \vec{k}\cdot\vec{x}\right). \tag{2.36}$$

Some noteworthy results. The acoustic energy density and the acoustic energy fluxes are both quadratic in the pressure amplitude, and the energy flux is aligned with to which the wave propagates. Their magnitude is a function only of the pressure amplitude and the flow properties (mean density and speed of sound). They also vary in time, each having a mean value of

$$\langle\mathcal{E}\rangle = \frac{1}{2\overline{\rho}c_0^2}\hat{p}'^2 \tag{2.37}$$

$$\left\langle\vec{\mathcal{J}}\right\rangle = \hat{p}'^2\frac{\vec{k}}{||\vec{k}||}\frac{1}{2c_0\overline{\rho}} \tag{2.38}$$

One intuitive, but important result here, is that the energy flux direction coincides with the direction of the wave number $\vec{k}$ *for a media at rest.*

2.4 Phase velocitiy

When describing the velocity of an acoustic wave, one needs to define what he/she means by velocity. Phase velocity is one possibility. The core idea behind it is to track a pressure peak as the wave as time passes. This, however, can be ambiguous. Let's use the plane wave as an example. For simplicity, let us limit ourselves to a 2D case and use $\vec{k} = (k_x, k_y)$, $||\vec{k}|| = k_{ac}$.

Now we need to define along which line we want to track the pressure peak. The most natural choice is to do so in the propagation direction. Being s a coordinate in the direction of $\vec{k}$, i.e. $\vec{x} = \frac{\vec{k}}{k_{ac}}s$, the pressure field of a planar wave, (2.29), becomes

$$\begin{aligned} p'(s,t) &= \hat{p}'e^{-i(\omega t - \vec{k}\cdot\vec{x})} \\ &= \hat{p}'e^{-i(\omega t - ||\vec{k}||s)} \\ &= \hat{p}'e^{-i||\vec{k}||(c_0 t - s)}, \end{aligned} \tag{2.39}$$

which reduces to the same expression found in section 2.1.1, and corresponding with a wave travelling with speed c_0, i.e., the speed of sound. This feels intuitive, but it is not always the case. Let us assume now a line not aligned with the wave propagation velocity. Using $s = x$, the same analysis gives us

$$\begin{aligned} p'(s,t) &= \hat{p}'e^{-i(\omega t - k_x x)} \\ &= \hat{p}'e^{-i\frac{1}{k_x}\left(c_0\frac{||\vec{k}||}{k_x}t - s\right)}, \end{aligned} \tag{2.40}$$

as $||\vec{k}||/k_x$ is always larger than one, this would indicate a wave traveling faster than the speed of sound.

This gives us a general result. The phase speed of acoustic waves is always *equal or greater* than the speed of sound on a medium at rest. This result will have important applications later when we will discuss aeroacoustic sources of sound.

2.5 Impedance

Impedance another key concept in acoustics. It is defined as the relation between acoustic pressure and velocity,

$$\hat{z} = \frac{\hat{p}'}{\hat{\vec{v}} \cdot \vec{n}}, \tag{2.41}$$

along a surface, with $\vec{n}$ indicating the surface normal. Although it can also be defined in the time domain, the definition in the frequency domain, as above, is more commonly used.

Using (2.31), the specific impedance of plane wave impinging normally on plane ($\vec{n} = \vec{k}/||\vec{k}||$) can be computed as

$$\hat{z} = \frac{\hat{\hat{p}}' e^{-i(\omega t - \vec{k}\cdot\vec{x})}}{\dfrac{\vec{k}\cdot\vec{n}}{||\vec{k}||}\dfrac{\hat{\hat{p}}'}{c_0\bar{\rho}} e^{-i(\omega t - \vec{k}\cdot\vec{x})}} = c_0\bar{\rho}, \tag{2.42}$$

which again is a function of the local fluid properties only.

To motivate the definition of impedance, it is worthy imagining what the flow impedance is close to a walls. Lets consider two cases, a hard and a soft wall. Near a hard wall, i.e., the classical notion of a wall, something that does not move, the non-penetration condition implies that $\hat{\vec{u}}' \cdot \vec{n} = 0$, thus $\hat{z} = \infty$. An example of a soft wall is a light membrane that has neglegible mass and stiffness, and thus which moves with the flow without any resistance. Near this wall, the boundary condition becomes $\hat{p}' = 0$, as any pressure fluctuation would create an inifite displacement in the membrane. In this case $\hat{z} = 0$.

Impedance thus becomes an usefull concept, for example, to *model* the behaviours of different walls. What would happen to the acoustics of my jet engine if the jet walls were less rigid? This can be investigated modelling the wall dynamics by a scalar quantity, its impedance, instead of devellopping complex structural models. What is the effect of wall structural damping in the acoustics? Again, impedances can be used to investigate that. This example will be investigated in more detail in the next chapter.

2.5.1 The role of impedance as a boundary condition

Note that, from (2.26), the wave equation on the frequency domain becomes a Helmholtz equation. To solve it, one needs only to impose boundary conditions. Typical boundary conditions are

- $\hat{p}' = 0$ on a wall, i.e, a Diriclet condition,
- $\hat{\vec{u}}' \cdot \vec{n} = 0$ on a wall, which imples $\vec{\nabla}\hat{p}' \cdot \vec{n} = 0$, i.e, a Neumann condition, and,
- $a\vec{\nabla}\hat{p}' \cdot \vec{n} + b\hat{p}' = 0$ on a wall, i.e., a Robin condition.

From (2.41), we can not that the impedance is prportional to the ratio between the pressure and its gradient normal to the surface. Thus, on a Robin condition, we have that $\hat{z} \propto a/b$, that is, specifing the imperance at the walls is sufficient to fully determine the problem.

This has some intersting uses. For example, if one aims to study a complex duct configuration that connects ducts with different sized on each end, one can solve only for the complex region while specifying the impedance of the two side ducts to account for them.

In practice, this is not an easy task, but the possibilty highlights the physical role of impedance. By specifying the ratio between "forces" (pressure), and "resulting movement" (velocities), the impedance can fully characterise the acoustics properties of a given surface.

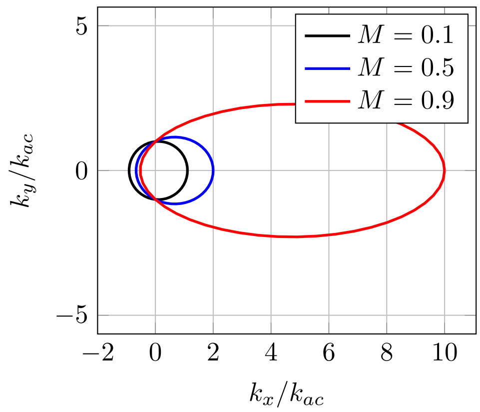


Figure 2.2: Dispersion relation for acoustic waves on a media with a moving media.

2.6 Effect of mean flow velocity

When the media is not at rest, but instead has a mean flow velocity $\vec{\overline{u}}$, (2.7)-(2.9) become

$$\left(\frac{\partial}{\partial t} + \vec{\nabla} \cdot \vec{\overline{u}}\right) \rho' + \overline{\rho} \vec{\nabla} \cdot \vec{u}' = 0, \tag{2.43}$$

$$\overline{\rho} \left(\frac{\partial}{\partial t} + \vec{\nabla} \cdot \vec{\overline{u}}\right) \vec{u}' = -\vec{\nabla} p', \tag{2.44}$$

$$\left(\frac{\partial}{\partial t} + \vec{\nabla} \cdot \vec{\overline{u}}\right) s' = 0. \tag{2.45}$$

and the wave equation becomes

$$\left(\frac{\partial}{\partial t} + \vec{\nabla} \cdot \vec{\overline{u}}\right) p' - c_0^2 \vec{\nabla}^2 p' = 0. \tag{2.46}$$

and the dispersion relation,

$$\left(\left(\omega + \vec{\overline{u}} \cdot \vec{k}\right)^2 + c_0 \vec{\nabla}^2\right) \hat{p}' = 0. \tag{2.47}$$

Assuming that the flow moves in the x direction with a Mach number $M = \frac{||\vec{\overline{u}}||}{c_0}$, the dispersion relation can be expressed as

$$-\left(\frac{\omega}{c_0} + M k_x\right)^2 + ||\vec{k}||^2 = 0. \tag{2.48}$$

3

Acoustics in the presence of passive walls

In this chapter, we will consider the effect of walls. First, we will study how these walls affect incident waves, and then how wall movement can create acoustic waves.

3.1 Plane waves impinging on a hard wall

We will study the following problem: a planar harmonic acoustic wave impinges on a wall. The wall is located at $x = 0$. The acoustic wave is impinging from above and has a wave vector $\vec{k}_i = (k_{x,i}, k_{y,i})$, with $k_{y,i} < 0$, and complex amplitude $\hat{p}'_i$.

To solve this problem, we need to define what is the differential equation we need to solve, and what are the boundary conditions. Far from the wall, the boundary condition restricts the *incident waves*, with the *incident pressure* field given by

$$p'_i = \hat{p}'_i e^{-i(\omega_i t - \vec{k}_i \cdot \vec{x})}. \tag{3.1}$$

Note however that this does not fully specify the pressure at $y \to \infty$, as there can be reflected waves contributing to it as well.

Assuming a hard wall, we have that the vertical component of the acoustic velocity (v') must be zero. The implication of the boundary condition on the pressure field can be determined from (2.3) as

$$\frac{dp'}{dy} = 0. \tag{3.2}$$

Finally, as seen in the previous section, the acoustic pressure field needs to satisfy the wave equation (2.14). Away from the wall, plane waves are solutions of this equation. However, the presence of the wall can couple different plane waves, i.e., pure plane waves are not full solutions to the problem, as they do not satisfy the boundary conditions.

Let us thus assume that the full pressure field is the combination of two waves, the incident and a reflected wave,

$$p' = \hat{p}'_i e^{-i(\omega_i t - \vec{k}_i \cdot \vec{x})} + \hat{p}'_r e^{-i(\omega_r t - \vec{k}_r \cdot \vec{x})}, \tag{3.3}$$

and we will try to determine $\hat{p}'_r$, ω_r and $\vec{k}_r$ to satisfy all the boundary conditions. Note that this ansatz automatically satisfies the wave equation, so we don't need to use it anymore.

At $y = 0$, the boundary condition, (3.2), reads

$$-\vec{k}_{y,i}\hat{p}'_i e^{-i(\omega_i t-\vec{k}_{x,i}x)} - \vec{k}_{y,r}\hat{p}'_r e^{-i(\omega_r t-\vec{k}_{x,r}x)} = 0. \tag{3.4}$$

The right-hand side is zero for any t and x, and thus the equation can only be satisfied if the left-hand is also null. This imposes the following conditions

$$\omega_i = \omega_r, \tag{3.5}$$
$$\vec{k}_{x,i} = \vec{k}_{x,r}, \tag{3.6}$$
$$\vec{k}_{y,i}\hat{p}'_i = -\vec{k}_{y,r}\hat{p}'_r, \tag{3.7}$$

This tells us that the reflected wave has the same frequency and x wave number as the reflected wave. We need thus to solve only for $\vec{k}_{y,r}$ and $\hat{p}'_r$. We thus need another equation to complement (3.7). We can use the dispersion relation, (2.28),

$$k_{x,i}^2 + k_{y,i}^2 = \frac{\omega_i^2}{c_0^2}, \tag{3.8}$$
$$k_{x,r}^2 + k_{y,r}^2 = \frac{\omega_r^2}{c_0^2}. \tag{3.9}$$

Using (3.5) and (3.6),

$$k_{y,r} = \sqrt{\frac{\omega_i^2}{c_0^2} - k_{x,i}^2} = \pm k_{y,i}. \tag{3.10}$$

Here we have to discard the positive sign, otherwise, the reflected and incident waves would be the same wave. Keeping the negative sign, equation (3.7) gives

$$\hat{p}'_r = \hat{p}'_i. \tag{3.11}$$

In short, the incident wave is reflected as a wave with the same pressure amplitude, and the same wave-vector, only with the wall-normal component reversed. The total pressure field reads

$$\begin{aligned} p'(t,\vec{x}) &= \hat{p}'_i e^{-i(\omega_i t-k_{x,i}x)}\left(e^{-ik_{y,i}y} + e^{ik_{y,i}y}\right) \\ &= \hat{p}'_i e^{-i(\omega_i t-k_{x,i}x)}\cos(k_{y,i}y), \end{aligned} \tag{3.12}$$

from which, using (2.8), the acoustic velocity is obtained as

$$\vec{u}'(t,\vec{x}) = \frac{\hat{p}'_i}{i\omega\bar{\rho}}\Big(-ik_{x,i}\cos(k_{y,i}y), -k_{y,i}\sin(k_{y,i}y)\Big)e^{-i(\omega_i t-k_{x,i}x)}, \tag{3.13}$$

We can compute the energy flux and density of this solution. From (2.24) we have

$$\mathcal{J}_x = \frac{\hat{p}_i'^2}{\omega\bar{\rho}}k_{x,i}\cos^2(k_{y,i}y)\cos^2\left(\omega_i t - k_{x,i}x\right), \tag{3.14}$$
$$\mathcal{J}_y = \frac{\hat{p}_i'^2}{\omega\bar{\rho}}k_{y,i}\sin(k_{y,i}y)\cos(k_{y,i}y)\cos(\omega_i t - k_{x,i}x)\sin(\omega_i t - k_{x,i}x), \tag{3.15}$$

and

$$\langle \mathcal{J}_x \rangle = \frac{\hat{p}_i'^2}{2\omega\bar{\rho}} k_{x,i} \cos^2(k_{y,i} y), \tag{3.16}$$

$$\langle \mathcal{J}_y \rangle = 0. \tag{3.17}$$

That is, the net energy transport in the vertical direction is zero. This is a consequence of the incident and reflected waves carrying the same amount of energy in opposite directions, which thus cancels out. However, both waves carry energy in the same direction along x, achieving thus net transport along this direction.

Note 3: Note that in the discussion above, we implicitly assume that the energy fluxes can be simply added, i.e., a superposition principle. Although this picture here gives the correct image, it is not always correct as (2.24) is a non-linear expression. We will discuss this latter on an academic case and also show implications on a real aeroacoustic phenomenon: the noise coming from the interaction between a jet and a wing.

3.2 Acoustic channel

An acoustic channel consists of two infinte walls, which we will consider here to be located at $y = \pm h$. Following the same reasoning as for the previous sections, the pressure field in such a channel must satisfy the wave equation and the following boundary conditions:

$$\left.\frac{dp}{dy}\right|_{y=\pm 1} = 0. \tag{3.18}$$

In contrast with the previous example, we will not assume an incident wave. Instead, we will fix the angular frequency (ω) and explore what types of waves can exist in this problem.

In the previous sections, we noticed that all the waves present had the same wavenumber in x and that the two possible waves had opposite values of the wavenumber in y. We will thus assume a solution of the type

$$p'(t, x, y) = Ae^{-i(\omega t - k_x x - k_y y)} + Be^{-i(\omega t - k_x x + k_y y)}, \tag{3.19}$$

and the boundary conditions at $y = h$ and $y = -h$ read, respectivelly,

$$ik_y Ae^{-i(\omega t - k_x x - k_y h)} - ik_y Be^{-i(\omega t - k_x x + k_y h)} = 0, \tag{3.20}$$

$$ik_y Ae^{-i(\omega t - k_x x + k_y h)} - ik_y Be^{-i(\omega t - k_x x - k_y h)} = 0. \tag{3.21}$$

For these equations to be satisfied for any y and x, we need

$$Ae^{ik_y h} = Be^{-ik_y h}, \tag{3.22}$$

$$Ae^{-ik_y h} = Be^{ik_y h}, \tag{3.23}$$

which is satisfied for

$$e^{i4k_y h} = 1 \tag{3.24}$$

Which imposes a condition of the allowable values of k_y,

$$4k_y h = 2n\pi. \tag{3.25}$$

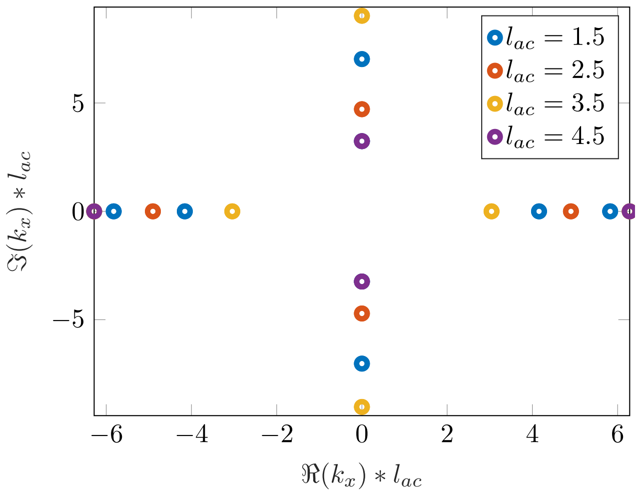


Figure 3.1: Dispersion relation for an acoustic channel. For all l_{ac}, there are points at $\Re k_x = \pm\pi$. As the l_{ac} increases, less modes have real wave numbers. For $l_{ac} > 4h$, only one mode is propagative.

The coefficients read,

$$A = Be^{in\pi} = \begin{cases} 1 & \text{for even n,} \\ -1 & \text{for odd n.} \end{cases}. \tag{3.26}$$

The pressure field is then Considering that ω, k_x, and k_y are connected by the dispersion relation, (2.28), we have

$$-\omega^2 + c_0^2\left(k_x^2 + \frac{n^2\pi^2}{4h^4}\right) = 0. \tag{3.27}$$

Now, differntly from (2.28), in (3.27) the wavenumber in y is discretize ($k_z = n\pi/2h$). Figure 3.1, illustrated this dispertion relation. Note that, while figure 2.1 had lines for a given frequency, in figure 3.1 has points. Again, this is a consequence of the discretization of the wavenumber in y.

As ω is fixed, we can solve for k_x,

$$k_x = \pm\sqrt{\frac{\omega^2}{c_0^2} - \frac{n^2\pi^2}{4h^2}} = \pm\sqrt{k_{ac}^2 - \frac{n^2\pi^2}{4h^2}}. \tag{3.28}$$

The pressure field reads

$$p'(t,x,y) = \begin{cases} Ae^{-i(\omega t - k_x x)}\cos\left(\frac{n\pi y}{2h}\right) & \text{for even } n \\ Ae^{-i(\omega t - k_x x)}\sin\left(\frac{n\pi y}{2h}\right) & \text{for odd } n \end{cases}. \tag{3.29}$$

3.2.1 Propagative waves

We will start the analysis of the different types of waves assuming that the square root argument in (3.28) is positive. A case where this is always true is when $n = 0$. In this case

$$k_x = \pm k_{ac}, \tag{3.30}$$

and

$$p'(t,x,y) = Ae^{-i(\omega t \pm k_{ac} x)}. \tag{3.31}$$

Comparing to (2.29), we see that this is a place wave traveling in the $\pm x$ direction within the acoustic duct.

For $n = 1$, we get

$$p'(t,x,y) = Ae^{-i(\omega t \pm k_x x)} \sin\left(\frac{\pi y}{2h}\right). \tag{3.32}$$

Compared to the solution of the wave reflected by a single wall, we see that we have a similar expression. This gives us a hint of what is going on: acoustic waves are bouncing in between the walls. As in that case, there is no net energy being carried in the y direction, but a net energy flux in the x direction.

We say that these waves are *progative waves*, as they propagate indefitetly inside the duct.

3.2.2 Evanescent waves

We will now consider the oposite scenario, when the square root argument in (3.28) is negative. In this case, k_y is a pure imaginary number, and the pressure field can be re-written as (assuming n even: the results are similar for odd n)

$$p'(t,x,y) = Ae^{\pm|k_x|x} e^{-i\omega t} \cos\left(\frac{n\pi y}{2h}\right). \tag{3.33}$$

Note that the amplitude of these wave ($Ae^{\pm|k_x|x}$) decay exponentially in the $\mp$ direction. They are thus said to be *evanescent waves*, as they eventually "disapear".

The critical condition for the sign change of square root argument in (3.28) is

$$\omega_{n,\text{cut-on}} > -\frac{n\pi}{2hc_0}. \tag{3.34}$$

For frequencies lower than $\omega_{n,\text{cut-on}}$ the n-th harmonic is evanescent, i.e., does not propagate in the duct. As the frequency is increased, more and more harmonic become propagative, and the pressure field becomes more complex.

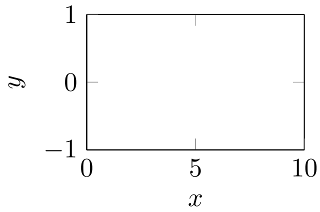
Note 4: Compute the energy flux of each wave $\pm$ indepentently, and when both are present. Even if one evanscent wave does not transports energy, the presence of two waves does.

3.2.3 Physical interpretation

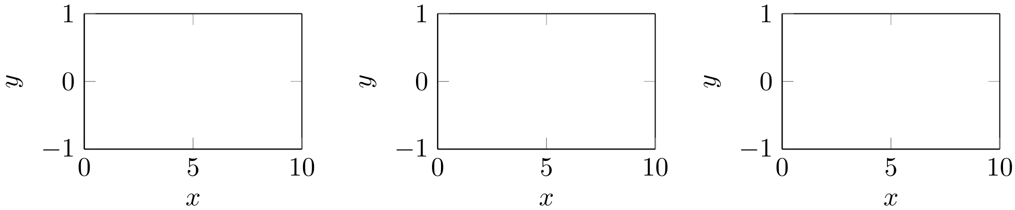
Figure 3.2 shows the first 4 harmonics on a duct. Frequency here was selected such that the first three modes are propagative, while the fourth is evanescent.

