Existence of Equilibrium in Large Matching Markets with Complementarities*

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Abstract

In matching markets, the existence of stable matchings can only be guaranteed under substantive restrictions on preferences. We investigate how these results change in large markets, which we model with a continuum of agents of each type, following the work of Aumann (1964) on general equilibrium theory. We find four main results: First, in many-to-many matching with contracts, stable outcomes do not exist in general. Second, if agents on one side of the market have substitutable preferences, then a stable outcome does exist. Third, with bilateral contracts and transferable utility, competitive equilibria always exist. Fourth, in more general settings with multilateral contracts (possibly including restrictions on transfers) and no structure on the set of agents, the core is nonempty.

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1 Introduction

Matching markets are markets for differentiated goods and services where parties care about characteristics of their partners. Examples span labor markets (such as hiring associates at a law firm), HMO networks in health care, and supply chain networks in manufacturing. Matching markets are also the topic of a large literature, from the Becker (1973) marriage model to more recent contributions with rich heterogeneous preferences,[1] such as the Kelso and Crawford (1982) labor market model and the Hatfield and Milgrom (2005) model of matching with contracts.

A central finding of this literature is that the existence of equilibrium depends on ruling out complementarities in preferences. A number of results show that substitutable preferences (which rule out such complementarities) are, in a certain sense, necessary to guarantee existence of an equilibrium.[2] This restriction is substantial, because complementarities are a central element of many matching markets. For example, in labor market clearinghouses, as in the work of Kojima, Pathak and Roth (2013), couples find positions in the same geographic area to be complementary. In models with one-dimensional heterogeneity, as in the Kremer (1993) O-ring theory, complementarities are often a central issue.

This paper investigates whether equilibrium existence can be guaranteed in large markets, which we model with a continuum of agents.[3] This is inspired by the literature on general equilibrium, where this large market approach was decisive in demonstrating the existence of an equilibrium without imposing convexity assumptions on preferences (Aumann, 1966).

[1] We use the term heterogeneous preferences to refer to preferences that may differ across agents. In particular, this includes models with rich heterogenous preferences such as those of Gale and Shapley (1962) and Kelso and Crawford (1982).

[2] Previous work in this literature typically shows that equilibria exist under the assumption that preferences are substitutable and that ruling out complementarities is "necessary" in a well-defined maximal domain sense to guarantee existence of an equilibrium. The particular papers vary in terms of generality, of whether utility is transferable, and whether the equilibrium concept is a stable matching or a competitive equilibrium. See, for example, the work of Kelso and Crawford (1982), Roth (1984b), Gul and Stacchetti (1999, 2000), Hatfield and Milgrom (2005), Sun and Yang (2006, 2009), and Hatfield et al. (2013).

[3] A contemporaneous paper by Che, Kim and Kojima (2018) considers a related model with a continuum of "workers" to be matched to a finite number of "firms"; we discuss the connection of their work with ours below.

More broadly, we investigate how the classic general equilibrium results relate to our results in matching theory, and how to apply general equilibrium techniques in matching settings.

We establish three theorems: Theorem 1 shows that, in a large two-sided matching market, a stable matching exists as long as one side of the market has substitutable preferences. Proposition 1 shows that this theorem is tight in a precise maximal domain sense, so that existence cannot be guaranteed under larger classes of preferences. By contrast, Theorem 2 shows that the core of a large matching market always exists, even with general trading networks and preferences. Finally, Theorem 3 shows that a competitive equilibrium always exists in a large matching market with transferable utility.

These results make three contributions: First, from a narrow perspective, our results contribute to the theory of matching markets. Theorems 1 and 3 show that equilibria exist under considerably more general conditions than in the analogous models with a finite number of consumers (Hatfield and Milgrom, 2005; Hatfield et al., 2013). The added generality is substantial because it includes natural classes of preferences that allow for complementarities.[4]

Second, from a broader perspective, our results demonstrate the similarities and differences between matching markets and general equilibrium models. We demonstrate two main differences: In general equilibrium theory, with large markets, there are mild conditions that guarantee both existence of equilibrium and core convergence (i.e., that the core coincides with the set of equilibria). This is in sharp contrast to matching markets: Proposition 1 and our examples show that stable outcomes can only be guaranteed to exist under substantial assumptions on preferences. And, whenever stable matchings do not exist, core convergence fails because Theorem 2 guarantees that the core is not empty. The intuition for this difference is that the definition of stability allows agents who form a block to keep some of their old contracts, making stability a more demanding concept than the core. However, there are also substantial similarities between matching theory and general

[4] In particular, we demonstrate the existence of stable outcomes in applied settings such as matching with couples (Roth and Peranson, 1999; Kojima et al., 2013; Ashlagi et al., 2014), and generalizations of commonly used empirical models of matching (Choo and Siow, 2006; Fox, 2010; Salanié and Galichon, 2011).

equilibrium theory. In general equilibrium and game theory, it is known that the core (or an approximation of the core) of a large game is non-empty under mild conditions, like in our Theorem 2. Results of this type were found by Aumann (1966, 1964) in exchange economies, Wooders (1983) and Wooders and Zame (1984) in general games with transferable utility, and Kaneko and Wooders (1986) in games with non-transferable utility and finite coalitions. Moreover, in club theory, it is known that equilibria exist in models similar to the transferable utility setting of Theorem 3 (Ellickson et al., 1999).

Third, from a technical perspective, our results show how to apply techniques from general equilibrium theory to matching markets.[5] Theorem 1 is based on a topological fixed-point theorem, following Arrow and Debreu (1954), in contrast to the order-theoretic fixed point theorems used in matching theory. Theorem 2 is an application of an existence theorem by Kaneko and Wooders (1986). Theorem 3 follows a proof technique by Gretsky et al. (1992, 1999).

This paper is organized as follows: In Section 2 we illustrate our main results with examples, and in Section 3 we describe the relationship of our work to the rest of the literature. In Section 4, we demonstrate the existence of stable outcomes in bilateral matching economies. We then show that core outcomes exist for continuum economies with multilateral contracting in Section 5. Finally, we consider continuum economies with discrete contracting but transferable utility in Section 6. Section 7 concludes.

2 Understanding Our Results

Before presenting our full analysis, we consider simple examples that illustrate our results. The first example considers entrepreneurs and programmers matching to form technology startups. There are three types of agents: entrepreneurs (e), generalist programmers (g), and database programmers (d), with a continuum of equal mass of each type. Entrepreneurs have an idea

[5]The literature following Hylland and Zeckhauser (1979) has used general equilibrium techniques to develop fair and efficient allocation procedures. Here we focus on issues related to the existence of equilibrium.

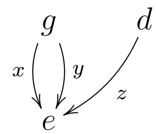

Figure 1: A simple economy. An arrow denotes a contract.

for a business, but need both types of programming services. For now, transfers between the agents are limited: There are only three contracts available, each of which specifies some standardized compensation for fulfillment of a programming task. Entrepreneurs may contract for general programming (x) and database programming (y) from the general programmers, or for database programming from the database experts (z), as depicted in Figure 1. Entrepreneurs need both types of programming to create a viable startup, and prefer that the database programming be performed by a specialist. Their preferences over bundles of contracts are $\{x, z\} \succ \{x, y\} \succ \varnothing$. Database programmers would rather contract than not, i.e., $\{z\} \succ \varnothing$, and generalist programmers are only willing to contract if they can sell both of their services, i.e., $\{x, y\} \succ \varnothing$.

The first surprising observation about this example is that there is no stable outcome, *even with a continuum of agents*. An outcome is stable if it is individually rational (i.e., no agent wishes to unilaterally withdraw from some contracts he currently signs) and there is no blocking set of contracts (i.e., a set of contracts each agent would choose given his current outcome, possibly dropping some contracts he is currently a party to). In this example, in any stable outcome every employed generalist programmer must be signing the only individually rational contract bundle $\{x, y\}$. However, if any entrepreneur is engaging in the bundle of contracts $\{x, y\}$, then that entrepreneur would rather drop the y contract and obtain specialized database services instead, moving to their preferred bundle $\{x, z\}$. Such an outcome is not stable, as the generalist programmers are not interested in only selling x, and would rather not transact; but if no agents were transacting, then $\{x, y\}$ would be a blocking set, as both the entrepreneur and the generalist prefer this bundle to nothing. Therefore, even with a continuum of agents, a stable outcome does not exist. Thus, assuming

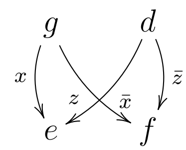

Figure 2: An economy with a stable outcome when there is a continuum of types. An arrow denotes a contract.

a large market is not sufficient to guarantee existence of stable outcomes without additional assumptions on preferences. However, the existence of stable outcomes can be shown under substantially more general conditions than in markets with a finite number of agents.

Theorem 1 shows that stable outcomes do exist in any large two-sided market where one side has substitutable preferences. This assumption is considerably more general than the standard assumption in the discrete matching literature that all agents have substitutable preferences (Alkan and Gale, 2003; Fleiner, 2003; Hatfield and Milgrom, 2005; Hatfield and Kominers, 2017).[6] In particular, this result implies that a stable outcome exists for any large two-sided many-to-one matching (with contracts) market, since unit-demand preferences are a special case of substitutable preferences.

Consider a variation of our previous example: There are still two programmer types, g and d, but now there are two firms, our entrepreneur e and another firm f, who desires at most one contract—either $\bar{x}$ with the generalist programmer, or $\bar{z}$ with the database programmer. We assume that the preferences of f are given by $\{\bar{x}\} \succ \{\bar{z}\} \succ \varnothing$. The generalist programmer now only desires one contract, either x or $\bar{x}$, and has preferences given by $\{x\} \succ \{\bar{x}\} \succ \varnothing$. The database programmer prefers $\bar{z}$ to z, i.e., has preferences given by $\{\bar{z}\} \succ \{z\} \succ \varnothing$. Finally, the preferences of the entrepreneur e are given by $\{x, z\} \succ \varnothing$.[7] The set of contracts is $\{x, \bar{x}, z, \bar{z}\}$. Figure 2 depicts the structure of this example.

[6] In finite market economies, substitutable preferences are often necessary to guarantee the existence of stable outcomes: see, e.g., Hatfield and Kojima (2008) on the setting of many-to-one matching and Hatfield and Kominers (2012) on the setting of matching in vertical networks. Substitutable preferences are not necessary to guarantee the existence of stable outcomes in the setting of many-to-one matching with contracts, but weakly substitutable preferences are: see Hatfield and Kojima (2008).

[7] In particular, e never desires the contract y, and so we drop it from the example.

In a discrete model with one agent of each type, there is no stable outcome. If both programmers are matched to e, then f can block the match by poaching the database programmer, i.e., $\{\bar{z}\}$ is a blocking set. But $\{\bar{z}, x\}$ is not stable, as it is not individually rational for e. The outcome $\{\bar{z}\}$ is not stable, as f will then wish to lay off the database programmer and hire the generalist programmer, i.e., $\{\bar{x}\}$ is a blocking set. Finally, the outcome $\{\bar{x}\}$ is not stable, as then both programmers would be willing to work for the entrepreneur, i.e., $\{x, z\}$ is a blocking set.

By contrast, if there is a continuum of agents of each type, a stable outcome exists. Suppose there are an equal mass of each type of agent. Half of the programmers of each type work for the entrepreneur, and the other programmers work for the firm. Thus, firms of type f are at capacity, and so do not wish to acquire any more programmers from other firms. Likewise, an entrepreneur of type e will not block the match as he is unable to attract a database programmer.

Theorem 2 shows that the core of a matching market is always non-empty. Returning to the first startup example, consider the outcome where all entrepreneurs sign the set of contracts $\{x, y\}$ with generalist programmers. Although this allocation is not stable, it is in the core. There is no coalition of agents that can do better, as for an entrepreneur to move to the z contract with the specialized database programmer would require withdrawing from his entire relationship with the generalist programmer. In fact, the core is always non-empty in a class of models allowing multilateral contracting, general trading networks, and limitations on transfers.

Finally, Theorem 3 shows that, if agents have quasilinear preferences over a numeraire commodity, a competitive equilibrium always exists. Thus, in both of the examples given above, if entrepreneurs and programmers have quasilinear preferences over a numeraire, then prices can adjust so that the market clears. In fact, when agents have quasilinear preferences over a numeraire commodity, competitive equilibria exist in general trading networks, not just two-sided (i.e., buyer–seller) economies. Hence, with a continuum of agents and a numeraire, only mild conditions are required to guarantee the existence of equilibrium. By contrast, in models with a discrete number of agents significant restrictions must be placed on preferences in order to ensure the existence of a competitive equilibrium. These restrictions are necessary in models of exchange economies with indivisible goods (Gul and Stacchetti, 1999, 2000; Sun and Yang, 2006, 2009; Baldwin and Klemperer, 2018) and in matching models with discrete contractual relationships (Kelso and Crawford, 1982; Hatfield et al., 2013).[8] Theorem 3 generalizes the existence result of Azevedo et al. (2013), who demonstrate existence in a general equilibrium setting with indivisible goods, but without the rich set of contracts we consider. Ellickson et al. (1999) have shown existence of equilibria in related settings, but that do not include our quasilinear case. Moreover, our result implies the existence of stable matchings for the roommate problem (with transfers), first shown by Chiappori et al. (2014).

Our results for continuum economies raise the question of whether approximately stable outcomes exist in large, but finite, matching markets, for an appropriately defined concept of approximate stability. Consider again the example depicted in Figure 2. In the stable outcome in the continuum economy, half of the programmers of each type join the entrepreneur and the other half join the firm. Now suppose there is a large, but finite, number k of agents of each type. If k is even, then a stable outcome analogous to the continuum stable outcome exists: $\frac{k}{2}$ of the programmers of each type join the entrepreneur and the other programmers join the firm. If k is odd, no stable outcome exists, but an approximately stable outcome exists, in the sense that there exists an individually rational outcome for which every blocking set contains one particular agent. Proposition 2 generalizes this observation.[9]

[8] However, in the special case of an auction, i.e., a buyer–seller market with only one seller, a competitive equilibrium always exists: see Bikhchandani and Ostroy (2002).

[9] We caution readers that whether approximately stable outcomes are a useful solution concept depends on details of the institutional setting. First, if it is easy for agents to find and implement blocking sets, approximately stable outcomes may not be predictive of final market outcomes. Second, from a mechanism design perspective, legal or other constraints may preclude an approximately stable outcome from the set of feasible market outcomes. Finally, computation of approximately stable outcomes must be practical. Nevertheless, in many settings, approximate stability can be seen as either an intuitively plausible prediction of market outcomes or as a mechanism design goal.

