



Institut Supérieur de l'Aéronautique et de l'Espace



E N S M A

Fluid Mechanics

Course Notes

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Contents

1	Introduction: Billards Fluid Dynamics	1
1.1	Physics review	1
1.2	Conservation laws	4
1.3	Temperature	6
1.4	Take away	8
2	Thermodynamics Review	11
3	Governing equations	13
3.1	Conservation Laws	13
3.1.1	Integral form	13
3.1.2	Differential form	14
3.2	Fluid properties	15
3.2.1	Deformation	16
3.2.2	Fluid relation to a deformation	17
3.3	The Navier Stokes Equations	19
3.3.1	The kinetic energy equation	20
3.4	Types of flow	21
3.4.1	Perfect and real fluids	21
3.4.2	Compressible and incompressible flow	21
4	Perfect flows	23
4.1	The Euler equations	23
4.1.1	Bernoulli equation	24
4.1.2	Kelvin's theorem	24
5	Incompressible flows	27
5.1	The vorticity field	28
5.2	Some hidden dynamics	30
5.2.1	The diffusion equation	30
5.2.2	The convection equation	31
5.3	Time scales	31
5.4	The Reynolds number	33
5.4.1	Low Reynolds number flows	33
5.4.2	Moderate Reynolds number flows	34
5.4.3	High Reynolds number flows	34
5.5	The boundary layer	35
5.5.1	Time-scales arguments	35
5.5.2	Characterising boundary layers	36
5.5.3	Boundary layer models: Prandtl's equations	39

5.5.4	Boundary layer models: Integral formulation	41
5.6	Flow stability	47
6	Turbulence	49
6.1	Modelling turbulence	50
6.1.1	Turbulence as a stochastic process	50
6.1.2	Reynolds averaged models	51
6.1.3	The closure problem	52
6.2	Turbulent dynamics	53
6.2.1	Turbulent flow scenes : Kolmogorov theory of turbulence	55
6.2.2	Reynolds dependence of the scales	56
6.3	Turbulence models	57
6.4	Near wall turbulence	58
6.4.1	Wall units	59
6.4.2	Flow in channels	59
6.4.3	Turbulent boundary layer	62
A	Different forms of the Navier-Stokes equations	67
A.1	Incompressible flow	67
A.1.1	Vector form	67
A.1.2	Cartesian coordinates	68

List of Figures

1.1	Collision of a single particle.	2
1.2	Collision of many particles.	3
1.3	Force for an increasing number of particles. Blue and red lines show the total and mean forces. The black lines indicate the individual collision forces.	4
1.4	Convervation laws. The particles entering and leaving the domain Ω are illustrated.	5
1.5	Illustration of a particle trajectory in a gas.	7
1.6	Illustration of confined particles in a piston.	9
3.1	Illustration of the different types of deformation.	16
4.1	Control volume for the derivation of the Bernouli equation.	24
4.2	Illustration of a boundary layer near the leading edge of a flat plate (left) and the generation turbulence after a grid.	26
5.1	Illustration of vortices in different flow conditions.	29
5.2	Illustration of a diffusive process.	30
5.3	Illustration of the solution of the convection equation.	31
5.4	Examples of cases where the flow dynamics show different Reynolds numbers.	34
5.5	Illustration of a time-reversible flow.	35
5.6	Illustration of a boundary layer over a flat plate. The y scale here is expanded for visualization purposes normally the boundary layer is thin.	36
5.7	Illustration of differnet boundary layer profiles.	36
5.8	Illustration of different boundary layer measures. (* Same momentum for once the mass flow is corrected.)	38
5.9	Control volume for the integral formulation of the boundary layer.	41
5.10	Polhausen profiles	47
6.1	Illustration of two energy cascades. The first has a larger dissipation scale, and thus a shorter energy cascade. The second has a smaller dissipation scale, and thus the energy cascade reaches higher wavenumbers (shorter length-scales). Note that both cascades have the same energy transfer rate.	54
6.2	Illustration of the near wall velocity profile.	61
6.3	Comparison of laminar (i) and turbulent (ii) flow for a given mass flow rate (a) or pressure gradient (b).	62
6.4	Velocity profile of a turbulent channel flow.	63

6.5	Illustration of a turbulent boundary layer.	63
6.6	Laminar and turbulent boundary layer profiles. The intermittency factor, γ , is shown in (b).	64
6.7	Velocity profile of a turbulent boundary layer	65

Support Material

Other than these lecture notes, please see also

- Fundamentals of Fluid Mechanics - Patrick Chassaing
- Fluid Mechanics - James Liggett

I also strongly recommend Feynman's lectures on physics, which are available online for free. See the chapters on the “Flow of dry water”¹ and “Flow of wet water”².

¹Richard Feynman. *The Feynman Lectures on Physics Vol. II Ch. 40: The Flow of Dry Water.* URL: https://www.feynmanlectures.caltech.edu/II_40.html.

²Richard Feynman. *The Feynman Lectures on Physics Vol. II Ch. 41: The Flow of Wet Water.* URL: https://www.feynmanlectures.caltech.edu/II_41.html.

If, in some cataclysm, all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generation of creatures, what statement would contain the most information in the fewest words? I believe it is the atomic hypothesis that all things are made of atoms — little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another. In that one sentence, you will see, there is an enormous amount of information about the world, if just a little imagination and thinking are applied.

Richard Feynman



Introduction: Billards Fluid Dynamics

In this course, we will study fluid dynamics from scratch. We will cover the basic concepts from the bottom up, developing the equations that predict flow dynamics from the smallest number of assumptions. We will focus on physics, resorting as little as possible to complex mathematical formulations or postponing their use until intuition for the system is acquired.

One of our starting points will be that liquids are formed by particles moving around. We need to review the basic concepts of moving particles to understand what that implies.

1.1 Physics review

A building block of our course is Newton's second law

$$m\vec{a} = \vec{f}, \quad (1.1)$$

which relates changes in velocity with the force acting on the particle. Integrating this equation in time, we get

$$\vec{I} = \int \vec{f} dt = m\vec{v}, \quad (1.2)$$

where \vec{I} is known as the impulse. The notion of impulse is particularly useful when the forcing acts on very short time scales, as when a ball hits a wall.

Consider the case of a ball hitting a wall a vertical wall, as in figure 1.1a, which we assume is heavy enough not to move. We also assume that the collision

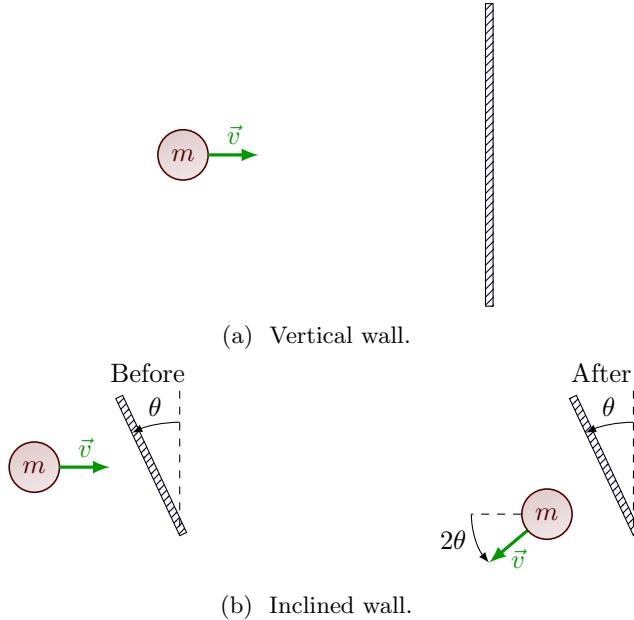


Figure 1.1: Collision of a single particle.

is elastic, so there is no energy loss. The initial and final conditions' velocities are thus

$$\vec{v}_i = v_0 \vec{x}, \quad (1.3)$$

$$\vec{v}_o = -v_0 \vec{x}, \quad (1.4)$$

and we can compute the particle's impulse to be $\vec{I} = 2mv_x \vec{i}$. That is, we know that the total force applied to the particle changed its momentum by this amount. Now, we know that the particle applies an equal but opposite force to the wall; thus, the wall receives a total momentum of $-\vec{I}$. The wall does not move either because it is very heavy or because an extra force prevents it from moving, e.g., if it is attached to the ground.

It is easy to show now that if the wall is inclined with an angle θ , as in 1.1b, the following happens

$$\vec{v}_i = v_0 \vec{x}, \quad (1.5)$$

$$\vec{v}_o = -\cos(2\theta)v_0 \vec{x} - \sin(2\theta)v_0 \vec{j}, \quad (1.6)$$

and the total impulse to the wall is

$$I = (1 + \cos(2\theta))v_0 \vec{x} + \sin(2\theta)v_0 \vec{j}. \quad (1.7)$$

Let us see what happens if many particles collide with the wall, as in figure 1.2. Let us assume that we have n particles in a given volume and that the particles don't interact among themselves. Let only consider the particles in the region Ω . After all the collisions occur, the total impulse at the wall is simply eq (1.7) multiplied by nm .

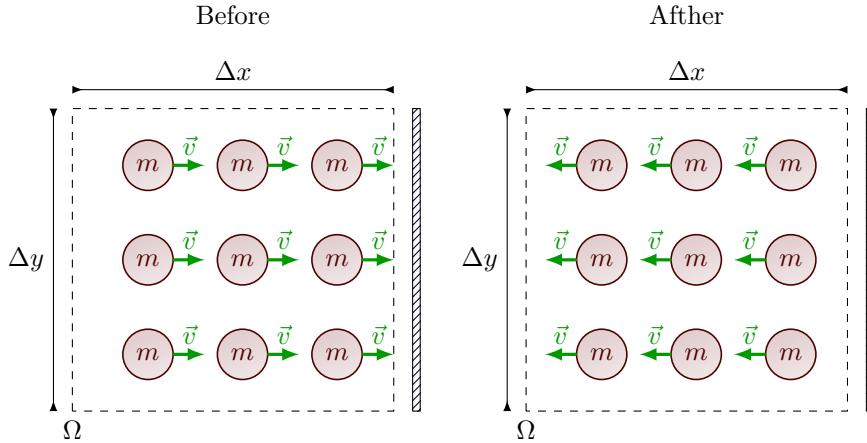


Figure 1.2: Collision of many particles.

Note that it takes $\Delta t = \Delta x/v_0$ for all the particles to hit the wall. If we make the assumption that \vec{f} is constant in (1.2), we get

$$\begin{aligned}\vec{f}_{\text{mean}} &= \frac{\vec{I}}{\Delta t} = nm \frac{(1 + \cos(2\theta))v_0 \vec{x} + \sin(2\theta)v_0 \vec{j}}{\Delta x/v_0} \\ &= \left((1 + \cos(2\theta))\vec{x} + \sin(2\theta)\vec{j} \right) \frac{nm}{\Delta x} v_0^2,\end{aligned}\quad (1.8)$$

and thus

$$\frac{\vec{f}_{\text{mean}}}{\Delta y \Delta z} = \left((1 + \cos(2\theta))\vec{x} + \sin(2\theta)\vec{j} \right) \rho v_0^2, \quad (1.9)$$

where $\rho = nm/\Delta x \Delta y \delta z$ is the average density of Ω . Note that $\frac{\vec{f}_{\text{mean}}}{\Delta y \Delta z}$ is a force by a length, i.e., a pressure: the pressure the wall felt due to all the collisions to it.

Look at the right-most term in (1.10), ρv^2 . It is somewhat surprising that the force is proportional to v^2 , when the impulse of a particle is proportional to v ? Not really; it just means that the faster the particles are, the more momentum every single collision transfers, and the more particles collide. Nevertheless, pay attention to this term. It, and others like it, will appear frequently throughout the course!

Before continuing, it is worth remembering that we made a big assumption by considering the average force in the above equation. How good is it? Figure 1.3 illustrates the force collisions created when there are very few, a bunch, and many particles. The more collisions we have, the better the approximation is.

I think everyone could foresee that this is a first naive model for a gas. It is interesting to see how well it works (or not). Figure 1.1b remembers something, no? What if our wall models an airplane wing? Then, the positive term in (1.10) indicates that there is a force pulling the wing up. Can we explain flight with this model?

Let's see. Consider the air standing still (all particles are at rest). The air density is $\rho = 1.2 \text{ kg/m}^3$. An aircraft flies with speeds as low as $v \approx 100 \text{ m/s}$,

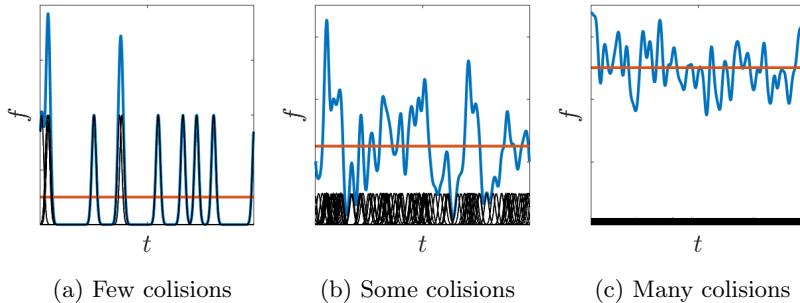


Figure 1.3: Force for an increasing number of particles. Blue and red lines show the total and mean forces. The black lines indicate the individual collision forces.

weighing around 350 tons (e.g., an Airbus A340). Let us assume $\theta = 10 \text{ deg}$, around the maximum angle a wing can take. The total vertical force is then

$$\frac{f_y}{\Delta y \Delta z} = \underbrace{\sin(2\theta)}_{0.34} \underbrace{\rho}_{1.2} \underbrace{v_0^2}_{150^2=22500} \approx 918 N/m^2, \quad (1.10)$$

To compensate for the gravitational force, we need a total wing area of

$$A = \Delta y \Delta z = \frac{350\,000 \text{ kg} * 9.81 \text{ m/s}^2}{918 \text{ N}} = 3750 \text{ m}^2. \quad (1.11)$$

This aircraft has a wingspan of 60m, and the wings are about 5m long. Our calculation clearly overestimates the lift we can produce by a lot.

Two key ingredients are missing in our model: the interaction between the particles and their self-movement (even at “rest”, the particles are moving). These two things are essential for our airplane. In the rest of this course, we will study how to account for these effects and their implications (there are a *lot* of them).

However, the model we developed here is not that stupid. In some situations, it is a pretty good model and works really well. When? When the “two missing ingredients” are not important. This happens high-up in the atmosphere, when the air is very thin, which means that the particles don’t interact that much because there are not a lot of them, and the objects (e.g., spacecraft) are moving very fast, so the particles are, for all purposes, standing still.

1.2 Conservation laws

To introduce the concept of conservation laws, we will revisit the problem investigated in the previous section. Figure 1.4 shows a generalized version, i.e., many particles colliding with an inclined plate. We want now to calculate how much mass, momentum, and energy is entering and leaving the domain.

The mass conservation law is always one of the most intuitive ones, so let us start there. How much mass is entering the domain via $\partial\Omega_1$? Every Δt seconds, the total mass passing by $\partial\Omega_1$ is (we will neglect the direction z to

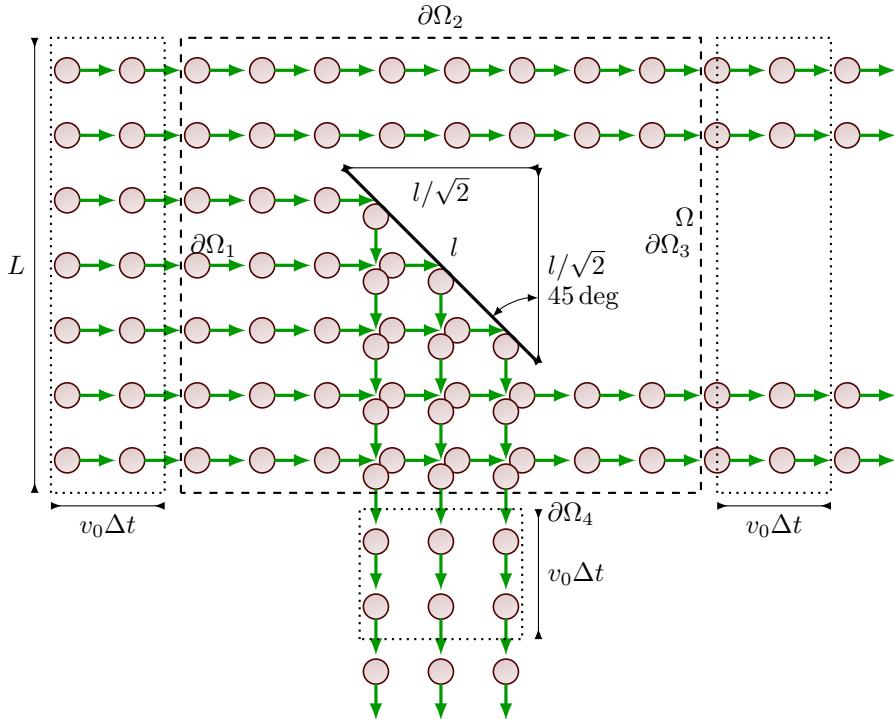


Figure 1.4: Conservation laws. The particles entering and leaving the domain Ω are illustrated.

keep it simpler, so in all below, imagine that ρ really means $\rho\Delta z$),

$$q_{M,\partial\Omega_1} = -(Lv_0\Delta t)\rho, \quad (1.12)$$

where the minus sign is used to mean an influx. Typically, positive values mean “stuff coming out”.

The mass leaving the domain by $\partial\Omega_3$ is

$$q_{M,\partial\Omega_3} = \left(\left(L - \frac{l}{\sqrt{2}} \right) v_0\Delta t \right) \rho, \quad (1.13)$$

and the mass leaving $\partial\Omega_4$ is

$$q_{M,\partial\Omega_4} = \left(\frac{l}{\sqrt{2}}v_0\Delta t \right) \rho, \quad (1.14)$$

and clearly $q_{M,\partial\Omega_2} = 0$.

We can see that $q_{M,\partial\Omega} = q_{M,\partial\Omega_1} + q_{M,\partial\Omega_2} + q_{M,\partial\Omega_3} + q_{M,\partial\Omega_4} = 0$: all the mass entering the domain is leaving it. This is what we call a conservation law: mass is conserved. If we had more mass entering the domain than leaving it, we would increase the mass in Ω .

How does this work for energy? The energy of each particle is $mv_0^2/2$, so the

total energy entering via each face is

$$q_{E,\partial\Omega_1} = -(Lv_0\Delta t)\rho v_0^2/2, \quad (1.15)$$

$$q_{E,\partial\Omega_2} = 0, \quad (1.16)$$

$$q_{E,\partial\Omega_3} = \left(\left(L - \frac{l}{\sqrt{2}} \right) v_0 \Delta t \right) \rho v_0^2 / 2, \quad (1.17)$$

$$q_{E,\partial\Omega_4} = \left(\frac{l}{\sqrt{2}} v_0 \Delta t \right) \rho v_0^2 / 2, \quad (1.18)$$

and again, $q_{E,\partial\Omega} = q_{E,\partial\Omega_1} + q_{E,\partial\Omega_2} + q_{E,\partial\Omega_3} + q_{E,\partial\Omega_4} = 0$.

Great. Let's do this for the x -component of the momentum. Near $\partial\Omega_{1,3}$, $v_x = v_0$, and $p_x = mv_x = mv_0$. So,

$$q_{p_x,\partial\Omega_1} = -(Lv_0\Delta t)\rho v_0, \quad (1.19)$$

$$q_{p_x,\partial\Omega_2} = 0, \quad (1.20)$$

$$q_{p_x,\partial\Omega_3} = \left(\left(L - \frac{l}{\sqrt{2}} \right) v_0 \Delta t \right) \rho v_0. \quad (1.21)$$

Around $\partial\Omega_4$, $v_x = 0$, which leads to

$$q_{p_x,\partial\Omega_4} = 0. \quad (1.22)$$

Now, if we do,

$$q_{p_x,\partial\Omega} = q_{p_x,\partial\Omega_1} + q_{p_x,\partial\Omega_2} + q_{p_x,\partial\Omega_3} + q_{p_x,\partial\Omega_4} = -\frac{l}{\sqrt{2}}\Delta t\rho v_0^2, \quad (1.23)$$

we don't have momentum conserved!

Let's quickly do it for p_y , remembering that $v_y = v_0$ at $\partial\Omega_4$ and zero otherwise.

$$q_{p_y,\partial\Omega_1} = q_{p_y,\partial\Omega_2} = q_{p_y,\partial\Omega_3} = 0, \quad (1.24)$$

$$q_{p_y,\partial\Omega_4} = \frac{l}{\sqrt{2}}\Delta t\rho v_0^2. \quad (1.25)$$

This does not add up, either! However, don't we have a law that conserves momentum? Yes, but remember that an applied force changes the momentum of the particles, and it is exactly the collision force that deflects the particles downward. In fact, we can see that (1.23)-(1.25) are equivalent to (1.10) multiplied by $v_0 n \Delta T$.