The first mode is clearly a plane wave, which travels int he $x+$ direction. The second mode can also be understood as a plane wave, but now travelling with an angle with respect to the x axis. This wave is thus reflected on the walls, creating a standing wave pattern in y. The third mode follows the same trend, but now, the wave travells with a larger angle with respect to the x axis. Finally the fourth mode, which is evanescent.

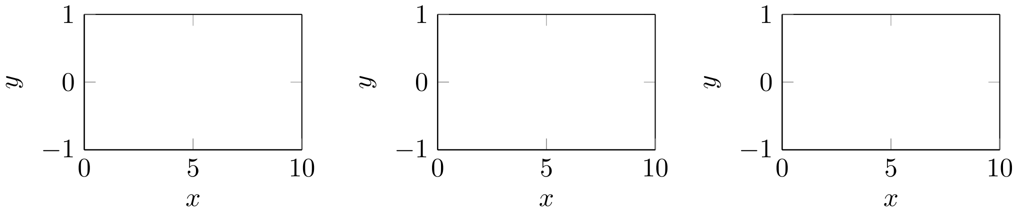
The first harmonic can be understood as a mode that, during each period in x, bounces once in the wall, while the second harmonic bounces twice. While, by inducion, we assume that the third harmonic should bounce three times, the wavenumber in the y direction is larger



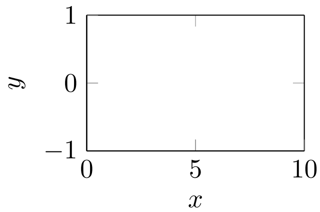
(a) Plane wave



(b) First harmonic



(c) Second harmonic



(d) Third harmonic

Figure 3.2: Illustration of acoustic modes in a duct. The first mode is a plane wave, while the second and third modes (first and second harmonics) are standing waves. The fourth mode (third harmonic) is an evanescent wave.

than the acoustic wavenumber, and thus the wavenumber in x becomes complex, and the wave evanescent. This happens as the wavelenght in the y direction becomes smaller than the acoustic wavelength, and thus cancelationg between the positive and negative pressure peaks occur.

3.2.4 Aeroacoustic applications

Acoustic ducts are present in many acoustic problems. For jet engines they are particularly important element in rotor noise. The rotor blades are typically located in a duct, and the noise generated by the rotor is transmitted through the duct. If the system can be built in such an way that all, or most of, the noise generated comes in the form of evanescent waves, this would drastically minimize the noise that leaves the engine. In fact, this phenomena is the basis for some ventilators found on the market: by encolusing the fan, and some clever engineering, the noise generated by the fan can be minimized.

In practice, this can only be achived when the fans blade velocity is slower than the speed of sound, which is not the case for jet engines. Alternative strategies consist in designing the duct walls to absorb part of the acoustic energy.

3.3 Plane wave approximation

The results above have one important consequence. Consider the case where h is small. More precisely, when

$$h < \frac{\pi}{2hc_0\omega}. \tag{3.35}$$

In this regime, the condition (3.34) is only satisfied for $n = 0$. This means that only the fundamental harmonic, i.e., plane waves can propagate in the duct, while the higher harmonics are evanescent. That means that even if force the flow with a complex pressure field (examples of that will be given later), only the fundamental harmonic will propagate. This is the basis of the plane wave approximation.

More generally, no matter what happens in the channel, such as a sudden change in the duct, the pressure field will converge back a plane wave . This simplification allows for many simplifications, which allow for analytical solutions of some problems.

In practice, there is also a lower limit for the duct height, below which the plane wave approximation starts to break apart. As mentioned in chapter 1, viscous effects are typically neglegible in acoustic phenomenon, as the acoustic length scales are typically much larger than the viscous length scales.

However, when the duct height becomes comparable to the viscous length scales, the viscous effects become important. At the walls, the non-slit conditions imply that there is always a scale on which they become important. This is analougous to the formation of classical boundary layers. While over a flat plate, the boundary layer thickness occurs due to a balance of the viscous and convective time-scales, in oscilating flows it is due to a balance in the oscilating and time-scales.

The penetration depth,

$$\delta = \sqrt{\frac{2\nu}{\omega}}, \tag{3.36}$$

is the characteristic length scale of the viscous effects on an oscillating flow. When the duct height is smaller than this, the viscous effects become important, leading to dissipation of acoustic energy.

Place wave approximations are thus valid when

$$\delta \ll h < \frac{\pi}{2hc_0\omega}. \tag{3.37}$$

However, for most practical applications, δ is small, and thus can be neglected. In air, for a frequency of $100Hz$, δ is of the order of $0.1mm$.

3.4 Finite channels

We wil consider a channel with one open and one closed end, as illustrated in figure 3.3. The boundary conditions can thus be written as

$$p'(t, x = 0, y) = 0 \tag{3.38}$$

$$\frac{dp'}{dx}(t, x = L, y) = 0 \tag{3.39}$$

$$\frac{dp'}{dy}(t, x = 1, y) = 0 \tag{3.40}$$

$$\frac{dp'}{dy}(t, x = -1, y) = 0 \tag{3.41}$$

The solutions found for the infinite channel can be leveraged to solve the finite channel. When we passed from a free acoustic wave to the reflection by an infinite wall, down- and up-traveling waves become coupled due to the presence of the wall. Here, a similar effect will occcur between left- and right-traveling waves.

Using as ansatz two duct waves (for even n), travelling at differnet directions, we have

$$p'(t, x, y) = Ae^{-i(\omega t - k_x x)} \cos\left(\frac{n\pi y}{2h}\right) + Be^{-i(\omega t + k_x x)} \cos\left(\frac{n\pi y}{2h}\right), \tag{3.42}$$

where k_x is given by (3.28). This solution already satisfies (3.40) and (3.41). But we still need to impose (3.38) and (3.39). The latter boundary conditions can be written as

$$Ae^{-i(\omega t)} = -Be^{-i(\omega t)}, \tag{3.43}$$

$$ik_x Ae^{-i(\omega t - k_x L)} = ik_x Be^{-i(\omega t + k_x x)}. \tag{3.44}$$

which is satisfied if

$$A = -B, \tag{3.45}$$

$$e^{-2ik_x L} = 1. \tag{3.46}$$

The second equation imposes that k_x is quantized as

$$k_x = \frac{m\pi}{2L}, \tag{3.47}$$

while the former implies that the pressure field reads

$$p'(t, x, y) = Ae^{-i\omega t} \sin\left(\frac{m\pi}{2L} x\right) \cos\left(\frac{n\pi}{2h} y\right) \tag{3.48}$$

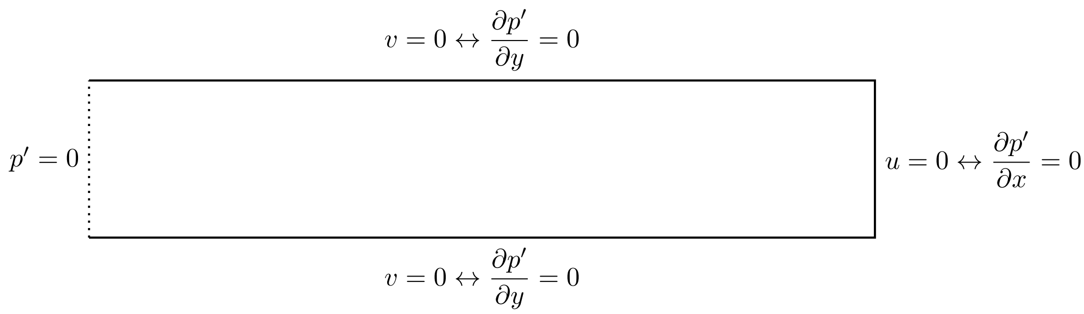


Figure 3.3: Illustration of a one-sided open acoustic channel.

The acoustic velocity components can be computed from the pressure field using (2.8), and they read

$$u'(t,x,y) = A\frac{m\pi}{2L}e^{-i\omega t}\cos\left(\frac{m\pi}{2L}x\right)\cos\left(\frac{n\pi}{2h}y\right) \tag{3.49}$$
$$v'(t,x,y) = -A\frac{n\pi}{2h}e^{-i\omega t}\sin\left(\frac{m\pi}{2L}x\right)\sin\left(\frac{n\pi}{2h}y\right). \tag{3.50}$$

The dispersion relation (2.28) becomes

$$\omega^2 = c_0^2\left(\frac{m^2\pi^2}{4L^2} + \frac{n^2\pi^2}{4h^2}\right) = c_0^2\left(k_x^2 + k_y^2\right), \tag{3.51}$$

where, in the later, we are interpreting the term $\frac{n\pi}{2h}$ as the vertical component, k_y, of the wave vector, in an analogy with (3.47).

Note that the presence of wall in both directions imply that, contrary to waves in free media or in an infinite duct, where any frequency is allowed, the frequency is now quantized (i.e., only discrete values are allowed).

> **Note 5:** What is the impedance of the duct at each end, i.e., at $x = 0$ and $x = L$? You can find the results relying only on the boundary conditions used.

3.4.1 Imposed pressure

We can also consider the case where the pressure is imposed at the walls. This is a common situation in aeroacoustics, where the pressure field is imposed, e.g., by the flow. The boundary condition (3.38) becomes

$$p'(t, x = 0, y) = p_0(t, y), \tag{3.52}$$

where we will assume, for simplicity, that

$$p_0(t,y) = \hat{p}_0 e^{-i\omega t}\cos\left(\frac{n\pi y}{2h}\right). \tag{3.53}$$

We will already assume that solution has a vertical wave number number (n) and angular frequency as the one imposed by the pressure field at $x = 0$. The boundary condition (3.38) and (3.39) then becomes

$$A + B = \hat{p}_0, \tag{3.54}$$
$$Ae^{ik_xL} = Be^{-ik_xL}. \tag{3.55}$$

Using the second equation, we can write B in terms of A. From the second equation,

$$A = \frac{B}{e^{-2ik_xL}}, \tag{3.56}$$

which, after substitution in the first equation, gives

$$A = \frac{\hat{p}_0}{1 + e^{-2ik_xL}}, \quad B = \frac{\hat{p}_0}{1 + e^{2ik_xL}}. \tag{3.57}$$

The pressure thus reads

$$p'(t,x,y) = \hat{p}_0\left(\frac{e^{ik_xx}}{1 + e^{-2ik_xL}} + \frac{e^{-ik_xx}}{1 + e^{2ik_xL}}\right)\cos\left(\frac{n\pi}{2h}y\right)e^{-i\omega t}. \tag{3.58}$$

Using (2.8), the acoustic velocity is found out to be

$$u'(t,x,y) = ik_x\hat{p}_0\left(\frac{e^{ik_xx}}{1+e^{-2ik_xL}} - \frac{e^{-ik_xx}}{1+e^{2ik_xL}}\right)\cos\left(\frac{n\pi}{2h}y\right)e^{-i\omega t} \tag{3.59}$$

$$v'(t,x,y) = \frac{n\pi}{2h}\hat{p}_0\left(\frac{e^{ik_xx}}{1+e^{-2ik_xL}} + \frac{e^{-ik_xx}}{1+e^{2ik_xL}}\right)\sin\left(\frac{n\pi}{2h}y\right)e^{-i\omega t} \tag{3.60}$$

3.4.2 Imperance of a finite acoustic duct

As the impedance is defines as the ratio between the velocity and pressure, (2.41), we can compute the impedance of the finite duct from (3.58) and (3.59) as

$$\hat{z} = \frac{\hat{p}'}{\hat{u}'} = -\frac{1}{k_x\tan(k_xL)}. \tag{3.61}$$

First thing to note is that the imperance here is always a real quantity. As the tangent function has a range from $-\infty$ to $+\infty$, the impedance can take any real value, for a given k_x, by correct tunning L.

Note 6: Find the values of L for which the impedance is infinite or minimum, and try to reason why that is the case.

3.4.3 Aeroacoustic applications

Suppose that we want to engineer a duct wall such that it behaves as a wall with a given, finite inpedance. The results above suggest that we can add small tubes on the wall, and tunne their lenghts such that, for a given frequency, the impedance of the wall is the desired one. In particular, the result of the previous section shows that any (real) impedance value can be achived in this manner.

This is the basis of the design of acoustic liners, which are used in aeroacoustics to reduce the noise generated by engines. The liners are typically made of a porous material, behaving as a tunned acoustic duct. On top of the dynamics indicated above, linears are also designed to absorb energy, which corresponds to a non-zero imaginary part of the impedance, which can attenuaae propagative sound waves.

4
Wave Reflection and Transmission

In this section, we will analyse the effect of flow disconuities in sound propagation. We will consider sudden change in flow properties such as density, temperature, fluid, and flow velocity. We will first treat density, temperature, and fluid variations, as their treatment is similar. We will then consider the effect of flow velocity changes.

4.1 Density, temperature, and fluid changes

Figure 4.1 illustrates the problem under study. The plus and minus signs will be used to denote the region above and bellow the discontinuity.

In the spirit of the previous sections, we will consider following ansatz for the acoustic pressure field

$$p'(x,y,t) = \begin{cases} Ie^{i(k_x x - k_{y+} y - \omega t)} + Re^{i(k_x x + k_{y+} y - \omega t)} & \text{if } y > 0 \\ Te^{i(k_x x - k_{y-} - \omega t)} & \text{if } y < 0 \end{cases}, \tag{4.1}$$

where we already assumed that all waves share the same frequency ω and the same wave number in the x direction, i.e., same k_x. The wave number in the y direction is k_{y+} for the incident and reflected waves and k_{y-} for the transmitted wave. The incident, reflected, and transmitted waves (complex) amplitudes are denoted by I, R, and T, respectively. Finally, we assume that $k_{y\pm}$ are positive, such that only the incident wave is propagating towards the discontinuity, while the transmited and reflected waves are propagating away from it.

As before, we need to impose continuity of pressure and velocity at the discontinuity. The interface condition thus becomes

$$I + R = T \tag{4.2}$$

$$-I\frac{k_{y+}}{\overline{\rho}_+ \omega} + R\frac{k_{y+}}{\overline{\rho}_+ \omega} = -T\frac{k_{y-}}{\overline{\rho}_- \omega}, \tag{4.3}$$

After some algebra, we can find the reflection ($\mathcal{R}$) and transmission ($\mathcal{I}$) coefficients as

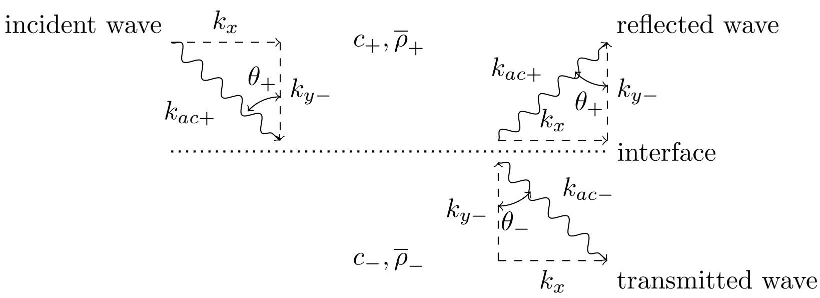


Figure 4.1: Illustration of incident, reflected, and transmitted waves at a flow discontinuity.

$$\mathcal{R} = \frac{R}{I} = \frac{\dfrac{k_{y+}}{\overline{\rho}_+\omega} - \dfrac{k_{y-}}{\overline{\rho}_-\omega}}{\dfrac{k_{y+}}{\overline{\rho}_+\omega} + \dfrac{k_{y-}}{\overline{\rho}_-\omega}}, \qquad \mathcal{T} = \frac{T}{I} = \frac{2\dfrac{k_{y+}}{\overline{\rho}_+\omega}}{\dfrac{k_{y+}}{\overline{\rho}_+\omega} + \dfrac{k_{y-}}{\overline{\rho}_-\omega}}. \tag{4.4}$$

From figure 4.1, we have

$$\cos(\theta_+) = \frac{k_{y+}}{k_{ac,+}}. \tag{4.5}$$

Recalling that $c_{0+} = \omega/k_{ac+}$, we can rewrite

$$\frac{k_{y+}}{k_{ac}} = \frac{k_{y+}}{\overline{\rho}_+\omega} = \frac{\cos(\theta_+)}{\overline{\rho}_+ c_{0+}} = \frac{\cos(\theta_+)}{z_+} = \frac{1}{z_{\theta_+}}, \tag{4.6}$$

and the transmission and reflected coefficients as

$$\mathcal{R} = \frac{1 - \dfrac{z_{\theta_+}}{z_{\theta_-}}}{1 + \dfrac{z_{\theta_+}}{z_{\theta_-}}}, \qquad \mathcal{T} = \frac{2}{1 + \dfrac{z_{\theta_+}}{z_{\theta_-}}}. \tag{4.7}$$

Note that reflection and tranmission coefficients are not only dependent of the fluid properties on each side, but also on the incident/transmitted angles.

The incident and reflected angles are the same. To compute the transmitted angle, we can use the dispersion relations, (2.28), on each side

$$k_{ac+}^2 = k_x^2 + k_{y+}^2, \text{ and} \tag{4.8}$$

$$k_{ac-}^2 = k_x^2 + k_{y-}^2, \tag{4.9}$$

from which

$$k_{ac-}^2 - k_{y-}^2 = k_{ac+}^2 - k_{y+}^2 \tag{4.10}$$

$$1 - \cos^2(\theta_-) = \frac{k_{ac+}^2}{k_{ac-}^2}\left(1 - \cos^2(\theta_+)\right), \tag{4.11}$$

using trigonometric identities, notticing that $\dfrac{k_{ac+}^2}{k_{ac-}^2} = \dfrac{c_{0-}^2}{c_{0+}^2}$, and taking the square root, we arrive at

$$\sin(\theta_+)c_{0-} = \sin(\theta_-)c_{0+}, \tag{4.12}$$

known as *Snell's law*.

4.2 Total internal reflection

Re-aranging (4.12), we can write

$$\sin(\theta_-) = \frac{c_{0+}}{c_{0-}} \sin(\theta_+). \tag{4.13}$$

Assuming that the speed of sound on the tranmitted side is smaller than on the incident side, we can see that there is a critical angle θ_c above which (4.13) requires $\sin(\theta_-) > 1$, but there is no (real) angle that satisfies this. To understand what happens for $\theta_+ > \theta_c$, the dispersion relations, can be written as

$$\frac{1}{c_{0+}^2} = \frac{1}{c_x^2} + \frac{1}{c_{y+}^2} \tag{4.14}$$

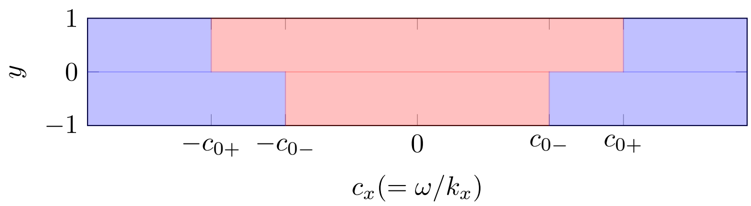
$$\frac{1}{c_{0-}^2} = \frac{1}{c_x^2} + \frac{1}{c_{y-}^2} \tag{4.15}$$

where $c_x = \omega/k_x$ and $c_{y\pm} = \omega/k_{y\pm}$ are the phase speeds along the x and y axis, respectively.

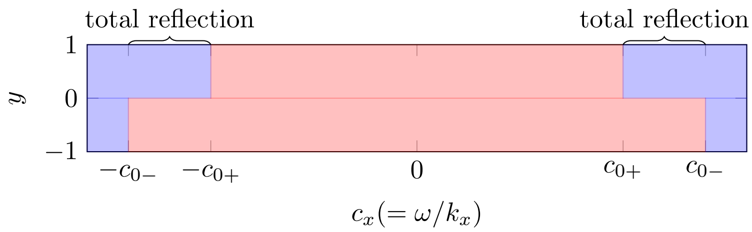
As we assume that the incident wave is propagative, we must have c_x and c_{y+} real, and thus both are larger than c_{0+}. As both domains share c_x and k_x, if $c_x < c_{0-}$, then c_{y-} and c_{y-} are imaginary: the transmitted wave is evanescent.

As evanescent waves do not carry energy, the reflected wave carries the same energy as the incident wave, i.e., it has the same magnitude. This can be seen in (4.4): if k_{y-} is a pure imaginary number, the numerator and the denumerator have the same norm, and thus $|\mathcal{R}| = 1$.

Figures 4.2 and 4.3 illustrates the conditions for total internal reflection in terms of c_x and/or k_x.

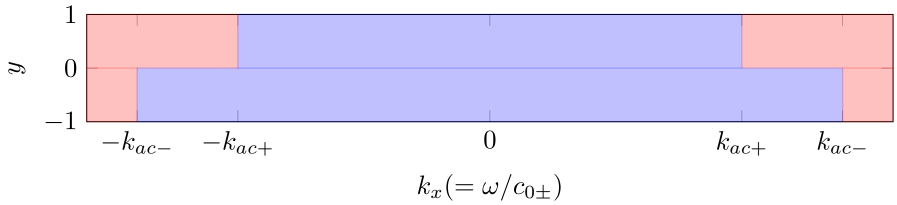


(a) $c_{0-} < c_{0+}$, no total internal reflection

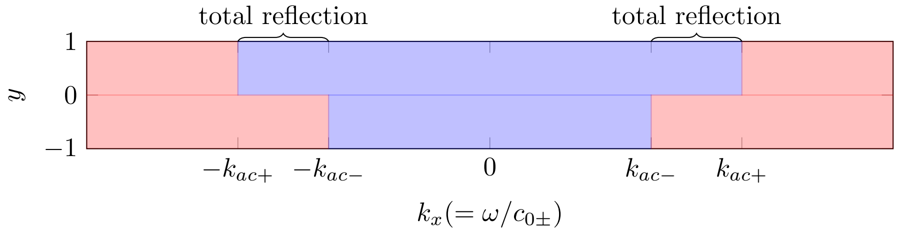


(b) $c_{0-} > c_{0+}$, potential total reflection

Figure 4.2: Illustration of impact of differnet sounds speed on total internal reflection of waves impinging from the above. Blue/red regions indicate the presence of propagative/evanescent waves.