3 Relationship to the Literature

In addition to the previously cited works in matching, general equilibrium, club theory, and game theory, our work is related to the literature on large matching markets. A number of recent papers that have applied large market ideas to matching (e.g., Immorlica and Mahdian (2005), Kojima and Pathak (2009), Lee (2016), Kojima et al. (2013), Ashlagi et al. (2014), Lee and Yariv (2014), and Ashlagi et al. (2017)). Some papers explicitly consider a model with a continuum of agents. Bodoh-Creed (2013), Echenique et al. (2013), and Menzel (2015), like us, consider a model with a continuum of agents on both sides, while Azevedo and Leshno (2016) and Abdulkadiroğlu et al. (2015) have a finite number of firms being matched to a continuum mass of workers. The key difference between our work and these papers is that we focus on the existence of equilibrium in a very general setting. By contrast, other works consider settings without complementarities, where existence was well-known in the discrete case. The focus of Azevedo and Leshno (2016), Bodoh-Creed (2013), Abdulkadiroğlu et al. (2015), and Menzel (2015) is instead in building tractable models in those settings, and applying them to specific problems. Echenique et al. (2013) investigate testable implications of stability.

The most closely related paper to our work here is the contemporaneous work by Che et al. (2018). They considered a model of a finite number of "large" firms matching to a continuum of workers. Like us, they use a topological fixed-point theorem to show that stable outcomes exist even if some workers are complements. Thus, their model is more similar to the classic Arrow and Debreu (1954) model, where "firms" play the role of consumers and "workers" play the role of divisible goods. By contrast, our model is more similar to that of Aumann (1964), in which there is a continuum of consumers. In fact, the key assumption in Che et al. (2018) is a convexity assumption similar to that of Arrow and Debreu (1954). However, the work of Che et al. (2018) is not a particular case of ours, because they consider an infinite number of types, incorporating rich preference structures that are not allowed in our setting. Thus, we view their work as complementary to our contributions in Section 4 as each considers the economy becoming large in distinct ways. Finally, in subsequent work, Jagadeesan (2017) built upon our model by incorporating large firms; he uses extensions of our arguments to establish some of the results in Che et al. (2018).

Nguyen and Vohra (2018) consider existence of approximately stable outcomes in a setting with a large but finite number of agents. Their main application is resident-hospital matching with couples. They show that it is possible to change the capacities of hospitals so that a stable outcome always exists. In particular, they use Scarf's Lemma and combinatorial optimization techniques to tightly bound how much it is necessary to change capacities. For example, in the matching with couples context the total hospital capacity in the market has to increase by at most nine. However, their results can be generalized far beyond the couples example. The most substantive assumptions are that all agents are acceptable matches, and that hospital preferences satisfy a generalized responsiveness condition. Our work here differs in that we show the existence of stable outcomes in very general settings, while Nguyen and Vohra (2018) impose more structure but give tight bounds on the capacity increases necessary to restore stability.

4 Stable Outcomes in Large Economies

4.1 Framework

There is a finite set B of *buyer types* and a finite set S of *seller types*; for each agent type $i \in I \equiv B \cup S$, there exists a mass θ^i of agents of type i. There also exists a finite set X of *contracts*, and each contract $x \in X$ is associated with a buyer type $\mathsf{b}(x) \in B$ and a seller type $\mathsf{s}(x) \in S$.

For a set of contracts $Y \subseteq X$, we let $\mathsf{b}(Y) \equiv \cup_{y \in Y}\{\mathsf{b}(y)\}$ and $\mathsf{s}(Y) \equiv \cup_{y \in Y}\{\mathsf{s}(y)\}$. We also let $Y_i \equiv \{y \in Y : i \in \{\mathsf{b}(y), \mathsf{s}(y)\}\}$ denote the set of contracts in Y associated with agents of type i.

4.1.1 Preferences

Each type of agent $i \in I$ has strict preferences $\succ_i$ over sets of contracts involving that agent. We naturally extend preference relations to subsets of X: for $Y, Z \subseteq X$, we write $Y \succ_i Z$ if and only if $Y_i \succ_i Z_i$.

For any agent type $i \in I$, the preference relation $\succ_i$ induces a *choice function*

$$C^i(Y) \equiv \max_{\succ_i}\{Z : Z \subseteq Y_i\}$$

for any $Y \subseteq X$.[10]

The notion of substitutability has been key in assuring the existence of stable outcomes in settings with a finite number of agents.[11] An agent type $i \in I$ has substitutable preferences if, when presented with a larger choice set, any previously rejected contract is still rejected.

Definition 1. An agent type $i \in I$ has *substitutable* preferences if for all $x, z \in X$ and $Y \subseteq X$, if $z \notin C^i(Y \cup \{z\})$, then $z \notin C^i(\{x\} \cup Y \cup \{z\})$.

4.1.2 Outcomes

We let m^i_Z denote the mass of agents of type $i \in I$ who engage in contracts $Z \subseteq X_i$; thus, $m^i \in [0, \theta^i]^{\wp(X_i)}$, where $\wp(X_i)$ is the power set of X_i. The *supply* of a contract $x \in X$ is given

[10]Here, we use the notation $\max_{\succ_i}$ to indicate that the maximization is taken with respect to the preferences of agent i.

[11]In the setting of many-to-many matching with contracts, substitutable preferences are both sufficient (Roth, 1984b; Echenique and Oviedo, 2006; Klaus and Walzl, 2009; Hatfield and Kominers, 2017) and necessary in the maximal domain sense (Hatfield and Kominers, 2017) to guarantee the existence of stable outcomes. (Here, by necessary in the maximal domain sense we mean that if one agent's preferences are not substitutable, there exist substitutable preferences for the other agents such that no stable outcome exists. See, e.g., the work of Pycia (2012) for a setting where stable matches exist even in the presence of complementarities.) In the setting of many-to-one matching with contracts, substitutability of preferences is sufficient (Hatfield and Milgrom, 2005), but not necessary in the maximal domain sense (Hatfield and Kojima, 2008, 2010; Hatfield et al., 2018); however, if each contract specifies a unique buyer-seller pair, preference substitutability is necessary in the maximal domain sense (Hatfield and Kojima, 2008). Similarly, in settings with transferable utility, substitutability is both sufficient to guarantee the existence of competitive equilibria (Kelso and Crawford, 1982; Gul and Stacchetti, 1999; Sun and Yang, 2006; Hatfield et al., 2013) and necessary in the maximal domain sense (Gul and Stacchetti, 1999; Hatfield and Kojima, 2008; Hatfield et al., 2013). See the work of Baldwin and Klemperer (2018) for further discussion of when competitive equilibria exist in settings with discrete goods.

by

$$m_x^{\mathsf{s}(x)} \equiv \sum_{\{x\} \subseteq Z \subseteq X_{\mathsf{s}(x)}} m_Z^{\mathsf{s}(x)}$$

while *demand* is given by

$$m_x^{\mathsf{b}(x)} \equiv \sum_{\{x\} \subseteq Z \subseteq X_{\mathsf{b}(x)}} m_Z^{\mathsf{b}(x)}.$$

We may now define an outcome for this economy as a vector of contract allocations for each type of agent such that supply equals demand.

Definition 2. An *outcome* is a vector $((m^b)_{b \in B}, (m^s)_{s \in S})$, where $m^i \in [0, \theta^i]^{\wp(X_i)}$ for each $i \in I$, such that

1. For all $i \in I$, $\sum_{Z \subseteq X_i} m_Z^i = \theta^i$, and

2. For all $x \in X$, $m_x^{\mathsf{s}(x)} = m_x^{\mathsf{b}(x)}$.

The first condition of Definition 2 ensures that the total mass of type i agents participating in some set of contracts is equal to the total mass of those type of agents in the economy; note that an agent does not participate in any contract if he participates in the empty set of contracts. The second condition ensures that for each contract x, the mass of sellers participating in x is the same as the mass of buyers participating in x.

4.2 Existence of Stable Outcomes

As is standard in matching theory, we define an equilibrium as a *stable* outcome. An outcome m is *individually rational* if, for all $i \in I$ and $Z \subseteq X_i$, $Z \neq C^i(Z)$ implies that $m_Z^i = 0$. An outcome m is *blocked* by a set of contracts $Z \subseteq X$ if:

1. There exists a cover[12] $\{Z^{\mathbf{b}}\}_{\mathbf{b} \in \mathbf{B}}$ of Z such that for each $Z^{\mathbf{b}}$ there exists a buyer type $b \in B$ such that $\{b\} = \mathsf{b}(Z^{\mathbf{b}})$ and an associated set $Y^{\mathbf{b}} \subseteq X_b \smallsetminus Z^{\mathbf{b}}$ such that

   (a) $m_{Y^{\mathbf{b}}}^b > 0$, and

[12] A *cover* of a set Z is a collection of sets $\{Z^{\mathbf{i}}\}_{\mathbf{i} \in \mathbf{I}}$ such that $\cup_{\mathbf{i} \in \mathbf{I}} Z^{\mathbf{i}} = Z$.

(b) $Z^{\mathbf{b}} \subseteq C^b(Z^{\mathbf{b}} \cup Y^{\mathbf{b}})$.

2. There exists a cover $\{Z^{\mathbf{s}}\}_{\mathbf{s}\in\mathbf{S}}$ of Z such that for each $Z^{\mathbf{s}}$ there exists a seller type $s \in S$ such that $\{s\} = \mathsf{s}(Z^{\mathbf{s}})$ and an associated set $Y^{\mathbf{s}} \subseteq X_s \smallsetminus Z^{\mathbf{s}}$ such that

   (a) $m^s_{Y^{\mathbf{s}}} > 0$, and

   (b) $Z^{\mathbf{s}} \subseteq C^s(Z^{\mathbf{s}} \cup Y^{\mathbf{s}})$.

3. There exist positive integer quantities $q^{\mathbf{b}}$ for each $\mathbf{b} \in \mathbf{B}$ and $q^{\mathbf{s}}$ for each $\mathbf{s} \in \mathbf{S}$ such that the supply and demand of each contract $z \in Z$ is the same:

$$\sum_{\mathbf{b}:z\in Z^{\mathbf{b}}} q^{\mathbf{b}} = \sum_{\mathbf{s}:z\in Z^{\mathbf{s}}} q^{\mathbf{s}}.$$

Definition 3. An outcome m is *stable* if it is individually rational and it is not blocked.

This definition of stability is equivalent to the standard definition from the matching literature (see, e.g., Hatfield and Milgrom (2005)). Individual rationality requires that no agent can do strictly better by dropping some of his contracts. An outcome not being blocked means that it is impossible for a positive measure of agents to do strictly better by forming new contracts with each other, while possibly keeping some of their contracts with other agents.

Stability is a natural analogue to competitive equilibrium in settings without transfers.[13] In a stable outcome, every agent is choosing an optimal set of contracts given the "prices," i.e., the set of contracts that other agents would be willing to accept—this corresponds to the standard competitive equilibrium requirement that each agent chooses an optimal bundle given prices. Market clearing is also satisfied in any stable outcome, as a contract is chosen by its buyer if and only if that contract is also chosen by its seller.

We illustrate the model with a simple example.

[13] Hatfield et al. (2013) show that in settings with transfers and quasilinear utility, every competitive equilibrium corresponds to a stable outcome.

$$\begin{tikzcd} s \arrow[d, bend right, "x"'] \arrow[d, bend left, "y"] \\ b \end{tikzcd}$$

Figure 3: A simple economy. An arrow denotes a contract.

Example 1. Consider a simple economy where $B = \{b\}$, $S = \{s\}$ and $\theta^b = \theta^s = 1$. Let $X = \{x, y\}$ where $\mathsf{b}(x) = \mathsf{b}(y) = b$ and $\mathsf{s}(x) = \mathsf{s}(y) = s$, illustrated in Figure 3. Let preferences be given by

$$b : \{x, y\} \succ \varnothing,$$

$$s : \{x\} \succ \{y\} \succ \varnothing.$$

The only stable outcome is given by $m^b_{\{x,y\}} = \frac{1}{2}, m^b_\varnothing = \frac{1}{2}, m^s_{\{x\}} = \frac{1}{2}, m^s_{\{y\}} = \frac{1}{2}$, with all other entries of m being zero. Note that to show the outcome m where $m^b_\varnothing = m^s_\varnothing = 1$ (and $m^b_Y = m^s_Y = 0$ for all $Y \neq \varnothing$) is not stable in our setting requires the full generality of Definition 3, in the sense that we need to use different covers of the same blocking set for buyers and sellers; we let $Z = \{x, y\}$ and consider the cover $\{\{x, y\}\}$ for buyers and the cover $\{\{x\}, \{y\}\}$ for sellers.

We now state the main theorem of this section.

Theorem 1. *If buyers' preferences are substitutable, then a stable outcome exists.*

To prove Theorem 1, we construct a generalized Gale-Shapley operator, as is standard in the matching with contracts literature. Let $O^B \in [0, \infty)^X$ denote an *offer vector* for the buyers, i.e., the mass of each contract the buyers have access to.

Suppose that b has preferences given by

$$Y^K \succ_b \ldots \succ_b Y^k \succ_b \ldots \succ_b Y^1 \succ_b \varnothing \equiv Y^0$$

over all individually rational subsets of X_b. We define $h^b_{Y^k}(O^B)$ inductively, $k = K, \ldots, 0$ as

$$h^b_{Y^k}(O^B) \equiv \min\left\{\theta^b - \sum_{\tilde{k}>k}^{K} h^b_{Y^{\tilde{k}}}(O^B), \min_{x \in Y^k}\left\{O^B_x - \sum_{\tilde{k}>k}^{K} h^b_{Y^{\tilde{k}}}(O^B)\mathbb{1}_{\{x \in Y^{\tilde{k}}\}}\right\}\right\}. \tag{1}$$

The first term of the minimand is the remaining mass of agents of type b who are not yet assigned via the inductive process. The second term of the minimand is the amount of the set Y^k still available from the offer vector O^B given the mass of each contract in Y^k taken in earlier steps of the inductive process. Intuitively, each buyer type is assigned the maximum possible amount of that buyer type's favorite set of contracts Y^K given the offer vector O^B; having done so, that buyer type is then assigned the maximal amount of that buyer type's second favorite set of contracts Y^{K-1} from what is left, and so on. We may then define the choice function for buyers of type b as

$$\bar{C}^b_x(O^B) \equiv \sum_{\{x\} \subseteq Y \subseteq X_b} h^b_Y(O^B)$$

given an offer vector O^B, i.e., $\bar{C}^b_x(O^B)$ is the mass of x contracts chosen by buyers of type b when these buyers have access to O^B.[14] We define $h^s(O^S)$ for each seller and $\bar{C}^s_x(O^S)$ for each seller analogously. This formulation is equivalent to the usual formulation in finite economies, where O^B is an offer set and the choice function of the buyers is just the union of the choice function of each buyer.

