CHAPTER 3: Sets, Combinatorics, and Probability

Most students have seen sets and set operations before, and the material in Section 3.1 presents little difficulty, with the exception of the short section on countable and uncountable sets. Cantor's diagonalization method usually requires several passes before the light dawns. This material is not central, and can be omitted if desired, although it's nice for computer science students to know that there are different kinds of infinity.

Counting problems (Sections 3.2-3.4) are something that students usually think are fun. Occasional frustration occurs because there are sometimes very plausible-sounding wrong approaches. I've made a special effort to address this difficulty in the subsection "Eliminating Duplicates" in Section 3.4. Another stumbling block is the ability to distinguish whether the problem is a *combinations* problem or a *permutations* problem (hence determining what formula to use). I reiterate numerous times that if there is ordering involved, the problem is a permutations problem, not a combinations problem. Decision trees are used in Chapter 5 to establish worst-case lower bounds for searching and sorting. The use of decision trees in Section 3.2 for counting outcomes provides an easy introduction.

Section 3.5 introduces probability as a natural extension of counting. It covers the basics of finite probability, probability distributions, conditional probability, and expected value using, I am afraid, the usual coins and dice. An example of average case analysis of algorithms ties this material back to computer science.

The binomial theorem, Section 3.6, offers a nice occasion to see both an inductive proof and a combinatorial proof, but nonetheless I generally omit this section. The students all know how to square a binomial, and that's really about all they need.

EXERCISES 3.1

*1. a. T b. F c. F d. F

2. a. T b. F c. T d. F ($\sqrt{2} \notin Q$)

3. Four:
$\{2, 3, 4\} = \{x | x \in N \text{ and } 2 \le x \le 4\} = \{3, 4, 2\}$
{a, b, c} = {x|x is the first letter of cat, bat, or apple}
$\varnothing$ = {x|x is the first letter of cat, bat, and apple}
{2, a, 3, b, 4, c}

*4. a. {0, 1, 2, 3, 4}
b. {4, 6, 8, 10}
c. {Washington, Adams, Jefferson}
d. $\varnothing$
e. {Maine, Vermont, New Hampshire, Massachusetts, Connecticut, Rhode Island}
f. {-3, -2, -1, 0 , 1, 2, 3}

5. a. $\{2, 3\}$
   b. $\{\sqrt{7}, -\sqrt{7}\}$
   c. $\{4\}$

6. a. $\{x | x \in N \text{ and } 1 \le x \le 5\}$
   b. $\{x | x \in N \text{ and } x \text{ is odd}\}$
   c. $\{x | x \text{ is one of the Three Wise Men}\}$
   d. $\{x | x \text{ is a nonnegative integer written in binary form}\}$

7. a. $\{4, 6\}$
   b. $\{1, 2, 3\}$
   c. $\{x | x \in N \text{ and } x \text{ is an odd number}\}$

*8. If $A = \{x | x = 2^n \text{ for } n \text{ a positive integer}\}$, then $16 \in A$. But if $A = \{x | x = 2 + n(n - 1) \text{ for } n \text{ a positive integer}\}$, then $16 \notin A$. In other words, there is not enough information to answer the question.

9. a. 2 b. 2 c. 1 d. 3 e. 3

10. a. T b. T c. F d. T e. T f. F g. F h. T

11. *a. F; $\{1\} \in S$ but $\{1\} \notin R$
    *b. T
    *c. F; $\{1\} \in S$, not $1 \in S$
    *d. F; 1 is not a set; the correct statement is $\{1\} \subseteq U$
    *e. T
    *f. F; $1 \notin S$
    g. T
    h. T
    i. T
    j. F; $3 \notin U$ and $\pi \notin U$
    k. T
    l. T
    m. T
    n. F

12. a. T
    b. T
    c. F; neither member of C is a member of A
    d. T
    e. T
    f. F; this 2-element set is not an element of A.
    g. T
    h. F; $a \notin C$
    i. T

13. Let $x \in A$. Then $x \in R$ and $x^2 - 4x + 3 < 0$ or $(x - 1)(x - 3) < 0$. The possible solutions are:

    $x - 1 < 0$ and $x - 3 > 0$
    $x < 1$ and $x > 3$, which cannot be satisfied
    or
    $x - 1 > 0$ and $x - 3 < 0$
    $x > 1$ and $x < 3$
    $1 < x < 3$

But if $x \in R$ and $1 < x < 3$, then $x \in R$ and $0 < x < 6$, so $x \in B$. Therefore $A \subseteq B$. The number $5 \in B$ but $5 \notin A$, so $A \subset B$.