(a) $c_{0-} < c_{0+}$, no total internal reflection



(b) $c_{0-} > c_{0+}$, no total internal reflection

Figure 4.3: Same as figure 4.2, but in terms of the wave number k_x.

4.3 The effect of flow velocity variations

When the flow velocity changes, the dispersion relation becomes (see eq. 2.48)

$$\left(\frac{\omega}{c_{0+}} + Mk_x\right)^2 = k_x^2 + k_y^2. \tag{4.16}$$

Naturally, this changes the wave numbers for which k_y is real/imaginary, and thus the conditions for total internal reflection. The critical values of k_x, denoted $k_{x,c}$, are

$$k_{x,c} = \frac{\omega}{M \pm c_0} \tag{4.17}$$

or the corresponding phase velocity

$$c_{x,c} = M \pm c_0. \tag{4.18}$$

The effect of different Mach numbers on the total internal reflection is illustrated in figure 4.4.

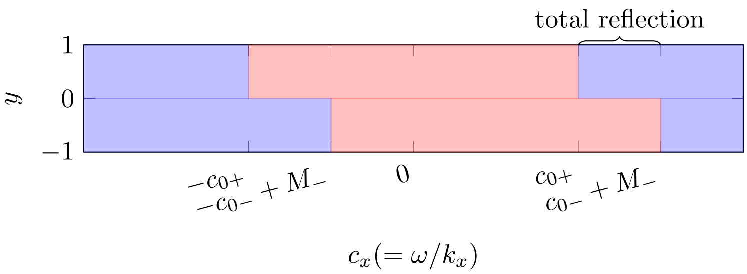


Figure 4.4: Illustration of the impact of mean flow in the conditions for total internal reflection of waves impinging from the above. Blue/red regions indicate the presence of propagative/evanescent waves.

So, where different temperatures/speed of sounds shrink or expand the regions waves are evanescent (and thes where total internal reflection are possible), different Mach numbers shift these regions. Of course, both effects can be combined.

This effect creates a directionality to the flow: waves travelling in the direction of the flow on the tranmission side are more prone to being totally reflected. This has important consequences for perseived noise.

4.3.1 Aeroacoustic applications

Wave reflection and refractions are another crucial component for aeroacoustic applications.

Consider the case of a wind turbine. Modelling it as a point noise source. The acoustic intensity is expected to decay with $1/r^2$. However, not only the ground acts as a reflection plane, but also can an stratified atmosphere. The acoustic energy thus can be trapped within these two planes, having a decay rate of $1/r$ instead. This is a particularly important effect for offshore turbines, where, on a calm day, the sea surface can act as a perfect reflector.

The effect is also important for aircrafts, and even road noise (see page 82).

5 Wall originated noise

In the previous sections, we have discussed "passive" acoustics: we assume that sound waves are present and we study their propagation and interaction with obstacles. In this section, we will discuss "active" acoustics: we will study the generation of sound waves. More precisely, sound created by wall movement.

5.1 Travelling wave on a wall

5.1.1 Harmonic wall

The simplest problem to study is the generation of sound by a wall moving in a fluid. We will consider a wall with a small sinosoidal pattern, moving with a velocity v_w in a fluid at rest. For simplicity, we will assume a 2D configuration, and an infinite wall in the x direction, and the flow to be at rest.

For simplicity, we will assume that the wall vertical displacement to be described as

$$y_w(t,x) = y_a e^{ik_x(x-t/v_w)} = y_a e^{ik_x x - \omega t}, \tag{5.1}$$

where y_a is the amplitude of the wall movement, k_x is associated wave number, and $\omega = k_x v_w$ is the angular frequency resulting from the wall movement. The above corresponds to a wall with a harmonic profile, moving in the x direction with a velocity v_w.

The solution of this problem is similar to that presented in section 3.1. The difference is that now we will not assume an incident wave, and the boundary conditions at the wall now read

$$v'(t,x,y=0) = \frac{\partial y_w}{\partial t}(t,x), \tag{5.2}$$

i.e., the fluid velocity at the wall is equal to the wall velocity.

The pressure field can be expressed as

$$p'(t,x,y) = Ae^{ik_x x + ik_y y - i\omega t}, \tag{5.3}$$

where we are keeping only one upwards-moving wave as we are assuming no incident wave.

Computing v' from p', (2.8), and enforcing the boundary condition at the wall, (5.2), gives

$$A\frac{k_y}{\omega\overline{\rho}} = i\omega y_a \quad \Rightarrow \quad A = \frac{\omega^2\overline{\rho}y_a}{k_y}i. \tag{5.4}$$

We also need to satisfy the disperion relation, (2.28),

$$k_y = \sqrt{\frac{\omega^2}{c_0^2} - k_x^2}. \tag{5.5}$$

Just as in the acoustic duct, there are two regimes. If $\omega < ck_x$, then k_y is imaginary, and if $\omega > ck_x$, then k_y is real. Again, these regimes correspond to evanescent and propagating waves, respectively. Here, we can obtain a more physical interpretation of these regimes by rewriteing the dispersion relation as

$$k_y = k_x\sqrt{\frac{v_w^2}{c_0^2} - 1}. \tag{5.6}$$

From this equation, we can deduce that that the wave will be propagative (k_y will be real) only if the wall is supersonic ($v_w > c_0$), i.e., it is moving faster than the speed of sound.

This has a direct connectio with the discussion in section 2.4. As the minimum phase velocity of propagative velocity is c_0, this is the minimum phase velocity that a source needs to have in order to generate propagative waves. This will be a key concept to understand sound generation in aeroacoustics in chapter 6.

As an introduction to the next section, note that although we used a complex value for the wall position in (5.1), this can easily be remedied using the superposition principle. We can write a real-valued the wall position as

$$y_w(t, x) = y_a cos(k_x x - \omega t) = \frac{y_a}{2}\left(e^{i(k_x x - \omega t)} + e^{-i(k_x x - \omega t)}\right). \tag{5.7}$$

We can then superpose the solutions for the terms with $e^{i(k_x x - \omega t)}$ and $e^{-i(k_x x - \omega t)}$ to obtain

$$\begin{aligned} p'(t, x, y) &= \frac{\omega^2\overline{\rho}y_a}{2k_y}ie^{i(k_x x + k_y y - \omega t)} - \frac{\omega^2\overline{\rho}y_a}{2k_y}ie^{-i(k_x x + k_y y - \omega t)}, \\ &= \frac{\omega^2\overline{\rho}y_a}{k_y}\sin\left(k_x x + k_y y - \omega t\right). \end{aligned} \tag{5.8}$$

5.1.2 Wall mounted loudspeaker: superposition principle

Lets now solve a more intersting problem. Let's assume a loud speaker of size L mounted on a wall, and that the speaker is generating a sound wave with a given frequency ω. The wall+speaker movement can be described as

$$y_w(t, x) = y_a cos(\omega t)\left(h\left(x - \frac{L}{2}\right) - h\left(x + \frac{L}{2}\right)\right), \tag{5.9}$$

where h is the step function.

In the previous cases, the harmonic nature of the boundary condition imposed the harmonic nature of the pressure field, with the pressure field and boundary condition sharing frequencies (ω) and horizontal wavenumbers (k_x). Now, we have a more complex boundary condition. Here, to solve the problem, we will again use the superposition principle and, for such, decompose the wall movement into harmonic components components. This can be achieved writting

$$\begin{aligned} y_w(t,x) = & \underbrace{\frac{y_a}{2}\left(h\left(x-\frac{L}{2}\right)-h\left(x+\frac{L}{2}\right)\right)e^{i\omega t}}_{y_{w,+}} \\ & + \underbrace{\frac{y_a}{2}\left(h\left(x-\frac{L}{2}\right)-h\left(x+\frac{L}{2}\right)\right)e^{-i\omega t}}_{y_{w,-}}, \end{aligned} \tag{5.10}$$

and then expressing $y_{w,-}$ with a Fourier transform,

$$y_{w,-} = e^{-i\omega t}\int_{-\infty}^{\infty} \hat{y}_{w,-}(k_x)e^{ik_x x}dk_x. \tag{5.11}$$

The quantity $\hat{y}_{w,-}(k_x)$ gives us the "amount" of hamornic signal with wavenumber k_x needed to construct $y_{w,-}$. Each of these components will generate a corresponding pressure field, and the total pressure field will be the sum of all these components. The individual pressure field components are already known from the previous section, so we can re-use those results.

The term $\hat{y}_{w,-}(k_x)$ is computed as

$$\begin{aligned} \hat{y}_{w,-}(k_x) &= \int_{-\infty}^{\infty}\left(h\left(x-\frac{L}{2}\right)-h\left(x+\frac{L}{2}\right)\right)e^{-ik_x x}dx \\ &= \frac{y_a L}{2}\frac{\sin(\pi L k_x)}{\pi L k_x}, \end{aligned} \tag{5.12}$$

and using (5.8), we can write the total presure field as

$$p'(t,x,y) = \frac{\omega^2\overline{\rho}L y_a}{2}\int_{-\infty}^{\infty}\frac{1}{k_y}\frac{\sin(\pi L k_x)}{\pi L k_x}e^{i(k_x x+k_y y-\omega t)}dk_x. \tag{5.13}$$

where again, k_y is given by (5.4). This is analytical solution for the problem. The integral can be computed analitically or numerically, but even without doing this there are differnet effects that can be noted by inspecting it.

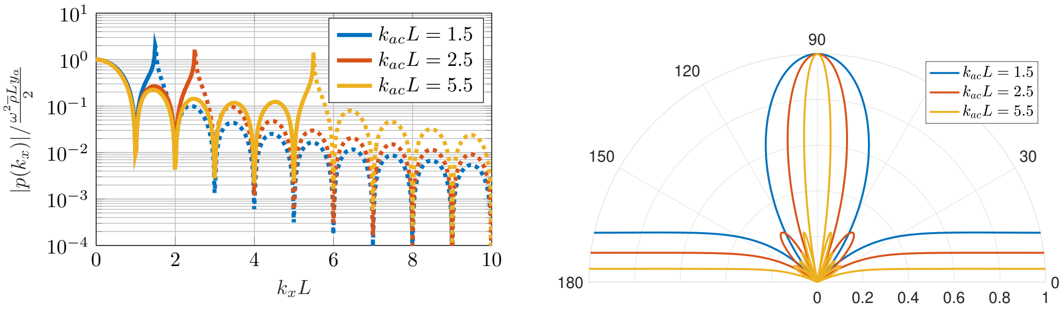
First, k_y is a function of k_x, and will be real for high values of k_x, and or imaginry for low values. The treshold, again is whether this specific component is supersonic or subsonic (see again eq. 5.5). We can thus intutivelly understand what is the range of wave numbers which contribute to the far field: k_xs for which k_y is real, keep their support far from the wall (large y), while k_xs for which k_y is imaginary, will have their amplitude decreasing exponentially as we move far from the wall.

Also, remembering that a plane wave with wavenumber $\vec{k} = (k_x, k_y)$ travels in the $\vec{k}$ direction, we can from (5.13) immediately identify the directivity of the acoustic field, i.e., what is the sound amplitude emited on each direction. The sound emited on an angle $\theta = \tan^{-1}(k_y/k_x)$, is proportional to $\frac{1}{k_y}\frac{\sin(\pi L k_x)}{\pi L k_x}$, and is illustrated in figure 5.1.

5.1.3 The role of the speaker shape

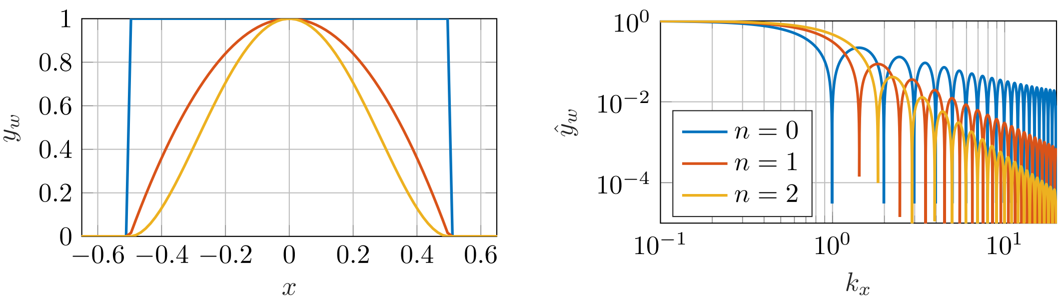
Let us now consider what is the role of the speaker shape on the sound field. While previously we considered a square function, lets examine what happens if we consider a different shape, namely:

$$y_w(t,x) = y_a\left(1-\left(\frac{|x|}{L/2}\right)^n\right). \tag{5.14}$$



(a) (b)

Figure 5.1: Directivity of the sound field emited by a wall-mounted loudspeaker, indicated by k_x, on the left, and by $\theta = \tan^{-1}(k_y/k_x)$, on the right. On the left, solid lines indicate that the waves are propagattive, while dashed lines indicate that they are evanescent.



(a) (b)

Figure 5.2: Smooth speakers shapes (a) and spectra (b).

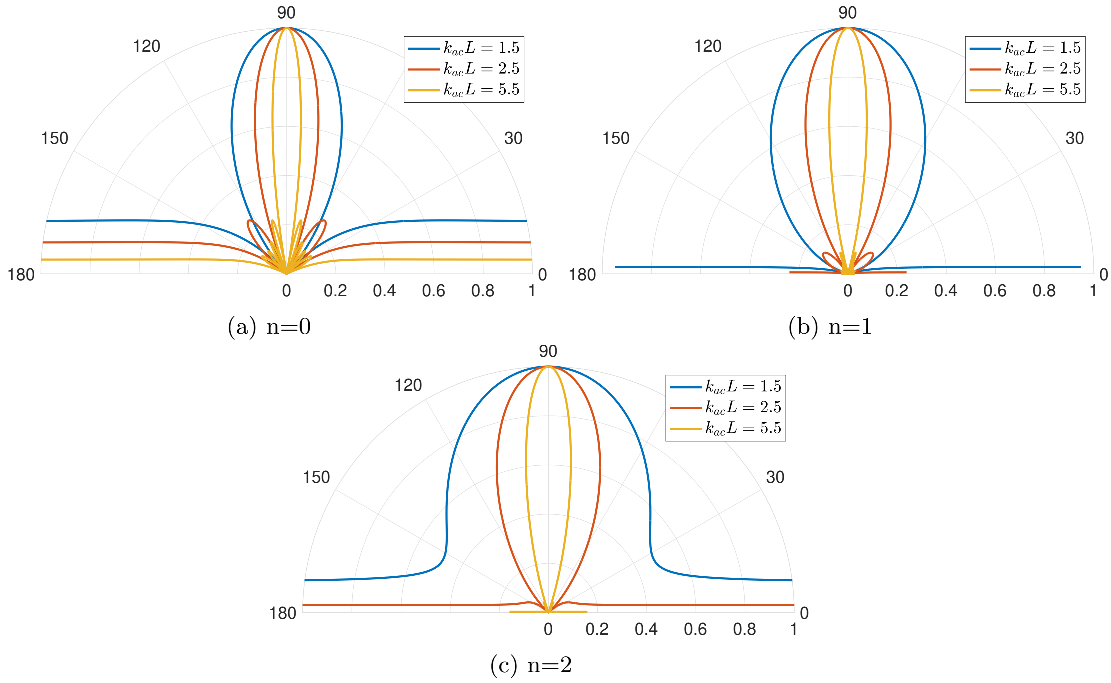
Here we do not seek an analytical expression for the sound field, but instead, explore trends numerically. Figure 5.2 illustrates the shape of different speakers, and the corresponding spectra (the coefficient amplitude of its Fourier transform).

The corresponding directivities of these speakers are shown in figure 5.2. Note that the side lobes are considerably reduced. This is due to the faster decay of the Fourier transform of the speaker shape, seen in 5.2b.

Typically, the faster/slower decay of the spectra is related to the smoothness of the signal. The square function ($n = 0$) is not continuous, thus its Fourier transform decays slowly. For $n = 1$, the function continuous, and thus its spectra decays faster. However, its first derivative is not continuous. For $n = 2$, the function and its first derivative are continuous, and thus its spectra decays even faster.

The spectral content of the spekaer shape for high values of k_x dictates the sound emission at swallow angles. Thus, the smoother the speaker shape, the sound field tends to be more directional.

Note that nearly all the solutions shown here have a strong emission parallel to the plates ($\theta = 0°$ and $180°$). Clearly, this is not what you expect for a usuall speaker. In fact, here it is just a consequence of the assumption of inifite walls.



(a) n=0

(b) n=1

(c) n=2

Figure 5.3: Directivities for different speaker shapes.

5.2 Aeroacoustic application : trailling edge noise

The results above allows us to understand the basic mechanisms behind trailling-edge noise. This is the dominant mechanism, for example, in wind turbines.

The trailling edge noise is generated by the interaction of the turbulent flow around the airofoil with the trailling edge of a body, where turbulent energy gets scattered into the acoustic field. Figure 5.4 illustrates this using numerical results from Sandberg 2015. A detailed mathematical description of this problem is beyond the scope of this text, but we can provide a qualitative understanding of the problem using some simplying assimptions.

To understand the basic mechanisms behind trailling edge noise, we will assume that the near wall turbulent dynamics is dominated by convection from the meanflow. We can imagine it consisting from convected vortexes, and that these vortexes imprint their pressure signature along the wall.

Let first consider the case of an inifinte wall, with vortex convection speed of u_m. Model the turbulent vortices as a sum of periodic vortices, the pressure field on the wall can be approximated by a harmonic pressure, i.e.,

$$p_w(t, y) = Ae^{ik_x(x - t/u_m)}. \tag{5.15}$$

Using this pressure as a boundary condition, the acoustic pressure field can becomes

$$p'(t, x, y) = Ae^{i\left(k_x x + k_y y - \frac{k_x}{u_m} t\right)}, \tag{5.16}$$

and as in (5.6), we can write k_y as

$$k_y = k_x \sqrt{\frac{u_m^2}{c_0^2} - 1}, \tag{5.17}$$

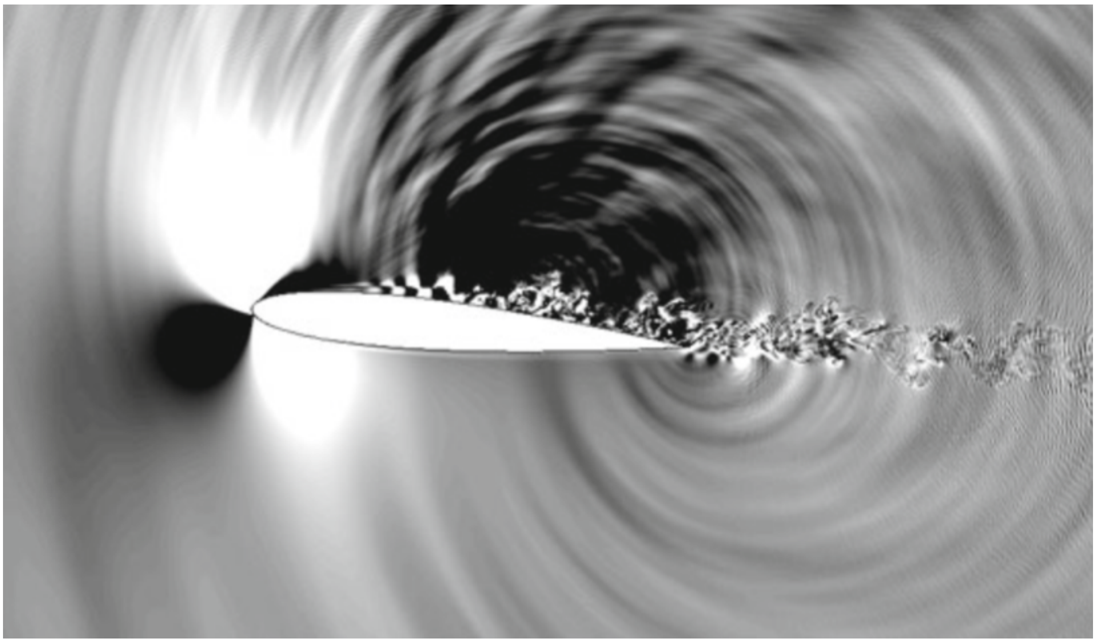


Figure 5.4: Pressure field of the turbulent flow around an airfoil. (Sandberg 2015)

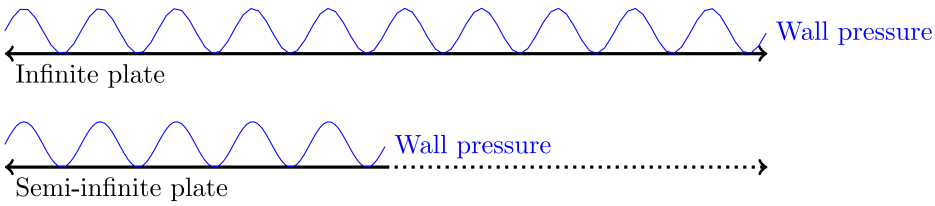


Figure 5.5: Illustration of the wall pressure on inifinte (above) and finite (below) plates due to vortex convetion.

and thus, for subsonic flows, k_y is always imaginary, and the acoustic pressure field is thus evanescent. Thus reflects a good rule of thumb: in the abscence of scattering mechanisms, homogenious subsonic turbulence tends to be silent, or at least not too noisy[1].