[14]We use the $\bar{C}^b$ notation, as opposed to C^b, to denote that the choice is with respect to all buyers of type b, not just one buyer of type b.

We can now define the following generalized Gale-Shapley operator[15]

$$\Phi(O^B, O^S) \equiv (\Phi^B(O^S), \Phi^S(O^B)) \tag{2}$$
$$\Phi^B_x(O^S) \equiv \bar{C}^{\mathsf{s}(x)}_x((O^S_{X \smallsetminus \{x\}}, \theta^{\mathsf{s}(x)}))$$
$$\Phi^S_x(O^B) \equiv \bar{C}^{\mathsf{b}(x)}_x((O^B_{X \smallsetminus \{x\}}, \theta^{\mathsf{b}(x)})).$$

Given an offer vector O^S available to the sellers, the mass of contract x available to the buyers, $\Phi^B_x(O^S)$, is defined by the mass of contract x that sellers would be willing to take if $\theta^{\mathsf{s}(x)}$ of the contract x (i.e., the maximum amount sellers could demand) was available and a mass of every other contract y equal to O^S_y was available.

Since $\bar{C}^b(\cdot)$ and $\bar{C}^s(\cdot)$ are continuous functions for all $b \in B$ and $s \in S$, it follows immediately that Φ is a continuous function from $(\times_{x\in X}[0, \theta^{\mathsf{s}(x)}]) \times (\times_{x\in X}[0, \theta^{\mathsf{b}(x)}])$ to $(\times_{x\in X}[0, \theta^{\mathsf{s}(x)}]) \times (\times_{x\in X}[0, \theta^{\mathsf{b}(x)}])$. Hence, by Brouwer's fixed point theorem, there exists a fixed point.

To complete the proof, all that is necessary is to ensure that fixed points of Φ do, in fact, correspond to stable outcomes, which is established by the following lemma.

Lemma 1. *Suppose that $(O^B, O^S) = \Phi(O^B, O^S)$. Then if buyers' preferences are substitutable,*

[15] This type of operator is standard in the matching with contracts literature. This particular operator is not a direct analogue of the generalized Gale-Shapley operator of Hatfield and Milgrom (2005) and Hatfield and Kominers (2012); rather, it is most closely related to the operator of Ostrovsky (2008), who considers whether a contract would be chosen by an agent given the other contracts that agent currently has access to. The analogue to our operator in the discrete setting is given by

$$\Phi(X^B, X^S) \equiv (\Phi^B(X^S), \Phi^S(X^B))$$
$$\Phi^B(X^S) \equiv \{x \in X : x \in C^S(X^S \cup \{x\})\}$$
$$\Phi^S(X^B) \equiv \{x \in X : x \in C^B(X^B \cup \{x\})\}.$$

When preferences of both buyers and sellers are substitutable, this operator is also monotonic, implying the existence of fixed points by Tarski's theorem. Furthermore, a stronger result regarding the relationship between fixed points and stable outcomes can be shown for this operator than the operator of Hatfield and Milgrom (2005) and Hatfield and Kominers (2012): In particular, there exists a *one-to-one* correspondence between fixed points and stable outcomes when all agents' preferences are substitutable. Moreover, if (X^B, X^S) is a fixed point, then $X^B \cap X^S$ is a stable outcome, $X^B \smallsetminus X^S$ is the set of contracts desired by the sellers but rejected by the buyers (at the outcome $X^B \cap X^S$), $X^S \smallsetminus X^B$ is the set of contracts desired by the buyers but rejected by the sellers (at the outcome $X^B \cap X^S$), and $X \smallsetminus (X^B \cup X^S)$ is the set of contracts rejected by both buyers and sellers (at the outcome $X^B \cap X^S$).

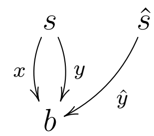

Figure 4: An economy without a stable outcome. An arrow denotes a contract.

$((h^b(O^B))_{b\in B}, (h^s(O^S))_{s\in S})$ *is a stable outcome.*

Proof. See Appendix A. $\square$

Stable outcomes correspond to fixed points of the generalized Gale-Shapley operator as for any fixed point (O^B, O^S), if Z blocks $((h^b(O^B))_{b\in B}, (h^s(O^S))_{s\in S})$, then for each $z \in Z$, some buyers of the associated type $\mathsf{b}(z)$ will choose z from their current set of contracts and z, as the preferences of each buyer type are substitutable. But then each seller must have access to all the contracts in Z; but if Z blocks $((h^b(O^B))_{b\in B}, (h^s(O^S))_{s\in S})$, then some measure of each of the associated seller types will choose all of the corresponding contracts in Z, implying that (O^B, O^S) is not a fixed point.

While substitutability of buyers' preferences is enough to ensure that a stable outcome exists, it is not sufficient for any of the standard structural results on the set of stable outcomes. It is straightforward to construct an example of a many-to-one market where the set of stable outcomes does not form a lattice (in the usual way) and in which the conclusion of the rural hospitals theorem of Roth (1986) does not hold.

However, if the preferences of both sides are not substitutable, then a stable outcome does not necessarily exist, even when there is a continuum of agents and contracts are bilateral. To see this fact, we formalize the example given in Section 2 as Example 2 and show that no stable outcome exists.

Example 2. Suppose that $S = \{s, \hat{s}\}$ and $B = \{b\}$ (with $\theta^s = \theta^{\hat{s}} = \theta^b = 1$) and suppose that $X = \{x, y, \hat{y}\}$, where $\mathsf{s}(x) = \mathsf{s}(y) = s$, $\mathsf{s}(\hat{y}) = \hat{s}$, and $\mathsf{b}(x) = \mathsf{b}(y) = \mathsf{b}(\hat{y}) = b$, which is depicted in Figure 4.

Let the preferences of the three agents be given by:

$$s : \{x, y\} \succ_s \varnothing$$

$$\hat{s} : \{\hat{y}\} \succ_{\hat{s}} \varnothing$$

$$b : \{x, \hat{y}\} \succ_b \{x, y\} \succ_b \varnothing$$

No stable outcomes exist. It is immediate that in any stable outcome, individual rationality imposes that $m^s_x = m^s_y$ and that $m^b_y + m^b_{\hat{y}} = m^b_x$. Suppose that $m^s_x = 0$; then $m^b_{\{x,y\}} = m^s_{\{x,y\}} = 0$ and $\{x, y\}$ is a block. Suppose that $m^s_x > 0$; then $m^s_{\{x,y\}} = m^b_{\{x,y\}} > 0$ and $\{\hat{y}\}$ is a block.

The above example shows that stable outcomes do not necessarily exist when preferences of agents on both sides of the market are not substitutable, even when a continuum of agents is present. In Example 2, the key issue is that, when considering blocking sets, we allow buyer b to break one of his contractual obligations (in this case, dropping y) without affecting the other contracts he has access to; however, since seller s has non-substitutable preferences, when seller s no longer has access to contract y, he also no longer wants to participate in x (which, since b does not have substitutable preferences, would imply that b no longer wishes to agree to y or $\hat{y}$).[16]

Furthermore, even when all agents' preferences are substitutable, our results rely on the acyclic nature of the network structure, i.e., the fact that no agent may both buy from and sell to another agent, even through intermediaries. In our setting, the acyclicity follows immediately from the two-sided nature of the market. However, consider the more general setting of Ostrovsky (2008) and Hatfield and Kominers (2012); in that setting, an agent can act as both a buyer and a seller. Redefine the contract x in Example 2 so that $\mathsf{s}(x) = b$ and $\mathsf{b}(x) = s$; in this case, the model is no longer acyclic, as b both buys from and *sells to* s. However, the preferences of b, s, and $\hat{s}$ are (fully) substitutable in the sense of

[16]However, it is not necessary that a buyer-seller pair have multiple possible contracts between them (as they do in Example 2) in order to construct an example where no stable outcome exists.

Hatfield and Kominers (2012). But since there is no stable outcome in Example 2, simply relabeling the buyer and seller of a particular contract should not induce a given outcome to become stable.

4.3 Tightness

A natural question is to what extent the requirement of substitutable preferences on one side is necessary for the existence of stable outcomes. It is easy to find examples without substitutable preferences on one side where stable outcomes exist. Therefore, substitutable preferences on one side are not strictly necessary for existence. We now state a maximal domain result that establishes a precise sense in which this condition is tight.

Proposition 1. *Suppose a buyer type b has preferences that are not substitutable over a set of contracts X_b, and suppose there is at least one other buyer type. Then there exists a substitutable preferences for that buyer type, a set of seller types, masses of each type, and a set of contracts X containing X_b such that no stable outcome exists.*

Proof. See Appendix A. □

Proposition 1 says that, if a single buyer does not have substitutable preferences, then it is possible to find other buyers with substitutable preferences and other sellers such that a stable outcome does not exist. The intuition for the result is a generalization of Example 2. Namely, if a buyer does not have substitutable preferences, it is possible to define the preferences of other agents so that a similar situation arises. The proof formalizes this construction.

We highlight two significant caveats about Proposition 1. First, maximal domain results are not the same as necessary conditions for existence. As we noted before, it is easy to construct examples where buyers and sellers have non-substitutable preferences but where stable outcomes exist. Second, Proposition 1 shows that the set of preferences where sellers have unrestricted preferences and buyers have substitutable preferences is a maximal domain, in the sense that including any additional preference profiles in this class makes it possible that a stable outcome does not exist. However, there can be other maximal domains. Maximal domain results are common in the matching literature (e.g., Kamada and Kojima (2018); Gul and Stacchetti (1999)) and these are standard caveats.

4.4 Differences Between Matching Theory and General Equilibrium Theory

Our results imply two substantial differences between matching theory and general equilibrium theory. In general equilibrium theory, in large markets equilibria exist under mild regularity conditions and the set of equilibria coincides with the core. Our results show that neither of these two conclusions hold in matching theory.

These differences can be seen in the setting of Example 2. In that example, a stable outcome does not exist, even with a continuum of agents. However, a core outcome, defined formally below, does exist. The core outcome m is given by $m^s_{\{x,y\}} = m^b_{\{x,y\}} = 1$ and $m^i_Z = 0$ (for $i \in \{b, s, \hat{s}\}$ and $Z \subseteq X$) otherwise.[17] Example 2 also helps us understand the intuition behind these differences: Recall that the definition of stability allows agents who form a block to keep some of their old contracts while signing new contracts with each other. By contrast, the definition of the core requires that a coalition do better only by signing contracts among itself. The core outcome m is not stable because a buyer b can drop the contract y with a seller s while retaining the contract x with that seller. This makes the existence of stable outcomes more demanding.

More generally, Theorem 1 and Proposition 1 show that the existence of stable outcomes depends on substantial restrictions on preferences even in large markets, unlike the findings of the general equilibrium literature. Theorem 2 below shows that the core is always non-empty. Thus, these differences between matching theory and general equilibrium theory are a more general phenomenon than the setting of Example 2.

[17]This outcome is in the core as no coalition can improve their joint outcome; b is only better off if he obtains both x and $\hat{y}$ (and drops contract y), but this requires s to agree even though such an outcome is not even individually rational for s.

4.5 Stable Outcomes in Large Finite Economies

We now extend our model to consider the case where there is a large but finite number of agents. Define a *finite economy* as a vector $n = (n^i)_{i \in I}$, specifying a non-negative integer of agents of each type. We denote by $|n|$ the number of agents in economy n. For any positive integer k, we will refer to the economy $k \cdot n$ as the *k-replica* of economy n.

A vector $(m_Z^i)_{i \in I, Z \subseteq X_i}$ is a *stable outcome* of the finite economy n if it is a stable outcome in the continuum model with $\theta = n$ and all of the coordinates of m are integers. We say that the finite economy n has an outcome that is *stable excluding α agents* if there exists a finite economy $\bar{n}$ with a stable outcome such that $\bar{n}^i \leq n^i$ for all $i \in I$ and $|n| \leq |\bar{n}| + \alpha$. Intuitively, n has an outcome that is stable excluding α agents if there exists another finite economy $\bar{n}$ created by excluding up to α agents from n such that $\bar{n}$ has a stable outcome. Essentially, if n has an outcome that is stable excluding α agents, then there exists a feasible outcome for n such that every agent receives an individually rational allocation and any blocking set must involve a contract with at least one of those α agents. We have the following result.

Proposition 2. *Consider a finite economy n, and assume that all buyers have substitutable preferences. There exist positive integers α and β such that:*

1. *Any replica of n has an outcome that is stable excluding α agents.*

2. *For any k that is an integer multiple of β, the k-replica of n has a stable outcome.*

Proof. See Appendix A. □

The first part of the proposition shows that, regardless of the size of a replica, it is always possible to exclude a fixed, finite number of participants and achieve a stable outcome. In particular, as the size of the replica grows, the fraction of agents who have to be excluded is of the order of $\frac{1}{k}$. The second part of the proposition shows that, in any replica that is a multiple of some integer, an exact stable outcome exists.

Proposition 3 follows by considering a continuum economy with the mass of each type i corresponding to n^i. Theorem 1 then implies the continuum economy has a stable outcome;

moreover, it can then be shown that a stable outcome exists with a rational number of agents engaging in each set of contracts. Thus, by taking a large enough k-replica of our economy, we can ensure that there exists an exact stable outcome, which establishes the second part of the proposition. The first part of the proposition then follows.

We now consider an example to illustrate three points: approximately stable outcomes are a reasonable equilibrium prediction in a large market, they can be a compelling market design objective, and they can be applied to situations where the existence of an exact stable outcome depends on restrictive assumptions.

Example 3. In this example, we apply our model to the setting of school choice matching with diversity constraints discussed by Abdulkadiroğlu (2005), Hafalir et al. (2013), Kominers and Sönmez (2016), and Fragiadakis and Troyan (2017), among others. There are two types of students: high-ability students, denoted by h, and low-ability ability students, denoted by ℓ. There are two schools: a school subject to diversity constraints, and a school not subject to diversity constraints. We model these schools by assuming there are two types of "school agents", c and $\bar{c}$, corresponding, respectively, to the two schools in this market; hence, as the market grows large the size of each school will increase (by allowing the number of agents of types c and $\bar{c}$ to increase). The set of contracts is the set of student–school agent pairs, i.e., $X = \{(h, c), (\ell, c), (h, \bar{c}), (\ell, \bar{c})\}$.

The constrained school, i.e., the school corresponding to agents of type c, is required to admit at least as many low-ability students as high-ability students, and so we model this requirement by letting the preferences of an agent of type c be given by

$$c : \{(h, c), (\ell, c)\} \succ \{(\ell, c)\} \succ \varnothing.$$

Meanwhile, the unconstrained school, i.e., the school corresponding to agents of type $\bar{c}$, has no restrictions on whom it may admit, and prefers high-ability students to low-ability students.