*14. Let $(x, y) \in A$. Then (x, y) lies within 3 units of the point $(1, 4)$, so by the distance formula, $\sqrt{(x-1)^2 + (y-4)^2} \leq 3$, or $(x-1)^2 + (y-4)^2 \leq 9$, which means $(x-1)^2 + (y-4)^2 \leq 25$, so $(x, y) \in B$. The point $(6, 4)$ satisfies the inequality $(x-1)^2 + (y-4)^2 \leq 25$, so $(6, 4) \in B$, but $(6, 4)$ is not within 3 units of $(1, 4)$, so $(6, 4)$ does not belong to A.

15. a. For $a = 1$, $b = -2$, $c = -24$, the quadratic equation is $x^2 - 2x - 24 = 0$ or $(x + 4)(x - 6) = 0$, with solutions 6 and -4. Each of these is an even integer between -100 and 100, so each belongs to E.

b. Here $Q = \{6, -4\}$, but $E = \{-4, -2, 0, 2, 4\}$, and $Q \not\subseteq E$.

16. Let $x \in A$. Then $\cos(x/2) = 0$. But $\cos(x/2) = 0$ if and only if $x/2 = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$, or $x = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$ and, for any multiple of π, the sine function is 0. Thus $x \in B$.

17. *a. T *b. F *c. F *d. T *e. T *f. F g. F h. T i. F j. F

18. Let $x \in A$. Then, because $A \subseteq B$, $x \in B$. Because $B \subseteq C$, $x \in C$. Thus $A \subseteq C$.

19. Assume that $A' \subseteq B'$ but that $B \not\subseteq A$. Then there is an element x such that $x \in B$ but $x \notin A$. Thus $x \in A'$ and $x \notin B'$, which contradicts $A' \subseteq B'$.

20. a. The proof uses mathematical induction.
$n = 2$: A set with 2 elements has exactly 1 subset with 2 elements, namely the set itself. Putting $n = 2$ into the formula $n(n-1)/2$ gives the value 1. This proves the base case.
Assume that any set with k elements has $k(k-1)/2$ subsets with exactly 2 elements. Show that any set with $k + 1$ elements has $(k + 1)k/2$ subsets with exactly 2 elements.

Let x be a member of a set with $k + 1$ elements. Temporarily removing x from the set gives a set of k elements that, by the inductive hypothesis, has $k(k-1)/2$ subsets with exactly 2 elements. These are all of the 2-element subsets of the original set that do not include x. All 2-element subsets of the original set that do include x can be found by pairing x in turn with each of the remaining k elements, giving k subsets. The total number of 2-element subsets is therefore

$$\frac{k(k-1)}{2} + k = \frac{k(k-1) + 2k}{2} = \frac{k^2 - k + 2k}{2} = \frac{k^2 + k}{2} = \frac{(k+1)k}{2}$$

b. Again, the proof is by induction.
$n = 3$: A set with 3 elements has exactly 1 subset with 3 elements, namely the set itself. Putting $n = 3$ into the formula $n(n - 1)(n - 2)/6$ gives the value 1. This proves the base case.
Assume that any set with k elements has $k(k - 1)(k - 2)/6$ subsets with exactly 3 elements.
Show that any set with $k + 1$ elements has $(k + 1)k(k - 1)/6$ subsets with exactly 3 elements.

Let x be a member of a set with $k + 1$ elements. Temporarily removing x from the set gives a set of k element that, by the inductive hypothesis, has $k(k - 1)(k - 2)/6$ subsets with exactly 3 elements. These are all of the 3-element subsets of the original set that do not include x. All 3-element subsets of the original set that do include x can be found by pairing x in turn with each of the 2-element subsets of the k-element set. From part (a), there are $k(k - 1)/2$ such subsets. The total number of 3-element subsets is therefore

$$\frac{k(k-1)(k-2)}{6}+\frac{k(k-1)}{2}=\frac{k(k-1)(k-2)+3k(k-1)}{6}=\frac{k(k-1)(k-2+3)}{6}=\frac{(k+1)k(k-1)}{6}$$

21. $\wp(S) = \{\emptyset, \{a\}\}$

*22. $\wp(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$; $2^4 = 16$ elements

23. $\wp(S) = \{\emptyset,\{\emptyset\}\}$

24. $\wp(S) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\emptyset, \{\emptyset\}\}\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}, \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}$

25. $\wp(\wp(S)) = \{\emptyset, \{\emptyset\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}, \{\emptyset, \{a\}\}, \{\emptyset, \{b\}\}, \{\emptyset, \{a, b\}\}, \{\{a\}, \{b\}\}, \{\{a\}, \{a, b\}\}, \{\{b\}, \{a, b\}\}, \{\emptyset, \{a\}, \{b\}\}, \{\emptyset, \{a\}, \{a, b\}\}, \{\emptyset, \{b\}, \{a, b\}\}, \{\{a\}, \{b\}, \{a, b\}\}, \{\emptyset, \{a\}, \{b\}, \{a, b\}\}\}$

*26. $A = \{x, y\}$

27. $\wp(A)$ contains 3 elements, and $3 \neq 2^n$ for any n, so such a set A does not exist.