5.2.1 Trailling edge modelling

Clearly, the flow near the traílling edge is not equivalent to the flow over an inifite flat plate. In particular, there is no non-slip boundary condition around it, so the flow dynamics are different. Nevertheless, a frequent model used to understand flow dynamics uses what is called the “fronzen turbulence” hypothesis. That is, we assume that turbulence is unaffected by the modification we impose on the problem (here the presence of the traílling edge).

Using this assumption, we will model an airfoil tralling edge with a semi-infinite plate, whose traílling edge is located at $x = 0$. On top of this plate, a frozen turbulent field is assumed. Using the assumptions of the previous section, we can write the pressure field on the

[1]Take this with a grain of salt. As will be explored in chapter 6, jet noise is mostly due to turbulence, without any obvious scattering mechanism. The cituation there is more subtle, among other things, turbulence is not homogenious...

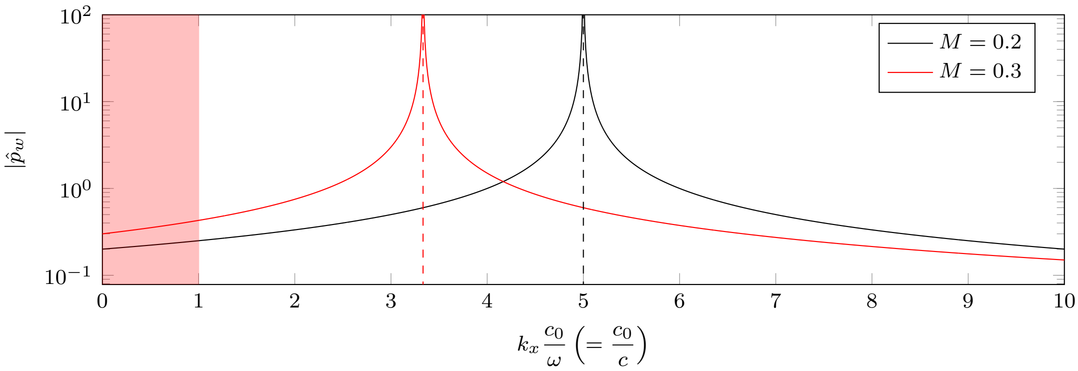


Figure 5.6: Illustration of the pressure spectra of the infinite (dahesed lines) and semi-infinite walls (solid lines). Where $c = \omega/k_x$ is the phase spped along the x direction. The red region indicates wavenumbers for which the phase speed is greater than the speed of sound, and thus suports propagative waves.

wall as

$$p_w(t,x) = Ae^{ik_{x,w}(x-t/u_m)}h(-x), \tag{5.18}$$

where $h(x)$ is the step function. That is: it is the same pressure field, but not restricted negative xs. The pressure field is illustrated in figure 5.5.

To analyse the spectra of p_w, we use the following relation

$$\mathcal{F}\Big(f(x)g(x)\Big) = \mathcal{F}\Big(f(x)\Big) \ * \ \mathcal{F}\Big(g(x)\Big), \tag{5.19}$$

i.e., the Fourier transform of a product of two functions is the convolution of the Fourier transforms of the functions.

In (5.18), the first term is a harmonic function, i.e., its Fourier transform is a delta function. The second term is a step function, and its Fourier transform is

$$\mathcal{F}\Big(h(x)\Big) = \frac{1}{ik_x} + \pi\delta(k_x), \tag{5.20}$$

where δ is the Dirac delta function.

The spectra of the wall pressure is thus

$$\hat{p}_w(k_x) = A\left(\pi\delta(k_x - k_{x,w}) + \frac{1}{i(k_x - k_{x,w})}\right), \tag{5.21}$$

which is illustrated in figure 5.6.

The main difference between the infinte and semi-infinite walls is that the latter has broader spectra. For a given frequency (ω), the presence of the edge creates pressure components with different phase velocities (recall that $c = k_x/\omega$). Thus, even if the convection speed is subsonic, the presence of the edge induces some supersonic harmonic components, creating propagative waves, i.e., noise. Another feature that we can observe is that, as Mach number increases, the energy contained in the propagative wavenumber range increases. That is: trailling edge noise increases with Mach number.

We can estimate the sound power trend with the Mach number. Lets consider only the wave number $k_{x,ac} = \omega/c_0$, i.e., the limiting wavenumber where waves become propagative. From figure 5.6, we can see that this is the monst energetic wavenumber with is also propagative, which justifies why we will focus on it.

The pressure convection velocity can be writen in terms of the Mach number as $v_w = Mc_0$. We can thus write

$$\omega = k_{x,w} v_w = k_{x,w} M c_0, \quad \text{and also} \tag{5.22}$$
$$k_{x,ac} = k_{x,w} M. \tag{5.23}$$

Thus, the amplide at the acoustic wave number is

$$\left|\hat{p}_w(k_{x,ac})\right| = A \frac{M}{(1-M)k_{x,ac}} \overset{M \ll 1}{\approx} A \frac{M}{k_{x,ac}}. \tag{5.24}$$

Using (2.8), the associated x-velocity amplitude is

$$|\hat{u}'(k_{x,ac}| \propto \frac{k_{x,ac}}{\omega} |\hat{p}_w(k_{x,ac})| = \frac{M}{v_w} |\hat{p}_w(k_{x,ac})|. \tag{5.25}$$

Thus, from (2.24), we can see that

$$\vec{\mathcal{J}} = p'\vec{u}' \propto M|p|^2 \propto M^3, \tag{5.26}$$

i.e., the acoustic intensity is proportional to the cube of the Mach number.

Although not shown here, free-stream turbulence generates a power intensity proportional to M^5. Showing this is beyond the scope of this text, but comparing this scaling to the one obtained in (5.26), we can see that trailling edge noise will be dominant for sufficiently small Mach numbers.

How small the Mach number needs to be for the trailling edge noise to be dominant is problem dependent, and is a function, for example, of the intensity of isolated sources compared to the that of the near wall turbulence. In maritime applications, where often $M > 0.01$, this is almost always the case. As previously mentioned, it is also often the case in wind turbines, unless if the turbine is on the wake of another turbine, thus feeling its wake, or when atmospheric turbulence is high. In these cases, leading edge noise is usually dominant.

5.2.2 Noise reduction strategies

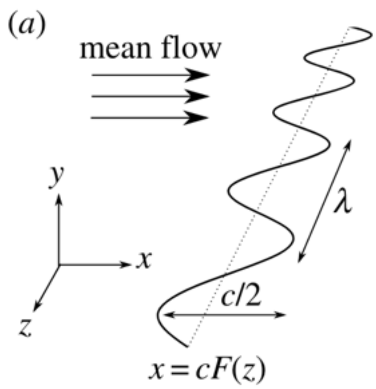
Figure 5.6 shows that the dominant mechanism behind noise is the broadening of the wall pressure spectra due to the disconuity caused by the sudden end of the wall. This suggests that, if this disconuity can be alliviated, the emmited noise can be reduced.

In this spirit, different strategied have been investigated. Serrated tralling edges are the most commom and well studied ones. In the spirit of the results of the previous section, we can understand it as providing a smoother transition between “wall” and “no-wall”. Another possibility is to use porous materials, making again a smoother transition bewtween the solid wall (porosity of 0%) and no wall (porosity of 100%). Finally, one can aim to achive similar results using flexible materials.

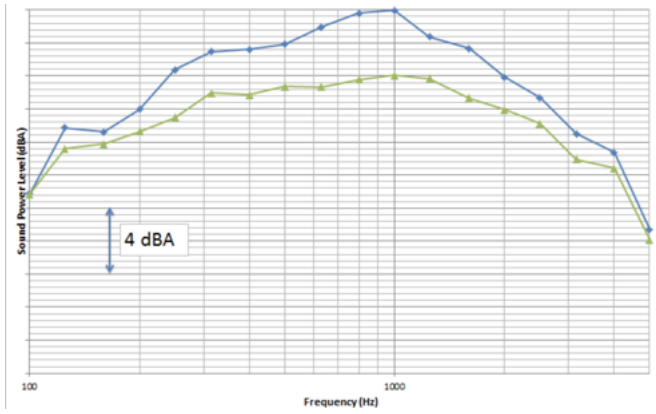
Out of these approaches, serrated trailling edges were the first to be investigated. Illustrated in figure 5.7 and they are already present as commercial products for wind turbines, promissing a noise reduction of the order of $3\,dB$s[2] (which ammonts to approximatly halfing the acoustic energy).

Another method to reduce noise which is currently being investigated is the use of porous and/or flexible materials, illustrated in figure 5.8. Typicall, flexibility tends to be more efficient for the reducion of higher frequency noise, while porosity is more efficient for lower frequency noise.

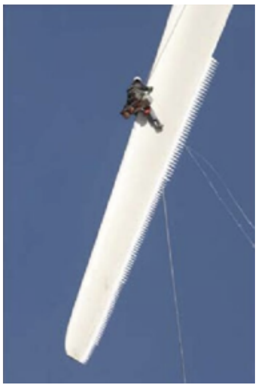
[2]see also: `https://nozebra.ipapercms.dk/Vestas/Communication/Productbrochure/TurbineOptions/sound-power-optimisation/?page=1`



(a) (Ayton 2019)



(b) (Hurault et al. 2015)

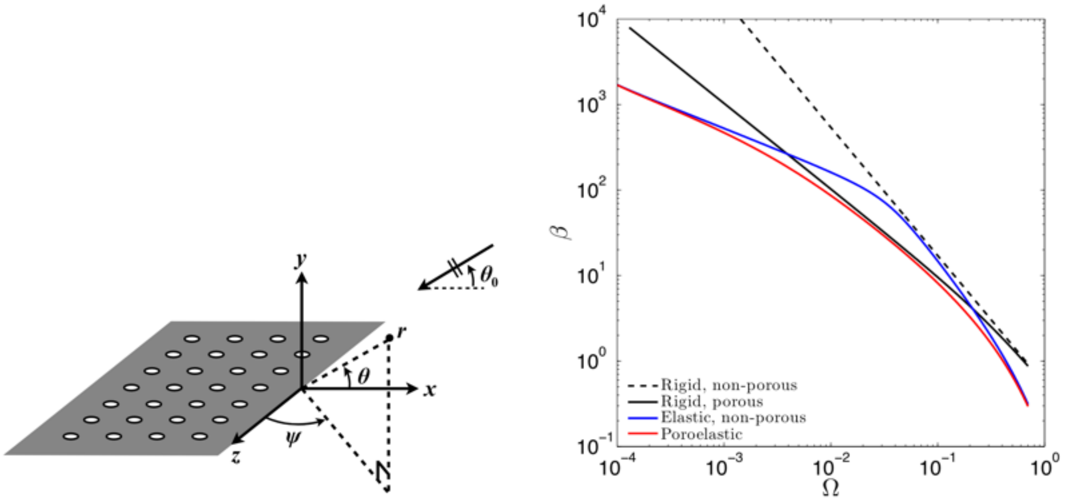


(c) (Yang 2013)

Figure 5.7: Theoretical academic studies (a), experimental validation (b), and on field application (c) of serrated trailing edges..

Note 7: The reasoning given here is not rigorous. It fails as it implicitly assumes that the pressure is zero when there is no wall, and this is not justified neither is true in general, and also does not impose a no penetration condition on the plate (i.e., $v = 0$). Enforcing the latter while not imposing the former requires much more sofisticated mathematical tools, which are beyond the scope of this text.

Despite these shortcommings, the correct M^3 is captured. You will find different reasonins and forumations for this problem in many books and papers, however they will invaribly require much denser mathematical treatments. Howe 1998 presents a correct treatment of the problem using the Wiener-Hopt method is given, where the same trend is obtained.



(a) Model (b) Predicted noise emissions

Figure 5.8: Devellopments in porous, elastic, and poroelastic materials for noise reduction. (Jaworski and Peake 2012)

6
Sound sources

We have note yet managed to write this section of the course. Below you will find the lecture notes.

An introduction to aeroacoustics
Lecture 2: Governing equations of acoustics

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Outline

The Euler equations

Acoustic Reynolds number:

$$Re_a = \frac{c_o \lambda}{\nu} \approx 8.10^6 \quad \text{at frequency,} \quad f = 1kHz$$

- Acoustics is essentially an inviscid phenomenon
- Navier-Stokes equations reduce to Euler equations
- This simplifies things

The Euler equations

Mass, momentum and energy conservation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i} = 0$$
$$\rho\Big(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\Big) = -\frac{\partial p}{\partial x_i}$$
$$\frac{\partial s}{\partial t} + u_j \frac{\partial s}{\partial x_j} = 0$$

Equation of state:

$$s = s(p, \rho) \quad \text{or} \quad p = p(\rho, s)$$

where ρ, p, s, u_i are, respectively, density, pressure, entropy and velocity.

Linearisation

- Acoustics involves small-amplitude fluctuations
- Euler equations can be further simplified by linearisation
- First step: decompose state variables into time-invariant and fluctuating components:

$$
\begin{aligned}
u_i(x_i,t) &= \overline{u}_i(x_i) + u_i'(x_i,t)\\
\rho(x_i,t) &= \overline{\rho}(x_i) + \rho'(x_i,t)\\
p(x_i,t) &= \overline{p}(x_i) + p'(x_i,t)\\
s(x_i,t) &= \overline{s}(x_i) + s'(x_i,t)
\end{aligned}
$$

Linearisation

- Consider a medium at rest, $\overline{u}_i = 0$,
- Substitute time-invariant-plus-fluctuation decomposition into governing equations,
- **Mass conservation:**

$$
\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \rho'}{\partial t} + \frac{\partial \overline{\rho}\,\overline{u}_i}{\partial x_i} + \frac{\partial \rho'\overline{u}_i}{\partial x_i} + \frac{\partial \overline{\rho} u_i'}{\partial x_i} + \frac{\partial \rho' u_i'}{\partial x_i} = 0
$$

Linearisation

- Consider a medium at rest, $\overline{u}_i = 0$,
- Substitute mean-plus-fluctuation decomposition into governing equations
- Remove terms equal to zero and non-linear terms
- **Mass conservation:**

$$
\begin{aligned}
\cancel{\frac{\partial \overline{\rho}}{\partial t}} + \frac{\partial \rho'}{\partial t} + \cancel{\frac{\partial \overline{\rho}\,\overline{u}_i}{\partial x_i}} + \cancel{\frac{\partial \rho'\overline{u}_i}{\partial x_i}} + \frac{\partial \overline{\rho} u_i'}{\partial x_i} + \cancel{\frac{\partial \rho' u_i'}{\partial x_i}} &= 0\\
\rightarrow \frac{\partial \rho'}{\partial t} + \overline{\rho}\frac{\partial u_i'}{\partial x_i} + \cancel{u_i'\frac{\partial \overline{\rho}}{\partial x_i}} &= 0\\
\rightarrow \frac{\partial \rho'}{\partial t} + \overline{\rho}\frac{\partial u_i'}{\partial x_i} &= 0
\end{aligned}
$$

Linearisation

- Consider a medium at rest, $\overline{u}_i = 0$,
- Substitute mean-plus-fluctuation decomposition into governing equations
- Remove terms equal to zero and non-linear terms
- **Momentum conservation:**

$$
\begin{aligned}
(\overline{\rho} + \rho')\left(\frac{\partial u_i'}{\partial t} + \cancel{u_j'\frac{\partial u_i'}{\partial x_j}}\right) &= -\frac{\partial p'}{\partial x_i}\\
\rightarrow \overline{\rho}\frac{\partial u_i'}{\partial t} + \cancel{\rho'\frac{\partial u_i'}{\partial t}} &= -\frac{\partial p'}{\partial x_i}\\
\rightarrow \overline{\rho}\frac{\partial u_i'}{\partial t} &= -\frac{\partial p'}{\partial x_i}
\end{aligned}
$$

Linearisation

- Consider a medium at rest, $\overline{u}_i = 0$,
- Substitute mean-plus-fluctuation decomposition into governing equations
- Remove terms equal to zero and non-linear terms
- **Energy conservation:**

$$\frac{\partial s'}{\partial t} = 0$$
$$\rightarrow p' = c_o^2 \rho'$$

Linearisation

- Linearised Euler equations,
- Suitable for describing acoustic phenomena:

$$\frac{\partial \rho'}{\partial t} + \overline{\rho}\frac{\partial u_i'}{\partial x_i} = 0 \tag{1}$$
$$\overline{\rho}\frac{\partial u_i'}{\partial t} = -\frac{\partial p'}{\partial x_i} \tag{2}$$
$$p' = c_o^2 \rho' \tag{3}$$

- One may solve these equations, for given initial and boundary conditions, to compute a sound field,
- But they can be reduced to a more convenient single equation, the wave equation, by doing, $\frac{\partial}{\partial x_i}$(2)-$\frac{\partial}{\partial t}$(1)

The wave equation

- $\frac{\partial}{\partial x_i}$(2)-$\frac{\partial}{\partial t}$(1):

$$\overline{\rho}\cancel{\frac{\partial^2 u_i'}{\partial t \partial x_i}} + \frac{\partial^2 p'}{\partial x_i^2} - \frac{\partial^2 \rho'}{\partial t^2} - \overline{\rho}\cancel{\frac{\partial^2 u_i'}{\partial t \partial x_i}} = 0$$

- Use energy equation, $\rho' = \frac{p'}{c_o^2}$, to obtain the wave equation:

$$\nabla^2 p' - \frac{1}{c_o}\frac{\partial^2 p'}{\partial t^2} = 0$$

The wave equation

- What do solutions of this equation look like?
- Consider 1D simplification:

$$\frac{\partial^2 p'(x,t)}{\partial x^2} - \frac{1}{c_o}\frac{\partial^2 p'(x,t)}{\partial t^2} = 0$$

- Perform change of variables:

$$\xi = x + c_o t$$
$$\eta = x - c_o t$$

The wave equation

$$\xi = x + c_o t$$
$$\eta = x - c_o t$$

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$
$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)^2$$
$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = c_o\frac{\partial}{\partial \xi} - c_o\frac{\partial}{\partial \eta}$$
$$\frac{\partial^2}{\partial t^2} = \left(c_o\frac{\partial}{\partial \xi} - c_o\frac{\partial}{\partial \eta}\right)^2$$

$\rightarrow$ Wave equation takes the form,

$$\frac{\partial^2 p'}{\partial \xi \partial \eta} = 0$$

The wave equation

The wave equation,

$$\frac{\partial^2 p'}{\partial \xi \partial \eta} = 0$$

has general solution,

$$p(\xi, \eta) = f(\xi) + g(\eta)$$
$$= f(x - c_o t) + g(x + c_o t)$$

- Waves propagating to the left ($f(x + c_o t)$) and to the right ($x - c_o t$) at speed c_o

The wave equation

- Propagation in three dimensions best adressed in spherical polar coordinates: r, ϕ, θ
- The Laplacian then becomes,

$$\nabla^2 p' = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2}{\partial \phi^2}$$

- But in free space p' is independent of ϕ and θ:

$$\nabla^2 p' = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$$
$$= \frac{1}{r}\frac{\partial^2}{\partial r^2}(rp')$$

The wave equation

The 3D wave equation in spherical polar coordinates,

$$\nabla^2 p' - \frac{1}{c_o}\frac{\partial^2 p'}{\partial t^2} = 0,$$

can thus be written,

$$\frac{1}{r}\frac{\partial^2 (rp')}{\partial r^2} - \frac{1}{c_o}\frac{\partial^2 p'}{\partial t^2} = 0$$
$$\frac{\partial^2 (rp')}{\partial r^2} - \frac{1}{c_o}\frac{\partial^2 (rp')}{\partial t^2} = 0,$$

whose form is identical to the 1D equation considered above. Solutions thus take the form,

$$rp' = f(r + c_o t) + g(r - c_o t)$$
$$\rightarrow p' = \frac{f(r + c_o t)}{r} + \frac{f(r - c_o t)}{r}$$

The wave equation

- Causality (things cannot happen before they are caused) leads to rejection of waves incoming from infinity

$$p' = \cancel{\frac{f(r + c_o t)}{r}} + \frac{f(r - c_o t)}{r}$$
$$p' = \frac{f(t - r/c_o)}{r}$$

- Solution tells us
  - Sound at time t was emitted at time $t - r/c$ (*retarded time*),
  - The shape of the wave, $f(\cdot)$, does not change in space or in time,
  - The pressure amplitude decays $(1/r)$ as the wave propagates spherically.

The wave equation

- In frequency space,

$$p' = \frac{Ae^{-i\omega(t-r/c_o)}}{r}$$

- Amplitude, A, is determined by boundary conditions

Wave equation with source terms

Mass-conservation equation with external source:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = q(x_i, t)$$

- $q(x_i, t)$ is a mass flux and has units $[Kg s^{-1}]$ (e.g. pulsating bubble).

Wave equation with source terms

Momentum-conservation equation with external source:

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) + \frac{\partial p}{\partial x_i} = f_i(x_j, t)$$

- $f_i(x_j, t)$ represents an applied force (vibrating surface).