We model this by letting the preferences of an agent of type $\bar{c}$ be given by

$$\bar{c} : \{(h, \bar{c})\} \succ \{(\ell, \bar{c})\} \succ \varnothing.$$

The preferences of the high-ability and low-ability students are given by

$$h : \{(h, c)\} \succ \{(h, \bar{c})\} \succ \varnothing$$
$$\ell : \{(\ell, \bar{c})\} \succ \{(\ell, c)\} \succ \varnothing.$$

Consider first a finite economy $(1, 1, 1, 1)$, i.e., an economy where there is one agent of each type.[18] Then no stable outcome exists. If both students are matched to the constrained school (i.e., the outcome $\{(h, c), (\ell, c)\}$), then the unconstrained school can block the outcome by attracting the low-ability student (i.e., $\{(\ell, \bar{c})\}$ is a blocking set). But the outcome $\{(h, c), (\ell, \bar{c})\}\}$ is also not stable, as once the low-ability student attends the unconstrained school, the constrained school violates its diversity constraint; hence, the constrained school then has to reject the high-ability student, i.e., $\{(h, c), (\ell, \bar{c})\}\}$ is not individually rational. Nor is the outcome $\{(\ell, \bar{c})\}$ stable, as then the high-ability student wishes to attend $\bar{c}$, i.e., $\{(h, \bar{c})\}$ is a blocking set. Finally, $\{(h, \bar{c})\}$ is not stable, as the large school can attract both students, i.e., $\{\{(h, c), (\ell, c)\}\}$ is a blocking set.[19]

Moreover, no stable outcome exists for any economy of the form $k \cdot (1, 1, 1, 1)$, where k is an odd integer. For example, if $k = 101$, the outcome where the constrained school has 51 students of each type and the unconstrained school has 50 students of each type is unstable, as the unconstrained school will wish to attract one more low-ability student, and such a student will wish to attend the unconstrained school.

Consider now the continuum model, where $\theta = (1, 1, 1, 1)$. Then a stable outcome m exists, where m is given by $m^h_{\{(h,c)\}} = m^h_{\{h,\bar{c}\}} = \frac{1}{2}, m^\ell_{\{(\ell,c)\}} = m^\ell_{\{\ell,\bar{c}\}} = \frac{1}{2}, m^c_{\{(h,c),(\ell,c)\}} = m^c_\varnothing = \frac{1}{2}$

[18] Note that this corresponds to there being two “seats” at the constrained school but only seat at the unconstrained school.

[19] The only other individually rational outcome, $\varnothing$, is also blocked by $\{(h, c), (\ell, c)\}$.

and $m^{\bar{c}}_{\{(h,\bar{c})\}} = m^{\bar{c}}_{\{(\ell,\bar{c})\}} = \frac{1}{2}$, with all other entries of the matrix m being zero. That is, it is a stable outcome for half of the students of each type to attend the school subject to diversity constraints, and for half of the students of each type to attend the unconstrained school. In this outcome, the unconstrained school is at capacity, and so does not wish to poach low-ability students from the constrained school; moreover, all of the high-ability students are either at the unconstrained school or prefer their current placement to the unconstrained school. The constrained school is under capacity, but is up against its diversity constraint; hence, it is only willing to accept low-ability students, but all such students are already either at the constrained school or they prefer their current placement to the constrained school.

Consider now a k-replica of the $(1, 1, 1, 1)$ economy. If k is even, the continuum stable outcome corresponds to an exact stable outcome of the k-replica economy. Hence, we have that $\beta = 2$ in Proposition 2 for this economy. Moreover, we can always find a feasible outcome for the finite economy for which at most one student is unmatched, i.e., $\alpha = 1$ in Proposition 2 for this economy. Hence, when k is odd, it is a plausible prediction that each school will be matched to about $\frac{k}{2}$ students of each type.

Moreover, an approximately stable outcome is also a reasonable allocation from a market design perspective. A clearinghouse could not ensure finding an exactly stable outcome in this setting; however, the clearinghouse could implement an approximately stable outcome. Which approximately stable outcome to implement depends on institutional details of the application in question: One option is to allow the constrained school to have an extra high-ability student. Another option would be to artificially reduce the capacity of the unconstrained school by one.[20]

Our Theorem 1 is also related to the issue of the existence of stable outcomes in matching markets with couples. Roth (1984a) first noted that couples may provide a challenge to the

[20] Budish (2011) proposes a similar idea for the setting of course allocation; he suggested slightly lowering the capacity for some classes in order to implement an approximate *competitive equilibrium from equal incomes*. Nguyen and Vohra (2018) proposed a similar idea for matching in a model with more structure than ours. In their setting, they are able to find sharp bounds on how much both aggregate capacity and the capacity of each school has to be changed.

National Resident Matching Program, as the preferences of couples are not substitutable, and hence the existence of a stable outcome in finite markets is not guaranteed.[21] Nevertheless, stable outcomes do exist in all the empirical instances studied Roth (2002). While our work does not imply the existence of exact stable matches even in large finite markets, it does show that at least approximately stable outcomes always exist.[22]

We also highlight two important caveats to our results on stability in large finite markets. Note that we find that existence of a stable outcome is much easier to guarantee in a continuum model. This implies that many, but not all, of the existence problems in the literature on finite markets are due to divisibility problems. However, these divisibility problems can still be important for two reasons: First, it may be that approximate stability is not adequate in certain market design applications. Second, our approximate results for large finite markets have two important limitations: The kind of limit that is relevant may not be our limit of replicas with a finite number of types of each agent, and it may be that our approximations are not quantitatively tight enough for certain applications. This points to the importance of both exact existence results with a finite number of agents and of tight approximations that depend on more structure, such as in the work of Nguyen and Vohra (2018).

5 The Core in Large Economies

5.1 Framework

In this section we show that the core of a large matching market exists under mild conditions. In particular, we allow for general trading networks and for contracts with multiple parties. There is a finite set I of *agent types*. For each $i \in I$, there exists a mass θ^i of agents of type i.

[21] Dutta and Massó (1997) consider more broadly the question of when stable outcomes exist when one side has preferences over colleagues. The particular difficulty of matching couples in the NRMP has generated an extensive literature on the types of preferences for couples for which stable matches may be guaranteed to exist: see Klaus et al. (2007), Klaus and Klijn (2007), and Haake and Klaus (2009).

[22] See also work by Kojima et al. (2013) and Ashlagi et al. (2014), who show that the probability of a stable outcome approaches 1 as the market grows large under certain assumptions on how the market grows.

There is a finite set of *roles*, $\mathcal{X}$, and each role $\chi \in \mathcal{X}$ is identified with a unique agent type $\mathsf{a}(\chi)$. For a set of roles $\mathcal{Y} \subseteq \mathcal{X}$, let $\mathsf{a}(\mathcal{Y}) = \cup_{y \in \mathcal{Y}}\{\mathsf{a}(y)\}$ and denote by $\mathcal{Y}_i$ the set of roles associated with agent i, i.e., $\mathcal{Y}_i \equiv \{y \in \mathcal{Y} : \mathsf{a}(y) = i\}$. Each agent type $i \in I$ has a weak preference order $\succsim_i$ over sets of roles in $\mathcal{X}_i$. A *contract* x is a set of roles, i.e., $x \subseteq \mathcal{X}$. Denote by X the set of all contracts. Each contract is composed of contract-specific roles, i.e., $x \cap y = \varnothing$ for all distinct $x, y \in X$.

Definition 4. An *outcome* is a vector $(m^i)_{i \in I}$, where $m^i \in [0, \theta^i]^{\wp(\mathcal{X}_i)}$, such that both:

1. For all $i \in I$, we have that $\sum_{\mathcal{Z} \subseteq \mathcal{X}_i} m^i_{\mathcal{Z}} = \theta^i$.

2. For all $x \in X$, for all roles $\chi, y \in x$, we have that

$$\sum_{\{\chi\} \subseteq \mathcal{Z} \subseteq \mathcal{X}_{\mathsf{a}(\chi)}} m^{\mathsf{a}(\chi)}_{\mathcal{Z}} = \sum_{\{y\} \subseteq \mathcal{Z} \subseteq \mathcal{X}_{\mathsf{a}(y)}} m^{\mathsf{a}(y)}_{\mathcal{Z}}. \tag{3}$$

The first condition of Definition 4 ensures that each type of agent is fully assigned to some set of roles (possibly including the empty set). The second condition of Definition 4 ensures that "supply meets demand"—that is, for each contract, each role has an equal mass of agents (of the appropriate type) performing that role.

We illustrate the model of Section 5 by embedding the economy of Example 1 into this framework. The set of contracts, set of agent types, and masses of each type of agent are the same as in Example 1. The roles available to type b agents are $\mathcal{X}_b = \{\mathcal{b}_x, \mathcal{b}_y\}$ and to type s agents are $\mathcal{X}_s = \{\mathcal{s}_x, \mathcal{s}_y\}$. The contract $x = \{\mathcal{b}_x, \mathcal{s}_x\}$ and the contract $y = \{\mathcal{b}_y, \mathcal{s}_y\}$. The preferences of agents of type b are given by $\{\mathcal{s}_x, \mathcal{s}_y\} \succ_b \varnothing$ and the preferences of agents of type s are given by $\{\mathcal{s}_x\} \succ_s \{\mathcal{s}_y\} \succ_s \varnothing$.

We now define the core. An outcome m is *blocked* by a non-zero vector $(\tilde{m}^i)_{i \in I}$, where $\tilde{m}^i \in [0, \theta^i]^{\wp(\mathcal{X}_i)}$ if both:

1. For all types i, for each set of roles $\mathcal{Z}$ such that $\tilde{m}^i_{\mathcal{Z}} > 0$, there exists a set of roles $\mathcal{Y}$ such that $m^i_{\mathcal{Y}} > 0$ and $\mathcal{Z} \succ_i \mathcal{Y}$.

2. For all $x \in X$, for all roles $\chi, y \in x$, we have that

$$\sum_{\{\chi\} \subseteq Z \subseteq X_{\mathsf{a}(\chi)}} \tilde{m}_Z^{\mathsf{a}(\chi)} = \sum_{\{y\} \subseteq Z \subseteq X_{\mathsf{a}(y)}} \tilde{m}_Z^{\mathsf{a}(y)}. \tag{4}$$

The first condition ensures that for each set of roles specified by $\tilde{m}$ there exists a positive mass of agents (of the appropriate type) willing to take on that set of roles. The second condition ensures that for each contract specified by $\tilde{m}$ there is an equal mass of agent (of the appropriate type) performing each role required by that contract.

Definition 5. An outcome m is in the *core* if it is not blocked.

This is the standard concept of the core; an outcome is in the core if there is no blocking coalition of agents who can do better on their own, forgoing relationships outside of the blocking coalition. The concept of stability differs from that of the core in two ways: First, under stability, agents in the blocking set may "hold onto" currently held contracts while taking on new relationships; this makes stability a harder condition to satisfy than the core. However, under the core, a blocking coalition need only make every agent in the coalition (strictly) better off, while, under stability, every agent in the coalition must be willing to choose every contract in the blocking set given his current set of contracts; this makes stability an easier condition to satisfy than the core. In general, there is no straightforward relationship between the core and the set of stable outcomes; for instance, Blair (1988) provides an example of a many-to-matching setting in which there is a unique core outcome and a different unique stable outcome.

However, in Example 1, the unique core outcome m is for half of the b-type agents to engage in the roles $\mathscr{b}_x$ and $\mathscr{b}_y$ while the other half does not engage in any roles; meanwhile, half of the s-type agents engage in the role $\mathscr{s}_x$ and half engage in the role $\mathscr{s}_y$. In this example, the unique core outcome corresponds to the unique stable outcome; however, this correspondence does not hold in general.

5.2 Existence

The main result of this section is that the core exists very generally.

Theorem 2. *A core outcome exists.*

Theorem 2 is consistent with the literature on the core in cooperative game theory and general equilibrium theory. That literature has shown that the core does not necessarily exist in games with a finite number of players; existence typically depends on a type of assumption called balancedness, introduced by Scarf (1967) (in the case without transferable utility). While balancedness is a substantial assumption, there are several results on the non-emptiness of the core, or of an approximation of the core, in large games; these results demonstrate non-emptiness under mild assumptions. These include classic results for exchange economies such as those of Aumann (1964, 1966) and results for coalition-formation games with both transferable and non-transferable utility, such as those of Wooders (1983), Wooders and Zame (1984), and Kaneko and Wooders (1986). The proof of our existence theorem is an application of Theorem 1 of Kaneko and Wooders (1986) to our setting, in which each agent may engage in many different contracts (instead of joining just one coalition as in the setting of Kaneko and Wooders).

The intuition for our core existence theorem is natural given these earlier results. However, the existence theorem is surprising when compared to our results on stable outcomes; the existence of stable outcomes even in large markets depends on substantial assumptions on preferences. The reason for this difference is that, in the definition of stability, agents can block an outcome while keeping some of their old contracts. By contrast, in the core definition, a coalition of agents can only form a block by trading with each other. This additional flexibility when considering blocks makes the existence of stable outcomes much harder to guarantee, consistent with our results.

5.3 Large Finite Economies

As with stability, the existence of the core in a continuum economy can be used to derive an approximate existence result for large finite economies.

Define a *finite economy* as a non-negative integer vector $n = (n^i)_{i \in I}$ specifying the number of agents of each type. The number of agents in economy n is denoted by $|n|$. A k-replica of economy n is denoted $k \cdot n$. A vector $(m^i_\chi)_{i \in I, \chi \in X_i}$ is a *core outcome of a finite economy* if it is a core outcome of the continuum model with $\theta = n$ where each coordinate of m is an integer. A finite economy n has a *core outcome excluding α agents* if there exists a finite economy $\bar{n}$ with a core outcome such that $\bar{n}^i \le n^i$ for all $i \in I$ and $|n| \le |\bar{n}| + \alpha$.

Proposition 3. *For every finite economy n there exist positive integers α and β such that:*

1. *Any replica of n has a core outcome excluding α agents.*

2. *For any k that is an integer multiple of β, the k-replica of n has a core outcome.*

Proof. See Appendix A. □

Proposition 3 guarantees that, in a large replica economy, there is always an allocation that is an approximate core outcome. Intuitively, since an economy with a continuum of agents has a core outcome, it is possible to arrange most agents in any large finite replica into this outcome with only a bounded number of agents being assigned to different bundles of contracts.