28. Let $X \in \wp(A)$. Then X is a subset of A, hence, since $A \subseteq B$, X is a subset of B and $X \in \wp(B)$.

*29. Let $x \in A$. Then $\{x\} \in \wp(A)$, so $\{x\} \in \wp(B)$ and $x \in B$. Thus $A \subseteq B$. A similar argument shows that $B \subseteq A$ so that $A = B$.

30. a. $x = 1, y = 5$
    b. $x = 8, y = 7$
    c. $x = 1, y = 4$

31. a. If $x = u$ and $y = v$, then clearly $\{\{x\},\{x, y\}\} = \{\{u\},\{u, v\}\}$.
    Now assume $\{\{x\},\{x, y\}\} = \{\{u\},\{u, v\}\}$. Then $\{x\} = \{u\}$ or $\{x\} = \{u, v\}$. If $\{x\} = \{u, v\}$, then $u = v = x$; also $\{u\} = \{x, y\}$, and $x = y = u$. Thus $x = u$ and $y = v$. If $\{x\} = \{u\}$, then $x = u$; also $\{x, y\} = \{u, v\}$ and, since $x = u$, $y = v$.

    b. For example, the ordered triples (1, 1, 2) and (1, 2, 1) expressed in set form would be
    $\{\{1\},\{1, 1\},\{1, 1, 2\}\} = \{\{1\},\{1, 2\}\}$ and
    $\{\{1\},\{1, 2\},\{1, 2, 1\}\} = \{\{1\},\{1, 2\}\}$, respectively,
    and distinct ordered triples would have the same representation.

32. *a. binary operation
    *b. no; $0 \circ 0 \notin N$
    *c. binary operation
    d. no; ln x undefined for $x \le 0$
    e. unary operation
    f. no; closure fails
    g. no; uniqueness fails (two different fractions could have the same denominator)
    h. binary operation

33. a. no; operation undefined for $x = 0$
    b. binary operation
    c. unary operation
    d. binary operation
    e. no; $x \circ y$ is undefined - there is no "greatest" common multiple because a larger common multiple can always be found
    f. no; closure fails because the sum of two random Fibonacci numbers is not always a Fibonacci number. For example, 1 + 3 = 4 and 4 is not a Fibonacci number although 1 and 3 are.
    g. unary operation
    h. no; closure fails. The sum of two irrational numbers can be rational, for example, $\sqrt{2} + (-\sqrt{2}) = 0$

34. n^{n^2}; each of the n^2 entries in the n x n matrix has n possible answers. We multiply the number of possibilities. Thus there are $\underbrace{n \cdot n \cdot \ldots \cdot n}_{n^2} = n^{n^2}$ ways to complete the table.

	x_1	x_n
x_1		
x_n		

35. a. (A + B)*(C - D) = ((A + B)*(C - D)) $\rightarrow$ AB+CD-*
    b. A**B - C*D = ((A**B) - (C*D)) $\rightarrow$ AB**CD*-
    c. A*C + B/(C + D*B) = ((A*C) + (B/(C + (D*B)))) $\rightarrow$ AC*BCDB*+/+

*36. a. 13 b. 2 c. 28

37. a. {t} b. {p, q, r, s, t, u} c. {q, r, v, w} d. ∅ e. {r, v} f. {u, w}
g. {(p, r), (p, t), (p, v), (q, r), (q, t), (q, v), (r, r), (r, t), (r, v), (s, r), (s, t), (s, v)}
h. {q, r, v}

38.*a. {1, 2, 4, 5, 6, 8, 9} *b. {4, 5}
*c. {2, 4} d. {1, 2, 3, 4, 5, 9}
e. {2, 6, 8} f. {0, 1, 3, 7, 9}
g. ∅ *h. {0, 1, 2, 3, 6, 7, 8, 9}
i. {2, 3} j. {0, 1, 3, 4, 7, 9}
k. {2, 6, 8} l. {2, 3}
m. {(1, 2), (1, 3), (1, 4), (4, 2), (4, 3), (4, 4), (5, 2), (5, 3), (5, 4), (9, 2), (9, 3), (9, 4)}