The wave equation with source terms

Governing linearised equations with source terms,

$$\frac{\partial \rho}{\partial t} + \overline{\rho}\frac{\partial u_i}{\partial x_i} = q(x_i, t) \tag{4}$$

$$\overline{\rho}\frac{\partial u_i}{\partial t} + \frac{\partial p}{\partial x_i} = f_i(x_j, t) \tag{5}$$

To derive associated wave equation: $\frac{\partial}{\partial x_i}(5) - \frac{\partial}{\partial t}(4)$, giving,

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x_i^2} = \frac{\partial q}{\partial t} - \frac{\partial f_i}{\partial x_i}$$

Sources of sound are rate-of-change of mass flux and divergence of applied forces.

An introduction to aeroacoustics
Lecture 3: Sources of sound

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Outline

Sound generation by pulsating sphere

Consider a small sphere (could be a bubble) of radius a oscillating with velocity amplitude V at frequency ω.

From linearised momentum equation, pressure and velocity related by

$$\overline{\rho}\frac{\partial u_i'}{\partial t} = -\frac{\partial p'}{\partial x_i},$$

Radial velocity at sphere surface,

$$v = V\mathrm{e}^{-j\omega t}.$$

Associated pressure fluctuation must have same frequency,

$$p = P\mathrm{e}^{-j\omega t}.$$

Linear momentum balance therefore,

$$\nabla P\mathrm{e}^{-j\omega t} = j\omega\rho_o V\mathrm{e}^{-j\omega t}.$$

Sound generation by pulsating sphere

$$\nabla P\mathrm{e}^{-j\omega t} = j\omega\rho_o V\mathrm{e}^{-j\omega t}.$$

But pressure fluctuations are solution of the wave equation, so,

$$p = \frac{f(t-r/c)}{r} = \frac{A\mathrm{e}^{-j\omega(t-r/c)}}{r}.$$

A can be found from boundary conditions at sphere surface, a. First compute Laplacian of pressure,

$$\nabla p = \frac{A}{r^2}\left[\frac{j\omega r}{c} - 1\right]\mathrm{e}^{-j\omega(t-r/c)}.$$

Apply boundary condition,

$$\frac{A}{a^2}\left[\frac{j\omega a}{c} - 1\right]\mathrm{e}^{-j\omega(t-a/c)} = j\omega\rho_o V\mathrm{e}^{-j\omega t}$$

$$\rightarrow A = \frac{(ka)(ka-j)\rho_o Vca}{(ka)^2+1}\mathrm{e}^{-jka},$$

Giving solution for the pressure,

$$p = \frac{ka}{r}\frac{ka-j}{(ka)^2+1}(\rho_o Vca)\mathrm{e}^{-jk(r-a)}\mathrm{e}^{-j\omega t}$$

Sound generation by pulsating sphere

Pressure field of pulsating sphere,

$$p = \frac{ka}{r}\frac{ka - j}{(ka)^2+1}(\rho_o V c a) e^{-jk(r-a)} e^{-j\omega t}.$$

In limit of small sphere, or large wavelength (low frequency), $ka \ll 1$, we refer to an acoustically compact source,

$$p \approx -j\frac{\rho_o c k a^2}{r} V e^{jkr} e^{-j\omega t}.$$

Green's functions

$$p \approx -j\frac{\rho_o c k a^2}{r} V e^{jkr} e^{-j\omega t}.$$

This fundamental solution, known as a point-source solution, can be used as a building block for more complex problems.

The source strength, $q(x_i, t)$, is a mass flux,

$$q = 4\pi a^2 \rho_o V e^{-j\omega t},$$

but the source term (RHS of inhomogeneous wave equation) comprises the rate of change of mass flux,

$$\frac{\partial q}{\partial t} = -j\omega\rho_o V 4\pi a^2 e^{-j\omega t} = -jkc\rho_o V 4\pi a^2 e^{-j\omega t}$$

Green's functions

Compare expressions for pressure field,

$$p \approx -j\frac{\rho_o c k a^2}{r} V e^{jkr} e^{-j\omega t},$$

and source term,

$$\dot{q} = -j\rho_o c k a^2 4\pi V e^{-j\omega t},$$

$$\rightarrow p(\mathbf{x}, \omega) = \frac{\dot{q} e^{jkR}}{4\pi R},$$

where $R = |\mathbf{x} - \mathbf{y}|$ is the distance between the source position, $\mathbf{y}$ and the observer position, $\mathbf{x}$.

Choosing the general form for the source strength, rather than a single frequency, ω,

$$p(\mathbf{x}, t) = \frac{\dot{q}(\mathbf{y}, t - R/c)}{4\pi R}.$$

Green's functions

Sound pressure at $(\mathbf{x}, t)$ generated by source at $(\mathbf{y}, t - R/c)$,

$$p(\mathbf{x}, t) = \frac{\dot{q}(\mathbf{y}, t - R/c)}{4\pi R}.$$

$\rightarrow$ Sound at time t generated by source at *retarded time* $\tau = t - R/c$,

Now consider source *field*: a distribution of point sources of varying strengths. Their individual contributions can be summed to give the total sound field.

To do this: (1) Chop each point source up into an infinity of tiny temporal segments and sum their contributions; (2) Sum the contributions from all points of the source field.

For a given point source, the contribution of each temporal segment to the pressure field is,

$$p = \frac{\delta\left[\tau - (t - R/c)\right]}{4\pi R} = \frac{\delta(\tau - t + R/c)}{4\pi R}$$

where δ is the Delta Dirac and here imposes that we consider only the specific temporal segment $\tau = t - R/C$.

Green's functions

$$p = \frac{\delta(\tau - t + R/c)}{4\pi R},$$

is the impulse response of the wave equation. It is more formally written as,

$$G(\mathbf{x}, t; \mathbf{y}, \tau) = \frac{\delta\left[\tau - (t - R/c)\right]}{4\pi R} = \frac{\delta(\tau - t + R/c)}{4\pi R},$$

and is known as the *Green's function* of the wave equation. It describes the sound heard at $(\mathbf{x}, t)$ by an impulse at $(\mathbf{y}, \tau)$.

It allows us to sum up, over space and time, the infinity of tiny temporal source segments form the entire field of point sources, $\dot{q}(\mathbf{y}, t)$:

$$p(\mathbf{x}, t) = \int_\tau \int_\mathcal{V} \dot{q}(\mathbf{y}, \mathbf{t}) G(\mathbf{x}, t; \mathbf{y}, \tau) \mathrm{d}\mathcal{V} \mathrm{d}\tau,$$

i.e. it provides a solution to the problem,

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \dot{q}(\mathbf{x}, t).$$

Sound from a point force

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \cancel{\frac{\partial q}{\partial t}} - \frac{\partial f_i}{\partial x_i} \tag{1}$$

One way to find a solution to this problem is to use the fundamental solution considered in the last section.

To do this: write $p = \nabla \cdot \mathbf{s}$. Equation (3) then becomes,

$$\frac{1}{c^2}\frac{\partial^2 \nabla \cdot \mathbf{s}}{\partial t^2} - \nabla^2 \nabla \cdot \mathbf{s} = -\frac{\partial f_i}{\partial x_i} \tag{2}$$

$$\rightarrow \nabla \cdot \left[\frac{1}{c^2}\frac{\partial^2 s}{\partial t^2} - \nabla^2 s = -\mathbf{f}\right]. \tag{3}$$

We know the solution to the equation inside the square brackets,

$$\mathbf{s} = -\frac{\mathbf{f}(\mathbf{y}, t - R/c)}{4\pi R},$$

whence,

$$p = \nabla \cdot \mathbf{s} = \frac{\nabla R}{4\pi R} \cdot \left(\frac{\mathbf{f}(\tau)}{R} + \frac{1}{c}\frac{\partial \mathbf{f}(\tau)}{\partial \tau}\right)$$

Sound from a point force

$$p = \frac{\nabla R}{4\pi R} \cdot \left(\frac{\mathbf{f}(\tau)}{R} + \frac{1}{c}\frac{\partial \mathbf{f}(\tau)}{\partial \tau}\right)$$

Noting that $\nabla R = \hat{\mathbf{r}}$ is a unit vector that points from source to observer:

$$p = \frac{\hat{\mathbf{r}}}{4\pi R} \cdot \left(\frac{\mathbf{f}(\tau)}{R} + \frac{1}{c}\frac{\partial \mathbf{f}(\tau)}{\partial \tau}\right),$$

and, given that $\hat{\mathbf{r}} \cdot \mathbf{f} = f \cos\theta$ where θ is the angle between the source-observer vector and the direction of the force,

$$p = \frac{\cos\theta}{4\pi R}\left(\frac{f}{R} + \frac{\dot{f}}{c}\right),$$

which is the point-source solution to the equation,

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = -\frac{\partial f_i}{\partial x_i}. \tag{4}$$

Multipole sources

We have seen that acoustic problems involving sources of unsteady mass flux, or unsteady forces exerted on a medium, are described by,

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \dot{q} - \frac{\partial f_i}{\partial x_i}, \tag{5}$$

whose fundamental solution (for point sources) is,

$$p = \frac{\dot{q}(\tau)}{4\pi R} - \frac{\cos\theta}{4\pi R}\left(\frac{f(\tau)}{R} + \frac{\dot{f}(\tau)}{c}\right).$$

The acoustic field of the mass source is omnidirectional; that of the force source is directive (dependence on $\cos\theta$).

Both components of the sound field decay with distance (due to spherical spreading). For the mass source: $1/R$; for the force source, there are two components: $1/R^2$ and $1/R$. Close to the source (*nearfield*) the $1/R^2$ terms dominates; far from the source (*farfield*) the $1/R$ term is dominant.

Multipole sources

Directivity: volume source drives fluctuations with a spherical symmetry; force source has a preferred direction.

Consider force source to comprise two mass sources pulsating in antiphase, very close together. Their sound fields will interfere destructively to produce cancellation and $\cos\theta$ directivity.

As this source comprises two symmetric sources it is called a *dipole* (cf. Fig 1(b)). And the mass source, involving a single, symmetric source, is referred to as a *monopole*.

By arranging more and more elementary symmetric sources, higher-order, multipole sources can be constructed: quadrupoles (cf. Fig 1(c)), octopoles,...

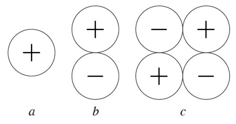

Figure 1: Multipole sources

Multipole sources

Building a dipole field from two monopoles: consider sound field radiated by two monopoles very close together,

$$\begin{aligned} p &= p^+ + p^- \\ &= \frac{q(\tau^+)}{4\pi R^+} - \frac{q(\tau^-)}{4\pi R^-}, \end{aligned}$$

where,

$$\begin{aligned} \tau^{\pm} &= t - R^{\pm}/c, \\ R^{\pm} &= \left[(x \mp a/2)^2 + y^2 + z^2\right]^{1/2}. \end{aligned}$$

For $a \ll 1$, expand pressure field in Taylor series,

$$p \approx p\big|_{a=0} + \left.\frac{dp}{da}\right|_{a=0} a + \dots$$

Multipole sources

$$p \approx p\big|_{a=0} + \left.\frac{dp}{da}\right|_{a=0} a + \dots$$

Computing $\frac{dp}{da}$ requires application of chain rule, because τ is a function of R and R is a function of a:

$$p^{\pm} = \frac{q(\tau(a))}{R(a)} \qquad \longrightarrow \qquad \begin{aligned} \frac{dp^{\pm}}{da} &= \frac{R\frac{dq}{da} - q\frac{dR}{da}}{R^2}, \\ \frac{dq}{da} &= \frac{dq}{d\tau}\frac{d\tau}{da} \end{aligned}$$

The first term in the Taylor series is zero. Using the chain rule as above to obtain $\left.\frac{dp}{da}\right|_{a=0}$ leads to,

$$p \approx a\frac{x}{4\pi R}\left(\frac{\dot{q}(\tau)}{Rc} + \frac{q(\tau)}{R^2}\right)$$

Considering that $f = aq$ (*dipole moment*) and $x = R\cos\theta$, we have,

$$p \approx \left(\frac{\dot{f}(\tau)}{c} + \frac{f(\tau)}{R}\right)\frac{\cos\theta}{4\pi R}, \qquad \text{as before.}$$

Lighthill's wave equation for Aeroacoustics

Mass and momentum conservation equations, again in non-linear form (turbulence is non-linear), neglecting viscosity and assuming isentropic processes,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\rho u_i = 0 \tag{6}$$

$$\frac{\partial}{\partial t}\rho u_i + \frac{\partial}{\partial x_j}\rho u_i u_j + \frac{\partial p}{\partial x_i} = 0 \tag{7}$$

Doing, $\frac{\partial}{\partial x_i}$(7)-$\frac{\partial^2}{\partial t^2}$(6) gives,

$$\begin{aligned} \frac{\partial^2 \rho}{\partial t^2} - \frac{\partial^2 p}{\partial x_i} &= \frac{\partial^2 \rho u_i u_j}{\partial x_i \partial x_j}, \\ \rightarrow \frac{1}{c_o^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x_i} &= \frac{\partial^2 \rho u_i u_j}{\partial x_i \partial x_j}, \end{aligned}$$

which is known as **Lighthill's acoustic analogy**.

An introduction to aeroacoustics
Lecture 4: Sound generated aerodynamically

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Outline

Lighthill's wave equation

Begin with mass and momentum conservation equations,

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0, \tag{1}$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j + p_{ij}) = 0, \tag{2}$$

where $p_{ij=}p\delta_{ij} - \sigma_{ij}$ is the compressive stress tensor, comprised of $p\delta_{ij}$, associated with the hydrodynamic pressure, and σ_{ij} associated with viscous effects that cause deviations from hydrostatic behaviour.

Consider that p_{ij} can be written in terms of an acoustic stress, $c_o^2\rho'\delta_{ij}$ and a correction, $p_{ij} - c_o^2\rho'\delta_{ij}$,

$$p_{ij} = c_o^2\rho'\delta_{ij} + (p_{ij} - c_o^2\rho'\delta_{ij}),$$

The momentum equation then takes the form,

$$\frac{\partial \rho u_i}{\partial t} + c_o^2\frac{\partial \rho'}{\partial x_i} = -\frac{\partial}{\partial x_j}(\rho u_i u_j + p_{ij} - c_o^2\rho'\delta_{ij}) = -\frac{\partial T_{ij}}{\partial x_j}$$

Lighthill's wave equation

So, the governing equations take the form,

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0, \tag{3}$$

$$\frac{\partial \rho u_i}{\partial t} + c_o^2\frac{\partial \rho'}{\partial x_i} = -\frac{\partial T_{ij}}{\partial x_j} \tag{4}$$

where,

$$T_{ij} = \rho u_i u_j + p_{ij} - c_o^2\rho'\delta_{ij}$$

is known as Lighthill's stress tensor, and by comparison with the inhomogeneous linearised equations of acoustics, can be understood as a source term.

This becomes clear by doing, $\frac{\partial}{\partial t}$(3)-$\frac{\partial}{\partial x_i}$(4), to give Lighthill's wave equation,

$$\frac{\partial^2 \rho'}{\partial t^2} - c_o^2\frac{\partial^2 \rho'}{\partial x_i^2} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}.$$

Lighthill's wave equation

We saw in the last lecture how to solve such an equation, using the Green's function,

$$\begin{aligned}
\rho'(\mathbf{x},t) &= \int_{-\infty}^{\infty}\int_{\mathcal{V}} \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j}(\mathbf{y},\tau) G(\mathbf{x},t;\mathbf{y},\tau) \mathrm{d}\mathbf{y}\mathrm{d}\tau \\
&= \frac{1}{4\pi c^2}\int_{-\infty}^{\infty}\int_{\mathcal{V}} \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j}(\mathbf{y},\tau)\delta\left(\tau-\left(t-\frac{R}{c}\right)\right)\frac{\mathrm{d}\mathbf{y}}{R}\mathrm{d}\tau \\
&= \frac{1}{4\pi c^2}\int_{\mathcal{V}} \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j}\left(\mathbf{y},t-\frac{R}{c}\right)\frac{\mathrm{d}\mathbf{y}}{R}
\end{aligned}$$

Equation can be simplified by using reciprocity property of the Green's function (source and observer can be interchanged),

$$\rho(\mathbf{x},t) = \frac{1}{4\pi c^2}\frac{\partial^2}{\partial x_i \partial x_j}\int_{\mathcal{V}} T_{ij}\left(\mathbf{y},t-\frac{R}{c}\right)\frac{\mathrm{d}\mathbf{y}}{R}.$$

Lighthill's wave equation

To progress, we need to manipulate the differentiation,

$$\frac{\partial^2}{\partial x_i \partial x_j}\frac{T_{ij}}{R}$$

bearing in mind that $T_{ij}\Big(\tau\big(R(x_i)\big)\Big)$ and $R(x_i)$.

$$\frac{\partial}{\partial x_i}\left[\frac{\partial}{\partial x_j}\frac{T_{ij}}{R}\right] = \frac{\partial}{\partial x_i}\left[\frac{R\frac{\partial T_{ij}}{\partial x_j} - T_{ij}\frac{\partial R}{\partial x_j}}{R^2}\right]$$

$$\frac{\partial T_{ij}}{\partial x_j} = \frac{\partial T_{ij}}{\partial \tau}\frac{\partial \tau}{\partial R}\frac{\partial R}{\partial x_j} = -\frac{(x_j-y_j)}{cR}\frac{\partial T_{ij}}{\partial \tau}$$

$$\rightarrow \frac{\partial}{\partial x_i}\left[\frac{\partial}{\partial x_j}\frac{T_{ij}}{R}\right] = \frac{\partial}{\partial x_i}\left[-\frac{(x_j-y_j)}{cR^2}\frac{\partial T_{ij}}{\partial \tau} - \cancel{\frac{T_{ij}}{R^2}\frac{(x_j-y_j)}{R}}\right]$$

Term in $1/R^3$ is a nearfield term and is therefore neglected.

Lighthill's wave equation

$$\begin{aligned}
\frac{\partial}{\partial x_i}\left[\frac{\partial}{\partial x_j}\frac{T_{ij}}{R}\right] &= -\frac{(x_j-y_j)}{c}\frac{\partial}{\partial x_i}\left[\frac{1}{R^2}\frac{\partial T_{ij}}{\partial \tau}\right] \\
&= -\frac{(x_j-y_j)}{c}\left[\frac{R^2\frac{\partial}{\partial \tau}\frac{\partial T_{ij}}{\partial x_i} - \frac{\partial T_{ij}}{\partial \tau}\frac{\partial R^2}{\partial x_i}}{R^4}\right] \\
&= \frac{(x_i-y_i)(x_j-y_j)}{c^2 R^3}\frac{\partial^2 T_{ij}}{\partial \tau^2} + \text{H.O.T.}
\end{aligned}$$

Solution to Lighthill's equation can thus be written,

$$\rho(\mathbf{x},t) = \frac{1}{4\pi c^2}\int_{\mathcal{V}}\frac{(x_i-y_i)(x_j-y_j)}{R^3}\frac{1}{c^2}\frac{\partial T_{ij}}{\partial t^2}\left(\mathbf{y},t-\frac{R}{c}\right)\mathrm{d}\mathbf{y}$$

If the observer is far from the source, $R = |\mathbf{x}-\mathbf{y}| \approx |\mathbf{x}|$, whence,

$$\rho(\mathbf{x},t) = \frac{1}{4\pi c^2}\frac{x_i x_j}{|\mathbf{x}|^3}\int_{\mathcal{V}}\frac{1}{c^2}\frac{\partial T_{ij}}{\partial t^2}\left(\mathbf{y},t-\frac{R}{c}\right)\mathrm{d}\mathbf{y}$$

Lighthill's wave equation

If and only if the source is acoustically compact, the retarded time can also be simplified, whence,

$$\rho(\mathbf{x},t) = \frac{1}{4\pi c^2}\frac{x_i x_j}{|\mathbf{x}|^3}\int_{\mathcal{V}}\frac{1}{c^2}\frac{\partial T_{ij}}{\partial t^2}\left(\mathbf{y},t-\frac{|\mathbf{x}|}{c}\right)\mathrm{d}\mathbf{y}. \quad (5)$$

If the source is not acoustically compact, the small differences in retarded time between different regions of the source field are critical in correctly determining the sound field.

Written in this form the equation provides the sound radiated into the farfield by a distribution of acoustically compact quadrupoles.

Without detailed knowledge of the source function T_{ij} a more precise expression can not be obtained. But the equation is suitable for a dimensional analysis.

Lighthill's wave equation

Consider turbulence characterised by velocity and length scales, u and l, dimensional analysis of equation 5 leads to,

$$\rho \sim \rho_o \frac{l}{|\mathbf{x}|} M^4$$

Acoustic intensity is defined as,

$$I = \frac{c^3}{\rho_o} \rho^2,$$

so the acoustic intensity associated with a compact turbulent eddy can be approximated as,

$$\begin{aligned} I &\sim \rho_o u^3 l^2 M^5, \\ &\sim \rho_o u^8 l^2 c^{-5}. \end{aligned}$$

Lighthill's wave equation

$$\begin{aligned} I &\sim \rho_o u^3 l^2 M^5, \\ &\sim \rho_o u^8 l^2 c^{-5}. \end{aligned}$$

Lighthill's eighth-power law, showing that jet noise should scale with the jet propulsion velocity to the power of eight!

This knowledge was what led to the main change in jet-engine design and that has led to substantial noise reduction from airline traffic.

Figure 1: 1970s Boeing 707

Sound from foreign bodies

If the volume of fluid considered contains solid bodies, the volume integral,

$$\frac{\partial^2}{\partial x_i \partial x_j} \int_{\mathcal{V}} \frac{T_{ij}(\mathbf{y}, \tau)}{R} \mathbf{dy},$$

implies a stress field within the solid boundary. To correctly account for the body, this non-physical distribution must be replaced by a physical source distribution on the surface of the body.