6 Competitive Equilibria in Large Economies

6.1 Framework

We now consider the setting of Hatfield et al. (2013), where agents have quasilinear utility with respect to a numeraire commodity in ample supply. There is a set of *agent types* I, and a finite set of *trades* Ω. An agent of type $i \in I$ is endowed with the *valuation function* $u^i(\Phi, \Psi)$, where $\Phi \subseteq \Omega$ represents the trades for which agent i is a buyer, and $\Psi \subseteq \Omega$ represents the trades for which agent i is a seller. We allow $u^i(\Phi, \Psi)$ to take on any value in $[-\infty, \infty)$ for each $\Phi \subseteq \Omega$ and each $\Psi \subseteq \Omega$. We normalize the outside option as $u^i(\varnothing, \varnothing) = 0$ for each $i \in I$.[23]

A price vector $p \in \mathbb{R}^\Omega$ assigns a price p_ω for each trade $\omega \in \Omega$. Given a vector of prices p, define expenditure as the vector $e_p \in \mathbb{R}^{\wp(\Omega)\times\wp(\Omega)}$ such that

$$e_p(\Phi, \Psi) = \sum_{\varphi\in\Phi} p_\varphi - \sum_{\psi\in\Psi} p_\psi.$$

That is, $e_p(\Phi, \Psi)$ is the net transfer paid by an agent buying Φ and selling Ψ. Hence, the *utility* of a type i agent who buys contracts $\Phi \subseteq \Omega$ and sells contracts $\Psi \subseteq \Omega$ at prices p is given by

$$u^i(\Phi, \Psi) - e_p(\Phi, \Psi).$$

An *economy* is given by a Lebesgue measurable distribution η over I, defined over a σ-algebra, and with $\eta(I) < \infty$. u^i is a measurable function of i.[24]

An *allocation* is a measurable map

$$A : I \to \Delta(\wp(\Omega) \times \wp(\Omega))$$

specifying for each type $i \in I$ a distribution A^i over bundles of trades bought and sold. The space of allocations is denoted $\mathscr{A}$. Denote by $A^i(\Phi, \Psi)$ the proportion of agents of type i

[23]Note that, unlike in Hatfield et al. (2013), we allow here for any type of agent to buy (or sell) any contract. The constraint that a type cannot transact a contract is incorporated by setting the utility of buying (selling) that contract to $-\infty$.

[24]This model is closely related to Ellickson et al. (1999), who consider a very general model of club formation in general equilibrium. One simple way to embed our model in the Ellickson et al. (1999) notation (pp. 1192-1194) is as follows. Consider the case where no agent is allowed to buy and sell the same trade, in the sense that the utility of doing so is $-\infty$. Then we can define the set of external characteristics as $\{b, s, n\}^\Omega$ recording whether each agent can be a buyer, seller, or neither in each trade. Then define the set of clubs as having one club for each combination of a trade, a personal characteristic that allows to buy the trade, and a personal characteristic that allows to sell the trade. Define this club as having a single activity, and a profile with one member who is a buyer and one member who is a seller.

who buy the bundle of trades $\Phi \subseteq \Omega$ and sell the bundle of trades $\Psi \subseteq \Omega$.

Given an allocation A, we define the *excess demand* for each $i \in I$ and for each trade $\omega \in \Omega$ as

$$\mathsf{Z}^i_\omega(A) \equiv \sum_{\{\omega\}\subseteq\Phi\subseteq\Omega} \sum_{\Psi\subseteq\Omega} A^i(\Phi,\Psi) - \sum_{\{\omega\}\subseteq\Psi\subseteq\Omega} \sum_{\Phi\subseteq\Omega} A^i(\Phi,\Psi).$$

Define the *excess demand* for each trade $\omega \in \Omega$ for the entire economy as

$$\mathsf{Z}_\omega(A) \equiv \int \mathsf{Z}^i_\omega(A)\,\mathrm{d}\eta.$$

An allocation A is *feasible* if $\mathsf{Z}(A) = 0$. An *arrangement* $[A;p]$ is comprised of an allocation A and a price vector $p \in \mathbb{R}^\Omega$.

Definition 6. An arrangement $[A;p]$ is a *competitive equilibrium* if it satisfies two conditions:

1. Each agent obtains an optimal bundle given prices p, i.e., for all $i \in I$, $A^i(\Phi,\Psi) > 0$ only if
$$(\Phi,\Psi) \in \operatorname*{arg\,max}_{(\tilde\Phi,\tilde\Psi)\in\wp(\Omega)\times\wp(\Omega)} u^i(\tilde\Phi,\tilde\Psi) - e_p(\tilde\Phi,\tilde\Psi).$$
If this is the case we say that A is *incentive compatible* given p.

2. A is a feasible allocation, i.e., $\mathsf{Z}(A) = 0$.

This is the standard notion of competitive equilibrium: the first condition ensures that each agent is optimizing given the prices p, and the second condition ensures that markets clear.

Finally, we will require some technical conditions in order to ensure the existence of a competitive equilibrium. An economy is *regular* if

1. The integral of absolute values of utility is finite, as long as agents are not given bundles for which they have utility of $-\infty$. That is,
$$\int \max_{\Phi,\Psi\subseteq\Omega,\,u^i(\Phi,\Psi)\neq-\infty} |u^i(\Phi,\Psi)|\,\mathrm{d}\eta < \infty.$$

2. Agents can supply any sufficiently small net demand for trades.[25] That is,

$$\{\mathsf{Y} \in \mathbb{R}^{\Omega} : \exists A \in \mathcal{A} \text{ such that } \mathsf{Y} = \mathsf{Z}(A) \text{ and } u^i(\Phi, \Psi) = -\infty \Rightarrow A^i(\Phi, \Psi) = 0\}$$

contains a neighborhood of 0.

These conditions are satisfied if, for instance, utility functions are uniformly bounded.

6.2 Existence

We now establish existence of an equilibrium. Our proof strategy is to first show that there exists an allocation that maximizes total surplus. We then show that a surplus maximizing allocation is an equilibrium, when coupled with a vector of prices equal to the marginal social values of increasing the supply of each trade.

Theorem 3. *Every regular economy has a competitive equilibrium.*

Given prices p and an allocation A, denote the average utility received and prices paid by agents of type i as

$$u^i \cdot A^i \equiv \sum_{\Phi, \Psi \subseteq \Omega} u^i(\Phi, \Psi) \cdot A^i(\Phi, \Psi)$$

$$e_p \cdot A^i \equiv \sum_{\Phi, \Psi \subseteq \Omega} e_p(\Phi, \Psi) \cdot A^i(\Phi, \Psi)$$

To prove the theorem, we introduce the social welfare function $\mathbf{W}(q)$, which denotes the maximal social welfare that may be attained by an allocation A such that $\mathsf{Z}(A) = q$. Formally,

$$\mathbf{W}(q) \equiv \sup_{\{A \in \mathcal{A} : \mathsf{Z}(A) = q\}} \int u^i \cdot A^i \, \mathrm{d}\eta.$$

[25] This condition rules out the case where there are no agents willing to sell a trade but demand is positive at any finite price. For example, if there is only one trade, $\Omega = \{\omega\}$, all agents have utility $-\infty$ when they are net sellers, and there are agents with arbitrarily high utility from being a net buyer, there is no equilibrium. The assumption rules out this example and similar cases involving sets of trades.

$\mathbf{W}(q)$ attains its supremum as the argument of the supremum is a continuous function and the supremum is taken over a compact space for a suitably defined topology. We note this as a claim.

Claim 1. $\mathbf{W}(q)$ *attains its supremum. Formally,*

$$\mathbf{W}(q) = \max_{\{A \in \mathcal{A}: \mathsf{Z}(A)=q\}} \int u^i \cdot A^i \, \mathrm{d}\eta.$$

Proof. See Appendix A. □

The social welfare function also satisfies the following properties:

1. $\mathbf{W}$ is uniformly bounded above:

$$\mathbf{W}(q) \leq \int \max_{\Phi, \Psi \subseteq \Omega} u^i(\Phi, \Psi) \, \mathrm{d}\eta$$

   for any $q \in \mathbb{R}^\Omega$, and this latter quantity is finite in a regular economy.

2. $\mathbf{W}(q) > -\infty$ for all q in a neighborhood of 0: By parts 1 and 2 of the definition of a regular economy, for any vector q with small enough norm there are enough agents to absorb the excess of any trades while incurring only finite disutility, and hence $\mathbf{W}(q) > -\infty$.

3. $\mathbf{W}$ is concave: Consider any two stocks q and $\tilde{q}$ in $\mathbb{R}^\Omega$, and let

$$A \in \operatorname*{arg\,max}_{\hat{A} \in \{\dot{A} \in \mathcal{A}: \mathsf{Z}(\dot{A})=q\}} \int u^i \cdot \hat{A}^i \, \mathrm{d}\eta$$
$$\tilde{A} \in \operatorname*{arg\,max}_{\hat{A} \in \{\dot{A} \in \mathcal{A}: \mathsf{Z}(\dot{A})=\tilde{q}\}} \int u^i \cdot \hat{A}^i \, \mathrm{d}\eta.$$

   For each $\alpha \in [0,1]$, we have that $\mathsf{Z}(\alpha A + (1-\alpha)\tilde{A}) = \alpha q + (1-\alpha)\tilde{q} \equiv \bar{q}$. Hence, letting

$\bar{A} \equiv (\alpha A^i + (1-\alpha)\tilde{A}^i)$, we have that

$$\begin{aligned}
\mathbf{W}(\bar{q}) &\geq \int u^i \cdot \bar{A}^i \,\mathrm{d}\eta \\
&= \int u^i \cdot (\alpha A^i + (1-\alpha)\tilde{A}^i) \,\mathrm{d}\eta \\
&= \alpha \int u^i \cdot A^i \,\mathrm{d}\eta + (1-\alpha) \int u^i \cdot \tilde{A}^i \,\mathrm{d}\eta \\
&= \alpha \mathbf{W}(q) + (1-\alpha)\mathbf{W}(\tilde{q}).
\end{aligned}$$

These properties imply that $\mathbf{W}(0)$ attains a maximum at some allocation A. Moreover, there exists at least one supergradient p of $\mathbf{W}$ at $q = 0$.[26] The arrangement $[A; p]$ satisfies the market clearing condition of Definition 6, that is, $\mathsf{Z}(A) = 0$. Let J be the set of types such that some agents of that type do not get optimal bundles given prices p.

We will now show that $\eta(J) = 0$. To demonstrate this, we will show that $\eta(J) > 0$ leads to a contradiction. Let $\tilde{A}$ be an allocation such that $\tilde{A}^i = A^i$ for $I \setminus J$ and $\tilde{A}^i$ is a distribution over optimal bundles given p otherwise, and let $\tilde{q} = \mathsf{Z}(\tilde{A})$. We have

$$\begin{aligned}
\mathbf{W}(\tilde{q}) &\geq \int u^i \cdot \tilde{A}^i \,\mathrm{d}\eta \\
&> \int u^i \cdot A^i - (e_p \cdot A^i - e_p \cdot \tilde{A}^i) \,\mathrm{d}\eta \\
&= \mathbf{W}(0) - p \cdot \mathsf{Z}(A) + p \cdot \mathsf{Z}(\tilde{A}) = \mathbf{W}(0) + p \cdot \tilde{q}.
\end{aligned}$$

The first inequality follows from the definition of $\mathbf{W}$. The second inequality from the fact that all agents prefer to buy $\tilde{A}$ to A, and the strictness of the inequality follows as $\eta(J) > 0$. In the third line, the first equality follows from the optimality of A and the second equality follows from the definition of $\tilde{q}$. The result that $\mathbf{W}(\tilde{q}) > \mathbf{W}(0) + p \cdot \tilde{q}$ contradicts the fact that p is a supergradient. The contradiction implies that $\eta(J) = 0$.

To complete the proof of Theorem 3, note that $\eta(J) = 0$ implies that $\mathsf{Z}(\tilde{A}) = 0$. Moreover,

[26]This follows from Theorem 23.4 of Rockafellar (1970) because $\mathbf{W}$ is concave, and properties 1 and 2 on page 33 imply that $\mathbf{W}$ is proper and that 0 is in the relative interior of the domain of $\mathbf{W}$.

$\tilde{A}$ is incentive compatible given p by definition. Therefore, $[\tilde{A}; p]$ is an equilibrium.

Our setting is related to models of general equilibrium with indivisible commodities and to the club theory literature. In general equilibrium with indivisibilities and a finite number of agents, a number of papers (Gul and Stacchetti, 1999, 2000; Sun and Yang, 2006, 2009; Hatfield et al., 2013) show the existence of competitive equilibrium under the assumption that preferences are substitutable or other restrictions on preferences (Baldwin and Klemperer, 2018).[27] We do not impose substantive restrictions on preferences but instead assume that the set of agents is a continuum. The most closely related work is by Azevedo et al. (2013), who prove the existence of competitive equilibria in the setting of Gul and Stacchetti (1999) in a model with a continuum of agents. We generalize their result by allowing for relationship-specific utility and for the assumption that some agents cannot engage in some trades (as in our model utility may take on the value $-\infty$). Hence, our results require a proof technique that is quite different from that of Azevedo et al. (2013), who employ a fixed point argument. Their argument does not work in our setting because the tâtonnement process they consider does not take bounded sets into bounded sets. This occurs due to the possibility of $-\infty$ utility for some bundles, which means that even at very high prices there may be excess demand for some trades. Instead, our proof is based on constructing an equilibrium from a welfare-maximizing allocation, an idea pioneered by Gretsky et al. (1992, 1999) for the continuum assignment problem.

Our work in this section is also related to the club theory literature Buchanan (1965): it is possible to write a model of matching with contracts and transferable utility as a club

[27] Baldwin and Klemperer (2018) approach this problem with a novel method by using techniques from tropical geometry. They give necessary and sufficient conditions on preferences for the existence of equilibria that are different than the substitutes conditions in the literature. It is easier to understand their approach with a simple example, which we take from Azevedo et al. (2013). Consider a setting with two indivisible goods where a single unit of each good available. Sonia is willing to pay \$75 for either good (or both). Charlie is willing to pay \$100 for both goods but has no value for a single item. No competitive equilibrium exists. Most of the literature would attribute the non-existence of equilibria to the fact that Charlie views the goods as complements. Baldwin and Klemperer, by contrast, note that as a price changes Sonia's demand can change in the directions $(1, 0)$, $(0, 1)$, and $(1, -1)$, while Charlie's demand can only change in the direction $(1, 1)$. Baldwin and Klemperer demonstrate that the fact that not all matrices formed by pairs of these vectors have determinant 0 or ± 1 implies that this class of demand functions can preclude existence of equilibrium.

theory model. The work most closely related to ours is that of Ellickson et al. (1999), who established the existence of an equilibrium in very general club economies. Their framework includes very general settings, including settings that are closely related to the matching model that we consider, so that we make a small marginal contribution relative to their results. However, we cannot simply apply their result directly as their regularity conditions rule out the quasilinear case.[28] By contrast, we impose the regularity conditions from Section 6.1, which are tailored to the quasilinear case and allow us to consider the matching model of Hatfield et al. (2013).