39. a. {a} b. {∅, {a}, {a, {a}} c. {∅, a, {a}, {{a}}, {a, {a}} } = S
d. ∅ e. {a, {a}} f. {∅, {a, {a}} } g. {∅}

40. a. {Adams, Jefferson, Grant} b. {Washington} c. ∅

41. a. F b. T c. T d. T

*42. a. B' b. $B \cap C$
c. $A \cap B$ d. $B' \cap C$
e. $B' \cap C'$ or $(B \cup C)'$ or $B' - C$

43. a. A' b. $A \cap B$ c. $A - B$

44. *a. C' *b. $B \cap D$ *c. $A \cap B$
d. $A \cap D'$ e. $B' \cap D'$ f. $C \cap A'$
g. $C \cup D$

*45. $D \cap R'$

46. $O \cap (G \cup R) \cap V'$

47. $(N \cup P) \cap A$

48.*a. T
b. T
*c. F (Let A = {1, 2, 3}, B = {1, 3, 5}, S = (1, 2, 3, 4, 5). Then
$(A \cap B)'$ = {2, 4, 5} but $A' \cap B'$ = {4, 5} ∩ {2, 4} = {4}.)
d. T
*e. F (Take A, B, and S as in (c), then A - B = {2}, (B - A)' = {1, 2, 3, 4}.)
f. T
g. F
h. F (Order matters in ordered pairs, so if A = {1, 2}, B = {3, 4}, then
B × A = {(3, 1), (3, 2), (4, 1), (4, 2)} and A × B = {(1, 3), (1, 4), (2, 3), (2, 4)}.)
i. T
j. T

k. F (Let $A = \{1, 2\}$, $B = \{2, 3, 4\}$, $C = \{1, 3\}$. Then $(A - B) \cup (B - C) = \{1\} \cup \{2, 4\} = \{1, 2, 4\}$ but $A - C = \{2\}$.)
l. T

49.*a. $B \subseteq A$ b. $A \subseteq B$
c. $A = \varnothing$ d. $B \subseteq A$
e. $A = B$ f. $A = B$

50. a. 12 b. 9 c. 16 d. 3 e. 4

51. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$, so $x \in A$.

52. Let $x \in A$. Then $x \in A$ or $x \in B$ so $x \in A \cup B$.

*53. Let $C \in \wp(A) \cap \wp(B)$. Then $C \in \wp(A)$ and $C \in \wp(B)$, from which $C \subseteq A$ and $C \subseteq B$, so $C \subseteq A \cap B$ or $C \in \wp(A \cap B)$. Therefore $\wp(A) \cap \wp(B) \subset \wp(A \cap B)$. The same argument works in reverse.

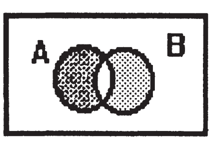
54. Let $C \in \wp(A) \cup \wp(B)$. Then $C \in \wp(A)$ or $C \in \wp(B)$, which means $C \subseteq A$ or $C \subseteq B$. In either case, $C \subseteq A \cup B$, or $C \in \wp(A \cup B)$.

55. Suppose $B \neq \varnothing$. Let $x \in B$. Then $x \in A \cup B$ but $x \notin A - B$, which contradicts the equality of $A \cup B$ and $A - B$.

*56. Suppose $A \cap B \neq \varnothing$. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$ so $x \in A \cup B$ but $x \notin A - B$ and $x \notin B - A$, so $x \notin (A - B) \cup (B - A)$, which contradicts the equality of $A \cup B$ and $(A - B) \cup (B - A)$.

57. i) Let $C \subseteq A$. Then $A \cup C = A$. Also,
$(A \cap B) \cup C = C \cup (A \cap B) = (C \cup A) \cap (C \cup B) = (A \cup C) \cap (B \cup C)$
$= A \cap (B \cup C)$
ii) Assume that $(A \cap B) \cup C = A \cap (B \cup C)$ and let $x \in C$. Suppose $x \notin A$. Then $x \in (A \cap B) \cup C$ but $x \notin A \cap (B \cup C)$, which is a contradiction. Therefore $x \in A$ and $C \subseteq A$.