Sound from foreign bodies

To begin, note that,

$$\begin{aligned} \frac{\partial}{\partial x_i}\left(\frac{T_{ij}(\mathbf{y},\tau)}{R}\right) &= \frac{1}{R}\frac{\partial T_{ij}}{\partial x_i} - \frac{T_{ij}}{R^2}\frac{\partial R}{\partial x_i} \\ &= \frac{1}{R}\frac{\partial T_{ij}}{\partial \tau}\frac{\partial \tau}{\partial x_i} - \frac{(x_i - y_i)T_{ij}}{R^3} \\ &= \frac{(x_i - y_i)}{R^2 c}\frac{\partial T_{ij}}{\partial \tau} - \frac{(x_i - y_i)T_{ij}}{R^3} \end{aligned}$$

whereas,

$$\begin{aligned} \frac{\partial}{\partial y_i}\left(\frac{T_{ij}(\mathbf{y},\tau)}{R}\right) &= \frac{1}{R}\frac{\partial T_{ij}}{\partial y_i} - \frac{T_{ij}}{R^2}\frac{\partial R}{\partial y_i}, \\ &= \frac{1}{R}\left(\frac{\partial T_{ij}}{\partial y_i} + \frac{\partial T_{ij}}{\partial \tau}\frac{\partial \tau}{\partial y_i}\right) - \frac{(y_i - x_i)T_{ij}}{R^3} \\ &= \frac{1}{R}\frac{\partial T_{ij}}{\partial y_i} + \frac{(y_i - x_i)}{R^2 c}\frac{\partial T_{ij}}{\partial \tau} - \frac{(y_i - x_i)T_{ij}}{R^3} \end{aligned}$$

whence,

$$\frac{\partial}{\partial x_i}\left(\frac{T_{ij}(\mathbf{y},\tau)}{R}\right) = -\frac{\partial}{\partial y_i}\left(\frac{T_{ij}(\mathbf{y},\tau)}{R}\right) + \frac{1}{R}\frac{\partial T_{ij}}{\partial y_i}$$

Sound from foreign bodies

Recall that,

$$T_{ij} = \rho u_i u_j + p_{ij} - c_o^2 \rho' \delta_{ij},$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j + p_{ij}) = 0.$$

and, using square brackets to denote the value of the function at $(\mathbf{y}, \tau)$,

$$\frac{\partial}{\partial x_i}\left[\frac{T_{ij}}{R}\right] = -\frac{\partial}{\partial y_i}\left[\frac{\rho u_i u_j + p_{ij}}{R}\right] + c^2 \frac{\partial}{\partial y_i}\left[\frac{\rho' \delta_{ij}}{R}\right] + \frac{1}{R}\frac{\partial T_{ij}}{\partial y_i}$$

Consider second term on RHS,

$$\begin{aligned}\frac{\partial}{\partial y_i}\left[\frac{\rho' \delta_{ij}}{R}\right] &= \frac{\partial}{\partial y_j}\left[\frac{\rho'}{R}\right], \\ &= \frac{1}{R}\frac{\partial \rho'}{\partial y_j} + \frac{(y_j - x_j)}{R^2}\left(\frac{\partial \rho'}{\partial R} - \frac{\rho'}{R}\right).\end{aligned}$$

Compare with,

$$\begin{aligned}\frac{\partial}{\partial x_j}\left[\frac{\rho'}{R}\right] &= \frac{(x_j - y_j)}{R^2}\left(\frac{\partial \rho'}{\partial R} - \frac{\rho'}{R}\right), \\ \rightarrow \frac{\partial}{\partial y_j}\left[\frac{\rho'}{R}\right] &= -\frac{\partial}{\partial x_j}\left[\frac{\rho'}{R}\right] + \frac{1}{R}\frac{\partial \rho'}{\partial y_j}\end{aligned}$$

Sound from foreign bodies

And so,

$$\begin{aligned}\frac{\partial}{\partial x_i}\left[\frac{T_{ij}}{R}\right] &= -\frac{\partial}{\partial y_i}\left[\frac{\rho u_i u_j + p_{ij}}{R}\right] - c^2 \frac{\partial}{\partial x_j}\left[\frac{\rho'}{R}\right] + c^2 \frac{1}{R}\frac{\partial \rho'}{\partial y_j} + \frac{1}{R}\frac{\partial T_{ij}}{\partial y_i}, \\ &= -\frac{\partial}{\partial y_i}\left[\frac{\rho u_i u_j + p_{ij}}{R}\right] - c^2 \frac{\partial}{\partial x_j}\left[\frac{\rho'}{R}\right] - \frac{1}{R}\frac{\partial \rho u_j}{\partial t}.\end{aligned}$$

Now differentiate with respect to x_j,

$$\frac{\partial^2}{\partial x_i \partial x_j}\left[\frac{T_{ij}}{R}\right] = -\frac{\partial^2}{\partial y_i \partial x_j}\left[\frac{\rho u_i u_j + p_{ij}}{R}\right] - c^2 \frac{\partial^2}{\partial x_j^2}\left[\frac{\rho'}{R}\right] - \frac{\partial}{\partial x_j}\left[\frac{1}{R}\frac{\partial \rho u_j}{\partial t}\right].$$

Sound from foreign bodies

Given that,

$$\begin{aligned}\frac{\partial}{\partial x_j}\left[\frac{1}{R}\frac{\partial \rho u_j}{\partial t}\right] &= \frac{1}{R}\frac{\partial^2 \rho u_j}{\partial t \partial y_j} - \frac{\partial}{\partial y_j}\left[\frac{1}{R}\frac{\partial \rho u_j}{\partial t}\right], \\ &= -\frac{1}{R}\frac{\partial^2 \rho'}{\partial t^2} - \frac{\partial}{\partial y_j}\left[\frac{1}{R}\frac{\partial \rho u_j}{\partial t}\right],\end{aligned}$$

We have,

$$\begin{aligned}\frac{\partial^2}{\partial x_i \partial x_j}\left[\frac{T_{ij}}{R}\right] &= -\frac{\partial^2}{\partial y_i \partial x_j}\left[\frac{\rho u_i u_j + p_{ij}}{R}\right] - c^2 \frac{\partial^2}{\partial x_j^2}\left[\frac{\rho'}{R}\right] + \frac{\partial^2}{\partial t^2}\left[\frac{\rho'}{R}\right] + \frac{\partial}{\partial y_j}\left[\frac{1}{R}\frac{\partial \rho u_j}{\partial t}\right], \\ &= -\frac{\partial^2}{\partial y_i \partial x_j}\left[\frac{\rho u_i u_j + p_{ij}}{R}\right] + \frac{\partial}{\partial y_j}\left[\frac{1}{R}\frac{\partial \rho u_j}{\partial t}\right].\end{aligned}$$

Sound from foreign bodies

Integrating this over the volume of the solid body, V_s,

$$\frac{\partial^2}{\partial x_i \partial x_j}\int_{V_s}\left[\frac{T_{ij}}{R}\right]\mathbf{dy} = -\frac{\partial}{\partial x_j}\int_{V_S}\frac{\partial}{\partial y_i}\left[\frac{\rho u_i u_j + p_{ij}}{R}\right]\mathbf{dy} + \int_{V_S}\frac{\partial}{\partial y_j}\left[\frac{1}{R}\frac{\partial \rho u_j}{\partial t}\right]\mathbf{dy},$$

and applying the divergence theorem gives,

$$\frac{\partial^2}{\partial x_i \partial x_j}\int_{V_s}\left[\frac{T_{ij}}{R}\right]\mathbf{dy} = -\frac{\partial}{\partial x_j}\int_S n_i\left[\frac{\rho u_i u_j + p_{ij}}{R}\right]\mathbf{dy_s} + \frac{\partial}{\partial t}\int_S\left[\frac{\rho \mathbf{u} \cdot \mathbf{n}}{R}\right]\mathbf{dy_s}.$$

This shows that the sound source constituted by the sound body involves both dipole and monopole terms.

Sound from foreign bodies

The expression for the total sound field radiated by a turbulent flow that includes a solid body is, therefore,

$$\begin{aligned}\rho'(\mathbf{x},t) &= \frac{1}{4\pi c^2}\frac{\partial^2}{\partial x_i \partial x_j}\int_{\mathcal{V}}\left[\frac{T_{ij}}{R}\right]\mathbf{dy},\\ &-\frac{1}{4\pi c^2}\frac{\partial}{\partial x_j}\int_S n_i\left[\frac{\rho u_i u_j + p_{ij}}{R}\right]\mathbf{dy_s},\\ &+\frac{\partial}{\partial t}\int_S\left[\frac{\rho\mathbf{u}\cdot\mathbf{n}}{R}\right]\mathbf{dy_s}.\end{aligned}$$

The first integral is associated with *quadrupole* sound generated by free turbulence, the second describes *dipole* sound associated with stresses on the surface of the solid body, while the third is associated with eventual *monopole* radiation that would occur if the solid body were undergoing volume fluctuations, or if there were mass flux through the surface.

Sound from foreign bodies

For rigid bodies at rest, the expression reduces to,

$$\begin{aligned}\rho'(\mathbf{x},t) &= \frac{1}{4\pi c^2}\frac{\partial^2}{\partial x_i \partial x_j}\int_{\mathcal{V}}\left[\frac{T_{ij}}{R}\right]\mathbf{dy},\\ &-\frac{1}{4\pi c^2}\frac{\partial}{\partial x_j}\int_S n_i\left[\frac{p}{R}\right]\mathbf{dy_s},\end{aligned}$$

the dipole sound generated by the solid body being associated with the surface pressure fluctuations.

If the body in question is acoustically compact, the dipole sound contribution simplifies to,

$$\rho'_d(\mathbf{x},t) = \frac{1}{4\pi c^2}\frac{\partial}{\partial x_i}\left[\frac{F_i}{R}\right],$$

where $F_i = \int_S n_i p \mathbf{dy_s}$ is the force exerted on the fluid by the body.

Sound from foreign bodies

As the magnitude of the force is,

$$F \sim \rho_o U^2 D^2,$$

Dimensional analysis gives, for the intensity of dipole sound associated with a compact solid,

$$\overline{\rho_d'^2} \sim \rho_o^2 M^6 \frac{D^2}{R^2}.$$

Note that whereas the quadrupole sound of free turbulence scales with U^8, the dipole sound of foreign bodies immersed in unsteady fluid scales with U^6.

This shows that compact dipoles are more efficient than compact quadrupoles by a factor M^{-2}. It can be shown that monopoles are more efficient by a further factor of M^2. You can check that for compact mnopoles, we have,

$$\overline{\rho_m'} \sim \rho_o^2 M^4 \frac{D^2}{R^2}$$

Wavepackets and turbulent jet noise

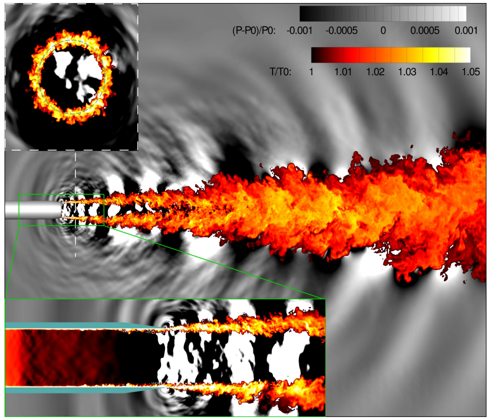


Figure 2: Large-eddy simulation of $M0.9$ turbulent jet

Wavepackets and turbulent jet noise

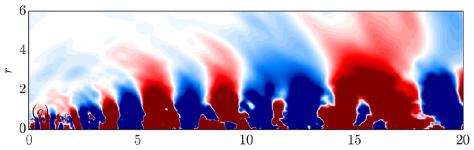


Figure 3: Instantaneous image of pressure field

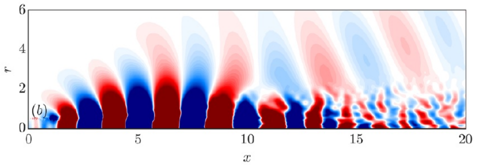
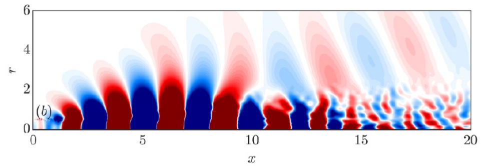


Figure 4: Filtered pressure field at Strouhal number, $St = 0.3$

Wavepackets and turbulent jet noise



- *"If you think you know some of the features of a random function, for goodness sake put them in"* [Mollo-Christensen 1967]
- Filtered LES data suggests a travelling wave with some amplitude envelope.
- Model source as,

$$S(y) = \frac{\partial^2 T_{11}}{\partial y_1 \partial y_1} = e^{ik_h y} e^{\frac{-y^2}{L^2}}$$

$\rightarrow$ a time-periodic, harmonic perturbation whose amplitude grows and decays at it evolves downstream with phase speed, $u_c = \omega / k_h$

Wavepackets and turbulent jet noise

$$S(y) = \frac{\partial^2 T_{11}}{\partial y_1 \partial y_1} = e^{ik_h y} e^{\frac{-y^2}{L^2}}$$

$\rightarrow$ a time-periodic, harmonic perturbation whose amplitude grows and decays at it evolves downstream with phase speed, $u_c = \omega / k_h$

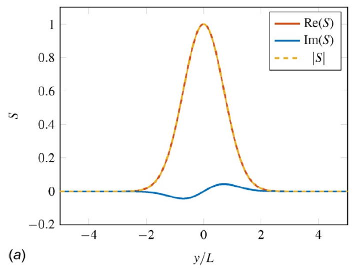


Figure 5: Compact source, $k_h L = 0.1$.

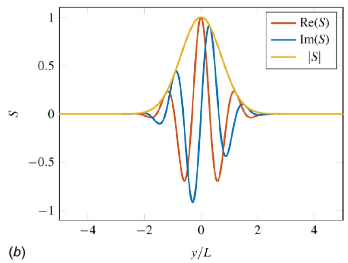


Figure 6: Non-compact source, $k_h L = 5$

Wavepackets and turbulent jet noise

Source,

$$S(y,\omega) = \frac{\partial^2 T_{11}}{\partial y_1 \partial y_1} = e^{ik_h y} e^{\frac{-y^2}{L^2}}$$

radiates sound field,

$$p(\mathbf{x},\omega) = \int S(y,\omega) G(\mathbf{x}, y, \omega) dy$$

where,

$$G(\mathbf{x}, y, \omega) = \frac{e^{ik|\mathbf{x}-y|}}{4\pi|\mathbf{x}-y|}$$

is the Green's function for the problem.

Evaluation of the integral gives, for the acoustic power spectral density in the farfield,

$$\langle p(\mathbf{x},\omega) p^*(\mathbf{x},\omega) = \frac{L^2}{16\pi|\mathbf{x}-y|^2} e^{-\frac{1}{2}k_h^2 L^2 \left(1 - M_c \cos\theta\right)^2}$$

Wavepackets and turbulent jet noise

PSD of farfield sound,

$$\langle p(\mathbf{x},\omega)p^*(\mathbf{x},\omega)\rangle = \frac{L^2}{16\pi|\mathbf{x}-y|^2}\mathrm{e}^{-\frac{1}{2}k_h^2L^2\left(1-M_c\cos\theta\right)^2}.$$

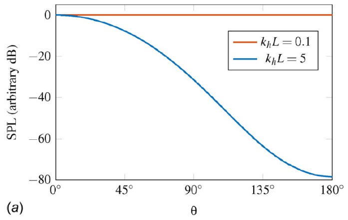


Figure 7: PSD of farfield for Mach number, $M = 0.6$

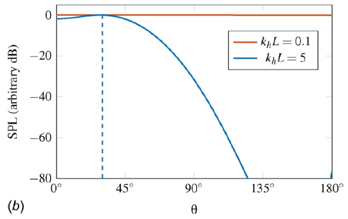


Figure 8: PSD of farfield for Mach number, $M = 2$

Wavepackets and turbulent jet noise

Comparison with experiments

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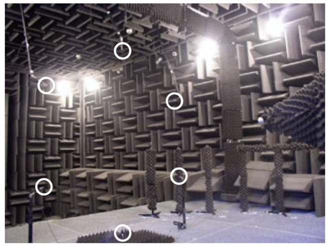

Figure 9: Experiment at *Bruit & Vent*

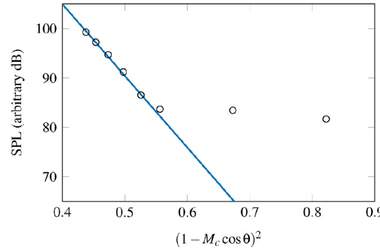

Figure 10: Comparison of measurement and model

Wavepackets and turbulent jet noise

Acoustic matching

- Not all source activity generates propagating waves,
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Consider observer (listener) in the farfield.

For the phase part of the source,

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and for the spherical decay term,

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Wavepackets and turbulent jet noise

With these assumptions the Green's function takes the form,

$$G(\mathbf{x},y,\omega) \approx \frac{1}{4\pi|\mathbf{x}|}\mathrm{e}^{ik(|\mathbf{x}|-y\cos\theta)},$$

and the solution for the sound field then becomes,

$$p(\mathbf{x}) = \frac{\mathrm{e}^{-ik|\mathbf{x}|}}{4\pi|\mathbf{x}|}\int S(y)\mathrm{e}^{iky\cos\theta}\mathrm{d}y.$$

The integral can be recognised as a Fourier transform with modified Fourier modes, $\mathrm{e}^{-iky\cos\theta}$.

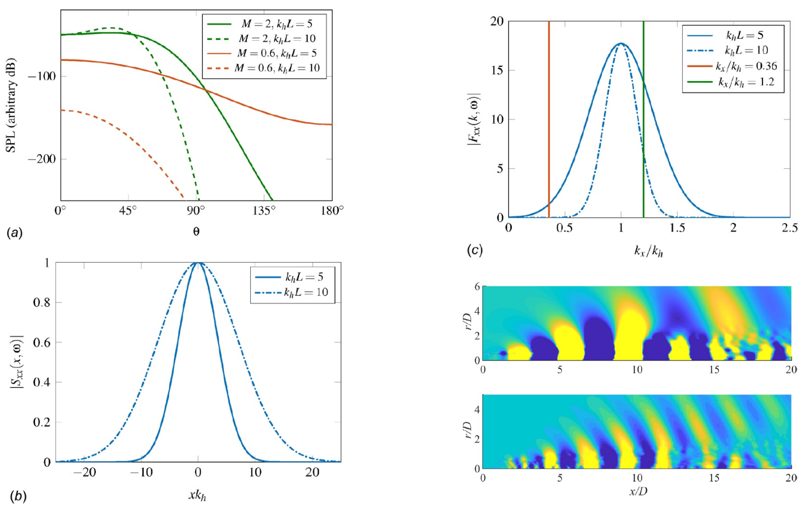
$\rightarrow$ only modes with non-zero projection onto this Fourier basis can contribute to the farfield, i.e.,

$$|k_y| = k\cos\theta, \qquad \rightarrow \frac{|k_y|}{k} \leq 1, \qquad \rightarrow \frac{|k_y|}{k_h} \leq M_c.$$

$$M_c = \frac{k}{k_h}, \qquad k = \frac{\omega}{c}, \qquad k_h = \frac{\omega}{U_c}.$$

Wavepackets and turbulent jet noise

Acoustic matching, $\frac{|k_y|}{k_h} \leq M_c$



Wavepackets and turbulent jet noise

Stochastic wavepackets

Turbulence is stochastic, which means that the Fourier transform of the source,

$$S(y,\omega) = \int_{-\infty}^{\infty} S(y,t)\mathrm{e}^{i\omega t}\mathrm{d}t$$

is ill-defined, as $S(y,t)$ is not square integrable.

To circumvent this problem, one must work with Power and Cross Spectral Densities (PSD and CSD). The PSD of the acoustic field can be written as,

$$\langle \hat{p}(\mathbf{x},\omega)\hat{p}^*(\mathbf{x},\omega)\rangle = \int_y \int_z \langle S(y,\omega)S^*(z,\omega)\rangle G(x,y,\omega)G(x,z,\omega)\mathrm{d}y\mathrm{d}z$$

So, the PSD of the sound field depends on CSD (two-point statistics) of the source.

This means that the streamwise desynchronisation of travelling wave maybe important for acoustic efficiency.