6.3 Efficiency and Uniqueness

A feasible allocation A is *efficient* if A maximizes welfare. That is, if, for any feasible allocation $\tilde{A}$,

$$\int u^i \cdot A^i \,\mathrm{d}\eta \geq \int u^i \cdot \tilde{A}^i \,\mathrm{d}\eta. \tag{5}$$

A competitive equilibrium $[A; p]$ is *efficient* if A is efficient.

Proposition 4. *Every competitive equilibrium is efficient.*

Proof. See Appendix A. $\square$

Economies with sufficiently rich preference heterogeneity have a unique equilibrium price vector. To state this result we define the following notion of preference heterogeneity.

Definition 7. The distribution η has full support if, for every open set $U \subseteq \mathbb{R}^{\wp(\Omega)\times\wp(\Omega)}$ we have $\eta(I^U) > 0$, where

$$I^U = \{i \in I : \text{the vector } u^i \in U\}.$$

[28]Their key assumption is club irreducibility. An economy is *club irreducible* in the one-good, quasilinear case if, for any allocation, there exists some amount of the numeraire such that almost every consumer would rather receive that amount of numeraire and not trade, instead of participating in the allocation. This precludes the quasilinear setting: Consider an allocation in which no consumers trade, each consumer receives a positive amount of the numeraire, and the distribution of the numeraire is unbounded. In this allocation, for any finite amount of the numeraire, there is a positive mass of consumers who would rather participate in the allocation than receive the fixed amount of the numeraire and not trade. This contradicts club irreducibility.

We can now state the uniqueness result.

Proposition 5. *A regular economy where η has full support has a unique vector of competitive equilibrium prices.*

Proof. See Appendix A. $\square$

This result is analogous to the uniqueness result in Azevedo et al. (2013). Intuitively, in a market with sufficiently rich preferences, in equilibrium there are always agents who are close to indifferent between engaging in a contract or not, and some of these marginal agents are engaging in the contract, and some are not. This implies that, if external agents were to supply or demand a small quantity of this contract, the gain or loss in social welfare would be proportional to the equilibrium price. Mathematically, this implies that the function $\mathbf{W}$ is differentiable at 0, and therefore that $\mathbf{W}$ has a unique supergradient. The fact that every equilibrium price is a supergradient then implies that equilibrium prices are unique.

6.4 Large Finite Economies

A *finite economy* is defined as a vector $n \equiv (n^i)_{i \in I}$ specifying the number of agents of each type i. Each $n^i \in \mathbb{Z}_{\geq 0}$ and $n^i = 0$ for all but a finite set of types. The total number of agents in the finite economy n is denoted $|n|$. Given a natural number k, the *k-replica* of the finite economy n is the finite economy $k \cdot n$, which has k copies of each agent present in n. The *∞-replica* is the continuum economy η_n given by

$$\eta_n = \sum_i \frac{n^i}{|n|} \cdot \delta^i,$$

where δ^i is Dirac delta function placing mass 1 on i. A finite economy is *regular* if its ∞-replica is regular.

An *allocation* of a finite economy n is an allocation A of the ∞-replica such that, for all i with $n^i \neq 0$, the coordinates of A^i are integer multiples of $\frac{1}{n^i}$. A *competitive equilibrium* of a finite economy is a pair $[A;p]$ such that A is an allocation of the finite economy and $[A;p]$ is an equilibrium of the ∞-replica. We say that a finite economy n has a *competitive equilibrium excluding α agents* if there exists a finite economy $\bar{n}$ with a competitive equilibrium such that $\bar{n}^i \leq n^i$ for all $i \in I$ and $|n| \leq |\bar{n}| + \alpha$. As in the earlier sections, the intuition is that it is possible to reach a competitive equilibrium by selecting α agents and either excluding them from trade or assigning them non-optimal, but individually rational, bundles.

The following proposition establishes an approximate existence result that is similar to that in the non-quasilinear case.

Proposition 6. *Consider a regular finite economy n. There exist positive integers α and β such that:*

1. *Any replica of n has a competitive equilibrium excluding α agents.*
2. *For any k that is an integer multiple of β, the k-replica of n has a competitive equilibrium.*

Proof. See Appendix A. □

7 Conclusion

This paper considers the existence of equilibria in large matching markets using ideas from general equilibrium theory. We follow Aumann (1964) in formalizing a large market as having a continuum of agents. We find that stable outcomes exist under conditions that are substantial, but much less restrictive than in the finite case. We show that, under mild assumptions, the core of a large trading network is always non-empty, and that with quasilinear preferences a competitive equilibrium always exists. Besides these contributions to matching theory, our results also show that there are two large differences between general equilibrium theory and matching theory: In matching theory, core convergence can fail and the existence of stable outcomes depends on substantial conditions on preferences even in large markets. Finally, from a technical perspective, we apply several ideas from general equilibrium theory to matching-theoretic contexts; we hope this approach will be fruitful in other problems in matching theory.

References

Abdulkadiroğlu, Atila, "College Admissions with Affirmative Action," *International Journal of Game Theory*, 2005, *33* (4), 535–549. (Cited on page 22.)

__ **, Yeon-Koo Che, and Yosuke Yasuda**, "Expanding "Choice" in School Choice," *American Economic Journal: Microeconomics*, 2015, *7* (1), 1–42. (Cited on page 9.)

Alkan, Ahmet and David Gale, "Stable Schedule Matching under Revealed Preference," *Journal of Economic Theory*, 2003, *112* (2), 289–306. (Cited on page 6.)

Arrow, Kenneth J. and Gerard Debreu, "Existence of an Equilibrium for a Competitive Economy," *Econometrica*, 1954, pp. 265–290. (Cited on pages 4 and 9.)

Ashlagi, Itai, Mark Braverman, and Avinatan Hassidim, "Stability in Large Matching Markets with Complementarities," *Operations Research*, 2014, *62* (4), 713–732. (Cited on pages 3, 9, and 25.)

__ **, Yash Kanoria, and Jacob D. Leshno**, "Unbalanced Random Matching Markets: The Stark Effect of Competition," 2017. (Cited on page 9.)

Aumann, Robert J., "Markets with a Continuum of Traders," *Econometrica*, 1964, *32* (1/2), 39–50. (Cited on pages 1, 4, 9, 28, and 38.)

__ , "Existence of Competitive Equilibria in Markets with a Continuum of Traders," *Econometrica*, 1966, pp. 1–17. (Cited on pages 2, 4, and 28.)

Azevedo, Eduardo M. and Jacob D. Leshno, "A Supply and Demand Framework for Two-Sided Matching Markets," *Journal of Political Economy*, 2016, *124* (5), 1235–1268. (Cited on page 9.)

__ **, E. Glen Weyl, and Alexander White**, "Walrasian Equilibrium in Large, Quasi-Linear Markets," *Theoretical Economics*, 2013, *8* (2), 281–290. (Cited on pages 8, 35, and 37.)

Baldwin, Elizabeth and Paul Klemperer, "Understanding Preferences: "Demand Types", and the Existence of Equilibrium with Indivisibilities," 2018. Mimeo, Oxford University. (Cited on pages 8, 11, and 35.)

Becker, Gary S., "A Theory of Marriage: Part I," *Journal of Political Economy*, 1973, *81* (4), 813–846. (Cited on page 2.)

Bertsimas, Dimitris and John N. Tsitsiklis, *Introduction to Linear Optimization*, Athena Scientific, 1997. (Cited on pages 50, 54, and 58.)

Bikhchandani, Sushil and Joseph M. Ostroy, "The package assignment model," *Journal of Economic Theory*, 2002, *107* (2), 377–406. (Cited on page 8.)

Blair, Charles, "The Lattice Structure of the Set of Stable Matchings with Multiple Partners," *Mathematics of Operations Research*, 1988, *13*, 619–628. (Cited on page 27.)

Bodoh-Creed, Aaron, "Efficiency and Information Aggregation in Large Uniform-Price Auctions," *Journal of Economic Theory*, 2013, *148* (6), 2436–2466. (Cited on page 9.)

Buchanan, James M., "An Economic Theory of Clubs," *Economica*, 1965, *32* (125), 1–14. (Cited on page 35.)

Budish, Eric, "The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Incomes," *Journal of Political Economy*, 2011, *119* (6), 1061–1103. (Cited on page 24.)

Che, Y.K., J. Kim, and F. Kojima, "Stable Matching in Large Economies," *Econometrica*, 2018, forthcoming. (Cited on pages 2, 9, and 10.)

Chiappori, Pierre-André, Alfred Galichon, and Bernard Salanié, "The Roommate Problem is More Stable Than You Think," 2014. CESifo working paper #4676. (Cited on page 8.)

Choo, E. and A. Siow, "Who Marries Whom and Why," *Journal of Political Economy*, 2006, *114* (1), 175–201. (Cited on page 3.)

Dutta, Bhaskar and Jordi Massó, "Stability of matchings when individuals have preferences over colleagues," *Journal of Economic Theory*, 1997, *75* (2), 464–475. (Cited on page 25.)

Echenique, Federico and Jorge Oviedo, "A Theory of Stability in Many-to-Many Matching Markets," *Theoretical Economics*, 2006, *1* (2), 233–273. (Cited on page 11.)

_ , Sangmok Lee, Matthew Shum, and M. Bumin Yenmez, "The Revealed Preference Theory of Stable and Extremal Stable Matchings," *Econometrica*, 2013, *81* (1), 153–171. (Cited on page 9.)

Ellickson, Bryan, Birgit Grodal, Suzanne Scotchmer, and William R. Zame, "Clubs and the Market," *Econometrica*, 1999, pp. 1185–1217. (Cited on pages 4, 8, 30, and 36.)

Fleiner, T., "A Fixed Point Approach to Stable Matchings and Some Applications," *Mathematics of Operations Research*, 2003, *28* (1), 103–126. (Cited on page 6.)

Fox, Jeremy T., "Identification in Matching Games," *Quantitative Economics*, 2010, *1* (2), 203–254. (Cited on page 3.)

Fragiadakis, Daniel and Peter Troyan, "Improving Matching under Hard Distributional Constraints," *Theoretical Economics*, 2017, *12* (2), 863–908. (Cited on page 22.)

Gale, David and Lloyd S. Shapley, "College Admissions and the Stability of Marriage," *American Mathematical Monthly*, 1962, *69* (1), 9–15. (Cited on page 2.)

Gretsky, Neil E., Joseph M. Ostroy, and William R. Zame, "The Nonatomic Assignment Model," *Economic Theory*, 1992, *2* (1), 103–127. (Cited on pages 4 and 35.)

_ , _ , and _ , "Perfect Competition in the Continuous Assignment Model," *Journal of Economic Theory*, 1999, *88* (1), 60–118. (Cited on pages 4 and 35.)

Gul, Frank and Ennio Stacchetti, "Walrasian Equilibrium with Gross Substitutes," *Journal of Economic Theory*, 1999, *87* (1), 95–124. (Cited on pages 2, 8, 11, 20, and 35.)

_ and _ , "The English Auction with Differentiated Commodities," *Journal of Economic Theory*, 2000, *92* (1), 66–95. (Cited on pages 2, 8, and 35.)

Haake, Claus-Jochen and Bettina Klaus, "Stability and nash Implementation in Matching Markets with Couples," *Theory and Decision*, 2009, *69* (4), 537–554. (Cited on page 25.)

Hafalir, Isa E., M. Bumin Yenmez, and Muhammed A. Yildirim, "Effective Affirmative Action in School Choice," *Theoretical Economics*, 2013, *8* (2), 325–363. (Cited on page 22.)

Hatfield, John William and Fuhito Kojima, "Matching with Contracts: Comment," *American Economic Review*, 2008, *98* (3), 1189–1194. (Cited on pages 6 and 11.)

_ and _ , "Substitutes and Stability for Matching with Contracts," *Journal of Economic Theory*, 2010, *145* (5), 1704–1723. (Cited on page 11.)

_ and Paul Milgrom, "Matching with Contracts," *American Economic Review*, 2005, *95* (4), 913–935. (Cited on pages 2, 3, 6, 11, 13, and 16.)

_ and Scott Duke Kominers, "Matching in Networks with Bilateral Contracts," *American Economic Journal: Microeconomics*, 2012, *4* (1), 176–208. (Cited on pages 6, 16, 18, and 19.)

_ and _ , "Contract Design and Stability in Matching Markets," *Games and Economic Behavior*, 2017, *101*, 78–97. (Cited on pages 6 and 11.)

_ , _ , Alexandru Nichifor, Michael Ostrovsky, and Alexander Westkamp, "Stability and Competitive Equilibrium in Trading Networks," *Journal of Political Economy*, 2013, *121* (5), 966–1005. (Cited on pages 2, 3, 8, 11, 13, 29, 30, 35, and 36.)

_ , _ , and Alexander Westkamp, "Stability, Strategy-Proofness, and Cumulative Offer Mechanisms," 2018. Harvard University working paper. (Cited on page 11.)

Hylland, Aanund and Richard Zeckhauser, "The efficient allocation of individuals to positions," *Journal of Political Economy*, 1979, *87* (2), 293–314. (Cited on page 4.)

Immorlica, Nicole and Mohammed Mahdian, "Marriage, Honesty, and Stability," in "Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms" Society for Industrial and Applied Mathematics 2005, pp. 53–62. (Cited on page 9.)

Jagadeesan, Ravi, "Complementary Inputs and the Existence of Stable outcomes in Large Trading Networks," in "Proceedings of the 2017 ACM Conference on Economics and Computation" ACM 2017, pp. 265–265. (Cited on page 10.)

Kamada, Yuichiro and Fuhito Kojima, "Fair Matching under Constraints," 2018. Mimeo, Stanford University. (Cited on page 20.)

Kaneko, M. and M.H. Wooders, "The Core of a Game with a Continuum of Players and Finite Coalitions: The Model and Some Results," *Mathematical Social Sciences*, 1986, *12* (2), 105–137. (Cited on pages 4, 28, 50, 51, and 52.)

Kelso, Alexander S. and Vincent P. Crawford, "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica*, 1982, *50* (6), 1483–1504. (Cited on pages 2, 8, and 11.)

Klaus, Bettina and Flip Klijn, "Paths to Stability for Matching Markets with Couples," *Games and Economic Behavior*, 2007, *58* (1), 154–171. (Cited on page 25.)

— and Markus Walzl, "Stable Many-to-Many Matchings with Contracts," *Journal of Mathematical Economics*, 2009, *45* (7-8), 422–434. (Cited on page 11.)

—, Flip Klijn, and Jordi Massó, "Some Things Couples Always Wanted to Know about Stable Matchings (but Were Afraid to Ask)," *Review of Economic Design*, 2007, *11* (3), 175–184. (Cited on page 25.)