58. a.



b. $\{2, 4, 6, 7, 9\}$

c. $x \in (A \cup B) - (A \cap B) \leftrightarrow x \in (A \cup B)$ and $x \in (A \cap B)'$
$\leftrightarrow (x \in A$ or $x \in B)$ and $x \notin A \cap B$
$\leftrightarrow (x \in A$ and $x \notin A \cap B)$ or $(x \in B$ and $x \notin A \cap B)$
$\leftrightarrow (x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A)$
$\leftrightarrow x \in (A - B) \cup (B - A)$

d. $\varnothing$; A

e. $A \oplus B = (A \cup B) - (A \cap B) = (B \cup A) - (B \cap A) = B \oplus A$

f. First note that $(A \oplus B)' = (A \cap B) \cup (A' \cap B')$. Then

$$\begin{aligned}(A \oplus B) \oplus C &= [(A \oplus B) - C] \cup [C - (A \oplus B)] \\ &= [(A \oplus B) \cap C'] \cup [C \cap (A \oplus B)'] \\ &= [((A - B) \cup (B - A)) \cap C'] \cup [C \cap ((A \cap B) \cup (A' \cap B'))] \\ &= (A \cap B' \cap C') \cup (B \cap A' \cap C') \cup (C \cap A \cap B) \cup (C \cap A' \cap B') \\ &= (A \cap B \cap C) \cup (A \cap B' \cap C') \cup (B \cap C' \cap A') \cup (C \cap B' \cap A') \\ &= A \cap [(B \cap C) \cup (B' \cap C')] \cup [(B \cap C') \cup (C \cap B')] \cap A' \\ &= [A \cap (B \oplus C)'] \cup [(B \oplus C) \cap A'] \\ &= (A - (B \oplus C)) \cup ((B \oplus C) - A) \\ &= A \oplus (B \oplus C)\end{aligned}$$

59. a. F b. T c. T d. F e. F

60. (1a) $x \in A \cup B \leftrightarrow x \in A$ or $x \in B \leftrightarrow x \in B$ or $x \in A \leftrightarrow x \in B \cup A$

(1b) $x \in A \cap B \leftrightarrow x \in A$ and $x \in B \leftrightarrow x \in B$ and $x \in A \leftrightarrow x \in B \cap A$

(2a) $x \in (A \cup B) \cup C \leftrightarrow x \in (A \cup B)$ or $x \in C \leftrightarrow (x \in A$ or $x \in B)$ or $x \in C$
$\leftrightarrow x \in A$ or $x \in B$ or $x \in C \leftrightarrow x \in A$ or $(x \in B$ or $x \in C) \leftrightarrow x \in A$ or $x \in (B \cup C)$
$\leftrightarrow x \in A \cup (B \cup C)$

(2b) $x \in (A \cap B \cap C) \leftrightarrow x \in (A \cap B)$ and $x \in C \leftrightarrow (x \in A$ and $x \in B)$ and $x \in C$
$\leftrightarrow x \in A$ and $x \in B$ and $x \in C \leftrightarrow x \in A$ and $(x \in B$ and $x \in C)$
$\leftrightarrow x \in A$ and $x \in (B \cap C) \leftrightarrow x \in A \cap (B \cap C)$

(3b) $x \in A \cap (B \cup C) \leftrightarrow x \in A$ and $x \in (B \cup C) \leftrightarrow x \in A$ and $(x \in B$ or $x \in C)$
$\leftrightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C) \leftrightarrow x \in (A \cap B)$ or $x \in (A \cap C)$
$\leftrightarrow x \in (A \cap B) \cup (A \cap C)$

(4b) $x \in A \cap S \rightarrow x \in A$ and $x \in S \rightarrow x \in A$
$x \in A \rightarrow x \in A$ and $x \in S$ because $A \subseteq S \rightarrow x \in A \cap S$

(5a) $x \in A \cup A' \rightarrow x \in A$ or $x \in A' \rightarrow x \in S$ or $x \in S$ because $A \subseteq S$, $A' \subseteq S \rightarrow x \in S$
$x \in S \rightarrow (x \in S$ and $x \in A)$ or $(x \in S$ and $x \notin A) \rightarrow x \in A$ or $x \in A' \rightarrow x \in A \cup A'$

(5b) For any x such that $x \in A \cap A'$, it follows that $x \in A$ and $x \in A'$, or x belongs to A and x does not belong to A. This is a contradiction, so no x belongs to $A \cap A'$, and $A \cap A' = \varnothing$

61. a. $x \in (A \cup B)' \leftrightarrow x \notin (A \cup B) \leftrightarrow$ x does not belong to either A or B
$\leftrightarrow x \notin A$ and $x \notin B \leftrightarrow x \in A'$ and $x \in B' \leftrightarrow x \in A' \cap B'$

b. $x \in (A \cap B)' \leftrightarrow x \notin A \cap B \leftrightarrow$ x does not belong to both A and B
$\leftrightarrow x \notin A$ or $x \notin B \leftrightarrow x \in A'$ or $x \in B' \leftrightarrow x \in A' \cup B'$