So here we can draw the following conclusion: *any unbalance in the amount of momentum leaving or entering a domain must be the consequence of a force acting inside it*. This is analogous to mass accumulating in the domain if the fluxes do not match up. The difference here is that the momentum accumulates in the plate (or whatever holds the plate still).

1.3 Temperature

So far, the particles have a simple trajectory: they move in straight lines unless they collide with a surface. The reality is way more complicated.

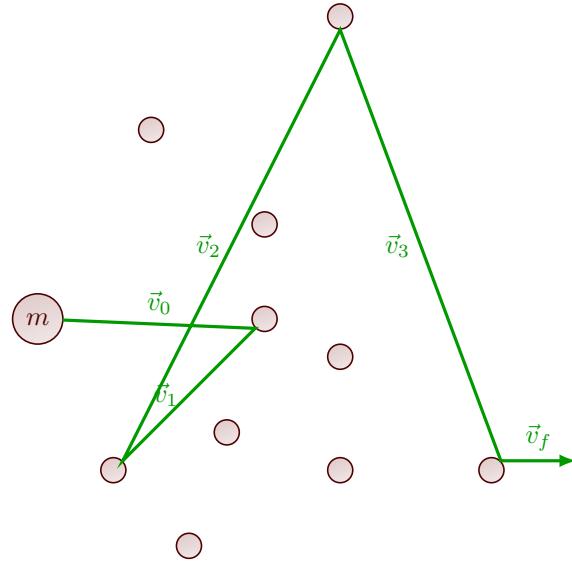


Figure 1.5: Illustration of a particle trajectory in a gas.

In a real gas¹, the particles interact a lot. In practice, they collide with each other all the time and thus change velocity and direction. This is illustrated in figure 1.5.

How can we model this effect while keeping the same spirit of the previous discussion? To start, let us assume that all the collisions are somewhat random, and thus that a particle velocity can be described as a mean velocity (\vec{v}_m) and a random component (\vec{v}_r),

$$\vec{v} = \vec{v}_m + \vec{v}_r. \quad (1.26)$$

The random part of this velocity has zero mean (any non-zero mean is accounted for in \vec{v}_m) and a variance, i.e., the mean of $|\vec{v}_r|^2$. We all probably know that temperature is related to the gas particles' velocity, so we write

$$m \langle |\vec{v}_r|^2 \rangle = k_b T. \quad (1.27)$$

There is much to be said about this formula that does not fit this course. It suffices to say that it is a statistical mechanics result (with many underlying assumptions)².

Let us return to figure 1.4 and re-write the flux terms, assuming $T > 0$. We will focus only on the fluxes over $\partial\Omega_1$.

The mass flux reads

$$\frac{q_{L,\partial\Omega_1}}{\Delta t} = Lv_0\rho = L(v_{0m,x} + v_{0r,x}). \quad (1.28)$$

¹Gases are a special case of a fluid. We focus most of the discussion on gases because they are, on a fundamental level, much simpler than, for example, liquids. Nevertheless, once we “zoom out” (we will define that soon enough), we find the exact same equations as for any other fluid!

²Trivia: The first to propose a model for this effect was none other than Einstein in 1905. Modeling the movement of dust particles in the air, he provided the first evidence that atoms are real!

Since \vec{v}_r is a random variable, so is q . We will then look at the *mean* flux (\bar{q}),

$$\frac{\bar{q}_{L,\partial\Omega_1}}{\Delta t} = L(\overline{v_{0m,x} + v_{0r,x}})\rho = L(\overline{v_{0m,x}} + \overline{v_{0r,x}})\rho = Lv_{0m,x}\rho, \quad (1.29)$$

where $\overline{\cdot}$ indicates the mean value of a random variable. We can see that the mean mass flux is the same as before, i.e., (1.12).

Let's do the same for the momentum flux. The momentum flux is

$$\frac{q_{p_x,\partial\Omega_1}}{\Delta t} = L(v_{0m,x} + v_{0r,x})^2\rho \quad (1.30)$$

Again, we will look at the mean flux,

$$\begin{aligned} \frac{\bar{q}_{p_x,\partial\Omega_1}}{\Delta t} &= L(\overline{(v_{0m,x} + v_{0r,x})^2})\rho \\ &= L(\overline{v_{0m,x}^2} + 2\overline{v_{0m,x}v_{0r,x}} + \overline{v_{0r,x}^2})\rho \\ &= Lv_{0m,x}^2\rho + Lv_{0r,x}^2\rho. \end{aligned} \quad (1.31)$$

We can see that the mean flux is not the same as before: compare it to (1.23). The difference is the term $Lv_{0r,x}^2\rho$, which is the mean of the random component of the velocity squared. We can see that this term is proportional to the temperature, (1.27)

We can do the same for the energy flux,

$$\frac{q_{E,\partial\Omega_1}}{\Delta t} = L(v_{0m,x} + v_{0r,x})^3\rho \quad (1.32)$$

Again, we will look at the mean flux,

$$\begin{aligned} \frac{\bar{q}_{E,\partial\Omega_1}}{\Delta t} &= L(\overline{(v_{0m,x} + v_{0r,x})^3})\rho \\ &= L(\overline{v_{0m,x}^3} + 3\overline{v_{0m,x}^2v_{0r,x}} + 3\overline{v_{0m,x}v_{0r,x}^2} + \cancel{\overline{v_{0r,x}^3}})\rho \\ &= Lv_{0m,x}^3\rho + 3Lv_{0m,x}\overline{v_{0r,x}^2} \end{aligned} \quad (1.33)$$

We can see that the mean flux is not the same as before, i.e., (1.23): again there is a term proportional to $Lv_{0r,x}^2$ ³.

Furthermore, we can see that the mean fluxes are the same as before, except for the terms proportional to the temperature. This is a general result: *the mean fluxes of mass, momentum, and energy are the same as before, except for terms proportional to the temperature.*

1.4 Take away

In this chapter, we saw a couple of important phenomena. First, the forces applied to a body result from particles impacting it. Consider this for a second. There is a debate over how airplanes fly, i.e., what creates lift. From what

³Note that we canceled out the term $\overline{v_{0r,x}^3}$. This is not fully justified here but comes from the assumption that the probability of finding a negative velocity is the same as finding a positive one, i.e., the probability density function (PDF) is symmetric.

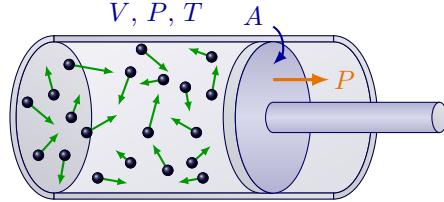


Figure 1.6: Illustration of confined particles in a piston.

we just saw, the explanation is simple: air particles collide with the wing and are deflected downwards, and thus, the airplane is pulled up, countering the gravitational force. Even if our simple estimate at the end of section 1.1 hugely underpredicts the lift force, the fundamental picture is still correct. We will return to this point later to understand what we missed in our estimate.

Another fundamental concept is that of conservation laws. This is the most fundamental concept in fluid mechanics (and in physics, for that matter). All macroscopic models used to describe flows are based on conservation laws. Why? Because they are much simpler and easier to work with than the Newtonian dynamics of individual particles.

Finally, we see two important dynamics to describe the conservation laws. The mean values (mean velocity, density, and energy) are macroscopic variables. We can see how much the flow moved after a small time interval, and there are internal variables, such as random particle fluctuations. The latter are typically reflected in thermodynamical quantities, such as temperature, but have relevant macroscopic effects.

As an exercise, you can show that these random fluctuations give rise to a net force on a wall, i.e., a pressure. For such, consider a flow at rest (mean velocity equals zero) and compute the impulse on the wall due to the impact of a particle. Then, estimate how many particles hit the wall at a given time and instant. Do you recognize the resulting equation? Figure 1.6 will give you an idea of what to expect!

Ludwig Boltzmann, who spent much of his life studying statistical mechanics, died in 1906, by his own hand. Paul Ehrenfest, carrying on the work, died similarly in 1933. Now it is our turn to study statistical mechanics.

David L. Goodstein

Thermodynamics is a funny subject. The first time you go through it, you don't understand it at all. The second time you go through it, you think you understand it, except for one or two points. The third time you go through it, you know you don't understand it, but by that time you are so used to the subject, it doesn't bother you anymore...

Arnold Sommerfeld

2

Thermodynamics Review

Thermodynamics is a broad field, and is particularly appealing as it deals with macroscopic quantities, such as temperature, pressure, and volume. This chapter will review the basic concepts of thermodynamics, focusing on those relevant to fluid mechanics. We will start with the basic concepts of temperature and pressure, and then move on to the first and second laws of thermodynamics. We will then discuss the concept of entropy and how it relates to the second law. Finally, we will discuss the concept of equilibrium and how it relates to entropy.

But before doing so, we need to define what is *macroscopic*, that is, at what size something can be considered macroscopic. Intuitively, we know that, let us say, a cubic meter of gas can be described by its pressure and volume, so it should be macroscopic. On the other hand, we know that gases are made of molecules (as in our previous chapter); thus, at the atomic level, we cannot model things purely in macroscopic quantities.

We can model a gas as a macroscopic quantity at length scales where details of the atomic state are no longer important. Imagine you have a gas particle which, by chance, is much faster than the others. As it travels through the gas, it will collide with other particles, and its energy will be transferred to them. Eventually, the particle will have the same energy as the others. So, if the scales we are interested in are much larger than the mean distance this particle travels between collisions, its specific condition will soon “dissolve” into the gas. We can this distance the mean free path, l_c . As previously illustrated in figure 1.5, individual distances between each collision can significantly vary, but what matters here is its mean value. Likewise, we can define a mean particle velocity, and from the mean velocity and mean distance, we can estimate the mean time between each collision.

For reference, at ambient conditions ($T = 300k$, $P = 1atm$), the mean free path of air is $l_c \approx 70nm$, the mean velocity if $v_c \approx 500m/s$, and the mean time between collisions is $\Delta t \approx 0.2ps$. So, for nearly all “human-sized” applications, we consider gases to be a macroscopic quantity.

TO BE CONTINUED

3

Governing equations

3.1 Conservation Laws

3.1.1 Integral form

In the previous chapter, we have seen that conservation laws can be helpful, for example, to determine the forces on a body without knowing all the flow details. This chapter will develop the conservation laws for mass, momentum, and energy. We will now investigate conservation laws in general and how they can be used to describe the dynamics of (most) fluids.

When working with conservation laws, there are two possible frameworks. The framework used in the previous section is called the “Eulerian” framework. In this framework, we look at a fixed point in space and see how the flow changes with time. The other framework is called “Lagrangian,” in which we follow a particle and see how its properties change with time. Both frameworks are equivalent, and we will use both in this course.

Lagrangian framework

Conservation laws in the Lagrangian framework are quite familiar:

$$\frac{dM_{\partial\Omega(t)}}{dt} = 0, \quad (3.1)$$

$$\frac{d\vec{P}_{\Omega(t)}}{dt} = \vec{f}_{\partial\Omega}, \quad (3.2)$$

$$\frac{dE_{\Omega(t)}}{dt} = P_{\partial\Omega}\cdot, \quad (3.3)$$

where the subscript indicates that the quantities refer to the total value inside the volume $\Omega(t)$. These equations can be spelled as *there is no change in the*

total mass inside the volume¹, the momentum change is due to the net force applied in it, and energy change is due to the net power provided to it.

We can re-write them in terms of local quantities, such as density and velocity, as

$$\frac{d}{dt} \int_{\Omega(t)} \rho(t) dV = 0, \quad (3.4)$$

$$\frac{d}{dt} \int_{\Omega(t)} \rho(t) \vec{u}(t) dV = \int_{\Omega(t)} \vec{f}(t) dV + \oint_{\partial\Omega(t)} \vec{f}_s(t) dS, \quad (3.5)$$

$$\frac{d}{dt} \int_{\Omega(t)} E(t) dV = \int_{\Omega(t)} \vec{f}(t) \cdot \vec{u} dV + \oint_{\partial\Omega(t)} (\vec{f}_s(t) \cdot \vec{u} + k \vec{\nabla} T \cdot \vec{n}) dS, \quad (3.6)$$

where \vec{f} represents forces acting inside the domain, e.g., gravity, and \vec{f}_s represents forces acting on the boundary, e.g., pressure, k is a thermal diffusion coefficient, used in the last term or account for heat entering or leaving Ω .

3.1.2 Differential form

Equations (3.4)-(3.6) can be written in differential form. To do so, we will need to use two calculus results. The first is the transport theorem, which states that

$$\frac{d}{dt} \int_{\Omega(t)} F(t) dV = \int_{\Omega(t)} \frac{\partial F}{\partial t} dV + \oint_{\partial\Omega(t)} F \vec{u} \cdot \vec{n} dS, \quad (3.7)$$

where \vec{n} is the normal vector to the surface $\partial\Omega(t)$. The second is the divergence theorem, also known as Green's theorem, which states that

$$\oint_{\partial\Omega(t)} \vec{F} \cdot \vec{n} dS = \int_{\Omega(t)} \nabla \cdot \vec{F} dV, \quad (3.8)$$

i.e., it converts a volume integral into a surface integral.

Using these theorems on (3.4)-(3.6) gives,

$$\int_{\Omega(t)} \left(\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) \right) dV = 0, \quad (3.9)$$

$$\int_{\Omega(t)} \left(\frac{\partial \rho \vec{u}}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u} \vec{u}) - \vec{f}(t) - \vec{\nabla} \cdot \overset{\leftrightarrow}{\sigma} \right) dV = 0, \quad (3.10)$$

$$\int_{\Omega(t)} \left(\frac{\partial E}{\partial t} + \vec{\nabla} \cdot (E \vec{u}) - \vec{f}(t) \cdot \vec{u} - \vec{\nabla} \cdot (\overset{\leftrightarrow}{\sigma} \cdot \vec{u}) + k \vec{\nabla}^2 T \right) dV = 0, \quad (3.11)$$

where $\overset{\leftrightarrow}{\sigma}$ is the stress tensor, which we will discuss later. For the discussion here, it suffices to know that $\overset{\leftrightarrow}{\sigma} \cdot \vec{n} = \vec{f}$, i.e., the tensor “multiplied” by the normal vector is the force acting on the surface.

These still seem to be integral equations. However, note that the integrals are over an arbitrary volume $\Omega(t)$. Thus, the integrands must be zero everywhere,

¹This is almost a definition, as the movement of Ω in time is such that it always keeps inside it the same particles

i.e.,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0, \quad (3.12)$$

$$\frac{\partial \rho \vec{u}}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u} \vec{u}) = \vec{f}(t) + \vec{\nabla} \cdot \overset{\leftrightarrow}{\sigma}, \quad (3.13)$$

$$\frac{\partial E}{\partial t} + \vec{\nabla} \cdot (E \vec{u}) = \vec{f}(t) \cdot \vec{u} + \vec{\nabla} \cdot \left(\vec{u} \cdot \overset{\leftrightarrow}{\sigma} \right) - k \vec{\nabla}^2 T. \quad (3.14)$$

Typically, the term \vec{f} represents external forces acting on the flow (e.g., gravity), and $\overset{\leftrightarrow}{\sigma}$ represents internal forces (e.g., pressure, viscous forces). While \vec{f} needs to somehow be prescribed to the system of equations (e.g., gravity is always acting downwards), $\overset{\leftrightarrow}{\sigma}$ is a function of the flow itself, and thus we need to model it explicitly. For this, we need to discuss the properties of the fluid.

Equations 3.12-3.14 are one form of the Navier-Stokes equations. Note, however, that they are not yet complete. We have a total of 5 equations (1 for mass, 3 for each component of the momentum, and 1 for energy), but we have 10 unknowns (density, 3 velocity components, energy, and 6 stress tensor components²).

What we need is, thus, to model the stress tensor to close our equations. For that, we need to discuss the properties of a fluid in general a bit.

3.2 Fluid properties

There are a few properties that are fundamental to describing a fluid. We will list them here and discuss them in more detail later; they are

- Velocity
- Density
- Temperature
- Viscosity
- Stress tensor, which includes
 - Pressure
 - Viscous stress

As seen in the previous section, velocity, density, temperature, and pressure can be understood from the movement of the fluid particles. The missing ingredient here is viscosity. Viscosity measures how much the fluid resists deformation and is related to the fluid's internal friction. It is intrinsically related to how the flow particles interact among them. Note that in the previous section, there was no interaction between the particles other than an eventual collision. In practice, they always exchange forces.

To discuss viscous forces, we need to discuss the different types of deformations a fluid can undergo.

²Although the tensor has 9 components, the stress tensor is symmetric so we have 6 “free” variables

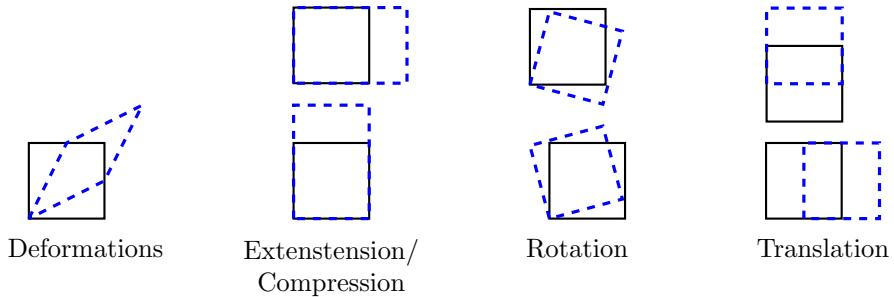


Figure 3.1: Illustration of the different types of deformation.

3.2.1 Deformation

There are four fundamental types of deformation: extension, shear, rotation, and compression. We will illustrate them in figure 3.1.

Shear is the deformation that happens when we move one part of the fluid relative to another. Rotation is the deformation that happens when we rotate a fluid. Finally, compression/extension is the deformation that happens when we try to change the volume of a fluid.

Deformations can be modeled by a function that maps a particle's final position to its initial position. For example, the examples found in figure 3.1 can be modeled by the following functions:

$$\begin{bmatrix} x \\ y \end{bmatrix}_{\text{Shear}} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad (3.15)$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_{\text{Rotation}} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad (3.16)$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_{\text{Comp/ext}} = \begin{bmatrix} 1.5 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad (3.17)$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_{\text{Translation}} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}. \quad (3.18)$$

Note that a linear function can describe almost all types of deformations. The only exception is the translation. Two movements in materials do not lead to forces or stress: translation and rotation. The reason is simple: these only take a body and put it somewhere else, possibly rotating it. They do not change the body's shape; thus, there is no stress.

We will thus investigate shear and compression. These two can be described as a linear function,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_M \begin{bmatrix} x \\ y \end{bmatrix}. \quad (3.19)$$

However, remember that not all linear functions are pure shear and/or compressions: rotations are also linear transformations. Rotations are characterized by an antisymmetric matrix; see (3.16). Thus, shear/compression can be expressed

by symmetric matrices. If we are given an arbitrary matrix M , we can decompose it into a symmetric and antisymmetric part,

$$M_{\text{sym}} = \frac{M + M^T}{2}, \quad (3.20)$$

$$M_{\text{asym}} = \frac{M - M^T}{2}. \quad (3.21)$$

From our previous discussions, any internal force in the material must depend on M_{asym} *only*.

It is important to distinguish between compression and shear of a given movement. You can check that the change in the cube volume is given by the trace of M . Thus, we can isolate the shear from a generic transformation matrix as

$$\begin{aligned} M_{\text{shear}} &= M_{\text{sym}} - n_{\text{dim}} \text{Tr}(M_{\text{sym}}) I \\ &= \frac{M + M^T}{2} - n_{\text{dim}} \text{Tr}(M) I. \end{aligned} \quad (3.22)$$

3.2.2 Fluid relation to a deformation

There are two main types of reactions to a deformation. There are static reactions, i.e., a reaction that depends only on the current state, and dynamical reactions, i.e., a reaction that depends on how the flow deforms.

We will thus write the stress tensor as a sum of the static and dynamical parts,

$$\overset{\leftrightarrow}{\sigma} = \overset{\leftrightarrow}{\sigma}_{\text{static}} + \overset{\leftrightarrow}{\sigma}_{\text{dynamic}}. \quad (3.23)$$

Static reaction

The most straightforward analog to a static reaction is a spring. No matter how fast a spring is compressed, it will always react with the same force, which depends only on its current length.

Note that different materials react to compression and shear deformations differently. We will start with compression: what happens when you try to compress a piece of wood or metal or compress a syringe full of water (or even air) while keeping the exit closed? We directly feel a resisting force. Thus, all materials, solids or liquids, resist compression.