Kojima, F. and P.A. Pathak, "Incentives and Stability in Large Two-Sided Matching Markets," *The American Economic Review*, 2009, *99* (3), 608–627. (Cited on page 9.)

Kojima, Fuhito, Parag A. Pathak, and Alvin E. Roth, "Matching with Couples: Stability and Incentives in Large Markets," *Quarterly Journal of Economics*, 2013, *128* (4), 1585–1632. (Cited on pages 2, 3, 9, and 25.)

Kominers, Scott Duke and Tayfun Sönmez, "Matching with Slot-Specific Priorities: Theory," *Theoretical Economics*, 2016, *11* (2), 683–710. (Cited on page 22.)

Kremer, M., "The o-Ring Theory of Economic Development," *The Quarterly Journal of Economics*, 1993, *108* (3), 551–575. (Cited on page 2.)

Lee, SangMok, "Incentive Compatibility of Large Centralized Matching Markets," 2016. (Cited on page 9.)

— **and Leeat Yariv**, "On the Efficiency of Stable Matchings in Large Markets," 2014. Mimeo, University of Pennsylvania. (Cited on page 9.)

Menzel, Konrad, "Large Matching Markets as Two-Sided Demand Systems," *Econometrica*, 2015, *83* (3), 897–941. (Cited on page 9.)

Nguyen, Thanh and Rakesh Vohra, "Near Feasible Stable Matchings with Complementarities," *American Economic Review*, 2018, forthcoming. (Cited on pages 10, 24, and 25.)

Ostrovsky, Michael, "Stability in Supply Chain Networks," *American Economic Review*, 2008, *98* (3), 897–923. (Cited on pages 16 and 18.)

Pycia, Marek, "Stability and preference alignment in matching and coalition formation," *Econometrica*, 2012, *80* (1), 323–362. (Cited on page 11.)

Rockafellar, R. Tyrrell, *Convex Analysis*, Princeton University Press, 1970. (Cited on page 34.)

Roth, Alvin E., "The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory," *Journal of Political Economy*, 1984, *92* (6), 991–1016. (Cited on page 24.)

_ , "Stability and Polarization of Interests in Job Matching," *Econometrica*, 1984, *52* (1), 47–57. (Cited on pages 2 and 11.)

_ , "On the Allocation of Residents to Rural Hospitals: A General Property of Two-Sided Matching Markets," *Econometrica*, 1986, *54*, 425–427. (Cited on page 17.)

_ , "The Economist as Engineer: Game Theory, Experimental Economics and Computation as Tools of Design Economics," *Econometrica*, 2002, *70* (4), 1341–1378. (Cited on page 25.)

_ **and Elliott Peranson**, "The Effects of the Change in the NRMP Matching Algorithm," *American Economic Review*, 1999, *89* (4), 748–780. (Cited on page 3.)

Salanié, B. and A. Galichon, "Cupid's Invisible Hand: Social Surplus and Identification in Matching Models," 2011. Mimeo, Columbia University, Department of Economics. (Cited on page 3.)

Scarf, Herbert E, "The Core of an N Person Game," *Econometrica*, 1967, pp. 50–69. (Cited on page 28.)

Sun, Ning and Zaifu Yang, "Equilibria and Indivisibilities: Gross Substitutes and Complements," *Econometrica*, 2006, *74* (5), 1385–1402. (Cited on pages 2, 8, 11, and 35.)

— **and** — , “A Double-Track Adjustment Process for Discrete Markets with Substitutes and Complements,” *Econometrica*, 2009, *77* (3), 933–952. (Cited on pages 2, 8, and 35.)

Wooders, Myrna Holtz, “The Epsilon Core of a Large Replica Game,” *Journal of mathematical Economics*, 1983, *11* (3), 277–300. (Cited on pages 4 and 28.)

— **and William R. Zame**, “Approximate cores of large games,” *Econometrica*, 1984, pp. 1327–1350. (Cited on pages 4 and 28.)

A Proofs

A.1 Proof of Lemma 1

We first show that $(((h^b(O^B))_{b\in B}, (h^s(O^S))_{s\in S})$ is an outcome. Condition 1 of Definition 2 is satisfied as, for each $b \in B$, $h^b_\varnothing(O^B) = \theta^b - \sum_{k=1}^K h^b_{Y^k}(O^B)$, and for each $s \in S$, $h^s_\varnothing(O^S) = \theta^s - \sum_{k=1}^K h^s_{Y^k}(O^S)$ by (1). To see that Condition 2 of Definition 2 is satisfied, suppose that $h_x^{\mathsf{s}(x)}(O^S) < h_x^{\mathsf{b}(x)}(O^B)$ for some $x \in X$.[29] There are two cases:

1. $h_x^{\mathsf{s}(x)}(O^S) = \bar{C}_x^{\mathsf{s}(x)}((O^S_{X\smallsetminus\{x\}}, \theta^{\mathsf{s}(x)}))$. Then $O^B_x = h_x^{\mathsf{s}(x)}(O^S)$ by (2), hence $h_x^{\mathsf{b}(x)}(O^B) \leq O^B_x = h^{\mathsf{s}(x)}(O^S)$, a contradiction.

2. $h_x^{\mathsf{s}(x)}(O^S) < \bar{C}_x^{\mathsf{s}(x)}((O^S_{X\smallsetminus\{x\}}, \theta^{\mathsf{s}(x)}))$. This implies by (1) that $h_x^{\mathsf{s}(x)}(O^S) = O^S_x$. But, by (2), $O^S_x = \bar{C}_x^{\mathsf{b}(x)}((O^B_{X\smallsetminus\{x\}}, \theta^{\mathsf{b}(x)})) \geq h_x^{\mathsf{b}(x)}(O^B)$, which implies that $h_x^{\mathsf{b}(x)}(O^B) \leq O^S_x = h^{\mathsf{s}(x)}(O^S)$, a contradiction.

We now show that the outcome $((h^b(O^B))_{b\in B}, (h^s(O^S))_{s\in S})$ is stable. It is immediate that it is individually rational by the definitions of $\bar{C}^b$ and $\bar{C}^s$. Suppose that there exists a blocking set Z (and associated covers $\{Z^{\mathbf{b}}\}_{\mathbf{b}\in\mathbf{B}}$ and $\{Z^{\mathbf{s}}\}_{\mathbf{s}\in\mathbf{S}}$, along with associated sets $\{Y^{\mathbf{b}}\}_{\mathbf{b}\in\mathbf{B}}$ and $\{Y^{\mathbf{s}}\}_{\mathbf{s}\in\mathbf{S}}$). Since the preferences of each buyer are substitutable, if $z \in Z^{\mathbf{b}}$, then $z \in C^{\mathsf{b}(z)}(\{z\} \cup Y^{\mathbf{b}})$. Hence, $z \in C^{\mathsf{b}(z)}(\{z\} \cup Y^{\mathbf{b}})$ for each $z \in Z$. Hence, by (2), it must

[29]The case where $h_x^{\mathsf{s}(x)}(O^S) > h_x^{\mathsf{b}(x)}(O^B)$ is analogous.

be that $O_z^S = \Phi_z^S(O^B) > h_z^{\mathsf{b}(z)}(O^B) = h_z^{\mathsf{s}(z)}(O^S)$ for all $z \in Z$ (where the equalities follow as (O^B, O^S) is a fixed point). But then any Z^{s} can be chosen by the corresponding seller s, and so (O^B, O^S) is not a fixed point.

A.2 Proof of Proposition 1

Since the preferences of buyer b are not substitutable, there exists a set of contracts $Y \subseteq X_b$ and contracts x and z in $X_b \setminus Y$ such that

$$z \notin C^b(Y \cup \{z\})$$

and

$$z \in C^b(Y \cup \{z\} \cup \{x\}).$$

Define the set of contracts as $X = X_b \cup \{\hat{x}\}$. Define the set of other types and masses as follows:

1. For each contract y in $X_b \setminus (Y \cup \{x\} \cup \{z\})$, there is a single seller type $\mathsf{s}(y)$ with $X_{\mathsf{s}(y)} = y$ and $\varnothing \succ_{\mathsf{s}(y)} \{y\}$. Set the mass $\theta^{\mathsf{s}(y)} = 0$.
2. For each contract y in Y, there is a single seller type $\mathsf{s}(y)$ with $X_{\mathsf{s}(y)} = y$ and $\{y\} \succ_{\mathsf{s}(y)} \varnothing$. Set the mass $\theta^{\mathsf{s}(y)} = 1$.
3. There is one other buyer type $\hat{b}$ with $X_{\hat{b}} = \{\hat{x}\}$, preferences $\{\hat{b}\} \succ_{\hat{b}} \varnothing$, and mass $\theta^{\hat{b}} = 1$.
4. There is a seller s with $X_s = \{x, z\}$, preferences $\{\hat{x}, z\} \succ_s \{x, z\} \succ_s \varnothing$, and mass $\theta^s = 1$.
5. Set the mass $\theta^b = 1$.

We now show that no stable outcome exists. To reach a contradiction, assume that m is stable outcome.

We first prove that $m^s_{\{\hat{x},z\}} = 0$ by contradiction. If $m^s_{\{\hat{x},z\}} > 0$, then $m_z > m_x$ since each seller of type s must be taking an individually rational set of contracts. Therefore, there is a positive mass of b types who receive a set of contracts $Y_0 \cup \{z\}$, with $Y_0 \subseteq Y$, that includes z but does not include x. Moreover, every contract y in Y is either in Y_0 or $m_y < 1$—but this contradicts stability: By our assumption on preferences of agents of type b, $C^b(Y \cup \{z\}) = C^b(Y)$ does not include z. So either $C^b(Y)$ is a strict subset of $Y_0 \cup \{z\}$, which violates individual rationality, or $C^b(Y) \smallsetminus Y_0$ is non-empty and blocks m.

We now prove that $m^s_{\{x,z\}} = 0$ by contradiction. If $m^s_{\{x,z\}} > 0$, then $\{\hat{x}, z\}$ would block m, with the s types assigned to $\{x, z\}$ and the unmatched $\hat{b}$ types both choosing $\hat{x}$.

This implies that $m_x = m_y = 0$. Therefore, there is a (potentially empty) subset Y_0 of Y such that $m^b_{Y_0} > 0$. Moreover, for every y in $Y \smallsetminus Y_0$, we have that $m^{\mathsf{s}(y)}_{\{y\}} < 1$. This implies that $C^b(\{x, z\} \cup Y) \smallsetminus Y_0$ is non-empty (because it includes z) and blocks m, as

1. $C^b(\{x, z\} \cup Y)$ is chosen from $(\{x, z\} \cup (Y \smallsetminus Y_0)) \cup Y_0$ by agents of type b assigned to Y_0,

2. $\{x, z\}$ is chosen from $\{x, z\} \cup \varnothing$ by agents of type s, and

3. y is chosen by its associated seller type for each contract y in $Y \smallsetminus Y_0$ from $\{y\} \cup \varnothing$.

Thus m can is not stable.

A.3 Proof of Proposition 2

Consider the continuum model with $\theta = n$. By Theorem 1 the continuum model has a stable outcome $\bar{m}$. Consider the set M^* of all outcome vectors m in the continuum model such that the support of m is contained in the support of $\bar{m}$. Note that, by the definition of stability, every outcome in M^* is stable. The set M^* can be written as the set of all vectors $(m^i_Z)_{i \in I, Z \in \wp(X_i)}$ such that

$$
\begin{aligned}
&m \geq 0, \\
&\sum_{\{x\} \subseteq Z \subseteq X_{\mathsf{s}(x)}} m_Z^{\mathsf{s}(x)} = \sum_{\{x\} \subseteq Z \subseteq X_{\mathsf{b}(x)}} m_Z^{\mathsf{b}(x)} \text{ for all } x, \\
&\sum_{Z \in \wp(X_i)} m_Z^i = \theta^i \text{ for all } i, \text{ and} \\
&m_Z^i = 0 \text{ if } \bar{m}_Z^i = 0 \text{ for all } i \text{ and } Z.
\end{aligned}
$$

Because M^* is a bounded and non-empty polytope, it has an extreme point m^*. Theorem 2.3 of Bertsimas and Tsitsiklis (1997) implies that the extreme point m^* is a basic feasible solution to these constraints. Theorem 2.2 of Bertsimas and Tsitsiklis (1997) implies that this basic feasible solution can be written as the product of the inverse of a matrix with integer entries and an integer vector. By Cramer's rule, all the entries of m^* are rational numbers. Consequently, there exists an integer β such that $k\beta \cdot m^*$ is an integer vector for all integer k. Moreover, because m^* is a stable outcome of the continuum model, $k\beta \cdot m^*$ is a stable outcome of the finite economy $k\beta \cdot n$. This proves the second part of the proposition.

As for the first part of the proposition, consider an arbitrary replica $k \cdot n$. Let k' be the smallest multiple of β that is no greater than k. The economy $k' \cdot n$ has a stable outcome, by part 2 of the proposition. Moreover, $k' \cdot n$ only excludes $(k - k')|n| \leq (\beta - 1)|n|$ agents. This establishes part 1 of the proposition taking $\alpha = (\beta - 1)|n|$.

A.4 Proof of Theorem 2

The proof is a direct application Theorem 1 of Kaneko and Wooders (1986). The work of Kaneko and Wooders shows that the core is non-empty for games with a continuum of players and a finite number of types who can form finite coalitions. Unfortunately, we cannot simply apply their theorem because their framework is different than our matching setting. Therefore, we proceed as follows: First, starting from our model, we write down a model in the notation used by Kaneko and Wooders. We then use their Theorem 1 to show the existence of an "f-core outcome" in their setting. Finally, we use this to construct a core outcome in our setting. The following proof makes heavy use of the notation and definitions on pages 108–114 of the work of Kaneko and Wooders.

We define the model in the Kaneko and Wooders (1986) framework as follows: The set of players (Kaneko and Wooders, 1986, page 108) is given by

$$N = \cup_{i \in I}(\{i\} \times [0, \theta^i]).$$

The σ-algebra $\mathcal{B}$ is the Borel σ-algebra and μ is the Lebesgue measure. That is, for each agent type i in our model there is a continuum of players of total mass θ^i in the associated Kaneko and Wooders model. Given a player n in N, we denote the first coordinate $n(1)$ of n as player n's type.