*c. $x \in A \cup (B \cap A) \leftrightarrow x \in A$ or $x \in (B \cap A) \leftrightarrow x \in A$ or $(x \in B$ and $x \in A) \leftrightarrow x \in A$

d. $x \in (A \cap B')' \cup B \leftrightarrow x \in (A \cap B')'$ or $x \in B \leftrightarrow x \in (A' \cup B)$ or $x \in B$ by part (b) and $(B')' = B \leftrightarrow x \in A'$ or $x \in B$ or $x \in B \leftrightarrow x \in A'$ or $x \in B \leftrightarrow x \in A' \cup B$

e. $x \in (A \cap B) \cup (A \cap B') \leftrightarrow x \in A \cap B$ or $x \in A \cap B'$
$\leftrightarrow$ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in B'$) $\leftrightarrow x \in A$

f. $x \in (A \cap (B \cup C))' \leftrightarrow x \notin (A \cap (B \cup C)) \leftrightarrow x \notin A$ or $x \notin (B \cup C)$
$\leftrightarrow x \in A'$ or ($x \notin B$ and $x \notin C$) $\leftrightarrow x \in A'$ or ($x \in B'$ and $x \in C'$)
$\leftrightarrow x \in (A' \cup (B' \cap C')$

62.*a. $((A \cup B) \cap (A \cup B') = A \cup (B \cap B')$ (3a)
$= A \cup \varnothing$ (5b)
$= A$ (4a)
The dual is $(A \cap B) \cup (A \cap B') = A$

b. $[((A \cap C) \cap B) \cup ((A \cap C) \cap B')] \cup (A \cap C)'$
$= [(A \cap C) \cap (B \cup B')] \cup (A \cap C)'$ (3b)
$= [(A \cap C) \cap S] \cup (A \cap C)'$ (5a)
$= (A \cap C) \cup (A \cap C)'$ (4b)
$= S$ (5a)
The dual is $[((A \cup C) \cup B) \cap ((A \cup C) \cup B')] \cap (A \cup C)' = \varnothing$

c. $(A \cup C) \cap [(A \cap B) \cup (C' \cap B)]$
$= (A \cup C) \cap [(B \cap A) \cup (B \cap C')]$ (1b)
$= (A \cup C) \cap [B \cap (A \cup C')]$ (3b)
$= (A \cup C) \cap [(A \cup C') \cap B]$ (1b)
$= [(A \cup C) \cap (A \cup C')] \cap B$ (2b)
$= [A \cup (C \cap C')] \cap B$ (3a)
$= (A \cup \varnothing) \cap B$ (5b)
$= A \cap B$ (4a)
The dual is $(A \cap C) \cup [(A \cup B) \cap (C' \cup B)] = A \cup B$

63. a. $A \cup A = \{x \mid x \in A \text{ or } x \in A\} = \{x \mid x \in A\} = A$
b. $A \cap A = \{x \mid x \in A \text{ and } x \in A\} = \{x \mid x \in A\} = A$
c. $A \cap \varnothing = \{x \mid x \in A \text{ and } x \in \varnothing\}$ but since no x is a member of $\varnothing$, this is the empty set.
d. $A \cup S \subseteq S$ because $A \subseteq S$; $S \subseteq A \cup S$ by Exercise 52.
*e. $x \in (A')' \leftrightarrow x \notin A' \leftrightarrow x \notin \{y \mid y \notin A\} \leftrightarrow x \in A$

64. *a. $A \cap (B \cup A') = (A \cap B) \cup (A \cap A')$ (3b)
$= (A \cap B) \cup \varnothing$ (5b)
$= A \cap B$ (4a)
$= B \cap A$ (1b)

b. $(A \cup B) - C = (A \cup B) \cap C'$ (defn. set diff.)
$= C' \cap (A \cup B)$ (1b)
$= (C' \cap A) \cup (C' \cap B)$ (3b)
$= (A \cap C') \cup (B \cap C')$ (1b)
$= (A - C) \cup (B - C)$ (defn. set diff.)

c. $(A - B) - C = (A - B) \cap C'$ (defn. set diff.)
$= (A \cap B') \cap C'$ (defn. set diff.)
$= C' \cap (A \cap B')$ (1b)
$= (C' \cap A) \cap B'$ (2b)
$= (A \cap C') \cap B'$ (1b)
$= (A - C) \cap B'$ (defn. set diff.)
$= (A - C) - B$ (defn. set diff.)