Things are not the same with shear. Imagine now that a piece of rubber is glued to two metal sheets. If you slide these sheets, the rubber will try to pull them back to their original position, i.e., it resists movement. This is not the same with liquids: once the plates have moved, no force is trying to return them to their original position. This is the definition of a fluid:

Definition 1 (Fluid) *A fluid is a material that does not resist static shear deformation.*

An equation of state gives the static reaction of a fluid. The most well-known one is the ideal gas law,

$$pV = nRT. \quad (3.24)$$

A simplified model for a liquid can be written as

$$p - p_0 = K(V - V_0) - \frac{\beta}{K}(T - T_0), \quad (3.25)$$

where the variables with a 0 subscript represent a reference state, and K and β are the bulk modulus and thermal expansion coefficient, respectively. This is a pretty good model for any fluid, as long as the temperature and volume variations are not large. This is often a good approximation for liquids, as the volume tends not to change too much (K is typically large).

Note that pressure exerts a force that is always normal to the surface of the fluid or a subdomain and is driven only by changes in the flow volume.

$$\overset{\leftrightarrow}{\sigma}_{\text{static}} = -p \overset{\leftrightarrow}{I}. \quad (3.26)$$

Thus, shear, rotation, or translations do not affect it.

Dynamical response

That does not mean that fluids do not have any shear force, just not static ones. The fluid does resist *movement*, i.e., as the plates move, it tries to stop them. It just does not care that, in the end, they are displaced.

Saying that a fluid/solid “resists” a given movement means that it exerts a force. This force is modeled as a stress (i.e., it is a force normalized by the surface, so we can more easily model it for infinitesimal volumes). We will now discuss how to relate the stress to the deformation. We will limit the discussion here to a linear relation between the stress and the deformation.

In general, we can write

$$\sigma_{ij} = \mathcal{E}_{ijkl} (\vec{\nabla} \vec{u})_{kl} \quad (3.27)$$

Let us break this down. The term $\vec{\nabla} \vec{u}$ is the gradient of the velocity field, i.e., it is a tensor that describes how the velocity changes in space. In two dimensions, it can be written as

$$\vec{\nabla} \vec{u} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} \end{bmatrix} \quad (3.28)$$

The term \mathcal{E}_{ijkl} is a tensor that describes how the stress changes with the deformation. It is called the elasticity tensor. It is a fourth-order tensor, i.e., it has four indices. What (3.27) means is that each stress component is an (independent) linear combination of each term of $\vec{\nabla} \vec{u}$.

This is the most general expression, but often, we can simplify it considerably based on physical arguments. Almost always, we assume that the fluid is isotropic, i.e., it behaves the same in all directions. As such, a deformation in x should be analogous to a deformation in y , for example. Also, we assume that fluid rotations do not generate stress (after all, they are not deforming the fluid). These two assumptions lead to the following general expression for the viscous part of the stress tensor,

$$\overset{\leftrightarrow}{\sigma}_{\text{dynamical}} = \overset{\leftrightarrow}{\tau} = 2\mu \overset{\leftrightarrow}{D} + \eta \overset{\leftrightarrow}{I} \vec{\nabla} \cdot \vec{u}. \quad (3.29)$$

Where $\overset{\leftrightarrow}{d}$ is the deformation rate tensor. Following (3.20), it reads

$$\overset{\leftrightarrow}{D} = \frac{\vec{\nabla}\vec{u} + (\vec{\nabla}\vec{u})^T}{2} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) / 2 \\ \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) / 2 & \frac{\partial u_y}{\partial y} \end{bmatrix}. \quad (3.30)$$

We can assign physical meaning to the terms in the viscous stress by manipulating (3.27) as

$$\overset{\leftrightarrow}{\tau} = 2\mu \left(\overset{\leftrightarrow}{D} - \frac{1}{3} \overset{\leftrightarrow}{I} \vec{\nabla} \cdot \vec{u} \right) + \left(\nu + \frac{2}{3}\mu \right) \vec{\nabla} \cdot \vec{u} \overset{\leftrightarrow}{I}. \quad (3.31)$$

Here, the first term is divergent-free, i.e., it is zero for a simple compression/expansion movement, while the second term is related only to the divergence of the velocity field. These are thus related to shear and bulk (compression) viscous stresses.

A very common assumption is that the bulk viscosity is zero, i.e.,

$$3\nu + 2\mu = 0, \quad (3.32)$$

which is known as the Stokes hypothesis. This is, in practice, a reasonable assumption, not precisely true thought: typically, pressure forces dominate bulk stress, so the viscous terms are somewhat negligible. This is not the case for the first term, which is related to shear: shear does not cause volume changes, so it does not affect the pressure term!

3.3 The Navier Stokes Equations

Putting together all that we have discussed, we arrive at the Navier-Stokes equations,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0, \quad (3.33)$$

$$\frac{\partial \rho \vec{u}}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u} \vec{u}) = -\vec{\nabla} p + \vec{\nabla} \cdot \overset{\leftrightarrow}{\tau} + \vec{f}, \quad (3.34)$$

$$\frac{\partial E}{\partial t} + \vec{\nabla} \cdot (E \vec{u}) = -\vec{u} \cdot \vec{\nabla} p + \vec{\nabla} \cdot \overset{\leftrightarrow}{\tau} + \vec{f} \cdot \vec{u} - k \vec{\nabla}^2 T, \quad (3.35)$$

$$p = R\rho T, \quad (3.36)$$

$$e = c_p T, \quad (3.37)$$

$$E = \frac{1}{2} \rho u^2 + \rho e \quad (3.38)$$

$$\overset{\leftrightarrow}{\tau} = 2\mu \left(\overset{\leftrightarrow}{D} - \frac{1}{3} \overset{\leftrightarrow}{I} \vec{\nabla} \cdot \vec{u} \right) + \left(\nu + \frac{2}{3}\mu \right) \vec{\nabla} \cdot \vec{u} \overset{\leftrightarrow}{I}. \quad (3.39)$$

where in the equation of state, we used $\rho \propto n/V$.

Note that the energy equation is written for the total energy here. Often, it is more interesting to re-write it in terms of the internal energy (e), as the kinetic energy (u^2) is implicit in the momentum equations (we will derive it later).

3.3.1 The kinetic energy equation

To derive the kinetic energy equation, we can use (3.33). The idea is to use the equality

$$\vec{u} \cdot \frac{d(\rho\vec{u})}{dt} = \frac{1}{2} \frac{d(\rho\vec{u} \cdot \vec{u})}{dt} = \frac{d\mathcal{K}}{dt}, \quad (3.40)$$

where \mathcal{K} is the kinetic energy density. An equation for the evolution of \mathcal{K} can be obtained by multiplying (3.33) by \vec{u} . After some algebraic manipulation, we get

$$\frac{\partial \mathcal{K}}{\partial t} + \vec{\nabla} \cdot (\mathcal{K}\vec{u}) = \vec{u} \cdot \left(\vec{\nabla} \cdot \overset{\leftrightarrow}{\sigma} \right) + \vec{u} \cdot \vec{f}. \quad (3.41)$$

To get a more physically interpretable result out of this, we can use

$$\vec{u} \cdot \left(\vec{\nabla} \cdot \overset{\leftrightarrow}{\sigma} \right) = \vec{\nabla} \cdot \left(\vec{u} \cdot \overset{\leftrightarrow}{\sigma} \right) - \left(\vec{\nabla} \vec{u} \right) : \overset{\leftrightarrow}{\sigma}. \quad (3.42)$$

The $:$ symbol is called the double dot product, and it just means that we multiply and add all the components of the matrices (like a dot product, but on matrices).

The advantage of this expression is that it tells us a lot if we interpret it in terms of the origin of the equation (the integral forms). The divergence term, $\vec{\nabla} \cdot (\vec{u} \cdot \overset{\leftrightarrow}{\sigma})$, corresponds to something happening on the surface of the domain Ω . Thus, it is typically a transfer of something (in this case, energy) between adjacent domains. Domain A is doing work on domain B ; thus, A loses kinetic energy, and B gains it. Nevertheless, the total kinetic energy is conserved.

The second term is something else. It happens inside the domain. So, it represents inner forces' creation/destruction of kinetic energy. Writting

$$\vec{\nabla} \vec{u} = \overset{\leftrightarrow}{D} + \overset{\leftrightarrow}{\omega}, \quad (3.43)$$

i.e., its symmetric and antisymmetric parts, and noting that, since $\overset{\leftrightarrow}{\omega}$ is symmetric and $\overset{\leftrightarrow}{\sigma}$ is antisymmetric, $\overset{\leftrightarrow}{\omega} : \overset{\leftrightarrow}{\sigma} = 0$, we can re-write (3.41) as

$$\frac{\partial \mathcal{K}}{\partial t} + \vec{\nabla} \cdot (\mathcal{K}\vec{u}) = \underbrace{\vec{\nabla} \cdot \left(\vec{u} \cdot \overset{\leftrightarrow}{\sigma} \right)}_{\text{Energy transfer}} - \underbrace{\overset{\leftrightarrow}{D} : \overset{\leftrightarrow}{\sigma}}_{\text{dissipation}} + \underbrace{\vec{u} \cdot \vec{f}}_{\text{ext. source/sink}}. \quad (3.44)$$

Note that we can say that $\overset{\leftrightarrow}{D} : \overset{\leftrightarrow}{\sigma}$ is a dissipation term as $\overset{\leftrightarrow}{D}$ is a linear combination of $\overset{\leftrightarrow}{D}$ and $\overset{\leftrightarrow}{I}$, and thus $\overset{\leftrightarrow}{D} : \overset{\leftrightarrow}{\sigma}$ is always positive. The negative sign in the equation thus means that it lowers the kinetic energy.

Since total energy is conserved, losing kinetic energy means the flow is heating. I.e., kinetic energy is lost throughout the flow due to viscous forces.

From (3.44) and (3.35), we can write an equation for the internal energy. For perfect gases, it is equivalent to an equation for temperature.

$$\frac{\partial \rho e}{\partial t} + \vec{\nabla} \cdot (\rho e \vec{u}) = \overset{\leftrightarrow}{\sigma} : \overset{\leftrightarrow}{D} - k \vec{\nabla}^2 T + \vec{u} \cdot \vec{f}. \quad (3.45)$$

Here again, we note that the term $\overset{\leftrightarrow}{\tau} : \overset{\leftrightarrow}{D}$, (hidden inside $\overset{\leftrightarrow}{\sigma} : \overset{\leftrightarrow}{D}$), can only increase the internal energy. Note that (3.45) and (3.35) can be used to model the flow. Although they are different, the difference between them is taken into account by the momentum equations.

3.4 Types of flow

3.4.1 Perfect and real fluids

Perfect fluids are fluids that do not undergo any irreversible dynamics. The most apparent irreversible process is heat diffusion. Heat always flows from hot to cold regions, never from cold to hot, i.e., the second law of thermodynamics, and thus is irreversible. Another irreversible process is due to viscous forces. These dissipate mechanical energy, converting it into heat. As heat cannot be (directly) converted back to kinetic energy, this is also an irreversible process.

The conditions for the fluid to be perfect are then,

$$k = 0 \quad (3.46)$$

$$\mu = 0 \quad (3.47)$$

$$2\nu + 3\mu = 0, \quad (3.48)$$

i.e., there is no thermal diffusion, bulk viscosity, or shear viscosity. Note that perfect fluids and gases are different things: an ideal gas can behave as a perfect fluid, depending on the circumstances.

Perfect fluid is a common assumption when dealing with complex compressible flows, such as supersonic flows, as this assumption greatly simplifies the equations. We will not cover this topic in this course, but it will be explored in the course in gas dynamics course (spoiler, everything in “just” manipulations of the integral conservation laws).

Real fluids are just the opposite: they have thermal diffusion, bulk viscosity, and/or shear viscosity.

Note that the same fluid may be well modeled as a perfect fluid in some scenarios and must be modeled by a real fluid in others. In fact, in the same problem, different flow regions may be treated differently. Understanding when to use each model requires a good understanding of the system, but it is also incredibly useful.

3.4.2 Compressible and incompressible flow

Another important distinction is that of compressible and incompressible flows. We talk about incompressible *flows*, not fluids. This is because the flow can be incompressible even if the fluid is compressible (it always is).

The nomenclature “incompressible” is an extrapolation of fluids that are very hard to compress, e.g., water. If we play around with water, it “feels” incompressible. However, if we apply sufficient force, we can compress it. But in any case, what that means is that in most situations, it *won’t* compress much. So, assuming incompressibility introduces very little error, with the uptake that it can significantly simplify the mathematical modeling.

If the flow is incompressible, we have that $\Delta V = 0$. The change in volume can be computed from the velocity field as

$$\frac{dV}{dt} = \vec{\nabla} \cdot \vec{u}, \quad (3.49)$$

and thus, the incompressibility assumption is equivalent to imposing that

$$\vec{\nabla} \cdot \vec{u} = 0. \quad (3.50)$$

Note that, from the mass conservation equation (3.33), this is equivalent to imposing that the density is constant in time, i.e., $\partial\rho/\partial t = 0$.

As incompressibility is often a good assumption, we get another good indication of why the Stokes hypothesis is a good one. If the flow is “incompressible” or almost, there is very little dissipation due to bulk viscosity.

Of course, the incompressibility assumption is not always valid. For example, if we deal with supersonic flows and have shock and expansion waves, both drastically change the fluid density. What happens here is that the inertia forces (how the flow resists acceleration) are large because the velocities are large. Thus, they are strong enough to compress the fluid. How fast does the flow have to be able to compress itself? A good rule of thumb is that if the flow is slower than 30% of the speed of sound ($M < 0.3$), it is incompressible. This is a very rough estimate, but it is a good starting point.

A simple example where this is *not* the case is a syringe. Fill a syringe with air, close the exit with your finger, and try to push the plunger. You can compress the air inside it quite a lot, even if the speed you push the plunger is low. This is a case where the abovementioned rule of thumb does not apply. So keep that in mind. The $M < 0.3$ rule of thumb is valid only for the flow inertial forces. If there are other forces in the systems, they may be able to compress the fluid significantly!

John von Neumann was well aware of the tremendous difference between what happens when you don't have the viscous terms and when you do, and he was also aware that, during most of the development of hydrodynamics until about 1900, almost the main interest was in solving beautiful mathematical problems with this approximation which had almost nothing to do with real fluids. He characterized the theorist who made such analyses as a man who studied "dry water." Such analyses leave out an essential property of the fluid.

Richard Feynman

4

Perfect flows

We say a gas is perfect when it does not have viscous effects. It is a good model for scenarios where these effects are small and can be neglected.

Without viscous forces, the fluid stress tensor becomes a diagonal tensor, i.e., all the forces acting on a given surface/domain are normal to it. This somewhat simple fact is the basis for all of the results we will discuss here.

4.1 The Euler equations

When the viscous terms are dropped from the Navier-Stokes equations, we get the Euler equations,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0, \quad (4.1)$$

$$\frac{\partial \rho \vec{u}}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u} \vec{u}) = -\vec{\nabla} p + \vec{f}, \quad (4.2)$$

$$\frac{\partial E}{\partial t} + \vec{\nabla} \cdot (E \vec{u}) = -\vec{\nabla} p \cdot \vec{u} + \vec{f} \cdot \vec{u}. \quad (4.3)$$

Although the assumption of perfect flow is unrealistic, it is very useful in practice. In many scenarios, the viscous forces are small: note that viscosity does not have to be close to zero; just the viscous forces must play a minor role.

These equations are the basis for a whole other course: gas dynamics, where you will investigate the dynamics of compressible flows in more detail, including the dynamics of supersonic flows in pipes, the formation of shocks, and some other cool phenomena.

These equations can be written in many forms, based on velocities, typically refereed as in primitive (variables) type, based on momentum, refereed as in conservative (variables) type, while the energy equation can be reformulated in terms of total or internal energy, temperature, enthalpy, or entropy. On top of that, they can be non-dimensionalized based on the free-stream velocity, speed

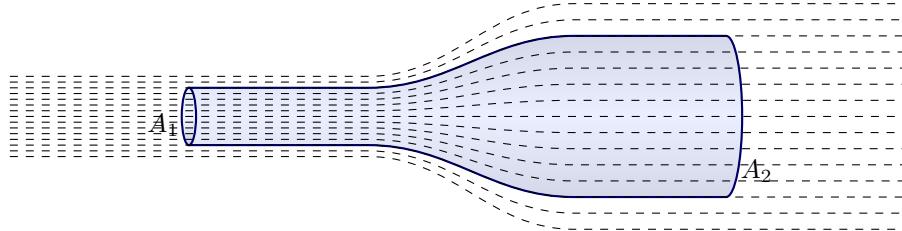


Figure 4.1: Control volume for the derivation of the Bernoulli equation.

of sound, etc. They are all equivalent, but some are more useful than others in different scenarios. For an exhaustive list, check [3]

4.1.1 Bernouli equation

Let us investigate a static perfect flow. Consider a domain as in figure 4.1: it has a cylindrical form, with the sides of the cylinder matching the flow streamlines. The cylinder is closed via surfaces from which flow enters/leaves the domain, on which we assume the velocities to be zero.

Consider first the conservation of mass. Using the integral form, and noting that since the lateral surface is tangent to the flow velocity, there is no mass flux through them, we get

$$\oint_{\Omega} \vec{u} \rho ds = 0 \implies A_1 u_1 \rho_1 = A_2 u_2 \rho_1 = Q_m, \quad (4.4)$$

where Q_m is the mass flow rate that passes through the domain.

Let us do the same now for the energy equation,

$$\oint_{\Omega} E \vec{u} \cdot \vec{n} ds = \oint_{\Omega} p \vec{u} \cdot \vec{n} ds + \int_{\Omega} \vec{u} \cdot \vec{f} dV \implies u_1 E_1 - u_2 E_2 = p_1 u_1 - p_2 u_2. \quad (4.5)$$

Dividing the equation by Q_m and using (4.4),

$$(u_1^2/2 - u_2^2/2) + (e_1 - e_2) + (p_1 - p_2) + g(h_1 - h_2) = P_f/Q_m, \quad (4.6)$$

where we considered the influence of the kinetic, internal, and gravitational potential energies in the total energy.

This is the well-known Bernouli equation, which is nothing more than the combination of the mass and energy conservation laws when applied to a well-crafted domain.

4.1.2 Kelvin's theorem

Kelvin's theorem states that the *circulation* on a line convected with the flow is conserved in time for perfect fluids. Circulation is defined as

$$\Gamma = \oint \vec{u} \cdot d\vec{l}. \quad (4.7)$$

Imagine that the integration line is a circle, and you can see that the circulation measures how much the flow revolves around the center of the circle (it “circulates” around it.)

To show that Γ is conserved, we need to show that $\frac{D\Gamma}{Dt} = 0$, for such we write

$$\frac{D\Gamma}{Dt} = \oint_c \frac{D\vec{u}}{Dt} d\vec{l} + \oint_c \vec{u} \frac{Dd\vec{l}}{Dt}. \quad (4.8)$$

Analysing each term we have

$$\frac{D\vec{u}}{Dt} = \vec{\nabla} p + \nu \vec{\nabla}^2 \vec{u} \xrightarrow{0} \quad (4.9)$$

$$\vec{u} \frac{Dd\vec{l}}{Dt} = \vec{u} \cdot \vec{\nabla} \vec{u} \cdot d\vec{l} = \vec{\nabla} \left(\frac{u^2}{2} \right) \cdot d\vec{l}. \quad (4.10)$$

Recalling that the line integral round a closed contour of a function’s gradient is zero, we get that $\frac{D\Gamma}{Dt} = 0$, and thus that Γ does not change in time.

This feels (and it is) a lot of mathematics, but it does have a neat physical interpretation, highlighted by Feynman: it is a consequence of conservation of angular momentum¹.

Think of a circular domain: the only way to change its angular momentum is to have tangential forces, and the only tangential force in flows is viscous forces, which, by definition, are absent in perfect flows.

On top of that, this theorem allows us to say a lot about some flows. Look at the boundary layer in figure 4.2. In the free stream, the fluid moves with a uniform velocity, meaning there are no, or at least small, frictional forces. The theorem then tells us that the circulation should remain constant (zero in this case). However, we can see that near the wall, $\Gamma \neq 0$. Thus, there must be a region where the flow is not perfect. This is the boundary layer, where velocity gradients are large enough to create substantial viscous forces and thus generate circulation (change the angular momentum of the fluid).

Another example is the generation of turbulence by a grid, as in figure 4.2. The flow is perfect upstream of the grid; thus, the circulation is zero. However, as the spacing between the wires is small at the grid, the viscous forces become important, generating a low circulation. Due to the turbulence, the flow cannot be modeled as perfectly downstream of the grid, as the velocity gradients created by the turbulent dynamics generate viscous forces.

¹We have discussed before many other conservation laws: mass, linear momentum, and energy, but we have not mentioned angular momentum. Could we further complete the equations of motion using it?