Wavepackets and turbulent jet noise

A stochastic line-source model

$$\langle S(y,\omega)S^*(z,\omega)\rangle = \mathrm{e}^{ik_h(y-z)}\mathrm{e}^{-\frac{y^2}{L^2}-\frac{z^2}{L^2}}\mathrm{e}^{-\frac{(y-z)^2}{L_c^2}}$$

Wavepackets and turbulent jet noise

Resolvent analysis

An introduction to aeroacoustics
Lecture 5: Wavepackets and turbulent jet noise

Peter Jordan, Vincent Jaunet & Guillaume Lehnasch

Institut PPRIME
CNRS · Université de Poitiers · ISAE-ENSMA

June 2020

Outline

Coherent structures in turbulent jets

Thoughts from the past

- *"[jet noise] is of interest as a problem in fluid dynamics in the class of problems which involve the interaction between instability, turbulence and wave emission" – Mollo-Christensen (1963)*
- *"The data suggest that one may perhaps represent the fluctuating [hydrodynamic] pressure field in terms of rather simple functions."* – Mollo-Christensen (1968)
- *". . . the eddies generating the noise seem to be much bigger than those eddies which have been the subject of intense turbulence study. They are very likely those large eddies which derive their energy from an instability of the mean motion*" – Bishop, Ffowcs-Williams & Smith (1971)
- *". . . the turbulence establishes an equivalent laminar flow profile as far as large-scale modes are concerned*" – Crighton & Gaster (1976)

Coherent structures in turbulent jets

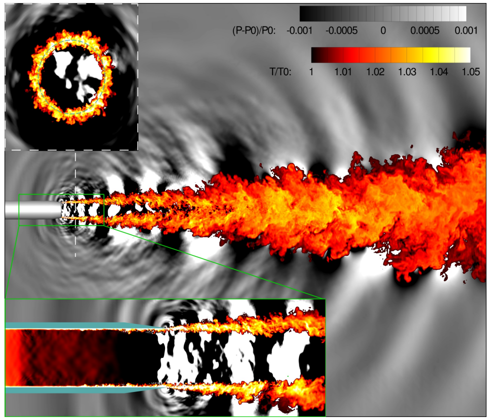


Figure 1: Large-eddy simulation of $M = 0.9$ turbulent jet

Coherent structures in turbulent jets

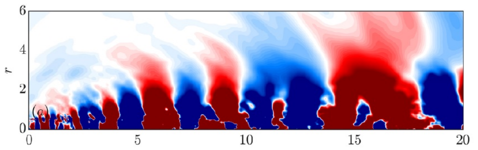


Figure 2: Instantaneous image of pressure field

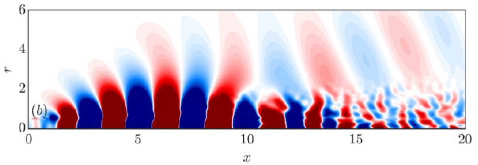
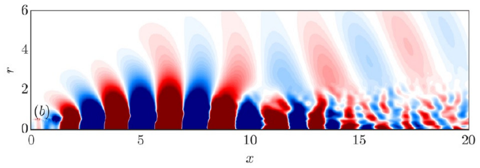


Figure 3: Filtered pressure field at Strouhal number, $St = 0.3$

The harmonic line-source



- *"If you think you know some of the features of a random function, for goodness sake put them in"* [Mollo-Christensen 1967]
- Filtered LES data suggests a travelling wave with some amplitude envelope.
- Model source as,

$$S(y) = \frac{\partial^2 T_{11}}{\partial y_1 \partial y_1} = e^{ik_h y} e^{\frac{-y^2}{L^2}}$$

$\rightarrow$ a time-periodic, harmonic perturbation whose amplitude grows and decays at it evolves downstream with phase speed, $u_c = \omega / k_h$

The harmonic line-source

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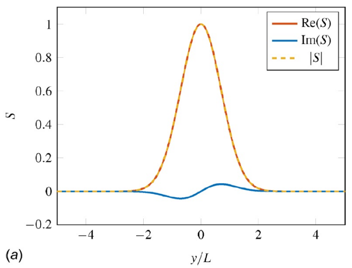

Figure 4: Compact source, $k_h L = 0.1$.

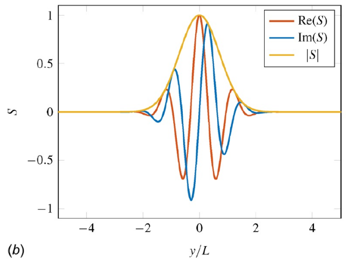

Figure 5: Non-compact source, $k_h L = 5$

The harmonic line-source

Source,

$$S(y, \omega) = \frac{\partial^2 T_{11}}{\partial y_1 \partial y_1} = e^{ik_h y} e^{\frac{-y^2}{L^2}}$$

radiates sound field,

$$p(\mathbf{x}, \omega) = \int S(y, \omega) G(\mathbf{x}, y, \omega) \mathrm{d}y$$

where,

$$G(\mathbf{x}, y, \omega) = \frac{e^{ik|\mathbf{x}-y|}}{4\pi|\mathbf{x}-y|}$$

is the Green's function for the problem.

Evaluation of the integral gives, for the acoustic power spectral density in the farfield,

$$\langle p(\mathbf{x}, \omega) p^*(\mathbf{x}, \omega) = \frac{L^2}{16\pi|\mathbf{x}-y|^2} e^{-\frac{1}{2}k_h^2 L^2 \left(1 - M_c \cos\theta\right)^2}$$

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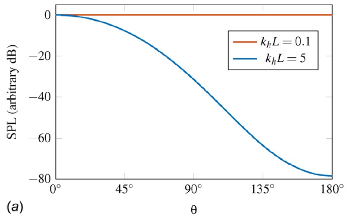

Figure 6: PSD of farfield for Mach number, $M = 0.6$

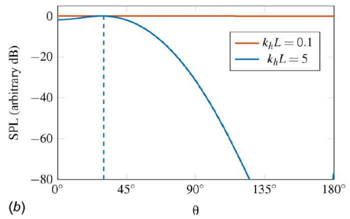

Figure 7: PSD of farfield for Mach number, $M = 2$

The harmonic line-source

Comparison with experiments

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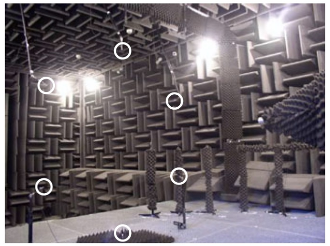
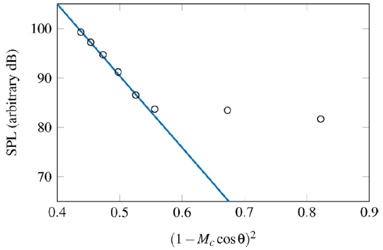


Figure 8: Experiment at *Bruit & Vent*



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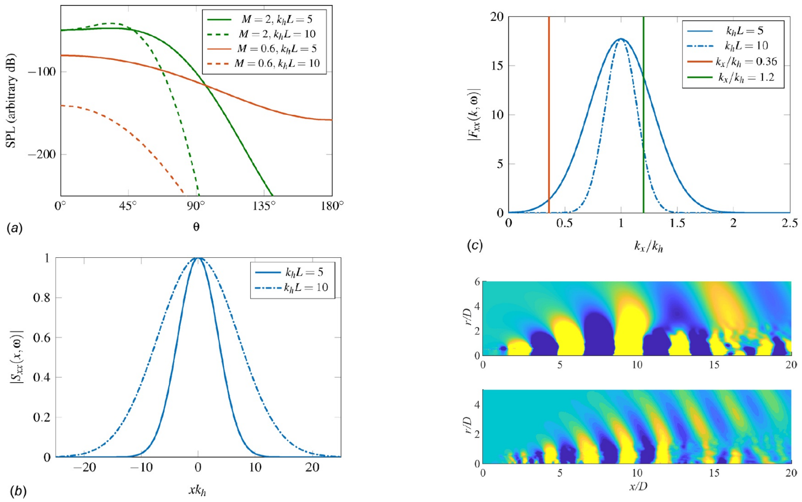
Acoustic matching, $\frac{|k_y|}{k_h} \leq M_c$



The stochastic line source

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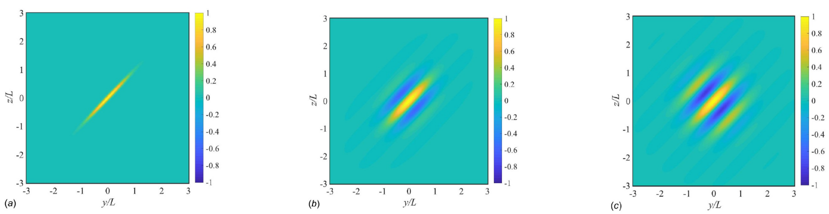
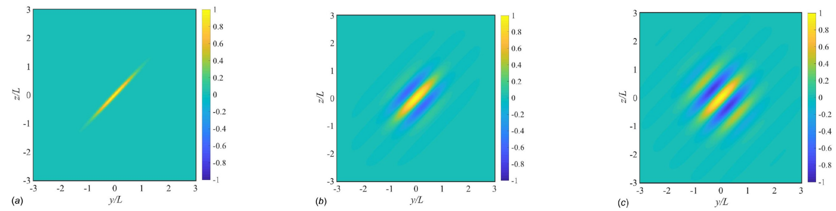


$k_hL = 5;\ L_c = L/10$ $k_hL = 5;\ L_c = L$ $k_hL = 5;\ L_c = 10L$

The stochastic line source



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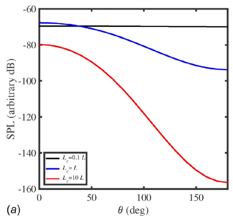
$M = 0.6$ $M = 2$

The stochastic line source



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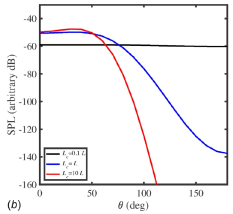
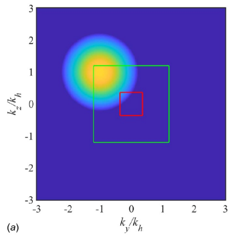


$M = 2$



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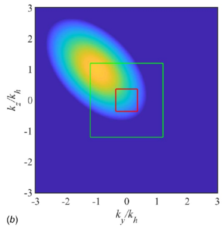

$M = 2$

The stochastic line source

Comparison with experiments

Model parameters informed by PIV measurements

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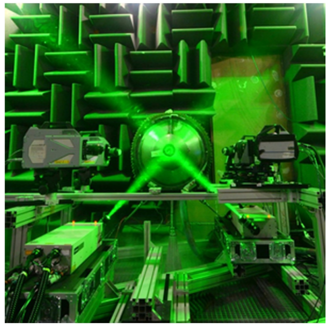

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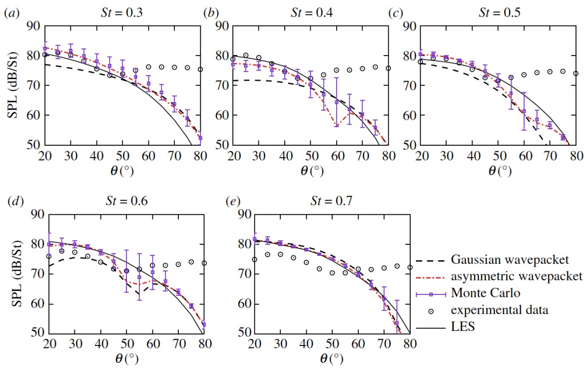


Resolvent analysis

Lighthill's acoustic analogy,

$$c_o^2\frac{\partial^2 p}{\partial t^2} - \Delta p = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j},$$

written in operator form is,

$$\mathcal{L}_w p = S.$$

Its solution is,

$$p(\mathbf{x},t) = \int S(\mathbf{y},\tau)G(\mathbf{x},t,\mathbf{y},\tau)\mathrm{d}\mathbf{x}\mathrm{d}\tau.$$

In frequency space,

$$\mathcal{L}_{Helmholtz}\hat{p}(\mathbf{x},\omega) = \hat{S}(\mathbf{x},\omega)$$

$$\hat{p}(\mathbf{x},\omega) = \int \hat{S}(\mathbf{y},\omega)\hat{G}(\mathbf{x},\mathbf{y},\omega)\mathrm{d}\mathbf{x}$$

Resolvent analysis

Writing,

$$\hat{p}(\mathbf{x},\omega) = \int \hat{S}(\mathbf{y},\omega)\hat{G}(\mathbf{x},\mathbf{y},\omega)\mathrm{d}\mathbf{x},$$

in operator notation,

$$p = \mathcal{R}_{Helmholtz} S,$$

we see that the resolvent operator,

$$\mathcal{R}_{Helmholtz}[\cdot] = \int [\cdot] G(\mathbf{x},\mathbf{y},\omega)\mathrm{d}\mathbf{x}.$$

With the spatial integration approximated using quadrature weights, W, the resolvent operator can be written in matrix forms as,

$$\mathbf{R}_{Helmholtz} = \mathbf{GW},$$

Solution to Lighthill's equation can thus be written in matrix (resolvent) form, as,

$$\mathbf{p} = \mathbf{R}_{Helmholtz}\mathbf{S}$$

Resolvent analysis

The Navier Stokes equations can be handled similarly.

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{1}{\mathrm{Re}}\nabla^2 u_i$$

Introduce Reynolds decomposition, Fourier transform, gather linear terms on the LHS and non-linear terms on the RHS,

$$-i\omega\hat{u}_i + U_j\frac{\partial \hat{u}_i}{\partial x_j} + \hat{u}_j\frac{\partial U_i}{\partial x_j} + \frac{\partial \hat{p}}{\partial x_i} - \frac{1}{\mathrm{Re}}\nabla^2\hat{u}_i = -\left[\hat{u}_j \star \frac{\partial \hat{u}_i}{\partial \hat{u}_j}\right]_{\omega\neq 0}$$

which in operator notation takes the form,

$$\mathcal{L}_{NS}\mathbf{q} = \mathbf{f},$$

and the linear operator can be inverted to give the resolvent operator, in which case the solution can be written,

$$\mathbf{q} = \mathcal{R}_{NS}\mathbf{f},$$

or, in matrix form,

$$\mathbf{q} = \mathbf{R}_{NS}\mathbf{f}$$

Resolvent analysis

Singular Value Decomposition (SVD) of resolvent operator,

$$\mathbf{R} = \mathbf{U\Sigma V}^H$$

implies that the solution,

$$\mathbf{q} = \mathbf{Rf}$$

can be written as,

$$\mathbf{q} = \mathbf{U\Sigma V}^H\mathbf{f}$$

U and **V** are unitary matrices and $\mathbf{\Sigma}$ is diagonal with diagonal entries organised be decreasing value.

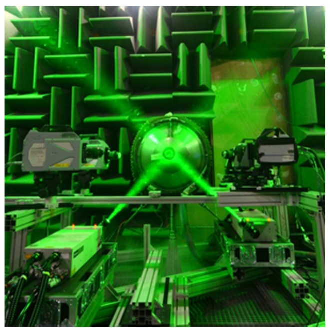
When we rearrange the equation to read,

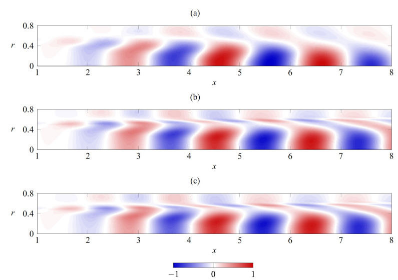
$$\mathbf{U}^H\mathbf{q} = \mathbf{\Sigma V}^H\mathbf{f},$$

we see that the projection of the response, $\mathbf{q}$, onto each of the vectors of the basis, $\mathbf{U}$ is equal to the projection of the forcing, $\mathbf{f}$ onto each of the vectors of the basis, $\mathbf{V}$, times the corresponding gain, σ.

Resolvent analysis

In flows with a strong instability mechanism, $\sigma_1 \gg \sigma_n$, $n \geq 2$, the linear operator is realtively insensitive to the specifics of the forcing (non-linear interactions) and the leading response mode may provide a good model for the most amplified structures: coherent structures, a.k.a. wavepackets.





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Appendix A : TDs

Tutorial 1 :
Reflection by a flexible wall

In this tutorial, the goal is to study the effect of wall impdance in the reflection of sound waves, and to derive some impedance models, based on simplified structural wall models.

1.1 Wave reflection

Consider a plane wave impinging on a wall with an angle of θ. Consider that the impedance is a local-only property (how the wall moves is a function only of the pressure at a point, not of what the wall on the vicinity of that point is doing), i.e.

$$\hat{v}(x, y = 0, \omega) = \hat{p}(x, y = 0, \omega)\hat{z}_{wall}(\omega). \tag{A.1}$$

1. Compute the reflection coefficient as a function of θ and $\hat{z}_{wall}(\omega)$.

2. Is there a value of $\hat{z}_{wall}(\omega)$ for which $\mathcal{R} = 0$ for $\theta = 0$? What happens for other values of θ?

1.2 Spring-mass wall

Consider a wall model as a spring-mass system, illustrated in figure A.1, were k is the spring coefficient k (force per unit lenght divided by the wall displacement), and ρ_{wall} the linear density (mass per unit lenght) of the wall.

3. Compute the impdance of the wall. Can this be used to reduce acoustic reflection? If not, what needs to be added?

4. Add a damping term to the spring-mass model, with a damping coefficient c. Re-compute the impedance of the wall. Can this be used to reduce acoustic reflection?

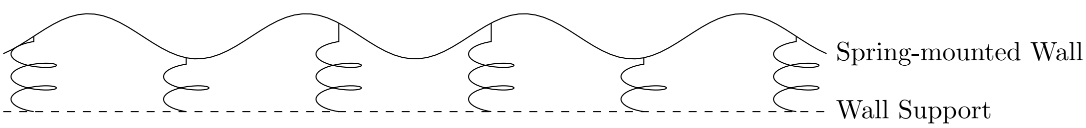

Figure A.1: Illustration of a spring-mounted wall.

1.3 Wall-mounted streched spring wall

Consider now a model where the wall is modelled as a streched spring, mounted on a viscoelastic subtrate. Wall is modelled as

$$\rho\frac{\partial^2 y_{wall}}{\partial t^2} + T\frac{\partial^2 y_{wall}}{\partial x^2} + c\frac{\partial y_{wall}}{\partial t} + k y_{wall} = 0 \tag{A.2}$$

5. Compute the impedance of the wall. Mention the potential advantages and disadvantages of this model.

Tutorial 2 :
Impedance effects on an acoustic duct

We will now consider the wall Impedance effects on the propagation of acoustic waves in ducts.

2.1 Wall impedance

Consider an 2D (infinitly extended) acoustic duct (as previously studied in chapter 3). We will now investigate the effect of finite wall impedance. That is the walls are flexible, not rigid as consdiered before. The wall impedance is denoted by $\hat{z}_{wall}(\omega)$.

1. Obtain the dispersion relation and the pressure field for the flexible duct.

1.1 Discuss the behaviours of $\hat{z}_{wall} \to 0$ and $\hat{z}_{wall} \to \infty$.

2. Consider a wall with a small, but finite impedance, i.e., $\hat{z}_{wall} = \epsilon \hat{z}_1$. Assuming expansions

$$k_y = k_{y,0} + \epsilon k_{y,1} + \dots, \tag{A.3}$$
$$k_x = k_{x,0} + \epsilon k_{x,1} + \dots, \tag{A.4}$$

estimate $k_{y,1}$ and $k_{x,1}$.

2.1 What is the effect of real and imaginary parts of $\hat{z}_{wall}$ on the wave numbers k_y and k_x?

2.2 Practical applications

Instead of a flexible wall, an effective wall impedance can be obtained with the use of liners. Liners are porous materials that are placed in the walls of ducts to absorb sound waves. The simplest liner model is to consider each hole as a ressonating tube, as in section 3.4.2.

3. Estimate the size of the liner required to have an impedance of $\hat{z}_{wall}$ of order 1. Is this pratical? Can it be used to reduce the amplitude of the sound waves?

4. Can you think of a practical application where the wall impedance of a duct is important? (See figure A.2)

5. Interpret figure A.2. What is the effect of small and large impedances? Do you expect large impedance limit to change if $\hat{z}_{wall}$ is real, imaginary, positive, or negative?

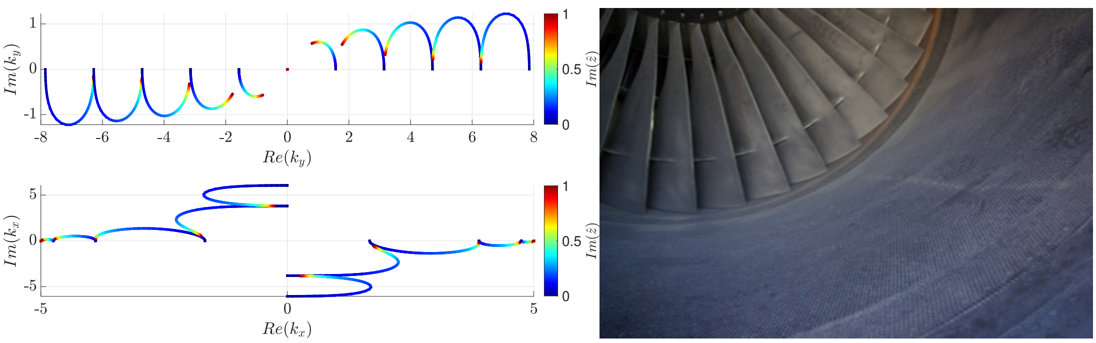

Figure A.2: On the top, k_x and k_y for increasing imaginary wall impedances. On the bottom, a real airplane engine.

Tutorial 3 : *Duct discontinuities*

3.1 Sudden duct change

Consider a 2D duct with a sudden change in its section, as indicated in figure A.3.

Consider now an acoustic wave with frequency ω and wavenumber k. The wave is incident from the left, with amplitude I. We aim to compute the amplitude of the reflected (R) and trasmitted (T) waves. We assume that the duct is small enough such that only planes waves are propagative inside it.

1. Write the boundary conditions at the duct discontinuity.
2. Compute the reflection and transmission coefficients.
3. What is the parameter that drives the reflection/transmission mechanism?
4. In this configuration, up to what degree can we relax the assumption that only plane waves are propagative? Does this makes the process more or less sensitive to geometrical variations?

3.2 Localized impedance

Now consider a duct with where, near $x = 0$, the duct has a small patch where the wall impedance is different, i.e., $\hat{z}_{wall}$, as illustrated in figure A.4. This impedance patch can model, for example, a Helmhotz ressonator, which we will discuss later. For now, the physics driving this impedance are not important.

1. Write the boundary conditions at the impedance patch.
2. Compute the reflection (R) and transmission (T) coefficients.
3. What is the impedance value which leads to no transmitted wave? How about no reflected wave?