d. $((A' \cup B') \cap A')' = ((A \cap B)' \cap A')'$ (DeMorgan's Laws)
$= ((A \cap B)')' \cup (A')'$ (DeMorgan's Laws)
$= (A \cap B) \cup A$ (part (a))
$= A \cup (A \cap B)$ (1a)
$= A \cup (B \cap A)$ (1b)
$= A$ (Exer. 56c)

e. $(A - B) - C = (A - B) \cap C'$ (defn. set diff.)
$= (A \cap B') \cap C'$ (defn. set diff.)
$= A \cap (B' \cap C')$ (2b)
$= A \cap (C' \cap B')$ (1b)
$= (A \cap C') \cap B'$ (2b)
$= ((A \cap C') \cap B') \cup \varnothing$ (4a)
$= ((A \cap C') \cap B') \cup (A \cap \varnothing)$ (Exercise 63c)
$= ((A \cap C') \cap B') \cup (A \cap (C \cap C'))$ (5b)
$= ((A \cap C') \cap B') \cup (A \cap (C' \cap C))$ (1b)
$= ((A \cap C') \cap B') \cup ((A \cap C') \cap C)$ (2b)
$= (A \cap C') \cap (B' \cup C)$ (3b)
$= (A \cap C') \cap (B' \cup (C')')$ (Exercise 63e)
$= (A \cap C') \cap (B \cap C')'$ (De Morgan's Laws)
$= (A \cap C') - (B \cap C')$ (defn. set diff.)
$= (A - C) - (B - C)$ (defn. set diff.)

f. $A - (A - B) = A \cap (A - B)'$ (defn. set diff.)
$= A \cap (A \cap B')'$ (defn. set diff.)
$= A \cap (A' \cup (B')')$ (De Morgan's Laws))
$= A \cap (A' \cup B)$ (Exercise 63e)
$= (A \cap A') \cup (A \cap B)$ (3b)
$= \varnothing \cup (A \cap B)$ (5b)
$= (A \cap B) \cup \varnothing$ (1a)
$= A \cap B$ (4a)

65. a. $A_1 \cup A_2 \cup \ldots \cup A_n = \{x | x \text{ belongs to some } A_i \text{ for } 1 \le i \le n\}$

b. $A_1 \cup A_2 = \{x | x \in A_1 \text{ or } x \in A_2\}$ $\quad n = 2$

$A_1 \cup A_2 \cup \ldots \cup A_n = (A_1 \cup \ldots \cup A_{n-1}) \cup A_n$ $\quad n > 2 \quad (1)$

*66. The proof is by induction on n. For n = 3,

$(A_1) \cup (A_2 \cup A_3) = (A_1 \cup A_2) \cup A_3$ by set identity 2a

$= A_1 \cup A_2 \cup A_3$ by Equation (1) of the answer to Exercise 65(b)

Assume that for $n = k$ and $1 \le p \le k - 1$,

$(A_1 \cup \ldots \cup A_p) \cup (A_{p+1} \cup \ldots \cup A_k) = A_1 \cup \ldots \cup A_k$

Then for $1 \le p \le k$,

$(A_1 \cup \ldots \cup A_p) \cup (A_{p+1} \cup \ldots \cup A_{k+1})$

$= (A_1 \cup \ldots \cup A_p) \cup [(A_{p+1} \cup \ldots \cup A_k) \cup A_{k+1}]$ by Equation (1)

$= [(A_1 \cup \ldots \cup A_p) \cup (A_{p+1} \cup \ldots \cup A_k)] \cup A_{k+1}$ by set identity 2a

$= (A_1 \cup \ldots \cup A_k) \cup A_{k+1}$ by inductive hypothesis

$= A_1 \cup \ldots \cup A_{k+1}$ by Equation (1)

67. a. $A_1 \cap A_2 \cap \ldots \cap A_n = \{x | x \text{ belongs to every } A_i \text{ for } 1 \le i \le n\}$

b. $A_1 \cap A_2 = \{x | x \in A_1 \text{ and } x \in A_2\}$ $\quad n = 2$

$A_1 \cap A_2 \cap \ldots \cap A_n = (A_1 \cap \ldots \cap A_{n-1}) \cap A_n$ $\quad n > 2 \quad (2)$

68. The proof is by induction on n. For n = 3,

$(A_1) \cap (A_2 \cap A_3) = (A_1 \cap A_2) \cap A_3$ by set identity 2b

$= A_1 \cap A_2 \cap A_3$ by Equation (2) of the answer to Exercise 67(b)