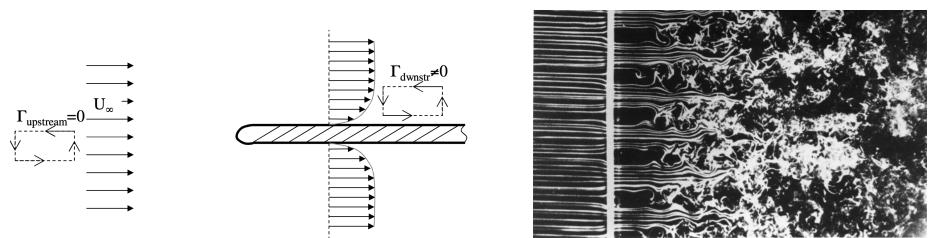


Figure 4.2: Illustration of a boundary layer near the leading edge of a flat plate (left) and the generation turbulence after a grid.

All models are wrong, but some are useful.

George Box

5

Incompressible flows

As mentioned before, many flows can be modeled as incompressible; thus, studying incompressible flows is interesting and valuable.

A flow is said to be incompressible if

$$\nabla \cdot \vec{u} = 0, \quad (5.1)$$

i.e., at all positions, there is no compression or expansion of fluid particles. Note that this does not imply that the density is constant throughout the flow: for example, a mixture of water and oil is incompressible, but the density varies throughout the flow. However, in this course, we will limit ourselves to cases where ρ is constant.

Using (5.1), we can simplify the Navier-Stokes equations, (3.33)-(3.35), to

$$\nabla \cdot \vec{u} = 0, \quad (5.2)$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} = \overset{\leftrightarrow}{D} : \overset{\leftrightarrow}{D} - \frac{1}{\rho} \vec{\nabla} p + \nu \vec{\nabla}^2 \vec{u} + \vec{f}, \quad (5.3)$$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \frac{k}{\rho c_p} \vec{\nabla}^2 T + \frac{1}{\rho c_p} \vec{f} \cdot \vec{u}, \quad (5.4)$$

where $\nu = \mu/\rho$ is the kinematic viscosity. To get to this equation, we made some simplifications. Notice that all the $\vec{\nabla} \cdot \vec{u}$ terms in the stress tensor vanish. On top of that, we can show that

$$\vec{\nabla} \cdot \overset{\leftrightarrow}{\tau} = \vec{\nabla} \cdot (\mu \vec{\nabla} \vec{u}) \stackrel{\mu=\text{const.}}{=} \mu \vec{\nabla}^2 \vec{u}. \quad (5.5)$$

There are a few interesting aspects of the incompressible equations. First, the momentum and mass conservation equations decouple from the temperature/energy equation: i.e., we do not need to know or solve for temperature to find the velocity field. Also, there is no dynamic equation for pressure.

In mathematical terms, pressure becomes a term that constrains the velocity field so that $\vec{\nabla} \cdot \vec{u} = 0$. Physically, this means that any volume change generates

a large pressure change, which quickly moves the fluid back to a compressed state. As the bulk compressibility is very large, this happens on a timescale much shorter than the flow dynamics. The flow is essentially incompressible for the latter, even if instantaneous volume changes can happen. Again, this is the physical interpretation; mathematically, we just imposed that $\vec{\nabla} \cdot \vec{u} = 0$. We can have a mental picture that the incompressible equations model all the “slow” dynamics of the flow, assuming that all the compressible phenomena dampen out quickly and do not affect the flow significantly.

To actually compute the pressure, we can take the divergent of (5.3), which gives

$$\vec{\nabla} \cdot \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} \right) = \vec{\nabla} \cdot \left(-\frac{1}{\rho} \vec{\nabla} p + \nu \vec{\nabla}^2 \vec{u} + \vec{f} \right), \quad (5.6)$$

which, after manipulation and using (5.2), gives

$$\frac{1}{\rho} \vec{\nabla}^2 p = \vec{\nabla} \cdot \left(\vec{f} - \vec{u} \cdot \vec{\nabla} \vec{u} \right), \quad (5.7)$$

which is as a Laplace equation for pressure. Note that all viscous terms drop out of this equation: i.e., viscosity does not affect the pressure field, at least not directly.

5.1 The vorticity field

A mathematical theorem states that any vector field can be decomposed into a sum of an irrotational field and a divergence-free field. This is called the Helmholtz decomposition.

In fluid mechanics, this is particularly interesting for incompressible flows. As we know that the velocity field is divergent-free (incompressible), we can try to write the equations in only terms of its rotational part.

Taking the curl of (5.3), we get

$$\frac{\partial \vec{\omega}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{\omega} = \nu \vec{\nabla}^2 \vec{\omega} + \vec{\nabla} \times \vec{f}, \quad (5.8)$$

where $\vec{\omega} = \vec{\nabla} \times \vec{u}$ is the vorticity field.

If we use this equation, pressure automatically drops out of the equation. Physically, this is due to pressure being unable to drive flow rotations. This is particularly nice, as it means that we can solve for the velocity field without having to compute the pressure field.

We would also hope that this equation is more concise than the originals: there are no vorticity restrictions as there were for the velocity. Unfortunately, this is not the case. If we compute the divergent of the vorticity field, we get

$$\vec{\nabla} \cdot \vec{\omega} = 0, \quad (5.9)$$

i.e., the vorticity field is also “incompressible”. This has little to do with physics; it is a mathematical consequence of the definition of the vorticity field. Nevertheless, it does have some interesting consequences.

Most notably, a vortex can never “start” or “end”. It can be diffused, but the total vorticity cannot change. So vortices can either loop around, forming

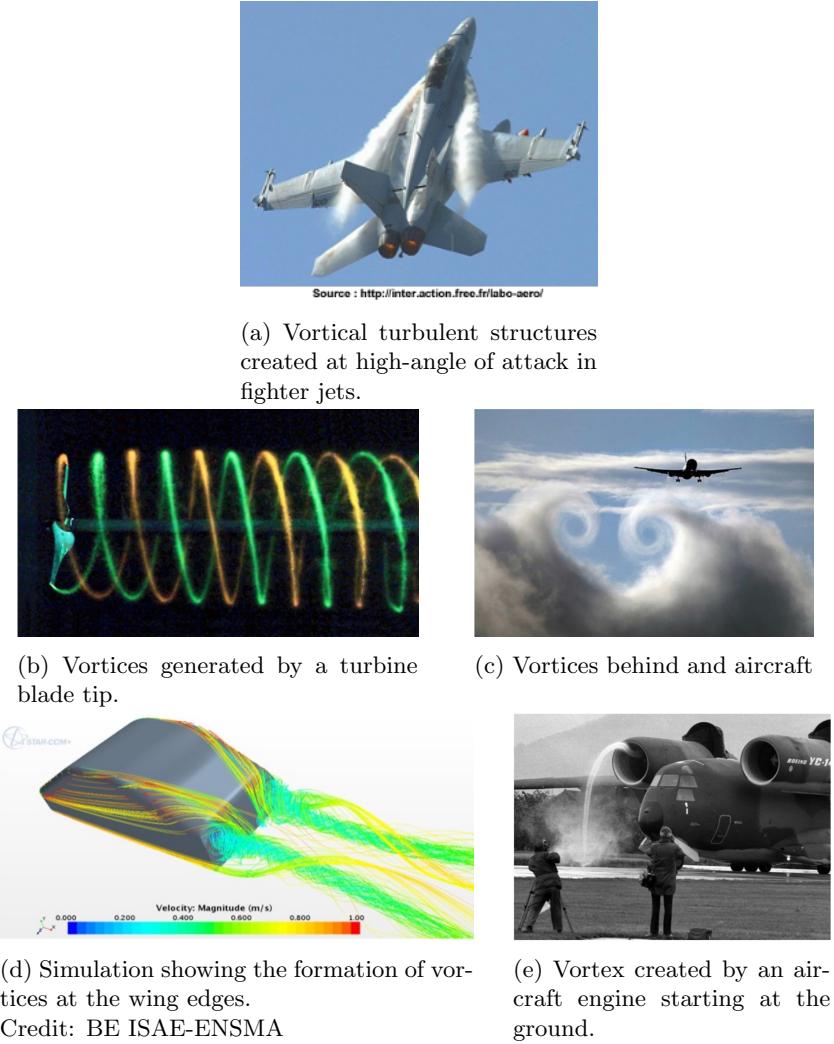


Figure 5.1: Illustration of vortices in different flow conditions.

vortex rings, extend to infinity, or start/end at a surface. This is an important property of vortices, as it means that they are, in a way, very stable structures.

Note that we threw in the word “diffusion”. It is thus worthy to discuss diffuse processes and try to understand more about them.

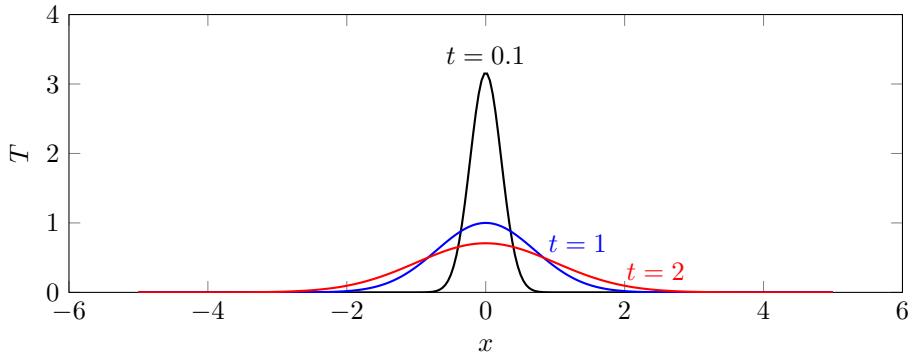


Figure 5.2: Illustration of a diffusive process.

5.2 Some hidden dynamics

5.2.1 The diffusion equation

Diffusive equations show up a lot in physics. For example, assuming that $\vec{u} = 0$ temperature equation, (5.4), becomes

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p} \vec{\nabla}^2 T. \quad (5.10)$$

This is known as a diffusion equation. Figure 5.2 illustrates the solution and why we call it a diffusion equation. In short, if we have a localized temperature spot, the temperature will spread out, but the total temperature(energy) is constant. But that rate with which the temperature spreads out changes considerably. Note that spreads much more between $t = 0.1$ and $t = 1$ than between $t = 1$ and $t = 2$.

This trend can be derived from (5.10). Let's simplify it to a single dimension and then scale the x axis by L .

$$\frac{\partial T}{\partial \tilde{t}} = \frac{k}{\rho c_p L^2} \frac{\partial^2 T}{\partial \tilde{x}^2} \quad (5.11)$$

$$x = L\tilde{x}. \quad (5.12)$$

Note that (5.11) is the same as (5.10), but now with the diffusion coefficient (what multiplied the laplacian), divided by L^2 . That means that the rate at which the temperature changes in time is proportional to $1/L^2$, i.e., it becomes much slower for large structures.

We could get to this conclusion via a simple dimensional analysis. If we look at the dimension of the new diffusion coefficient

$$\left[\frac{k}{\rho c_p L^2} \right] = T, \quad (5.13)$$

i.e., the diffusion process timescale scales as $1/L^2$: something twice as large takes *four* times longer to diffuse.

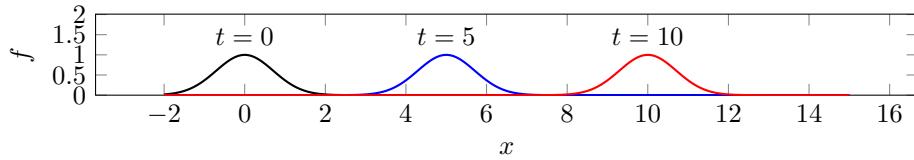


Figure 5.3: Illustration of the solution of the convection equation(5.15).

Interestingly, it is not only the temperature equation that reduces to a diffusion equation. Note that the momentum, (5.3), and vorticity, (5.8), equations also have a laplacian in them. So, in some circumstances, they also reduce to the diffusion equation. Even when the other terms are still present, we can say that a diffusive process is happening. In fact, we will later learn that there is always a scale where the diffusive processes are important.

Even if we have a diffusive process, keep in mind that other effects may lead to a quantity being *more* concentrated as time passes. We will explore one such case in a tutorial.

5.2.2 The convection equation

Consider the following equation

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f = 0, \quad (5.14)$$

where \vec{v} is a constant vector.

You can easily see that $f(\vec{x}, t) = g(\vec{x} - \vec{v}t)$ is a solution (try yourself). Note that this is a general solution. Given any initial condition, i.e., $f(\vec{x}, 0)$, the previous expression allows us to know the solution at any point at any given time.

Let us then take a look at the behavior of this solution. Figure 5.3 illustrates the solution. It corresponds to a simple translation of the initial condition in time, in the direction, and with the velocity of \vec{v} .

Re-writting (5.15) in 1D, calling the x by L , we get

$$\frac{\partial f}{\partial t} + \frac{v_x}{L} \frac{\partial f}{\partial x} = 0. \quad (5.15)$$

In a similar analysis as in the previous section, the time scales related to convection are proportional to the distance to be convected and inversely proportional to the convection velocity.

Note that a term like this is present in the temperature equation, (5.4). Thus, we can say that the velocity field convects the temperature. A similar term is also found on the momentum equations (5.3), but with a caveat. The velocity field convects itself. This is thus a non-linear term and *the* source of complexity in the dynamics of fluids.

5.3 Time scales

Given the different trends observed for the time scales of diffusion and convection process ($\propto L^{-1}$ and L^{-2}), it is expected that each process will be dominant at

a given length scale. Intuitively, the diffusion process is slower at larger length scales (large L), and thus convection will dominate. At smaller length scales, the opposite is true.

Besides the convective and diffusive time scales, we also need to consider the natural time scale of the flow. Imagine a car going from rest to full speed. Is the time it takes to fully accelerate comparable to one (or both) of the other time scales? Is it much faster? Much slower? Next, we will see that the relationship between these time scales gives rise to hugely different dynamics.

Returning to the momentum equation, (5.3). Let us rescale \vec{x} by a reference length L , \vec{u} by a reference velocity U , and t by the timescale T . This gives,

$$\frac{U}{T} \frac{\partial \vec{u}}{\partial \tilde{t}} + \frac{U^2}{L} \vec{u} \cdot \vec{\nabla} \vec{u} = -\frac{1}{\rho L} \vec{\nabla} p + \nu \frac{U}{L^2} \vec{\nabla}^2 \vec{u} + \vec{f}, \quad (5.16)$$

dividing by U , we get

$$\frac{1}{T} \frac{\partial \vec{u}}{\partial \tilde{t}} + \frac{U}{L} \vec{u} \cdot \vec{\nabla} \vec{u} = -\vec{\nabla} \tilde{p} + \frac{\nu}{L^2} \vec{\nabla}^2 \vec{u} + \vec{f}, \quad (5.17)$$

where $\tilde{p} = \frac{1}{\rho L U} p$ and $\tilde{f} = \frac{1}{\rho U} f$.

What (5.20) allows us to do, is to compare the time scales. In analogy to our discussion of the convection and diffusion equations, let us define the convective and diffusive timescales as

$$T_c = \frac{L}{U}, \quad (5.18)$$

$$T_d = \frac{L^2}{\nu}, \quad (5.19)$$

with which (5.20) becomes

$$\frac{1}{T} \frac{\partial \vec{u}}{\partial \tilde{t}} + \frac{1}{T_c} \vec{u} \cdot \vec{\nabla} \vec{u} = -\vec{\nabla} \tilde{p} + \frac{1}{T_d} \vec{\nabla}^2 \vec{u} + \vec{f}. \quad (5.20)$$

So now, our problem has three different time scales: convective, diffusive, and natural. We can now compare them. Note that the pressure term does not have an associated timescale with it. Physically, the timescale related to it is the acoustic timescale: i.e., how long does an acoustic wave take to travel a distance L , such that $T_a = L/c$, where c is the speed of sound. We do not see this timescale here because we assume that the flow is incompressible, and thus, the speed of sound is infinite. What we see is just the “quasi-static” pressure response.

In any case, the other time scales are there. What happens then when, let us say, the dissipative time scale is much shorter than the others, $T_d \ll T_c, T, T_f$? In this case, the dissipative term dominates, and the equations can be simplified to

$$\vec{\nabla}^2 \vec{u} - \vec{\nabla} p = 0. \quad (5.21)$$

which is the Stokes equation¹. We solve it for a given boundary condition (i.e., the velocity at ∞).

¹Here we re-scaled the pressure by T_d . We will do it in the following equations as well

There are many other options; what if $T_c \approx T \ll T_d, T$? In this case, the convective term dominates, and we get

$$\frac{\partial \vec{u}}{\partial \tilde{t}} + \vec{u} \cdot \vec{\nabla} \vec{u} = -\vec{\nabla} \tilde{p}, \quad (5.22)$$

which is a form of the Burgers equation. It is clear now that we can have different limits depending on the relevant timescales.

Note that T_d and T_c are the flow intrinsic timescales: they depend on the velocities and fluid properties only. The ratio between them is particularly telling:

$$\frac{T_c}{T_d} = \frac{\nu U}{L} = Re, \quad (5.23)$$

where Re is the Reynolds number. If Re is large, that means that the flow is convected a lot before dissipation becomes important. If its low, it dissipates a lot before it moves a lot. It is one of the most important numbers in fluid mechanics!

5.4 The Reynolds number

From (5.20) and (5.23), if we take $T = 1$, and re-scale the pressure and force terms, we get the following form of the Navier-Stokes equation

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} = -\vec{\nabla} p + \frac{1}{Re} \vec{\nabla}^2 \vec{u} + \vec{f}, \quad (5.24)$$

$$\vec{\nabla} \cdot \vec{u} = 0 \quad (5.25)$$

Written in this form, the Reynolds number's role in defining the viscous term's importance is clear.

The range of phenomena that can be modeled with this equation is quite remarkable. The scale of different behaviors that can be observed is also quite impressive.

5.4.1 Low Reynolds number flows

The convection term (the only non-linear term) is negligible at low Reynolds numbers, and the equation becomes linear. Linear equations have many properties, the most important, for this course are

- Additivity: if u_1 and u_2 are solutions, then $u_1 + u_2$ is also a solution.
- Uniqueness: only one solution exists for a given set of boundary conditions.

Additivity is reflected in the fact that if you apply an action to the system and then inverse this action, you can undo all the original effects. An example is illustrated in figure 5.5: check also a video of the phenomena².

This has some interesting consequences, for example, for bacteria. As they are tyne, bacteria swim in a flow with very low Reynolds numbers. What would

²https://fyfluidynamics.com/2012/02/the-reversibility-of-laminar-mixing-often-comes-as/?doing_wp_cron=1673857758.7231209278106689453125

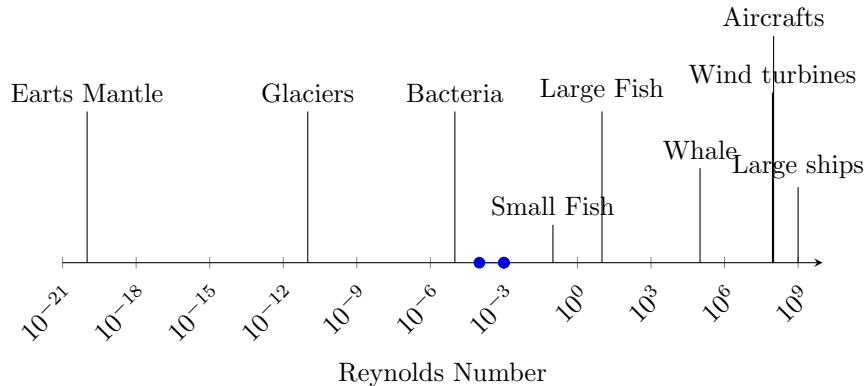


Figure 5.4: Examples of cases where the flow dynamics show different Reynolds numbers.

happen if they swam like fishes, flapping their tails to one side? They would move forward, but if they flapped their tails in the opposite direction, they would return to their original position. They would not be able to move at all!

Thus, bacteria have a different way of swimming, which is not time-reversible. They rotate their tails like a wine opener. This way, they are always doing “the same action”, and thus, they can move forward. A similar effect can be observed in small insects. Even if the Reynolds number is much larger than for bacteria, their flight pattern usually involves non-reversible movements. Dragonflies are a good example of this: they move their wings in a circular motion.

5.4.2 Moderate Reynolds number flows

The convection term becomes important at higher Reynolds numbers; thus, the equations (and the flow dynamics) are non-linear. The two properties mentioned above are no longer valid. In fact, we don’t even know if the equations are always solvable or if the solutions are unique or not!

On a practical level, the non-reversibility of the dynamics is, in fact, crucial for us. While breathing, we inhale and exhale. If the flow was reversible, we would inhale the same air we just exhaled. Soon, the air in front of us will be saturated, and we will not be able to get oxygen from it. However, the Reynolds number of the breathing processes is not negligible (I estimated it to be around 1000); the process of pushing and pulling the air is different. When exhaling, we form a jet, which tends to extend forward with slow lateral growth. When inhaling, we draw a lot of air from the surroundings. At each breath, we are thus collecting fresh air!

5.4.3 High Reynolds number flows

When the Reynolds number exceeds a critical threshold, the flow becomes unstable and form dynamic structures with different sizes. We call this turbulence, which we will study in a dedicated chapter.