3.3 Impedance matching

1. Can we combine a localized impedance near the duct discontinuity to avoid reflected waves?

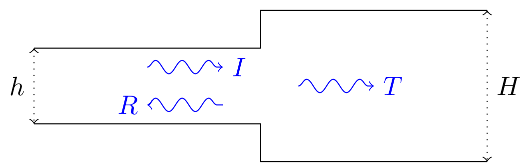

Figure A.3: Wave reflection and transmission in a sudden duct change.

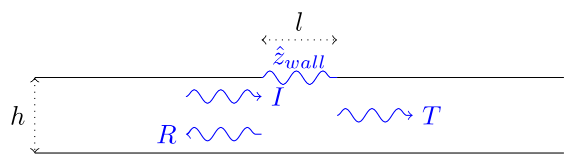

Figure A.4: Wave reflection and transmission in duct discontinuity.

Tutorial 4 :
Helmhotz resonator

The goal of this section is to derive a simplified model for a Helmhotz resonator. Helmhotz ressonators remove some of the drawbacks of the "acoustic pipe" liner explores in the previous tutorial. The Helmhotz resonator is a cavity with a neck, as illustrated in figure A.5. The cavity is connected to the main duct through the neck.

Here we will assume the plane wave assumption, i.e., only plane waves are propagative inside the resonator. Within this framework, the boundary conditions at the junction between the ressonator entry and the main duct are (a) pressure continuity and (b) mass conseevation.

Based on the geometrical parameters of the resonator, and the assumption that all the waves inside it are plane waves, derive

4.1 Resonator natural frequeny

1. Write the boundary conditions.
2. Write the interface boundary condition at the junction between the resonator and the main duct.
3. Derive the natural frequency of the resonator.
4. Compare with the ressonator studied in section 3.4.2.

4.2 Forced response

1. Assume a imposed pressure on the ressonator oppening with an arbitrary frequency ω. Solve for the pressure inside the resonator.
2. Compute the ressonator impedance.
3. What is the physics that explain the frequencies for which the impedance is 0 or ∞?

4.3 Real fluid effects

1. If $h \ll H$ what effect do you expect at the junction between the small and large sections of the duct? Can this be leveraged to reduce the amplitude of the sound waves?

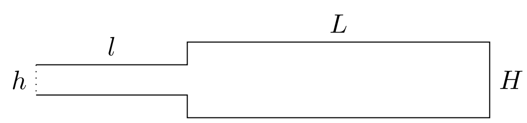

Figure A.5: Illustration of a Helmhotz resonator.

Tutorial 5 :
Three-dimensional acoustic ducts

Previously, we have focused on 2D ducts. However, in many practical applications, we face 3D (infinitly extended) tubes instead. In this tutorial, we will consider a 3D duct with a circular cross-section. The duct is infinite in the streamwise direction, and the walls are rigid.

5.1 Rectangular ducts

Consider a hardwalled rectangular duct, with sized l and L along the y and z directions, respectively.

1. Compute the dispersion relation for the duct.

2. What is the role of the wavenumbers along y and z? What is the physical interpretation of these values?

5.2 Circular ducts

5.2.1 Hard walled duct, $\hat{z}_{wall} = \infty$

1. Given the geometry, what is a suitable ansatz for the pressure field in the duct? Which characteristic number naturally appears in the problem?
2. Compute the dispersion relation for the duct.
3. Compute the eigenvalues and the associated eigenmodes. Which new number can be used to classify these modes?
4. What does the two numbers(n,m) represent?

5.2.2 Soft walled duct, $\hat{z}_{wall} = 0$

1. Adapt the above steps for a softwalled duct. What are the new boundary conditions ?

5.3 Modelling complex dynamics via imperance

Both during numerical simulations (G. Brès et al. 2014) and experimental campaings (Suzuki and Colonius 2006), tones in the near pressure field of jets were observed. Towne et al. 2017 shows some snapshots of frenquency and azymutally decomponsed pressure components. Figure A.6 shows the pressure spectra and some modes observed.

Based on the above, can you propose a model for the observed tones? How would you model the impedance of the jet?

(For a more detailed discussion on the impedance of jets, see Martini et al. 2019).

Bessel functions review

Bessel functions are like sines and cosines for cylindircal coordinates, but with different names.

Sine/cosine-like functions are called “Bessel functions of the first and second kinds”, and are illustrated in figure A.7. The points at which these functions are zero/have zero derivative, are listed in tables A.1 and A.2. I am calling them “sine/cosine-like” as they oscilate as sines

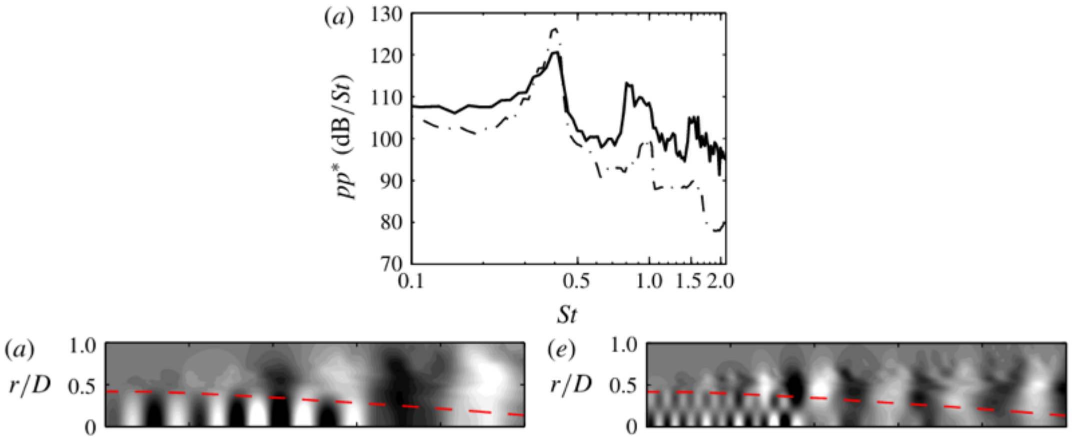

Figure A.6: (a) Near field pressure PSDs measured experimentalmy (solid lines), and obtained by numerical simulations (dashed lines) on a Mach 0.9 jet. (b) and (c) show two frequency and azymutal angle decomponsed pressure snapshots.

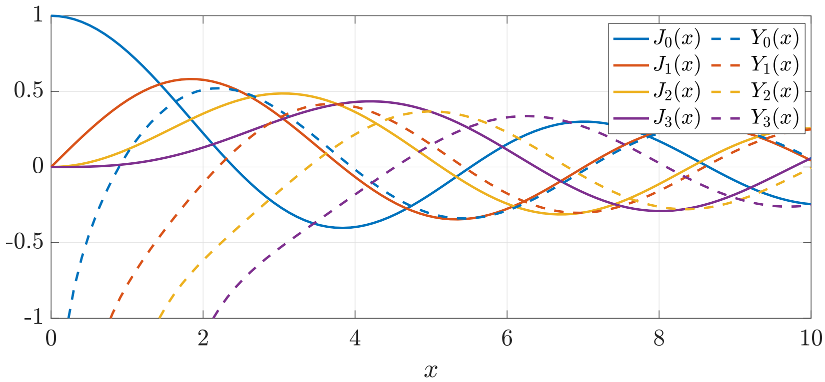

Figure A.7: Illustration of Bessel functions of the first (J_i) and second (K_i) kinds.

with cosines. The difference is that one of the functions diverges for $r \to 0$, and the amplitude of both is proportional to $1/r$ for large r.

These equations are solutions for

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dx} + \left(1 - \frac{i^2}{r^2}\right) f = 0. \tag{A.5}$$

We will use only the above functions here, but for reference, the “modified Bessel functions” (usually denoted I_i and K_i) are “exponensial-like”, and “Hankel functions” are “complex-exponential like”. Finally, spherical, instead of cylindrical, versions of those also exist.

k	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

Table A.1: Zeros of the Bessel function of the first kind (J_i).

k	$J_0'(x)$	$J_1'(x)$	$J_2'(x)$	$J_3'(x)$	$J_4'(x)$	$J_5'(x)$
1	3.8317	1.8412	3.0542	4.2012	5.3175	6.4156
2	7.0156	5.3314	6.7061	8.0152	9.2824	10.5199
3	10.1735	8.5363	9.9695	11.3459	12.6819	13.9872
4	13.3237	11.7060	13.1704	14.5858	15.9641	17.3128
5	16.4706	14.8636	16.3475	17.7887	19.1960	20.5755

Table A.2: Zeros of the derivative of the Bessel function of the first kind (J_i).

Tutorial 6 : *Multipoles*

The goal of this tutorial is to derive the expression for multipoles, and understand the relevance of their use. Here we will focus on the frequency domain representation of the sound field, and the Green's function for the free field.

6.1 Multipole expansion

The sound field generated by a monopole source, i.e., the Green's function, is given by

$$\hat{p}(x) = G(x, x') = S_m \frac{e^{ikr}}{4\pi r}, \tag{A.6}$$

where $\vec{r} = \vec{x} - \vec{x'}$, $r = \vec{r}$, and S_m is the monopole source term.

1. Starting from a monopole source, derive the expression for the dipole term.
2. Starting from a dipole source, derive the expression for the quadrupole term.

6.2 Higher- or lower-order poles?

From the greens function, we can consider an applied force either as a sum of monopole, or of dipoles.

1. For a force given by

$$f_x = e^{-x^2}\delta(y)e^{i\omega t}, \tag{A.7}$$

compute the equivalent monopole and dipole source terms.

2. What is the physical constrain that prevents a force term (usually a dipole), to act as a monopole?

6.3 Ill-conditioning of insuficient-order poles

To understand the ill conditioning associated of using monopoles to model dipoles, consider the following problem: two sound sources with amplitude a/ϵ and $-a/\epsilon$ are located at $x = \pm\epsilon$.

1. Compute the sound for $\epsilon \to 0$. Show that a dipole field is recovered.

2. Now consider that the sources amplitudes are $1.00001a/\epsilon$ and $-a/\epsilon$. Compute the sound for $\epsilon \to 0$. Comment on the result.

3. What does this result implies in practice? Imagine that you have numerical results from a DNS simulation, and you are trying to predict the sound field created by the flow. What can happen if you try to use (A.8) instead of (A.9) directly?

Pressure field resulting from $\hat{f}_x$

$$\text{Monopole representation:} \quad \hat{p}'(x) = \int_\Omega \frac{\partial \hat{f}_x}{\partial x'}(x')G(x, x')dx' \tag{A.8}$$

$$\text{Dipole representation:} \quad \hat{p}'(x) = -\int_\Omega \hat{f}_x(x')\frac{\partial G(x, x')}{\partial x'}dx' \tag{A.9}$$

Tutorial 7 :

Effect of source coherence in aeroacoustics (2 lectures)

The goal of this TD is to explore, and visualize, the role of source coherence in aeroacoustics. For such, we will use a python code to plot pressure fields. For simplicity, we will work in the frequency domain, such that the pressure field is given by (A.10).

For simplicity, take $c = 1$, and an acoustic wavenumber of $k_{ac} = 2\pi$ (acoustic wavelenght of 1).

7.1 Coherent source fields

1. Plot the acoustic field of a monopole source with intensity $a = 1$ located at $x = 0$.

2. Plot the acoustic field for two monopoles with intensities $a = 1$ and -1 located at $x = 0$ and $x = 0.5$.

3. Plot the acoustic field for n sources, uniformly distributed between $x = -5$ and $x = 5$, with intensities $a_i = e^{i\frac{2\pi}{l}x_i}$, for $L = 2$.

7.2 Incoherent source fields

4. Write the pressure at the point $(x, y) = (0, 3)$ as a function of n sources using the matrix formalism.

5. Write the pressure at this point in terms of the source cross-spectral density ($\mathcal{C} = a^H a$).

6. Compare the sound field from coherent sources to that of incoherent sources. Discuss the role of coherence/incoherence in the acoustic efficiency aeroacoustics.

7. How does this effect changes if $L = 2(> 1)$, as before, to $L = .5(< 1)$?

Expressions

$$\hat{p}'(x) = \sum \frac{a_i}{4\pi||\vec{x} - \vec{x}_i||} \tag{A.10}$$

Appendix B : Other recommended exercises

Aeroacoustics Tutorial 1

Eduardo Martini
Based on exercises by V. Fortuné and Y. Gervais.

September 17, 2024

Exercise 1: Efficiency of a Loudspeaker

Consider a speaker modeled as a point source emitting a spherical wave in the far field. Assume the electrical power consumed is 1 Watt with unit efficiency.

1. Calculate the acoustic pressure level L_0 at a distance of 1 meter.
2. The measured acoustic pressure level is $L_0 - 3\,\mathrm{dB}$ at 1 meter. Calculate the actual efficiency of the speaker. What would it be for a level of 90 dB?

Exercise 2: Sound Levels

1. Indicate the number of decibels to add to the pressure level Lp_1 of a source S_1, when there are 2, 10, and 100 sources with identical acoustic power to S_1.
2. The sound level associated with background noise in a machine-free factory is 80 dB. When a machine is running, it reaches 84 dB. Calculate the sound level in the factory when two machines are operating.
3. Record a sound wave $p(t) = A_p \cdot \sin(\omega t) + \epsilon(t)$ with $A_p = 250\,\mathrm{Pa}$ and $\epsilon(t)$ as random noise with standard deviation $\sigma = 20\,\mathrm{Pa}$. Calculate the sound level at the measurement point.

Exercise 3: Plane Waves in a Pipe

Consider a rigid pipe of length L, centered at $x = L/2$. Initially, assume one end at $x = 0$ is closed, and the other end is open to free air. Assume solutions to the problem are low-amplitude plane waves.

1. Write the boundary conditions for the problem.
2. Calculate the eigenmodes of the system.
3. What if both ends are closed?

Aeroacoustics Tutorial 4

Eduardo Martini
Based on exercises by V. Fortuné and Y. Gervais.

November 15, 2023

Sound Pipe

Consider a sound pipe along the $O\vec{x}$ axis, with cross-sectional area S, carrying a uniform flow with velocity U. The propagation velocities of incident and reflected waves are $c+U$ and $c-U$ respectively. We consider the case of a plane acoustic wave, meaning that the acoustic pressure field propagating in the duct is defined by:

$$p'(x,t) = Ae^{i\omega(t-\frac{x}{c+U})} + Be^{i\omega(t+\frac{x}{c-U})},$$

where A and B are the amplitudes of the two considered progressive waves with pulsation ω in the frame attached to the pipe.

1. Express the pulsations of the two waves in the frame attached to the moving fluid.
2. Show that the acoustic velocity, corresponding to the pressure $p'(x,t)$, has the following expression:
$$u' = \frac{1}{\rho c}\left(Ae^{i\omega(t-\frac{x}{c+U})} - Be^{i\omega(t+\frac{x}{c-U})}\right)$$
3. Deduce the expression of the impedance $Z(x) = \frac{p'}{u'S}$ at any point x in the pipe. Verify that at the origin $x=0$ of the pipe, the impedance is given by:
$$Z_0 = \frac{\rho c}{S}\frac{A+B}{A-B}$$
4. Suppose that at the abscissa $x=L$, the impedance is Z_L. Show that the ratio of amplitudes A and B is equal to:
$$\frac{B}{A} = \frac{\left(\frac{Z_L S}{\rho c} - 1\right)e^{-i\frac{\omega L}{c+U}}}{\left(\frac{Z_L S}{\rho c} + 1\right)e^{-i\frac{\omega L}{c-U}}}$$

5. Deduce the expression of Z_0, then show that Z_0 can be written in the form:

$$Z_0 = \frac{\rho c}{S} \frac{\frac{Z_L S}{\rho c} + i \tan\left(\frac{\omega L}{c(1-M^2)}\right)}{1 + i\frac{Z_L S}{\rho c} \tan\left(\frac{\omega L}{c(1-M^2)}\right)},$$

where $M = \frac{U}{c}$. Compare this expression to the one obtained without flow. What can be said about the influence of flow on the resonance frequency of an open-ended sound pipe of length L?

Velocity Measurement

Consider 3 points in the conduit: point A_2 at abscissa x, and points A_1 and A_3 located at abscissas $x - \Delta x$ and $x + \Delta x$, respectively.

- Provide the expression of the acoustic pressures p_1, p_2, and p_3 at points A_1, A_2, and A_3, as functions of x and Δx.
- Give the expressions for $p_1 + p_3$ and $p_1 - p_3$.
- Provide the expression of the ratio $\frac{p_1+p_3}{p_2}$. Specify the value of the ratio for the case without flow.
- Assuming a small Mach number, give the expressions for $p_1 + p_3$ and $p_1 - p_3$ in terms of M, then give the expression for $p_1 + p_3 + i\frac{M\omega\Delta x}{c}(p_1 - p_3)$. Choose the value of pulsation ω such that $\frac{\omega \Delta x}{c} = \frac{\pi}{2}$. Verify that the ratio $\frac{p_1+p_3}{p_1-p_3}$ is proportional to M, and provide the value of this ratio.
- Application: velocity measurement method for U.

Aeroacoustics TD6

Eduardo Martini

2023-2024

Previously in the course, we investigated acoustic wave reflection and transmissions due to temperature and density changes. We will now examine the effect of wind.

1 Wind refraction

Consider the following scenario: the atmospheric wind is stratified, i.e., below a given reference (we will use $y = 0$) there is no wind ($U = 0$). Above it, we have a constant wind speed U. The wind is blowing in the x direction. Consider an acoustic wave incident from the lower region, with a wave number k_0 and frequency ω_0. We want to find the intensity and the angles of the reflected and transmitted waves. The setup is illustrated in Figure 1.

1.1 Boundary conditions

The interface boundary conditions are pressure continuity, given by

$$p_+(x, y = 0^+, t) = p_-(x, y = 0^-, t) \tag{1}$$

and the deformation of the stratification line, which reads

$$\eta_-(x, t) = \eta_+(x, t) \tag{2}$$

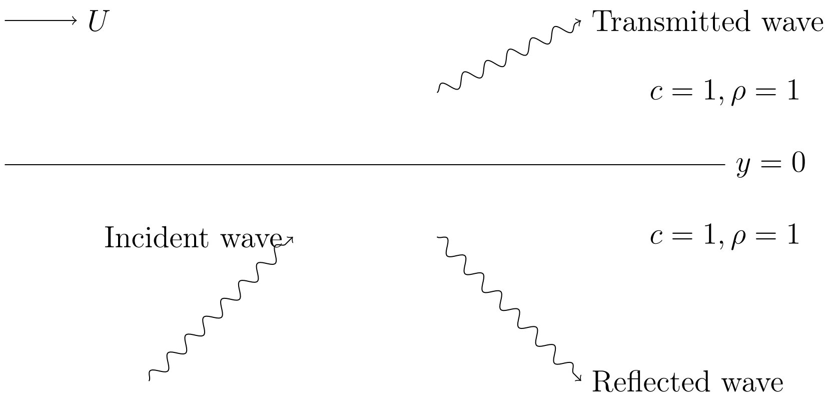

Figure 1: Stratified atmospheric condition

where η is the displacement of the stratification line. We can use the kinematic boundary condition to relate the pressure and the displacement, which reads

$$\frac{D\eta(x,t)}{Dt} = \frac{d\eta(x,t)}{dt} + U\frac{\partial\eta(x,t)}{\partial x} = v(x, y = 0) \tag{3}$$

Use the ansatz $p(x, y, t) = p_0 e^{i(k_x x + k_y y - \omega t)}$ and $\eta(x, t) = \eta_0 e^{i(k_x x - \omega t)}$ for the top and bottom domains to write down the interface condition in terms of pressure only.

1.2 Reflection and transmitted waves

Use the interface condition found above and the wave equation on both domains to find the reflected and transmitted waves.

Consider incoming wave, $p_I = p_0 e^{i(k_{x,i} x + k_{y,i} y - \omega_0 t)}$. Given that the angle of incidence is $\theta_i = atan(k_y/k_x)$, comment on the different behaviors of waves traveling in the direction of the wind and against it.

where θ is the propagation angle.

2 Road noise at ENSMA

Consider a truck cruising on the highway near ENSMA and assume the noise is dominated by its engine, which is at a constant 3000rmp. The distance between the road and ENSMA and the stratification layer is shown in Figure 2. At what wind speed will the noise be totally reflected? What is the distance d for the noise to reach ENSMA?

Is this number realistic? What factors can influence the final result?

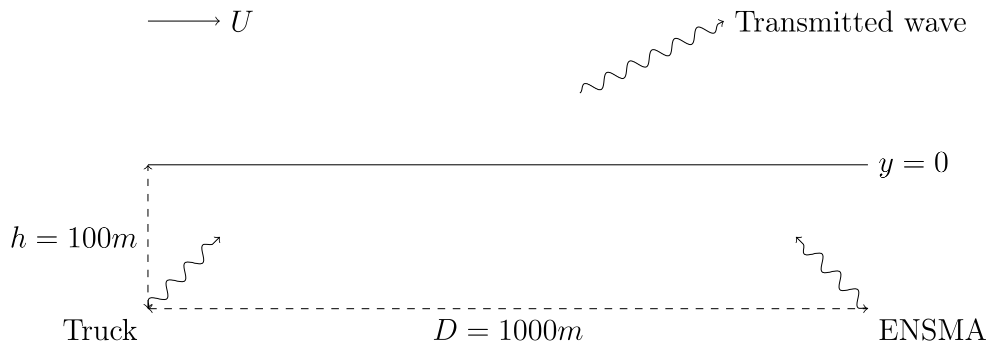

Figure 2: Stratified atmospheric condition