We now define the characteristic function (Kaneko and Wooders, 1986, page 110): For each type i in I, define a utility function $u_i : \wp(\mathcal{X}_i) \to \mathbb{R}$ that represents the preference relation $\succsim_i$ such that $u_i(\varnothing) = 0$. We now define the characteristic function V as the set of payoffs that can be attained by a coalition by signing contracts with each other. We also include any payoffs that are Pareto dominated by these, to satisfy Kaneko and Wooders' Property (5). Formally, given a finite coalition of players $S \subseteq N$, the vector $v \in \mathbb{R}^S$ is in $V(S)$ if:

1. For each s in S, $v_s \leq u_{s(1)}(\mathcal{Y}_s)$ for a set of roles $\mathcal{Y}_s \subseteq \mathcal{X}_{s(1)}$.

2. The sets of roles $\mathcal{Y}_s$ are consistent with players in the coalition signing contracts with each other. Formally, define the vector $(m^i_{\mathcal{Y}})_{i \in I, \mathcal{Y} \subseteq \mathcal{X}_i}$ where $m^i_{\mathcal{Y}}$ is the number of type i players in S with $\mathcal{Y}_s = \mathcal{Y}$. Then m satisfies the feasibility equation (3).

This characteristic function satisfies Kaneko and Wooders' Properties (2)–(5) by definition. It satisfies Property (6) because the $V(\{n\})$ are single-valued and therefore have an empty interior.

This game satisfies the r-property (Kaneko and Wooders, 1986, page 114) because the players of each of the finite number of types $i \in I$ are treated identically in the characteristic function. And it satisfies the per-capita bounded property (Kaneko and Wooders, 1986, page 114) because the characteristic function is bounded. Therefore, Theorem 1 of (Kaneko and Wooders, 1986, page 114) implies that the f-core is non-empty; this means that there is an outcome h in H^* that cannot be improved by any coalition. The definition of H^* implies that there is a sequence $(h^\nu)_{\nu\in\mathbb{N}}$ in H converging in measure to h (Kaneko and Wooders, 1986, page 111). Let p^v be a partition with $h^\nu \in H(p^\nu)$.

We now construct a core allocation m for our model. For each set S in partition p^ν, let $m_{\mathcal{Y}}^{S,\nu,i}$ be the number of type i players assigned to the set of roles $\mathcal{Y}$, as in the definition of the characteristic function, where they receive a payoff of at least h^ν. For each n in N, define $m_{\mathcal{Y}}^{n,\nu,i}$ as $m_{\mathcal{Y}}^{S,\nu,i}/|S|$ where S is the set that contains n in p^v. We can define the mass of type i players assigned to the set of roles $\mathcal{Y}$ in p^v as

$$m_{\mathcal{Y}}^{\nu,i} = \int m_{\mathcal{Y}}^{n,\nu,i}\, d\mu(n).$$

The fact that p^v is measure consistent (Kaneko and Wooders, 1986, page 108) and that each $m_{\mathcal{Y}}^{S,\nu,i}$ satisfies the feasibility equation (3) imply that m^ν is an outcome. Because the set of outcomes is compact, we can assume, passing to a subsequence, that $(m^\nu)_{\nu\in\mathbb{N}}$ converges to an outcome m.

Now we only have to show that m is in the core in our model. To reach a contradiction, assume that there is a vector $\tilde{m}$ that blocks m. By Claim 2 below, we can take $\tilde{m}$ to have integer coordinates. Because $\tilde{m}$ is a block, for each type i and set of contracts $\mathcal{Y}$ with $\tilde{m}_{\mathcal{Y}}^i > 0$, there exists a set of contracts $\mathcal{Z}$ with $m_{\mathcal{Z}}^i > 0$ and $u_i(\mathcal{Z}) < u_i(\mathcal{Y})$. This implies that $m_{\mathcal{Z}}^{\nu,i} > 0$ for sufficiently large ν. Therefore, there is a strictly positive measure of type i players n in N such that $h(n) \leq u_i(\mathcal{Z})$. Therefore, there is a coalition S with $m_{\mathcal{Y}}^i$ such players for each i and $\mathcal{Y}$ in $\mathcal{P}(\mathcal{X}_i)$ that can improve upon h (Kaneko and Wooders, 1986, page 112). This contradicts h being in the f-core, completing the proof.

Finally, we establish the claim that was used.

Claim 2. *If there exists a vector that blocks outcome m, then there exists a vector with integer entries that blocks m.*

Proof. Let $\tilde{m}$ be a vector that blocks m. The set of all vectors that have the support contained in the support of $\tilde{m}$ and that satisfy the feasibility constraint in equation (3) is a linear space defined by integer equations. Therefore, by Gaussian elimination, this set has a basis $b_1, b_2, \ldots, b_K$ formed by integer vectors. In particular, we can write $\tilde{m}$ as a linear combination

$$m = \alpha_1 \cdot b_1 + \cdots + \alpha_k \cdot b_k + \cdots + \alpha_K \cdot b_K,$$

with real coefficients α_k. If we take β_k to be rational numbers that approximate the α_k sufficiently well, then the vector

$$\sum_{k=1}^{K} \beta_k \cdot b_k$$

is weakly positive and has the same set of non-zero entries as the vector $\tilde{m}$. Because the coefficients are rational, we can multiply this vector by a suitable integer to obtain an integer vector $\bar{m}$. This vector is weakly positive, has the same set of non-zero entries as $\tilde{m}$, and satisfies the feasibility equation (3). Therefore, it blocks m, completing the proof. □

A.5 Proof of Proposition 3

Consider the continuum model with $\theta = n$. By Theorem 2 the continuum model has a core outcome $\bar{m}$. Consider the set M^* of all outcome vectors m in the continuum model such that the support of m is contained in the support of $\bar{m}$. Note that, by the definition of the core, every outcome in M^* is in the core. The set M^* can be written as the set of all vectors

$(m^i_\chi)_{i\in I,\chi\in X_i}$ such that

$$\begin{aligned}
&m \geq 0,\\
&m^{\mathsf{a}(\chi)}_\chi = m^{\mathsf{a}(y)}_y \text{ for all } x \in X, \text{ and all } \chi, y \in x,\\
&\sum_{\chi\in X_i} m^i_\chi = \theta^i \text{ for all } i, \text{ and}\\
&m^i_\chi = 0 \text{ if } \bar{m}^i_\chi = 0 \text{ for all } i \text{ and } \chi.
\end{aligned}$$

Because M^* is a bounded and non-empty polytope, it has an extreme point m^*. Theorem 2.3 of Bertsimas and Tsitsiklis (1997) implies that the extreme point m^* is a basic feasible solution to these constraints. Theorem 2.2 of Bertsimas and Tsitsiklis (1997) implies that this basic feasible solution can be written as the product of the inverse of a matrix with integer entries and an integer vector. By Cramer's rule, all the entries of m^* are rational numbers. Consequently, there exists an integer β such that $k\beta \cdot m^*$ is an integer vector for all integer k. Moreover, because m^* is a core outcome of the continuum model, $k\beta \cdot m^*$ is a core outcome of the finite economy $k\beta \cdot n$. This proves the second part of the proposition.

The first part of the proposition then follows from essentially the same argument as Proposition 2.

A.6 Proof of Claim 1

Let

$$\tilde{\mathcal{A}} = \{A \in \mathcal{A} : \mathsf{Z}(A) = q, \text{ and } u^i(\Phi, \Psi) = -\infty \implies A^i(\Phi, \Psi) = 0\}.$$

If $\mathbf{W}(q) = -\infty$, the claim is trivial. Consider the case where $\mathbf{W}(q) > -\infty$. The definition of $\mathbf{W}(q)$ implies that it is sufficient to take the supremum in $\tilde{\mathcal{A}}$. This set is compact in the product topology (pointwise convergence), by Tychonoff's Theorem. Consider a sequence of allocations A_k converging pointwise to an allocation A_∞. We can show that

$$\lim_{k\to\infty} \int u^i \cdot A_k^i \,\mathrm{d}\eta = \int u^i \cdot A_\infty^i \,\mathrm{d}\eta.$$

We have that $|u^i \cdot A^i|$ is bounded by $\max_{\Phi,\Psi} u^i(\Phi,\Psi)$. Moreover, regularity implies that

$$\int \max_{\Phi,\Psi} u^i(\Phi,\Psi)\,\mathrm{d}\eta$$

is finite. Convergence of the desired integrals then follows from the dominated convergence theorem.

The convergence above implies that

$$\int u^i \cdot A^i \,\mathrm{d}\eta$$

varies continuously with A in the compact set $\tilde{\mathscr{A}}$. Therefore, the supremum in the definition of $\mathbf{W}(q)$ attains its maximum.

A.7 Proof of Proposition 4

Consider a competitive equilibrium $[A;p]$ and any feasible allocation $\tilde{A}$. Individual optimization (Condition 1 of Definition 6) implies that, for all $i \in I$,

$$(u^i - e_p) \cdot A^i \geq (u^i - e_p) \cdot \tilde{A}^i.$$

Integrating this, we have that

$$\int (u^i - e_p) \cdot A^i \,\mathrm{d}\eta \geq \int (u^i - e_p) \cdot \tilde{A}^i \,\mathrm{d}\eta.$$

We have that $\mathsf{Z}(A) = \mathsf{Z}(\tilde{A}) = 0$ because both allocations are feasible. Hence, $\int e_p \cdot A^i \,\mathrm{d}\eta = \int e_p \cdot \tilde{A}^i \,\mathrm{d}\eta = 0$. Therefore, the above inequality is equivalent to (5), completing the proof.

A.8 Proof of Proposition 5

We first prove the following Lemma.

Lemma 2. *Every equilibrium price vector is a supergradient of* $\mathbf{W}$ *at* 0.

Proof. Consider an equilibrium $[A; p]$, and a vector $q \in \mathbb{R}^\Omega$. Let $\tilde{A}$ be an allocation with $\mathsf{Z}(\tilde{A}) = q$. Individual optimization (Condition 1 of Definition 6) implies that, for each $i \in I$,

$$(u^i - e_p) \cdot A^i \geq (u^i - e_p) \cdot \tilde{A}^i.$$

Integrating this we have

$$\int (u^i - e_p) \cdot A^i \,\mathrm{d}\eta \geq \int (u^i - e_p) \cdot \tilde{A}^i \,\mathrm{d}\eta.$$

Therefore,

$$\begin{aligned} \int u^i \cdot \tilde{A}^i \,\mathrm{d}\eta &\leq \int u^i \cdot A^i \,\mathrm{d}\eta + \int e_p \cdot (\tilde{A}^i - A^i) \,\mathrm{d}\eta \\ \int u^i \cdot \tilde{A}^i \,\mathrm{d}\eta &\leq \mathbf{W}(0) + p \cdot q. \end{aligned}$$

The inequality holds for any such $\tilde{A}$. This implies that $\mathbf{W}(q) \leq \mathbf{W}(0) + p \cdot q$, completing the proof. □

We now prove Proposition 5. Consider an equilibrium $[A; p]$. Fix a trade ω and $\epsilon > 0$. Define the marginal non-buyers of trade ω as the agent types i who do not buy trade ω at p, but who would gain utility of at least $p_\omega - \epsilon$ by adding trade ω to their bundle. Formally,

$$\begin{aligned} M(\epsilon) \equiv \{i \in I : & A^i(\Phi, \Psi) > 0 \implies \omega \notin \Phi, \omega \notin \Psi, \text{ and} \\ & u^i(\Phi \cup \{\omega\}, \Psi) - u^i(\Phi, \Psi) > p_\omega - \epsilon\}. \end{aligned}$$

By the full support assumption, $M(\epsilon)$ has positive measure. Consider a vector $q \in \mathbb{R}^\Omega$

such that $q_\omega = \delta > 0$ and $q_\psi = 0$ for all $\psi \in \Omega \setminus \{\omega\}$. For δ small enough, there exists an allocation $\tilde{A}$ with $\mathsf{Z}(\tilde{A}) = q$ such that $\tilde{A}^i = A^i$ for all $i \in I \setminus M(\epsilon)$ and that assigns the extra mass δ of trade ω to marginal non-buyers in the set $M(\epsilon)$. Therefore, by the definition of $M(\epsilon)$, we have that

$$\mathbf{W}(q) - \mathbf{W}(0) \geq \delta \cdot (p_\omega - \epsilon).$$

By Lemma 2 the price vector p is a supergradient, which implies that

$$p_\omega - \epsilon \leq \frac{\mathbf{W}(q) - \mathbf{W}(0)}{\delta} \leq p_\omega.$$

Moreover, the inequalities hold for all q with sufficiently small norm because can make an analogous argument for $q_\omega = \delta < 0$. Therefore, $\mathbf{W}$ has a directional derivative at 0, and $\partial_\omega \mathbf{W}(0) = p_\omega$. The fact that this directional derivative is well-defined implies that equilibrium prices are unique and equal to the marginal social value of each trade ω.

A.9 Proof of Proposition 6

The ∞-replica η_n is regular, and therefore has an equilibrium $[\bar{A}; p^*]$.

Consider the set $\mathcal{A}^*$ of all feasible allocations A such that, for all i with $n^i = 0$, $A^i = \bar{A}^i$, and for all i with $n^i > 0$ the support of A^i is contained in the support of $\bar{A}^i$. Note that $[A; p^*]$ is a competitive equilibrium of the ∞-replica for any A in the set $\mathcal{A}^*$.

Moreover, the bundles of contracts that agents with $n^i > 0$ receive in the allocations in $\mathcal{A}^*$,

$$\cup_{A \in \mathcal{A}^*} (A^i)_{i:n^i>0},$$

is the set of vectors $(A^i)_{i:n^i>0}$ that solve

$$A^i(\Phi,\Psi) \geq 0 \text{ for all } i,\ \Phi, \text{ and } \Psi,$$

$$\sum_{i,\Phi,\Psi:\omega\in\Phi} A^i(\Phi,\Psi)\cdot n^i = \sum_{i,\Phi,\Psi:\omega\in\Psi} A^i(\Phi,\Psi)\cdot n^i \text{ for all } x,$$

$$\sum_{\Phi,\Psi} A^i(\Phi,\Psi) = 1 \text{ for all } i, \text{ and}$$

$$A^i(\Phi,\Psi) = 0 \text{ if } \bar{A}^i(\Phi,\Psi) = 0.$$

Because $\mathscr{A}^*$ is a bounded and non-empty polytope, it has an extreme point $(A^*_i)_{i:n^i>0}$. Theorem 2.3 of Bertsimas and Tsitsiklis (1997) implies that the extreme point is a basic feasible solution to these constraints. Theorem 2.2 of Bertsimas and Tsitsiklis (1997) implies that this basic feasible solution can be written as the product of the inverse of a matrix with integer entries and an integer vector. By Cramer's rule, all the entries of $(A^*_i)_{i:n^i>0}$ are rational numbers. Thus, there exists an allocation A^* such that $[A^*;p]$ is an equilibrium of the ∞-replica, and all the coordinates of A^* with $n^i>0$ are rational numbers.

Consequently, there exists an integer β such that all the coordinates of A^* are integer multiples of $1/\beta$. Therefore, $[A^*;p]$ is a competitive equilibrium of any k-replica where k is a multiple of β.

The first part of the proposition then follows from essentially the same argument as Proposition 2.