Figure 5.4 illustrates a few examples of phenomena and the associated Reynolds numbers.

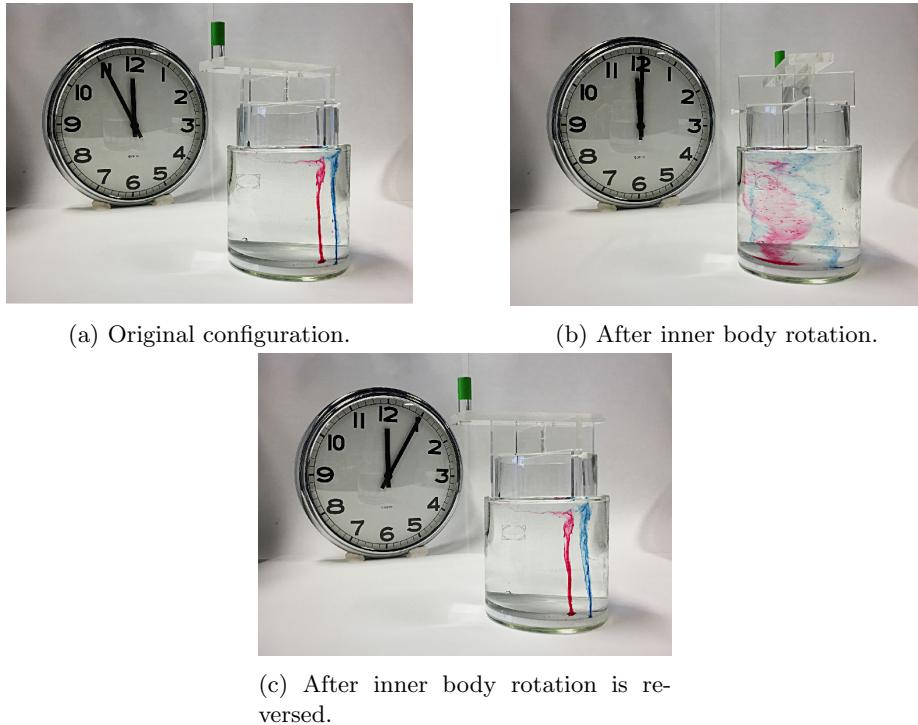


Figure 5.5: Illustration of a time-reversible flow. The colorant in the fluid is stretched by a rotation of the inner cylinder and disappears. When the rotation is reversed, it shows up again. Credits: Ved1123 - Own work, CC BY-SA 4.0, <https://commons.wikimedia.org/w/index.php?curid=67552518>.

5.5 The boundary layer

We now dig into a key element of nearly every wall-bounded flow. The formation of boundary layers. The canonical example of a boundary layer is the flow over a flat plate. Let us look into what we think happens.

We will assume that a zero-thickness flat plate is subjected to an external uniform flow with small viscosity, where the velocity is parallel to the plate. The first-order approximation we can make to this flow is a perfect flow: if we have zero viscosity, the flow just passes over the plate. This is a *really* good approximation for most of the flow. However, we know that, at the plate, we must have zero flow velocity (non-slip condition), so how do we reconcile the fact that a uniform velocity is a good approximation if it violates the boundary condition?

5.5.1 Time-scales arguments

One way to reason for this is to use the concept of time scales we discussed previously. The existence of the plate needs to “travel” through the flow to affect it. As the plate is parallel to the flow, it only affects it via shear forces, and the only mechanism by which the flow can transmit shear forces is via viscosity.

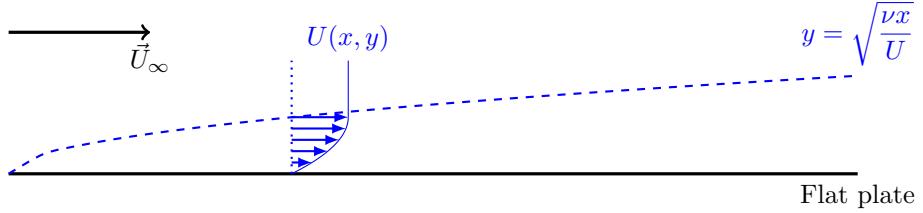


Figure 5.6: Illustration of a boundary layer over a flat plate. The y scale here is expanded for visualization purposes normally the boundary layer is thin.

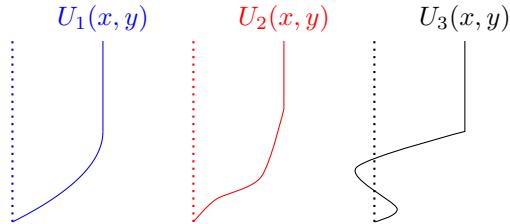


Figure 5.7: Illustration of different boundary layer profiles.

We saw earlier that the viscous effects over a length scale of L occur in a time scale of $T_d = L^2/\nu$. So it takes a time $T = L_y^2/\nu$ for a point located a distance L over the plate to be affected by it. However, the flow is moving at a velocity U , so this information is not just traveling “up” but being conveyed downstream. During the time T_d , a distance $L_x = UT_d$.

Expressing L_x in terms of L_y , we get

$$L_x = \frac{UL_y^2}{\nu}. \quad (5.26)$$

This means that the wall only affects a region further downstream. Figure 5.6 illustrates the expected trend. Since we assumed that viscosity, ν , was small, L_x is typically large non-vanishing values of L_y . Thus, the fluid may only feel most of the plate’s influence once it has passed it.

Based on this insight, we can develop a more formal approach to the problem.

5.5.2 Characterising boundary layers

There are several ways to characterize a boundary layer. One of them is the velocity profile: all the boundary layers in Figure 5.7 are different.

But before looking into the details of the boundary layer profiles (and we are yet to solve for one), it is interesting to define some macro quantities.

The boundary layer thickness, or height, is defined as the position on which the boundary layer velocity reached 0.99 of the edge value,

$$0.99U_e = U(x, \delta_{0.99}(x)). \quad (5.27)$$

This definition, although intuitive, has two problems. The value of 0.99 is arbitrary, and computing $\delta_{0.99}$ can be error-prone. Imagine that the velocity

profile is obtained from an experiment, then errors of 1% would already lead to errors. This is magnified by the fact that, close to its “end”, the velocity profile of the boundary layer changes very little, so small errors in the velocity would reflect in large errors in $\delta_{0.99}$.

It is thus useful to define *integral* quantities. These tend to be much less sensitive to errors, as errors tends to average out.

There are three main measures of boundary layers.

Boundary layer displacement thickness

is defined as

$$\delta^*(x) = \int \left(1 - \frac{U(x, y)}{U_e}\right) dy, \quad (5.28)$$

and measures the mass flow deficit induced by the boundary layer. That is, we compare a uniform velocity profile (what we would get from a perfect fluid), and compute how much mass flow we are missing due to viscosity. One typically mentions the formation of an “effective wall position”, i.e., by virtually displacing the wall upwards by δ^* , we obtain a wall that mimics, on a perfect fluid, the effects of viscosity.

To see why this is the case, consider the total mass flow rate over the plate. This is given by

$$\begin{aligned} \int_0^L \rho U(x, y) dy &= \int_0^L \rho U_\infty dy - \rho U_\infty \int_0^L \left(1 - \frac{U(x, y)}{U_\infty}\right) dy \\ &= \rho U_\infty \left(L - \int_0^L \left(1 - \frac{U(x, y)}{U_\infty}\right) dy\right) \\ &= \rho U_\infty (L - \delta^*). \end{aligned} \quad (5.29)$$

Boundary layer momentum thickness

is defined as

$$\theta(x) = \int \frac{U(x, y)}{U_e} \left(1 - \frac{U(x, y)}{U_e}\right) dy. \quad (5.30)$$

As before, we are measuring a deficit but here in the x-momentum. To see this, consider the momentum flux

$$\int_0^L \rho U(x, y)^2 dy = \int_0^L \rho U(x, y) dy - \rho U_\infty^2 \int_0^L \left(1 - \frac{U^2(x, y)}{U_\infty^2}\right) dy, \quad (5.31)$$

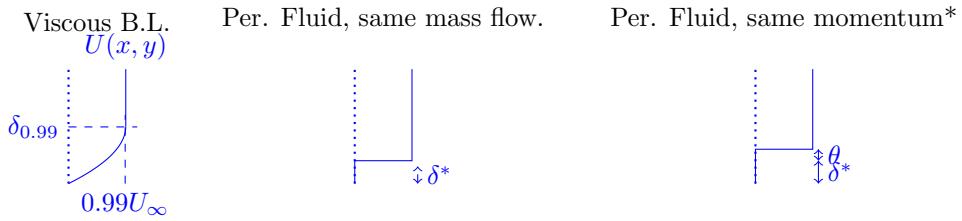


Figure 5.8: Illustration of different boundary layer measures. (* Same momentum for once the mass flow is corrected.)

we could have stopped here and defined a momentum thickness based on the right-most term in the previous equation. However, we typically define it as

$$\begin{aligned}
 \int_0^L \rho U(x, y)^2 dy &= \int_0^L \rho U_\infty^2 dy \\
 &\quad - \rho U_\infty^2 \left(\int_0^L \left(1 - \frac{U(x, y)}{U_\infty} \right) dy + \int_0^L \frac{U(x, y)}{U_\infty} \left(1 - \frac{U(x, y)}{U_\infty} \right) dy \right) \\
 &= \rho U_\infty^2 \left(L - \int_0^L \left(1 - \frac{U(x, y)}{U_\infty} \right) dy \right) \\
 &= \rho U_\infty^2 (L - \delta^* - \theta),
 \end{aligned} \tag{5.32}$$

that is, θ , gives the amount of momentum that is missing from the flow, *on top of what* would be missing simply due to reduced mass flow rate.

Boundary layer energy thickness

$$\delta_e(x) = \int \frac{U(x, y)}{U_e} \left(1 - \frac{U(x, y)^2}{U_e^2} \right) dy, \tag{5.33}$$

which has similarities to the momentum thickness, but is related to the energy deficit.

Shape factor

$$H = \frac{\delta^*}{\theta}, \tag{5.34}$$

is a measure of the shape of the boundary layer. It is typically close to 2/3 for most flows.

Wall friction

$$\tau_w(x) = \mu \frac{\partial U(x, 0)}{\partial y}, \tag{5.35}$$

gives directly the force per unit area that the wall is exerting on the fluid. It is a measure of the viscous effects between the wall and the flow.

5.5.3 Boundary layer models: Prandl's equations

Prandl's boundary layer equations are a set of equations that describe the flow in the boundary layer. They are based on the insights we developed at the end of the previous section (although he probably based his hypothesis on experimental observation).

We start with the incompressible Navier-Stokes equations. We assume the flow to be 2D, i.e., dependent only on x and y and $w = 0$, and steady, i.e., $\partial/\partial t = 0$.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (5.36)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad (5.37)$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right). \quad (5.38)$$

In the previous section, we saw that the wall effects are localized in a small region close to the wall, and that the changes in the streamwise direction occur slowly.

From the mass conservation equation, we notice that if $\frac{\partial u}{\partial t}$ is small, then so is $\frac{\partial v}{\partial y}$. Note that we cannot assume that either (or both) of these are zero, as this would imply that there is no change in u in the streamwise direction, which is not true. We also note that, whatever happens to v , happens in a very small region close to the wall. In this sense, " $\frac{\partial}{\partial y}$ " is large, and thus v is small.

Let us rescale our coordinate system such that the flow changes in both directions occur on the same scales. Let define L as the distance of a given location to the leading edge of the plate, and δ as the boundary layer height at this position. We thus define the following rescaled variables

$$\tilde{x} = \frac{x}{L}, \quad (5.39)$$

$$\tilde{y} = \frac{y}{\delta}. \quad (5.40)$$

$$\tilde{u} = \frac{u}{U}, \quad (5.41)$$

$$\tilde{v} = \frac{v}{V}, \quad (5.42)$$

$$\tilde{p} = \frac{p}{\rho U^2}, \quad (5.43)$$

where U is the magnitude of the free-stream velocity, and V a velocity scale for the changes in the y direction, to be defined. We want to define those such that the order of magnitude of \tilde{u} and \tilde{v} , and of " $\frac{\partial}{\partial \tilde{x}}$ " and " $\frac{\partial}{\partial \tilde{y}}$ " are the same.

Now, all the changes in \tilde{x} occur on the same scale as the changes in \tilde{y} . Re-writting the mass conservation equation with these variables gives

$$\frac{L}{U} \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\delta}{V} \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0. \quad (5.44)$$

Since the order of magnitude of $\frac{\partial \tilde{u}}{\partial \tilde{x}}$ and $\frac{\partial \tilde{v}}{\partial \tilde{y}}$ are the same, we must have that

$$\frac{V}{U} = \frac{\delta}{L} = \epsilon \ll 1. \quad (5.45)$$

Let us interpret this physically: u is “order 1”, i.e., is the reference velocity. All the changes in the flow are much faster in y than in x , and thus the mass conservation tells us that the velocity in v must be much smaller than u .

What are the implications for the momentum equations, i.e., can we identify the terms that are most important? Let us rewrite them in terms of \tilde{x} , \tilde{y} , and $\tilde{v} = \frac{L}{\delta}v$, so that all the variables have approximately the same order. What we obtain is

$$\frac{U^2}{L}\tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{x}} + \frac{UV}{\delta}\tilde{v}\frac{\partial\tilde{u}}{\partial\tilde{y}} = -\frac{U^2}{L}\frac{\partial\tilde{p}}{\partial\tilde{x}} + \nu\left(\frac{U}{L^2}\frac{\partial^2\tilde{u}}{\partial\tilde{x}^2} + \frac{U}{\delta^2}\frac{\partial^2\tilde{u}}{\partial\tilde{y}^2}\right), \quad (5.46)$$

$$\frac{UV}{L}u\frac{\partial\tilde{v}}{\partial\tilde{x}} + \frac{V^2}{\delta}\tilde{v}\frac{\partial\tilde{v}}{\partial\tilde{y}} = -\frac{U^2}{\delta}\frac{\partial\tilde{p}}{\partial\tilde{y}} + \nu\left(\frac{V}{L^2}\frac{\partial^2\tilde{v}}{\partial\tilde{x}^2} + \frac{V}{\delta^2}\frac{\partial^2\tilde{v}}{\partial\tilde{y}^2}\right). \quad (5.47)$$

Multiplying the first equation by L/U^2 and the second by δ/U^2 , we get

$$\tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{x}} + \frac{V}{U}\frac{L}{\delta}\tilde{v}\frac{\partial\tilde{u}}{\partial\tilde{y}} = -\frac{\partial\tilde{p}}{\partial\tilde{x}} + \frac{\nu L}{U\delta^2}\left(\frac{\delta^2}{L^2}\frac{\partial^2\tilde{u}}{\partial\tilde{x}^2} + \frac{\partial^2\tilde{u}}{\partial\tilde{y}^2}\right), \quad (5.48)$$

$$\frac{V}{U}\frac{\delta}{L}\tilde{u}\frac{\partial\tilde{v}}{\partial\tilde{x}} + \frac{V^2}{U^2}\tilde{v}\frac{\partial\tilde{v}}{\partial\tilde{y}} = -\frac{\partial\tilde{p}}{\partial\tilde{y}} + \frac{\nu V}{\delta U^2}\left(\frac{\delta^2}{L^2}\frac{\partial^2\tilde{v}}{\partial\tilde{x}^2} + \frac{\partial^2\tilde{v}}{\partial\tilde{y}^2}\right). \quad (5.49)$$

Using (5.45), we can simplify the equations to

$$u\frac{\partial u}{\partial\tilde{x}} + \tilde{v}\frac{\partial u}{\partial\tilde{y}} = -\frac{\partial p}{\partial\tilde{x}} + \frac{\nu L}{U\delta^2}\left(\cancel{\epsilon^2\frac{\partial^2 u}{\partial\tilde{x}^2}} + \frac{\partial^2 u}{\partial\tilde{y}^2}\right) \stackrel{\approx 0}{\sim}, \quad (5.50)$$

$$\cancel{\epsilon^2 u\frac{\partial v}{\partial x}} + \cancel{\epsilon^2 v\frac{\partial v}{\partial y}} \stackrel{\approx 0}{\sim} -\frac{\partial p}{\partial\tilde{y}} + \frac{\nu V}{\delta U^2}\left(\cancel{\epsilon^2\frac{\partial^2 v}{\partial\tilde{x}^2}} + \frac{\partial^2 v}{\partial\tilde{y}^2}\right). \quad (5.51)$$

We have not yet reached the final equations. Note that we have defined $V/U = \delta/L = \epsilon$, but we don't have a value for, e.g., V , just its relation to U . This is reflected in the terms $\nu L/U\delta^2$ and $\nu V/\delta U^2$. We can deal with the terms using pure mathematics and asymptotic methods, but instead, let us resort to a previous physical insight on the system.

Equation (5.26) suggests that $\delta = \sqrt{\nu L/U}$, and thus that $\nu = \delta^2 U/L = \epsilon \delta U$. We have thus

$$\frac{\nu L}{U\delta^2} = \epsilon \frac{\delta U L}{U\delta^2} = \epsilon \frac{L}{\delta} = \epsilon \frac{1}{\epsilon} = 1, \text{ and} \quad (5.52)$$

$$\frac{\nu V}{\delta U^2} = \epsilon \frac{U V \delta}{\delta U^2} = \epsilon^2. \quad (5.53)$$

So, up to the leading order,

$$u\frac{\partial u}{\partial\tilde{x}} + \tilde{v}\frac{\partial u}{\partial\tilde{y}} = -\frac{\partial p}{\partial\tilde{x}} + \frac{\partial^2 u}{\partial\tilde{y}^2}, \quad (5.54)$$

$$\frac{\partial p}{\partial\tilde{y}} = 0. \quad (5.55)$$

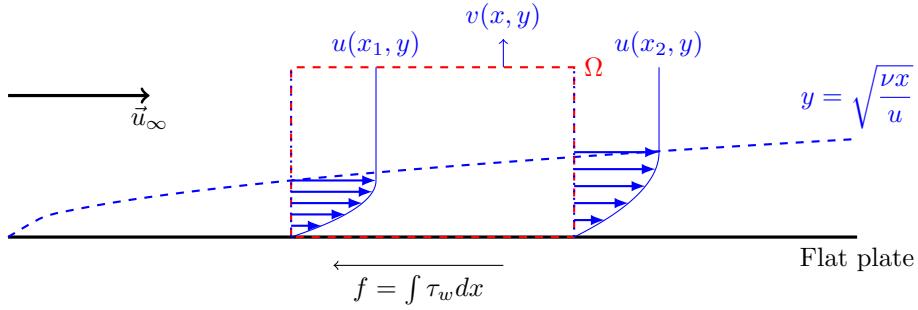


Figure 5.9: Control volume for the integral formulation of the boundary layer.

These are known as the (stationary) Prandl's equations. They share some similarities with the previous solutions of parallel flows discussed previously (in the tutorials). As before, pressure is constant in y . Note that the non-linear terms remain in the equations now, as we assume that both u varies with x , and v is (although small) non-zero. Another more mathematical detail is that the equations are now parabolized: i.e., we ignored all the second derivative terms in x .

As only terms with first derivatives in x remain, we can, in principle, solve these equations from an initial condition close to the plate's leading edge downstream. This not only makes the computation easier, as it is much simpler to march a differential equation in one way than to solve it directly on the whole domain but also gives us a physical insight: the boundary layer is mainly affected by what came before and is primarily ignorant to whatever comes next.

5.5.4 Boundary layer models: Integral formulation

The previously defined measures of the boundary layer show up quite naturally when we use mass and momentum balances on a boundary layer. Consider the control volume indicated in 5.9. Our first goal is to estimate the amount of flow leaving the control volume from the top (due to v). Using the mass balance, we get

$$\int_0^H \rho u(x + \Delta x, y) dy - \int_0^H \rho u(x, y) dy + \int_x^{x+\Delta x} \rho v(x, H) dx = 0. \quad (5.56)$$

Using (5.28), we can write

$$\int_x^{x+\Delta x} \rho v(x, H) dx = -\rho u_\infty(x)(H - \delta^*(x))|_x^{x+\Delta x}, \quad (5.57)$$

in the limit of $\Delta x \rightarrow 0$, we get

$$v(x, H) = -\frac{d}{dx}(u_\infty(x)(H - \delta^*(x))). \quad (5.58)$$

where the first term is the mass flow rate at the edge of the boundary layer, and the second term is the mass flow rate that is leaving the boundary layer.

Now, using momentum conservation,

$$\begin{aligned} \int_0^H (\rho u(x + \Delta x, y)^2 + p(x + \Delta x, y)) dy - \int_0^H (\rho u(x, y)^2 + p(x, y)) dy \\ + \int_x^{x+\Delta x} \rho u(x, H) v(x, H) dx = - \int_0^H \tau(x) dx. \end{aligned} \quad (5.59)$$

Using (5.30), and noting that $u(H, y) = u_\infty(x)$, we can write

$$\rho u_\infty(x)^2 (H - \delta^*(x) - \theta(x)) - H p(x) \Big|_x^{x+\Delta x} - \int_x^{x+\Delta x} \rho u_\infty(x) v(x, H) dx = \int_x^{x+\Delta x} \tau(x) dx. \quad (5.60)$$

Again, taking $\Delta x \rightarrow 0$,

$$\frac{d}{dx} \left(u_\infty(x)^2 (H - \delta^*(x) - \theta(x)) + H \frac{p(x)}{\rho} \right) + u_\infty(x) v(x, H) = -\frac{1}{\rho} \tau(x). \quad (5.61)$$

using (5.58), and noting that

$$\frac{d}{dx} (u_\infty^2 (H - \delta^*)) = u_\infty \frac{d}{dx} (u_\infty (H - \delta^*)) + u_\infty (H - \delta^*) \frac{d}{dx} (u_\infty),$$

we get

$$\frac{d}{dx} \left(-u_\infty^2 \theta + H \frac{p(x)}{\rho} \right) + u_\infty (H - \delta^*) \frac{d}{dx} (u_\infty) = -\frac{1}{\rho} \tau. \quad (5.62)$$

This expression depends on both $u_\infty(x)$ and $p(x)$. These parameters need to be specified in order to compute the boundary layer. They are how the “big picture” influences the “small world” of the boundary layer. E.g., is there an obstacle nearby, which makes the flow accelerate? Are we on the surface of a wing, where pressure and velocity change?

But we can simplify this further. At the edge of the boundary layer, the free-stream flow behaves as a perfect gas (viscous effects are negligible). We can thus use the Bernoulli equation (4.6) equation to write

$$\frac{1}{\rho} \frac{dp}{dx} = - \frac{d}{dx} \left(\frac{u_\infty^2}{2} \right) \quad \left(= -u_\infty \frac{d}{dx} u_\infty \right) \quad (5.63)$$

Thus, (5.62) simplifies to

$$\frac{d}{dx} (u_\infty^2 \theta) + u_\infty \delta^* \frac{d}{dx} (u_\infty) = \frac{\tau}{\rho}, \quad (5.64)$$

which is known as the *von-Karman integral equation*.

Assuming that we can relate δ^*, θ and τ , we can use equation (5.64) to predict the boundary layer growth. With some mild assumptions, we can already do some analysis.

Consider a case without a pressure gradient, i.e., $dp(x)/dx = dU_\infty/dx = 0$. As θ is a measure of the boundary layer thickness, it should scale with the its

height. As τ_w is the gradient of the velocity at the wall, it will be smaller for thicker boundary layers. So we can assume that $\tau \propto \nu/\theta$. We can then write

$$\frac{d\theta}{dx} \propto \frac{\nu}{\theta} \rightarrow d(\theta^2) \propto \nu dx \rightarrow \theta \propto \sqrt{\nu x}, \quad (5.65)$$

which is consistent with the previous time scale analysis, which allowed us to sketch figure 5.6, and the assumption in (??). This does not show that the results are correct, but at least that conclusion, up to this point, are consistent with the assumptions made to obtain them.

Some discussions on the physical interpretation of (5.64)

Equation (5.64) is both very easy and very trick to interpret. The easy interpretation is as follows: The first term is the change in momentum due to the thickening of the boundary layer, and the second term is the change in momentum due to an external pressure gradient. The right-hand side is the force per unit area that the wall is exerting on the fluid.

The interpretation can, however, be more subtle. Consider a constant u_∞ . The total momentum flux is given by $\rho u_\infty^2 (H - \delta^* - \theta)$. Why then only θ appear in the first term of (5.64)? Thus, a naive approach would suggest that the momentum balance should read

$$\frac{d}{dx} (u_\infty^2 (H - \delta^* - \theta)) = -\frac{\tau}{\rho}, \quad (5.66)$$

i.e., the loss in momentum flux should be due to the wall friction only. Why does the term δ^* disappear then?

What happens is that, as the boundary layer thickness, mass flux in x is reduced, which means that flow is leaving the control volume via the top surface (due to $v \neq 0$). This mass flux carries momentum with it. Thus, wall friction has a double effect on the boundary layer momentum: it slows down the fluid and pushes mass and momentum away. The second term is proportional to δ^* . This extra “effective force”, cancels out the term δ^* in the momentum balance (remember, the amount of flow leaving the domain is proportional to δ^*), and thus only θ appears.

Let us now consider the case with a pressure gradient. A naive approach (already considering the above) would suggest that the momentum balance should read

$$\frac{d}{dx} \left(u_\infty^2 (H - \theta) + H \frac{p}{\rho} \right) = -\frac{\tau}{\rho}. \quad (5.67)$$

However, here again, we have a similar effect. As the pressure gradient increases u_∞ , it increases the mass flow rate in x , which implies that there is flow entering Ω from above. Again, the pressure gradient has a two-fold effect on the boundary layer: it increases the momentum flux by directly accelerating the fluid, and it also brings more momentum from the flow above into our domain.

These two effects combined, give rise to (5.64).

Flat plate boundary layer : the Blasius solution

While (5.64) provides a physically interpretable equations that provides many insights into the boundary layer evolution, it is still incomplete. For example, there is no way a priori way to link θ , δ^* and τ_w .

Blausius proposed a solution to the boundary layer equations that is valid for a flat plate. To derive the solution he made the assumption that the boundary layer is self-similar. But what it means to be self-similar? Here is simply means that the boundary layer at any position x will “look like (itself)” at any other position. That is, the velocity profile at $x = 1m$ has a similar *shape* than the profile for $x = 3m$.

Shape is an important word here. We know by now (look again at figure 5.6) that the boundary layer grows, so the velocity profile cannot be the same. What we mean by the profile being self-similar is that it can be written as

$$u(x, y) = U_\infty f'(\eta), \quad \eta = U_\infty f\left(\frac{y}{g(x)}\right), \quad (5.68)$$

where η is a self-similar variable, and g is an unknown function. The function f' is the velocitu profile shape. We are defining it here as f' , not simply as f , as this will simplify somewhat the solution latter.¹ What this equation means is that the velocity profile is essencially the same at every position x , but it is scaled with a function g . If we can writte (5.54)-(5.55) in terms of η , we can aim to solve an ordinary, instead of a partial, differential equation. Concretelly, making the variable transformation $(x, y) \rightarrow (x', \eta)$, we can write

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x} \frac{\partial}{\partial x'} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} = \frac{\partial}{\partial x'} - \eta \frac{g'}{g} \frac{\partial}{\partial \eta} \quad (5.69)$$

$$\frac{\partial}{\partial y} = \frac{\partial x'}{\partial y} \frac{\partial}{\partial x'} + \frac{\partial \eta}{\partial y} \frac{\partial}{\partial \eta} = \frac{1}{g} \frac{\partial}{\partial \eta} \quad (5.70)$$

To further simplitly the mathematics, we can solve for the streamline function Ψ , defined as

$$u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x}, \quad (5.71)$$

which allows us to solve of a single unknown (Ψ), instead of two (U, V).

Now we proceed with the math. We have that

$$u = \frac{\partial \Psi}{\partial y} = \frac{1}{g} \frac{\partial \Psi}{\partial \eta} = U_\infty f'(\eta), \quad (5.72)$$

integrating in *eta* we have

$$\Psi(x, \eta) - \Psi(x, 0) = U_\infty g(x') \int_0^\eta f'(\eta') d\eta' = U_\infty g(x') (f(\eta) - f(0)), \quad (5.73)$$

Defining $\Psi(x, 0) = 0$ (only the derivatives of Ψ matter, so we can define it up to a constant without changing the physics), we have

$$\Psi(x, \eta) = U_\infty g(x') f(\eta) \quad (5.74)$$

$$u = U_\infty f'(\eta), \quad (5.75)$$

$$v = -U_\infty (g'(x') f(\eta) - \eta g'(x') f'(\eta)). \quad (5.76)$$

To use these in (5.54), we need to compute the derivatives of u and v with respect to x . We have

$$\frac{\partial u}{\partial x} = U_\infty f''(\eta) \frac{\partial \eta}{\partial x} = U_\infty f''(\eta) \left(-\eta \frac{g'}{g} \right), \quad (5.77)$$

$$\frac{\partial u}{\partial y} = \frac{1}{g} \frac{\partial}{\partial \eta} (U_\infty f'(\eta)) = \frac{U_\infty}{g} f''(\eta) \quad (5.78)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{g} \frac{\partial}{\partial \eta} \left(\frac{U_\infty}{g} f''(\eta) \right) = \frac{U_\infty}{g^2} f'''(\eta) \quad (5.79)$$

Replacing these into (5.54) and re-arranging, we get

$$\frac{U_\infty}{\nu} g g' = - \frac{f'''}{f f''}. \quad (5.80)$$

Note that, from our previous definitions, $g = g(x')$ and $f = f(\eta)$, and thus the left-hand side is a function of x' , and the right-hand side is a function of η . This means that both sides must be equal to a constant, which we will call K . We can then write

$$\frac{U_\infty}{\nu} g g' = K \rightarrow g = \sqrt{\frac{K \nu}{U_\infty}} (x' - x_0), \quad (5.81)$$

$$-\frac{f'''}{f f''} = K \rightarrow f''' + K f f'' = 0. \quad (5.82)$$

Although this is not obvious, the value of K here does not matter, as it turns out it is just a scaling factor. The fast explanation is, if we vary it, any changes it induces in g are compensated by the changes in f (remember that g only appears in the argument of f). You can try this for yourself. It is a customary choice to use $K = 1/2$.

Finally, to obtain the boundary layer profile, we need to solve

$$2f''' - f f'' = 0. \quad (5.83)$$

This equation does not have a known analytical solution, but it is quite straightforward to solve numerically, solving for it, we can compute the boundary layer parameters,

$$\delta_{0.99} = \frac{4.92}{\sqrt{Re_x}} x, \quad (5.84)$$

$$\delta^* = \frac{1.72}{\sqrt{Re_x}} x, \quad (5.85)$$

$$\theta(x) = \frac{0.664}{\sqrt{Re_x}} x, \quad (5.86)$$

$$\tau_w(x) = 0.332 \frac{\rho U_\infty^2}{\sqrt{Re_x}} \rightarrow C_f = \frac{0.664}{\sqrt{Re_x}}. \quad (5.87)$$

You can check that these solutions are consistent with (5.64). Note also that, although we made a strong assumption (self-similarity), we have not ignored any term in the equation, nor have we found any residual when solving it. This

means that the Blasius solution is a solution for Prandl's equations (although not necessarily the only one).

It is noteworthy that the Blasius solution has seen a lot of use in the literature, and has been so well validated that if experimental data contradicts it, the first assumption is that the experiment is wrong³.

The effect of pressure gradient: the Polhausen approximation

Despite the success of the Blasius solution, it has one inconvenience: It has no closed-form solution. This is still true if we study a case with a pressure gradient, and thus if we want to compare the profile for different pressure gradients, we need to solve the equation numerically for each case.

Today this is clearly not a big problem: it takes a fraction of a second to solve (5.83). But in the 1930s, when the Blasius solution was first proposed, this was a big problem. The solution was to propose an approximation that would allow us to solve the equation analytically. This approximation is known as the Polhausen approximation. Today, it can still be used in the pre-design stages, but our biggest interest in it is that it provides some physical insights.

Polhausen assumed that the boundary layer can be approximated by a 4th order polynomial.

$$\frac{U}{U_e} = a + by + cy^2 + dy^3 + ey^4. \quad (5.88)$$

To find the coefficients, he used the boundary conditions at the wall and at the edge of the boundary layer, i.e.,

$$\frac{U}{U_e} = 0, \quad \frac{dU}{dy} = 0 \quad \text{at } y = 0, \quad (5.89)$$

$$\frac{U}{U_e} = 1, \quad \frac{dU}{dy} = 0 \quad \text{at } y = \delta^*. \quad (5.90)$$

This gives us a system of 4 equations for 5 unknowns, which means that the solution is not unique. To complete the system of equations, he used the momentum equation at $y = 0$. As at the wall $U = V = 0$, we also have $\frac{dU}{dx} = \frac{dU^2}{dx^2} = 0$, this condition reads,

$$\mu \frac{d^2U}{dy^2} = \frac{1}{\rho} \frac{dP}{dx} = U_e \frac{dU_e}{dx}. \quad (5.91)$$

The solution reads

$$\frac{U}{U_e} = (2y - 2y^3 + y^4) + \frac{1}{6} \frac{U_e}{\nu} \frac{dU_e}{dx} (y - 3y^2 + 3y^3 - y^4). \quad (5.92)$$

You can check for yourself that it satisfies all the conditions (5.89), (5.90), (5.91). Figure 5.10 illustrates the profile for different values of Λ .

Note that for $\Lambda < -6$ we have $dU/dy = 0$ at the wall. Thus for stronger adverse pressure gradients, we start to have negative velocities, i.e., the flow

³This may sound unscientific: in science, it is the experiments that should guide what is or isn't "true". But keep in mind that the Blasius profile has been validated *experimentally* many times before. So in a sense, an experiment contradicting it is contradicting several previous experiments. Occam's razor forces us to doubt the most recent one.

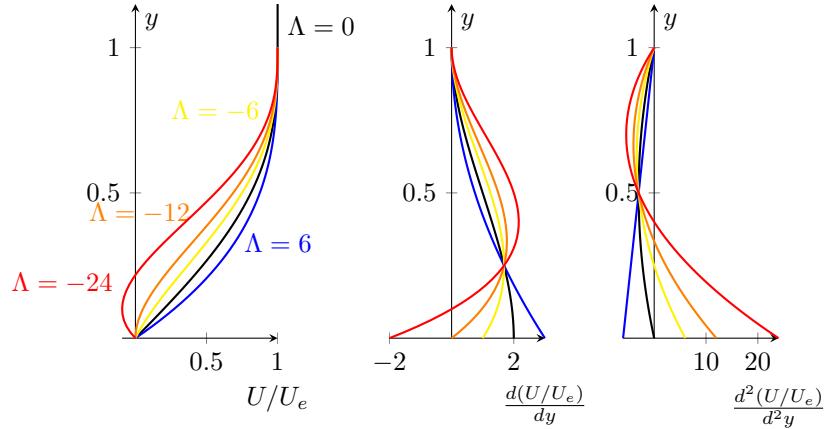


Figure 5.10: Polhausen profiles

will separate from the wall. Note also that for moderate negative values of Λ we have points where the second derivative is zero. This is called an inflection point and has important consequences for the stability of the boundary layer.

5.6 Flow stability

TBD

When I meet God, I am going to ask him two questions: Why relativity? And why turbulence? I really believe he will have an answer for the first.

Werner Heisenberg

6

Turbulence

So far we have dealt with laminar flows, i.e., where the flow is “well-behaved”. In most practical applications, however, the flow state is typically turbulent.

Turbulence is hard to define, and there are many proposed definitions. In general, turbulence is a flow state characterized by chaotic and unpredictable changes in pressure and velocity. This is not a very precise definition, but it captures the essence of the phenomena.

One feature observed in turbulent flows is that, even if the configuration has some symmetry, an instantaneous snapshot of the flow does not. This is called the *broken symmetry* of turbulence. So even in circumstances where we would expect the flow to be, for example, 2D, a snapshot of the flow will show a 3D behavior.

The difficulties in dealing with turbulent flows span different fields of science. From an engineering point of view, it is important to be able to model, or predict, some features of turbulent flows to be able to design and optimize different machines. From a physical point of view, turbulence forms complex structures, whose understanding may allow for novel technological and scientific developments. Finally, from a mathematical point of view, the very consistency of the Navier-Stokes equations is still unproven: it is one of the Millennium Prizes to show that the equations are solvable, or that the solution is unique!

Aside from the scientific aspects, from an engineering point of view, the only quantity of interest is mean quantities: we don’t need to predict the instantaneous drag on an aircraft to design it, and the mean value will suffice. For heat exchangers, the mean heat flux suffices. For power generation, the mean power output is what matters. Etc, etc...

6.1 Modelling turbulence

There are many ways to model turbulent flows. Take the flow on a turbulent pipe for example. One can measure quantities of interest in the flow empirically, e.g., the pressure drop, heat exchange, etc, and fit the data with some function. If the fitting is good enough, we have a powerful tool to design different machines. Let's call this the *empirical* approach.

Another strategy is to try to understand the physical mechanisms behind the turbulent state and infer from them some flow properties. These mechanisms can be obtained via first-principle analysis, or based on existing experimental observations. The advantage is that they provide insights into the dynamics of the flow, and thus can, potentially, suggest strategies to manipulate it. Let's call this a *physical based* approach.

Another strategy is the *stochastic* approach. This approach was responsible for the biggest advances in turbulent modeling of the past decades.

6.1.1 Turbulence as a stochastic process

When looked at from afar, a turbulent flow looks chaotic and unpredictable. If we probe the flow, for example using a hot wire to measure the velocity at some point, we end up with a time series that looks much like a random signal. One of the major advancements in turbulence modeling has been the approach we consider that a turbulent state is indeed random, or more precisely, stochastic.

To model a turbulent flow, we thus assume that the flow properties, on a given point, can be decomposed into a mean and a stochastic perturbation, i.e.,

$$\vec{U} = \langle \vec{U} \rangle + \vec{u}', \quad (6.1)$$

where $\langle \vec{U} \rangle$ is constant in time and \vec{u}' has, by definition, zero mean. This is known as the Reynolds decomposition.

It is important here to define what we mean by “mean”. Do we consider the average over time? Do we consider doing an experiment/simulation many times and averaging the results, also known as the ensemble average? All those are valid options, so which one should we choose?

Here an important assumption on turbulent flows is made: they are *ergodic*. Ergodicity is a concept from statistical mechanics. The definition of ergodicity is quite complex. But if the conditions for it are satisfied, then one can formally show that the time average of a quantity is equal to the ensemble average. Assuming ergodicity thus means that if we measure the velocity at a point in a turbulent flow for a long time, we will get the same result as if we measured the velocity at the same point in many different turbulent flows.

An important question is: are turbulent flow ergodic? The best answer is: we don't know. There are only a few examples for which we can formally prove ergodicity. One such case is an ideal gas in equilibrium: the model where the gas is formed by non-interacting particles that are moving around. If you add interaction between the particles, we already don't know if the system is ergodic or not. We think it is, but we cannot prove it.

The same is valid for turbulent flow: we think they are ergodic, but we cannot be sure. This can be alarming, but keep in mind that we have not been able to prove that the flow is not ergodic either. It would suffice to find a turbulent

flow where the temporal mean deviates from the ensemble mean. There are no documented cases so far, so even if turbulent flows are not ergodic, they are probably “almost ergodic”.

All of this discussion is to say that we can use almost any average in (6.1), and the final result will be equivalent.

As mentioned before, in engineering applications, often only the mean part of the flow needs to be modeled or predicted. What (6.1) allows is for one to (try) to derive an equation for predicting mean quantities.

6.1.2 Reynolds averaged models

One strategy to (try to) predict the mean quantities of a flow, is to apply the averaging operator directly on the Navier-Stokes equations. This approach gives rise to the *Reynolds-Averaged Navier-Stokes* (RANS) equations.

Let us start with the mass conservation equation. First, we use (6.1) into (5.2), and then we average the result. This reads

$$\left\langle \vec{\nabla} (\langle \vec{u} \rangle + \vec{u}') \right\rangle \vec{\nabla} \cdot \langle \vec{u} \rangle + \vec{\nabla} \cdot \langle \vec{u}' \rangle^0 \quad (6.2)$$

thus,

$$\vec{\nabla} \cdot \langle \vec{u} \rangle = 0. \quad (6.3)$$

So the mass conservation equation for the average velocity is the same as the original equation. If we manage to do the same for the momentum equations, we can maybe solve for the mean values without caring about the fluctuating ones.

Let us see how this goes for the momentum equation.

$$\left\langle \frac{\partial \rho (\langle \vec{u} \rangle + \vec{u}')}{\partial t} + \vec{\nabla} \cdot (\rho (\langle \vec{u} \rangle + \vec{u}') (\langle \vec{u} \rangle + \vec{u}')) \right\rangle = \left\langle -\vec{\nabla} p + \nu \vec{\nabla}^2 \vec{u} + \vec{f} \right\rangle, \quad (6.4)$$

Looking at each term:

$$\left\langle \frac{\partial \rho (\langle \vec{u} \rangle + \vec{u}')}{\partial t} \right\rangle = \rho \overline{\frac{\partial \langle \vec{u} \rangle}{\partial t}}^0 + \rho \overline{\frac{\partial \langle \vec{u}' \rangle}{\partial t}}^0 = 0 \quad (6.5)$$

For the second term we have,

$$\begin{aligned} \left\langle \vec{\nabla} \cdot ((\langle \vec{u} \rangle + \vec{u}') (\langle \vec{u} \rangle + \vec{u}')) \right\rangle &= \vec{\nabla} \cdot (\rho \langle \vec{u} \rangle \langle \vec{u} \rangle) + \vec{\nabla} \cdot \left(\rho \langle \vec{u} \rangle \langle \vec{u}' \rangle \right)^0 + \\ &\quad \vec{\nabla} \cdot \left(\rho \langle \vec{u}' \rangle \langle \vec{u} \rangle \right)^0 + \vec{\nabla} \cdot (\rho \langle \vec{u}' \vec{u}' \rangle) \\ &= \vec{\nabla} \cdot (\rho \langle \vec{u} \rangle \langle \vec{u} \rangle) + \vec{\nabla} \cdot (\rho \langle \vec{u}' \vec{u}' \rangle) \end{aligned} \quad (6.6)$$

For all the remaining terms the averaging is straightforward. We are then left with

$$\vec{\nabla} \cdot (\rho \langle \vec{u} \rangle \langle \vec{u} \rangle) = -\vec{\nabla} \langle p \rangle + \mu \vec{\nabla}^2 \langle \vec{u} \rangle + \left\langle \vec{f} \right\rangle - \vec{\nabla} \cdot (\rho \langle \vec{u}' \vec{u}' \rangle). \quad (6.7)$$

So we get *almost* the same equation as before for a stationary flow. But now we have the term on the right, which indicates that we cannot completely ignore the role of fluctuations, even when we only care about the mean values.

The term $\langle \vec{u}'\vec{u}' \rangle$ is known as the *Reynolds stress tensor*, as it appears in the equation in a form analogous to the fluid stress tensor. Physically the term corresponds to a net momentum flux due to the fluctuating part of the velocity field. To justify calling this term a stress tensor, note that the right-hand side can be written as

$$\vec{\nabla} \cdot \left(-\langle p \rangle \overset{\leftrightarrow}{I} + \mu \vec{\nabla} \langle \vec{u} \rangle - \langle \vec{u}'\vec{u}' \rangle \right) = \vec{\nabla} \cdot \underbrace{\left(-\langle p \rangle \overset{\leftrightarrow}{I} + \overset{\leftrightarrow}{\tau}_{\text{viscous}} + \overset{\leftrightarrow}{\sigma}_{\text{Reynolds}} \right)}_{\overset{\leftrightarrow}{\sigma}_{\text{total}}}, \quad (6.8)$$

i.e., it integrates seamlessly with the fluid stress tensor.

The Reynolds stress has a few properties. First, it is symmetric, i.e., the term $\langle uv \rangle$ is the same as $\langle vu \rangle$. Its trace is related to the total perturbation energy, k ,

$$\text{Tr}(\overset{\leftrightarrow}{\sigma}) = \langle u^2 \rangle + \langle v^2 \rangle + \langle k^2 \rangle = 2k. \quad (6.9)$$

We also often decompose it in the normal terms, i.e., $\langle u^2 \rangle$, $\langle v^2 \rangle$, $\langle k^2 \rangle$, which gives rise to forces normal to a given surface, and shear terms, $\langle uv \rangle$, $\langle uw \rangle$, $\langle vw \rangle$, which creates forces tangential to a given surface.

6.1.3 The closure problem

Equation (6.1) is, a priori, exact. Knowing the Reynolds stress tensor, we can solve for the mean values exactly. The problem is that we don't know the Reynolds stress tensor. This creates what is called a closure problem: we have a set of equations that we cannot solve because there are more variables than equations. This is similar to the scenario we faced at the end of section 3.1.2. Then, we were able to close our equations by modelling the fluid stress tensor. We need to do something similar here.

However, the situation is more complicated. We were able to close the equations before by studying how a fluid particle is deformed, and how it reacts to this deformation. We were able to use a lot of simplifying assumptions (isotropy, Newtonian behavior, Stokes's hypothesis) to obtain a model for the fluid stress tensor. On top of that, the fluid is typically the same everywhere: water is water all over the sea.

Here, we are dealing with a much more complex object: it is a function of the turbulent state, for which we know very little. On top of the Reynolds stress tensor, which depends on the local behavior of the flow, it can, and does, change its properties at each point of the flow. Modeling it is thus much more challenging.

In the impossibility of modeling the Reynolds stress tensor exactly, we must resort to finding some sort of approximation. To do so, we must first try to understand some properties of the turbulent state.

6.2 Turbulent dynamics

When looking at a turbulent flow, what we observe is that there is a hierarchy of structures: there are large structures, on the length-scale of the flow itself. We typically call these scales the *integral length-scales*. They are “surrounded” by smaller structures, which in turn are surrounded by even smaller structures, and so on. This is known as the *energy cascade*.

To understand why we talk about the energy cascade, we can investigate what is the role of the non-linear term. In the incompressible Navier-Stokes equations, the non-linearity is quadratic, i.e., it is a term of the form $\vec{u} \cdot \vec{\nabla} \vec{u}$. Let us consider a simpler toy problem,

$$\frac{dq}{dt}(x, t) = q^2(x, t), \quad (6.10)$$

which just keeps the aspect of having a quadratic non-linearity.

Now consider that we have an integral length-scale L . Let us just assume that $q(x, 0) = \cos(2\pi x/L)$. At $t = 0$, (6.10) reads

$$\frac{dq}{dt}(x, 0) = \cos^2(2\pi x/L) = \frac{1}{2} + \frac{1}{2} \cos(4\pi x/L). \quad (6.11)$$

We see that the non-linearity at a length-scale L is able to generate a signal at a length-scale $L/2$ and of a constant term. We can show that this type of non-linearity is conservative, i.e., it does not dissipate energy, only moves it around between different terms.

We thus have a picture of a length-scale L feeding energy into a length-scale $L/2$. Further analysis would lead to energy further being transferred to even smaller length scales. This is the essence of the energy cascade: the non-linearity transfers, on average, energy from large scales to small scales.

As a very first model, we can imagine that, in a turbulent state, there is a never-ending transfer of energy to smaller and smaller scales. This is an interesting picture. Assuming that when the scales are much smaller than the integral length scales they are very ineffective in affecting these large scales, we can picture the energy cascade as a mechanism that transfers energy to non-important scales, and thus effectively behaves as a dissipation mechanism, at least in what concerns the macroscopic dynamics of the flow. This is not correct, but not too far from the truth.

The reason it is false is that, when the scales become sufficiently small, the viscous terms become important. Remember that the dissipative timescales are of the order of $T_d \approx L^2/\nu$. So, at a given scale, viscous dissipation kills the energy cascade, and the energy cascade gets dissipated into heat.

Let us compare thus the net effect of the energy cascade by comparing the dynamics of a turbulent flow with a fictitious laminar flow at the same Reynolds number. For large Re , viscosity is small. For a laminar flow, this means that there is very little dissipation: the flow gradients are not strong enough to dissipate the energy, given that viscosity is very low. For a turbulent flow, the energy cascade transfers energy to smaller scales, where the gradients are large. Effectively, the energy cascade acts as an intermediary, a Robinwood-like, mechanism. It takes energy from the large scales of the flow and gives it to the smaller scales for it to be dissipated. This picture, in essence, shows that turbulent flows are much more effective in dissipating energy than laminar

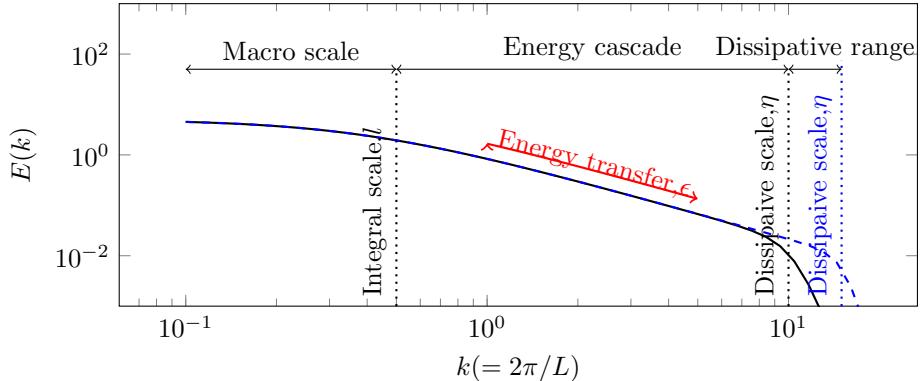


Figure 6.1: Illustration of two energy cascades. The first has a larger dissipation scale, and thus a shorter energy cascade. The second has a smaller dissipation scale, and thus the energy cascade reaches higher wavenumbers (shorter length-scales). Note that both cascades have the same energy transfer rate.

flows. We thus expect, e.g., drag, or pressure loss, to be proportionally larger in turbulent flows than in laminar flows.

This picture also raises some questions. What is the rate of energy transfer, i.e., how much energy is being dissipated by the turbulent flow? What is the total amount of energy stored in the energy cascade? And at what scale is energy being dissipated?

Some of these quantities are linked: the smaller the dissipation scale, the longer the energy cascade. Thus, the total amount of energy stored increases. The relation to others is less clear. We can imagine two energy cascades with the same dissipation, the first with a much smaller dissipation rate than the other, so the link between the dissipative scale and the energy dissipation rate is less direct.

Let us also consider some macroscopic implications of the turbulent dynamics. Imagine that we have a hot spot in the flow. On a laminar flow, the heat of this spot will diffuse due to thermal diffusion, and thus will slowly spread out. In a turbulent state, all the vortical structures present in the turbulent flow will move the heat around, almost randomly. So there is a mixing effect in the flow that, in some ways, mimics a stronger diffusion rate. This picture can make us predict that the heat transfer coefficient is much larger in turbulent flows than in laminar flows, which is indeed the case! But temperature is not the only quantity that can be diffused. As we discussed previously, momentum, or velocity, are also diffused in the flow. The momentum diffusion coefficient is the molecular viscosity. In line with the previous analogy, and the realisation that turbulence increases the energy dissipation of a flow, we come to realise that, maybe, we can model the effects of turbulence on the mean flow by means of an extra viscosity term. We call this an *eddy viscosity* term. We will discuss it shortly, as the concept is the basis of most turbulence modelling strategies. But first we need to discuss the different scales in a turbulent flow.

6.2.1 Turbulent flow scenes : Kolmogorov theory of turbulence

There are many scales in a turbulent flow. The largest scales is the scale of the problem: if we are talking about an aircraft, or a wing, its this size. Or the size of the windtunnele where we are testing them. The next scale is the integral scale. A turbulent flow behaves stochastically, but two point sufficiently close to each other will have similar dynamics, i.e., the flow dynamics at these points is correlated. The integral scale, typically indicated as l , is a measure of the distance on which the flow dynamics are correlated. The smallest scales are the dissipation scales, i.e., for scales smaller than this, molecular dissipation becomes dominant, and any non-stationary dynamics is quickly damped. In between, we have the inertial scales, which are the scales where the energy cascade takes place. They are called inertial because, at these scales, the inertial terms (acceleration and convection) are dominant over the viscous terms.

The identification, and quantification, of these scales is a major challenge in turbulence. The first and, to my knowledge, the only closed result in this field, is due to Kolmogorov, a russian mathematician. His theory makes a few assumptions. First, that the turbulent dynamics at a sufficiently small scale forget about the large scale structures. I.e., they are universal. If we observe a small enough flow region, there is no way to know if the turbulence was created by a jet, a wake, or a boundary layer. He also assumed that the turbulence is isotropic: it has no prefered directions.

What these assumptions allows is for us to perform a dimensional analysis. We first note that the energy transfer between scales in the energy cascade is constant: each length-scale receives the same amount of energy it sends to smaller scales. This a condition for the process to be stationary. Note also that this energy, at some point, reaches the dissipation scales, where it is dissipated by viscous terms and becomes heat. For this reason, ϵ is sometimes referred to as the energy transfer rate, and as the dissipation rate. Both need to be the same.

Note also that a given scale along the energy cascade, only knows about itself (its own size), and the amount of energy passing through it. Looking at figure 6.1, if we zoom in at any point in the cascade, they all look the same. So a given scale does not know if it is close to the integral scales or close to the dissipation scales.

If the only information it has is size, $[d] = m$, and the energy transfer rate $[\epsilon] = m^2 s^{-3}$. We can then use dimensional analysis to estimate most of the trends of the flow. The first one is the characteristic velocity, u at a length-scale L . We can interpret this as the tangential velocity of the vortexes with size L . Since $[u] = ms^{-1}$, the only way to combine ϵ and L to obtain a velocity is

$$u \propto \epsilon^{1/3} L^{1/3}. \quad (6.12)$$

Experiments show that the constant of proportionality is close to one ($\approx .95$), although it has been observed to change in some scenarios.

We can also measure the characteristic time scale, τ at a length-scale L . From u and L , we have

$$\tau \propto L/u \propto \epsilon^{1/3} L^{4/3}. \quad (6.13)$$

Note that this analysis is valid for all scales within the energy cascade, in particular, at $l = L$, i.e., the integral length scale. As this is the macroscopic limit of the energy cascade, is something we can observe, and in some cases, estimate. With measurements of u and L at the largest scales, allows us to estimate ϵ as

$$\epsilon \propto \frac{u^3}{l}, \quad (6.14)$$

and thus properties of all the smaller scales.

We can also estimate the smallest scale, η . Again we know that the amount of energy arriving at it is ϵ , and we know that it is dissipated into heat via the viscous term, proportional to ν . Since $[\epsilon] = m^2 s^{-3}$, and $[\nu] = m^2 s^{-1}$, dimensional analysis suggests that

$$\eta \propto \epsilon^{-1/4} \nu^{3/4}. \quad (6.15)$$

Finally, we can estimate how is energy distributed on each energy scale. The energy density function has units of $[E(k)dk] = m^2 s^{-2} \rightarrow [E(k)] = m^3 s^{-2}$. Since the energy can only depend on L and ϵ ,

$$E(k) \propto \epsilon^{2/3} L^{5/3}, \text{ or } \epsilon^{2/3} k^{-5/3}. \quad (6.16)$$

These results have been validated on many experiments and simulations, and are a testament on the power of a simple dimensional analysis, when wisely applied!

6.2.2 Reynolds dependence of the scales

There are many reasons why it is of interest to determine η . For example, when doing simulations, the mesh needs to be able to properly resolve this scale to accurately capture the dynamics of the system, i.e., $\delta x \approx \eta$. If the size of our domain is of the order l , the number of gridpoints in the domain is, roughly, $(L/\eta)^2$ for a 2D system and $(L/\eta)^3$ for a 3D system.

From (6.15) and (A.4), we can write

$$\frac{l}{\eta} = \epsilon^{1/4} \nu^{-3/4} l^1 = \left(\frac{ul}{\nu} \right)^{3/4} = Re_l^{3/4}. \quad (6.17)$$

Likewise, we can compare the velocities on the scales l and η ,

$$\frac{u_l}{u_\eta} = \frac{\epsilon^{1/3} l^{1/3}}{\epsilon^{1/3} \eta^{1/3}} = \left(\frac{l}{\eta} \right)^{1/3} = Re_l^{1/4}. \quad (6.18)$$

From a computational point of view, (6.17) indicates the cost, per time step, of a fully resolved simulation a flow in 2D increases with $Re_l^{6/4}$, and in 3D, with $Re_l^{9/4}$. However, due to numerical stability reasons, the time step length is proportional to $U/\delta x$, so, to simulate the same length of time, the cost scaling are $Re_l^{9/4} \approx Re_l^2$ and $Re_l^{12/4} = Re_l^3$. In numbers, going from $Re = 1000$ to $Re = 10000$ increases the cost of a 2D simulation by a factor of 100, and a 3D simulation by a factor of 1000!

6.3 Turbulence models

The crux of (most of) turbulence models is the idea of turbulence having a mixing effect on the flow, emulating a diffusive behavior.

Being l the distance at which the turbulent dynamics is coherent, and τ the associated time scale, we can use (5.19) to estimate the effective diffusion coefficient of the turbulent flow as

$$\nu_t \approx u_t l \approx \sqrt{k_c}/\epsilon, \quad (6.19)$$

where $k_c \approx u_t^2$ is a measure of the mean turbulent energy, and u_t a characteristic turbulent velocity (its r.m.s.). Any of the above expressions are valid and used in different contexts, to estimate ν_t .

Assuming that we know u_l and l (or ϵ , k_c) at every point of the flow, the RANS equations will read as

$$\langle \vec{u} \rangle \cdot \vec{\nabla} \langle \vec{u} \rangle = \vec{\nabla} \langle p \rangle + (\nu + \nu_t) \vec{\nabla}^2 \langle \vec{u} \rangle, \quad (6.20)$$

which we can then solve.

Of course we don't know, a priori u and l . Turbulence models thus try to estimate these quantities. Some classes of models are

- Algebraic models

These models try to link the turbulent parameters to *local* properties of the mean flow, typically its gradients.

- One-equation models

These models try to model some of the dynamics of turbulent properties. It assumes, for instance, that the turbulent kinetic energy is generated locally by the mean flow gradients. But this energy is then convected by the mean flow, and also dissipated by the viscous terms. This gives us one more equation to solve, but it models more of the dynamics of the flow. Typically, the other variable required to estimate ν_t , is constructed as in an algebraic model.

- Two-equation models

Finally, two-equation models use two equations to estimate the turbulent properties. The most famous of these models is the $k - \epsilon$ model. It assumes that the turbulent kinetic energy and the dissipation rate are the two most important quantities to estimate the turbulent viscosity and that their dynamics can be described by a combination of convection, production, and dissipation terms.

- Many other approaches

Many other approaches are also possible. Data-driven models, neural-network-based models, equations for the Reynolds stress tensors themselves, and many other approaches, have all been proposed.

6.4 Near wall turbulence

While Kolmogorov developed a theory of isotropic turbulence, often we are interested in the turbulent dynamics of the flow near a wall, and the very presence of the wall imposes that the flow cannot be isentropic. The flow is constrained by the wall, and the flow dynamics are very different from the bulk of the flow. But, as drag is often dominated by the flow near the wall, it is important to understand, and model, the dynamics of the flow in this region.

The difference between the “free” turbulence studied by Kolmogorov, and wall turbulence is that, at a distance d from the wall, we cannot expect vortices larger than d to be important. For example, we cannot be in the center of a vortex of size $2d$, as this vortex would have to penetrate the wall. The smallest vortex is roughly predicted by the viscous dissipation scale. As we move away from the wall, there is more and more vortexes that are important. We can then expect the turbulent intensity to increase with the distance from the wall.

Let us then try to understand the flow scales that can be present in the turbulent flow near a wall. At the wall, due to the non-slip conditions, velocity is zero, and thus are also the velocity perturbations, and therefore the Reynolds stress tensor. This hints at the fact that, at least close to the wall, molecular viscosity is the dominant mechanism of momentum transfer.

We don’t know much about what is going away from the wall, so let’s try to understand what is going on in this region using only local information. We know the molecular viscosity, fluid density, and the mean wall shear (we are at the wall after all). Dimensional analysis shows that

$$[\tau_w] = \frac{N}{m^2} = \frac{k\text{gm}/s^2}{m^2} = \frac{kg}{ms^2}, \quad [\nu] = ms, \quad [\rho] = \frac{kg}{m^3}. \quad (6.21)$$

We can manipulate these quantities to get something with velocity and length units:

$$u_\tau = \sqrt{\frac{\tau_w}{\rho}}, \quad (6.22)$$

$$\delta_v = \frac{\nu}{u_\tau}. \quad (6.23)$$

The first quantity is the *friction velocity*, and the second is the *viscous length scale*. Beyond names, let’s try to see if these quantities tell us something about the flow.

We argued that close to the wall, the flow is dominated by viscous effects. Since $\tau_u = \nu \frac{du}{dy}$, we can estimate that

$$u \approx \frac{\tau_w}{\nu} y = u_\tau \frac{y}{\delta_v}, \quad (6.24)$$

that is the velocity at a distance δ_v is u_τ . At this distance, we should expect viscosity to damp any vortexes, as they necessarily have sizes smaller than δ_v . Let us call this region the *viscous sub-layer*. As we move away from the wall, we the largest vortexes start to not be too damped and thus should start playing a role. However, there is a limited range of vortex sizes that are “allowed”, and viscosity is still important. We thus expect a transient behavior between the

viscous dynamics close to the wall, and the fully turbulent flow away from the wall. We call this a *buffer layer*.

Sufficiently far from the wall, there is a large range of vortexes present, and we can then expect the flow dynamics to be dominated by the turbulent dynamics, i.e., the viscous effects are no longer dominant, or even important. As we are very close to the wall, we can neglect any pressure variations. We thus know that the total shear stress is constant. Since the viscous stresses are negligible here, shear is dominated by the turbulent stress tensor. Thus $\langle uv \rangle = \tau_w^2$ (we will justify that for specific cases soon).

One more word about the friction velocity. We can compute the wall friction using (6.24), from which $\nu \frac{d\langle u \rangle}{dy} = \frac{u_\tau}{\delta_v}$. From the definition of δ_v , (6.23),

$$\tau_w = \rho u_\tau^2, \quad (6.25)$$

i.e., the friction velocity is a direct function of the wall shear. We can thus use one to compute the other.

6.4.1 Wall units

The definition of a velocity and a length-scale related to turbulent wall dynamics suggests the use of non-dimensional variables when studying the phenomena. These are typically referred to as *wall units* and identified with a $+$ superscript, i.e.,

$$y^+ = \frac{y}{\delta_v}, \quad u^+ = \frac{u}{u_\tau}, \quad t^+ = t \frac{u_\tau}{\delta_v}. \quad (6.26)$$

We will argue soon that whatever happens in the nearwall region is universal, i.e., is independent if the flow is a channel flow, a boundary layer, inside a turbine, etc. So using wall units allows an easy comparison of different flows.

To illustrate how this can simplify the notation, note that (6.24) can be written as

$$u^+ = y^+. \quad (6.27)$$

6.4.2 Flow in channels

As demonstrated in a tutorial, we mean stress equations once the simplifications of a mean parallel flow on a channel become

$$\frac{d\langle p \rangle}{dx} = \frac{d}{dy} \left(\nu \frac{d\langle u \rangle}{dy} - \langle u'v' \rangle \right) = \frac{d}{dy} (\tau_{\text{vis}} + \tau_{\text{turb}}) = \frac{d}{dy} (\tau_{\text{tot}}), \quad (6.28)$$

$$\frac{d\langle p \rangle}{dy} = \frac{d\langle v'v' \rangle}{dy}. \quad (6.29)$$

At the wall, $\langle v'v' \rangle = \langle u'v' \rangle = 0$, so close to the wall, the pressure is constant in y . From the previous section we discussed that there is a layer of size δ_c where this is the case. So integrating the first equation in y gives

$$\int_0^{\delta_v} \frac{d\langle p \rangle}{dx} dy = \int_0^{\delta_v} \frac{d}{dy} \left(\nu \frac{d\langle u \rangle}{dy} \right) dy, \quad (6.30)$$

$$\delta_v \frac{d\langle p \rangle}{dx} = \nu \frac{d\langle u \rangle}{dy} = \tau_w \rightarrow \frac{d\langle p \rangle}{dx} = \frac{\tau_w}{\delta_v}, \quad (6.31)$$

where we also used the fact that the velocity profile is linear inside δ_v .

We also have that $\frac{d\tau_{\text{tot}}}{dy}$ is a constant, i.e., τ_{tot} varies linearly with y . Working out the boundary conditions, we have, for a channel flow, that

$$\tau_{\text{tot}} = \tau_w, \quad (6.32)$$

while for a Poiseuille flow,

$$\tau_{\text{tot}} = \tau_w \left(1 - \frac{y}{h} \right) = \tau_w \left(1 - y^+ \right), \quad (6.33)$$

where h is half of the channel height.

In any of these cases, close to the wall ($y \ll h$, for the latter), the total stress is approximately constant, and equal to τ_w . As outside the viscous sublayer we have that τ_{vis} dominates the shear stress, and that $\tau_{\text{turb}} = \rho \langle u'v' \rangle$, we have

$$\rho \langle u'v' \rangle \approx \tau_w = \rho \rho u_\tau^2 \rightarrow \langle u'v' \rangle \approx u_\tau^2, \quad (6.34)$$

i.e., the turbulent velocity close to the wall is of the order of u_τ .

So, far enough from the viscous sublayer, we have that the flow dynamics should depende on δ_v , we are on a scale much larger than δ_v , and thus whatever effects δ_v has on it are small. Also, the flow should not depend on the large scale h , i.e., the channel height: as we are on a scalle much smaller than the channel height. Note that this argument depends on a large scale separation between h and δ_v , such that we can have $\delta_v \ll y \ll h$. For example,tThis is never satisfied if $\delta_v \approx h/10$. We are thus modelling what happens at large Reynolds numbers.

If the conditions above are satisfied, the local flow properties thus should only depend on u_τ , which gives the order of magnitude of the velocity fluctuation, and of y , i.e., our distance from the wall. If we aim to model how the mean velocity change in this region, i.e., $\frac{d\langle u \rangle}{dy}$, using as variables only u_τ and y , the only option is

$$\frac{d\langle u \rangle}{dy} = \kappa' \frac{u_\tau}{y}, \quad (6.35)$$

wher κ' is a constant known as the von-Karman constant.

Solving (6.35) we get

$$\langle u \rangle = \kappa' \ln y + c \rightarrow \langle u^+ \rangle = \frac{1}{\kappa'} \ln y^+ + B \quad (6.36)$$

We thus call this region the *log layer*. Measured empiricallt, we determine (with some variance), that $\kappa \approx 0.41$ and $B \approx 5.2$. The parameter κ is known as the *von – Karman* constant. Fiding out the precise value, or showing that there is a single value, is still a field of research. Nevertheless, the presence of a log region in near wall turbulence is unquestionable, having been seen in multiple experiments and simulations.

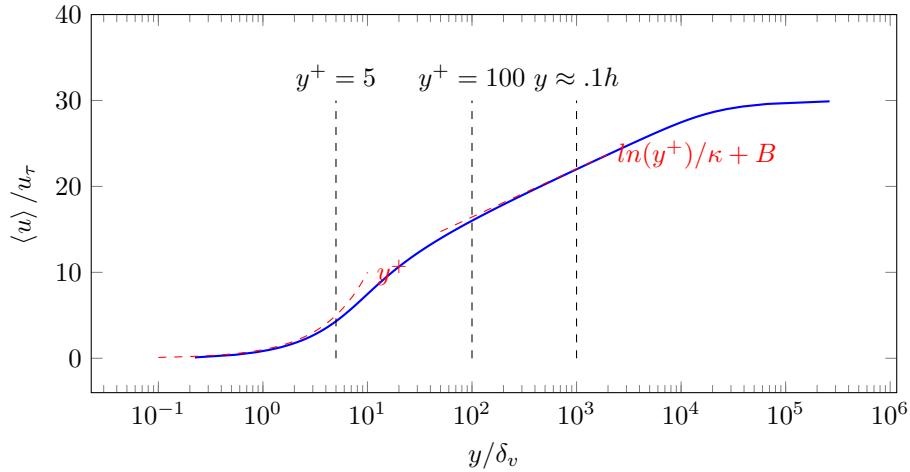


Figure 6.2: Illustration of the near wall velocity profile.

Clearly there is a transitional region between the viscous-sub layer and the log layer. We call this a buffer layer.

Finally, when y is of the order h , the scale-separation argument used to model the log layer is no longer valid, and the dynamics at the length-scale of the channel start to play an important role. The function that model the velocity profile at this region is known as the *wake function*.

Note that in all the discussion above, except the wake function, is based on the hypothesis that the flow depends only on τ_w and ν . These results are thus universal: the near wall turbulent regime from a channel is indistinguishable than that of, for example, a boundary layer.

Note that the velocity gradient at the log layer is much weaker than in the viscous sublayer. Mathematically, we can note that by writing

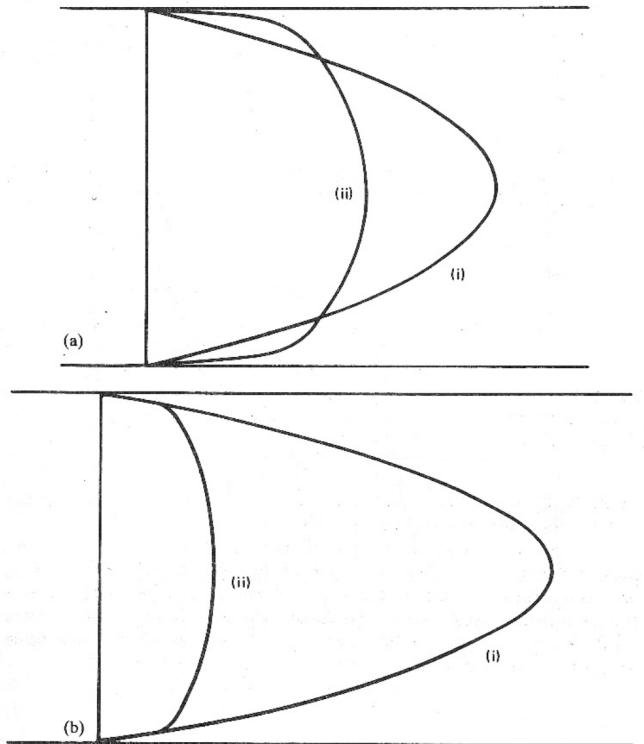
$$\frac{\left. \frac{du^+}{dy^+} \right|_{\text{vis. sub-layer}}}{\left. \frac{du^+}{dy^+} \right|_{\text{log -layer}}} = ky^+. \quad (6.37)$$

As in the log layer, typically, $y^+ > 100$, the ratio is much larger than one.

This can be interpreted physically recalling that the shear is constant near the wall. In the viscous sublayer, we only have molecular viscosity, which is weak. We thus need a sharp gradient to build up the shear. In the log layer, turbulent viscosity is present, requiring a smaller gradient to reach the same shear value. On top of that, as the flow becomes more turbulent far from the wall (larger vortices can be present), the turbulent mixing increases, and thus the gradient further decreases.

One important consequence of flow turbulence is its effect on the mean flow. We can have a channel flow which is driven by a pressure difference, i.e., we are imposing the pressure gradient, and thus the flow will accelerate up to the point where the wall shear exactly compensates for this gradient. We can also have a flow where the mean flow rate is imposed, e.g., imagine a piston, or a seringe.

Figure 6.3 illustrates how each of these cases affected by the state of the flow, i.e., turbulent or laminar. For the same pressure gradient, the velocity gradient



Comparison of (i) laminar and (ii) turbulent velocity profiles in a pipe for (a) the same mean velocity, and (b) the same pressure gradient. (The diagrams correspond to a turbulent flow Reynolds number of about 4000; for higher Re the contrast is more marked,)

Tritton (1988)

Figure 6.3: Comparison of laminar (i) and turbulent (ii) flow for a given mass flow rate (a) or pressure gradient (b).

at the wall is the same for both flows, but the laminar flow reaches higher speeds at the center of the channel, as there is not the effect of turbulent viscosity. For the same mass flow rate, we see that the turbulent profile is flatter, a result from all the mixing inside the channel. But, as a consequence, we have a stronger gradient close to the wall, meaning more friction, and thus more pressure loss.

6.4.3 Turbulent boundary layer

As mentioned, the near wall dynamics of a turbulent boundary layer are identical to that of the channel flow. But it is interesting to look at some of the details.

Perhaps the most important one is that, far from the wall. The flow is not turbulent. A classical scenario for boundary layer formation is when a uniform free-stream velocity meets a flat plate. Even if around the plate the flow is turbulent, far from it the flow does not know about this, so it is laminar. There thus a transition between the fully turbulent flow near the wall and the quiet external flow.

› Different Reynolds numbers

Wake function :
difference with the Log law is
very small in a channel flow

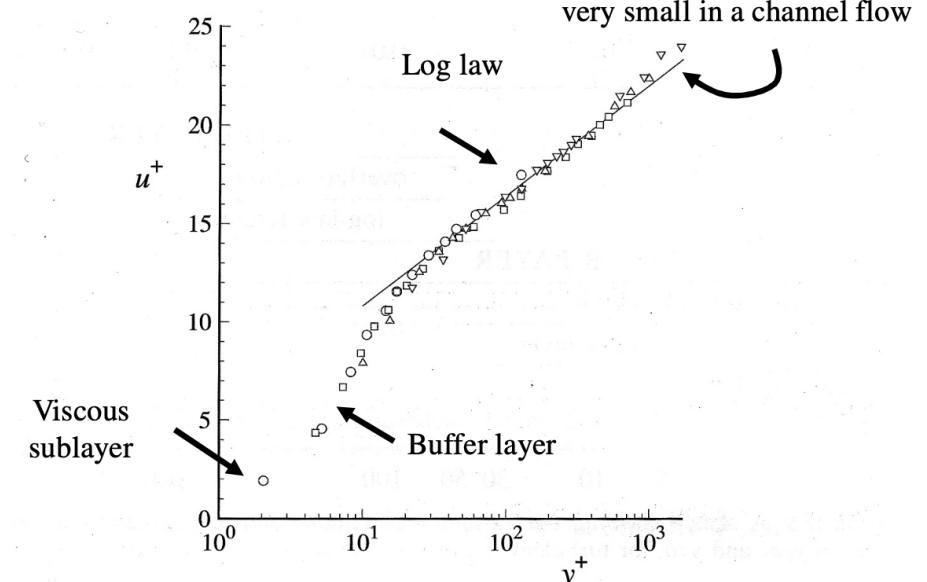
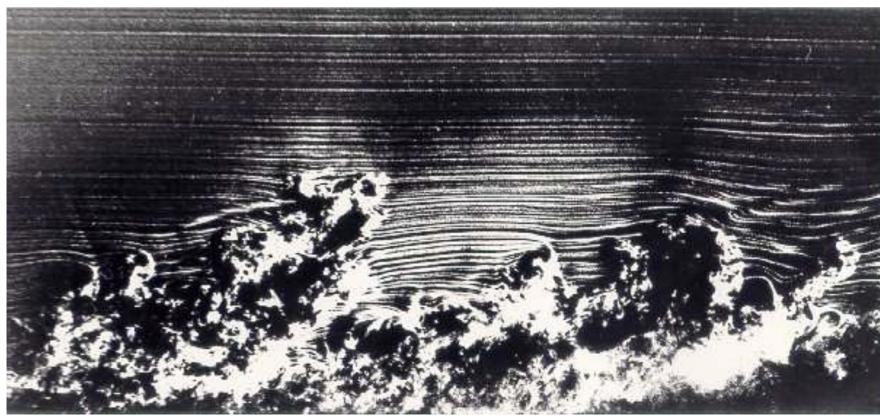


Figure 6.4: Velocity profile of a turbulent channel flow.



Origine: H. Naguib (IIT)

Figure 6.5: Illustration of a turbulent boundary layer.

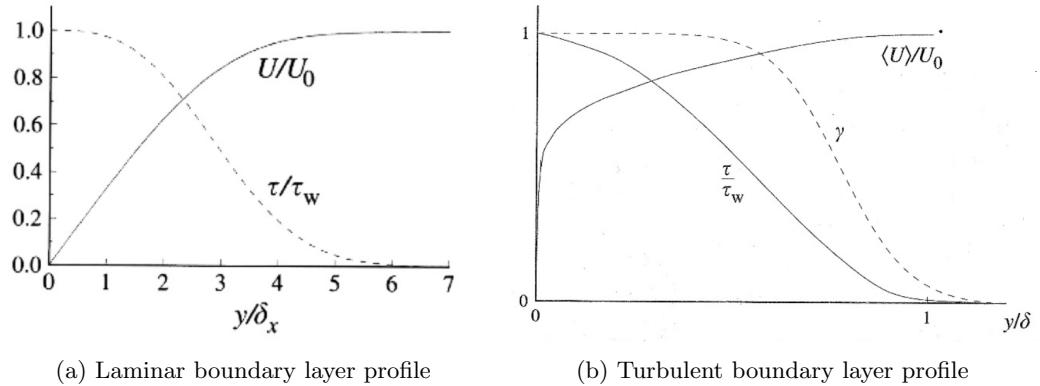


Figure 6.6: Laminar and turbulent boundary layer profiles. The intermittency factor, γ , is shown in (b).

Observation shows that the boundary layer, at a given height, is sometimes in a turbulent state, sometimes in a laminar state, see figure 6.5 for an illustration of that. We measure this by what is called an *intermittency factor*, which indicates what fraction of time is time the flow turbulent at a given position. Figure 6.6 shows some results from turbulent boundary layer measurements.

Note that this is different from a channel flow, where typically the whole flow is either laminar or turbulent. This implies that the wake function of the boundary layer deviates much more from the log law than the one for the channel, as can be observed comparing figures xx and 6.7.

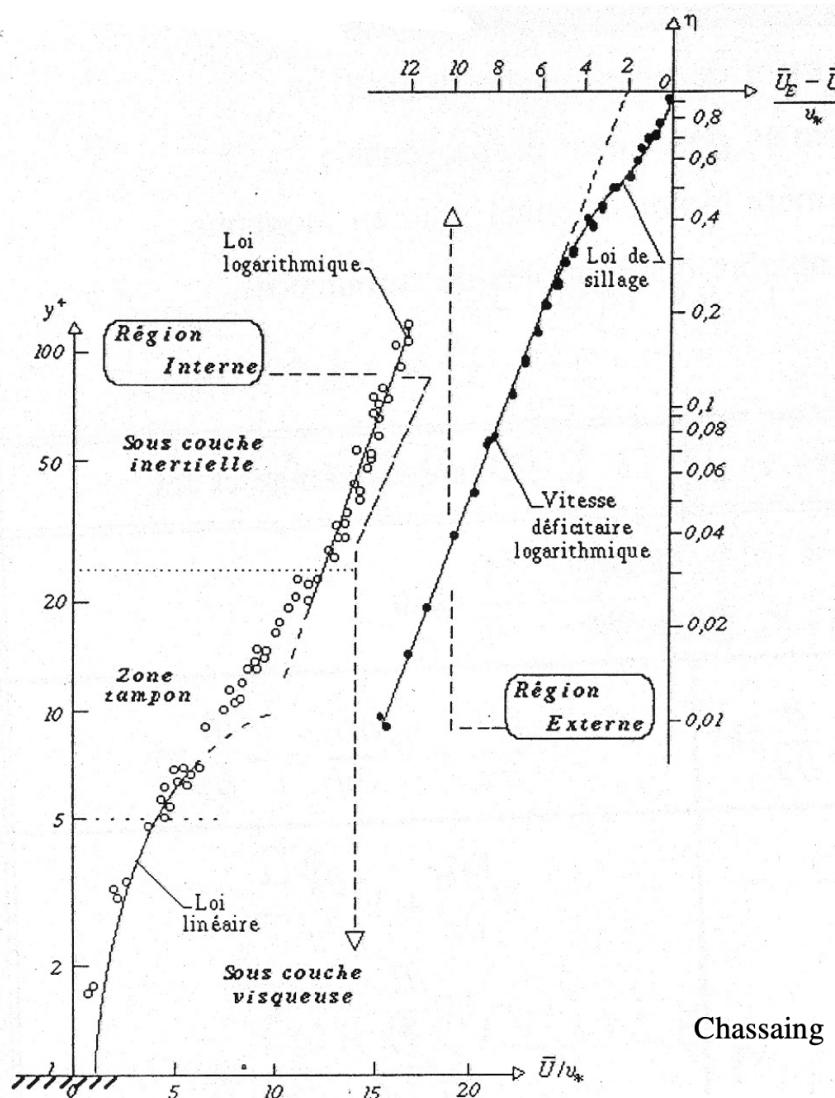


Figure 6.7: Velocity profile of a turbulent boundary layer

A

Different forms of the Navier-Stokes equations

This appendix is meant for a short description of some forms of the Navier-Stokes equations. For a full range of forms, look up the book by Masatsuka[3].

A.1 Incompressible flow

A.1.1 Vector form

$$\nabla \cdot \vec{u} = 0, \quad (\text{A.1})$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} = -\frac{1}{\rho} \vec{\nabla} p + \nu \vec{\nabla}^2 \vec{u} + \vec{f}, \quad (\text{A.2})$$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \overset{\leftrightarrow}{D} : \overset{\leftrightarrow}{D} + \frac{k}{\rho c_p} \vec{\nabla}^2 T + \frac{1}{\rho c_p} \vec{f} \cdot \vec{u}. \quad (\text{A.3})$$

where

$$\overset{\leftrightarrow}{D} = \frac{\vec{\nabla} \vec{U} + (\vec{\nabla} \vec{U})^T}{2}, \quad D_{ij} = \frac{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}}{2} \quad (\text{A.4})$$

A.1.2 Cartesian coordinates

In cartesian coordinates, the above equations reads

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (\text{A.5})$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \frac{1}{\rho} \vec{f}_x, \quad (\text{A.6})$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \frac{1}{\rho} \vec{f}_y, \quad (\text{A.7})$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \frac{1}{\rho} \vec{f}_z, \quad (\text{A.8})$$

Cylindrical coordinates

In cylindrical coordinates, the Navier-Stokes equations read

$$\frac{1}{r} \frac{\partial(ru)}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0, \quad (\text{A.9})$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \theta} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \right) + \frac{1}{\rho} \vec{f}_r, \quad (\text{A.10})$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \theta} + w \frac{\partial v}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial \theta} + \nu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial z^2} \right) + \frac{1}{\rho} \vec{f}_\theta, \quad (\text{A.11})$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + \frac{v}{r} \frac{\partial w}{\partial \theta} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial w}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2} \right) + \frac{1}{\rho} \vec{f}_z, \quad (\text{A.12})$$

where u , v , and w here corresponds to the velocity components in the r , θ and z directions, respectively.

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