Assume that for $n = k$ and $1 \le p \le k - 1$,

$(A_1 \cap \ldots \cap A_p) \cap (A_{p+1} \cap \ldots \cap A_k) = A_1 \cap \ldots \cap A_k$

Then for $1 \le p \le k$,

$(A_1 \cap \ldots \cap A_p) \cap (A_{p+1} \cap \ldots \cap A_{k+1})$

$= (A_1 \cap \ldots \cap A_p) \cap [(A_{p+1} \cap \ldots \cap A_k) \cap A_{k+1}]$ by Equation (2)

$= [(A_1 \cap \ldots \cap A_p) \cap (A_{p+1} \cap \ldots \cap A_k)] \cap A_{k+1}$ by set identity 2b

$= (A_1 \cap \ldots \cap A_k) \cap A_{k+1}$ by inductive hypothesis

$= A_1 \cap \ldots \cap A_{k+1}$ by Equation (2)

69. *a. Proof is by induction on n.

For n = 2, $B \cup (A_1 \cap A_2) = (B \cup A_1) \cap (B \cup A_2)$ by identity 3a.

Assume that $B \cup (A_1 \cap \ldots \cap A_k) = (B \cup A_1) \cap \ldots \cap (B \cup A_k)$

Then $B \cup (A_1 \cap \ldots \cap A_{k+1})$

$= B \cup ((A_1 \cap \ldots \cap A_k) \cap A_{k+1})$	by Exercise 67b
$= (B \cup (A_1 \cap \ldots \cap A_k)) \cap (B \cup A_{k+1})$	by identity 3a
$= ((B \cup A_1) \cap \ldots \cap (B \cup A_k)) \cap (B \cup A_{k+1})$	by inductive hyp.
$= (B \cup A_1) \cap \ldots \cap (B \cup A_{k+1})$	by Exercise 67b

b. Proof is by induction on n.
For $n = 2$, $B \cap (A_1 \cup A_2) = (B \cap A_1) \cup (B \cap A_2)$ by identity 3b.

Assume that $B \cap (A_1 \cup \ldots \cup A_k) = (B \cap A_1) \cup \ldots \cup (B \cap A_k)$

Then $B \cap (A_1 \cup \ldots \cup A_{k+1})$

$= B \cap ((A_1 \cup \ldots \cup A_k) \cup A_{k+1})$	by Exercise 65b
$= (B \cap (A_1 \cup \ldots \cup A_k)) \cup (B \cap A_{k+1})$	by identity 3b
$= ((B \cap A_1) \cup \ldots \cup (B \cap A_k)) \cup (B \cap A_{k+1})$	by inductive hyp.
$= (B \cap A_1) \cup \ldots \cup (B \cap A_{k+1})$	by Exercise 65b

70\. a. Proof is by induction on n
For $n = 2$, $(A_1 \cup A_2)' = A_1' \cap A_2'$ by Exercise 61a

Assume that $(A_1 \cup \ldots \cup A_k)' = A_1' \cap \ldots \cap A_k'$

Then $(A_1 \cup \ldots \cup A_k \cup A_{k+1})' = ((A_1 \cup \ldots \cup A_k) \cup A_{k+1})'$	by Exercise 65b
$= (A_1 \cup \ldots \cup A_k)' \cap A_{k+1}'$	by Exercise 61a
$= (A_1' \cap \ldots \cap A_k') \cap A_{k+1}'$	by inductive hyp.
$= A_1' \cap \ldots \cap A_k' \cap A_{k+1}'$	by Exercise 67b

b. Proof is by induction on n
For $n = 2$, $(A_1 \cap A_2)' = A_1' \cup A_2'$ by Exercise 61b

Assume that $(A_1 \cap \ldots \cap A_k)' = A_1' \cup \ldots \cup A_k'$

Then $(A_1 \cap \ldots \cap A_k \cap A_{k+1})' = ((A_1 \cap \ldots \cap A_k) \cap A_{k+1})'$	by Exercise 67b
$= (A_1 \cap \ldots \cap A_k)' \cup A_{k+1}'$	by Exercise 61b
$= (A_1' \cup \ldots \cup A_k') \cup A_{k+1}'$	by inductive hyp.
$= A_1' \cup \ldots \cup A_k' \cup A_{k+1}'$	by Exercise 65b

71\. a. $\bigcup_{i \in I} A_i = \{x \mid x \in (-1,1)\}$; $\bigcap_{i \in I} A_i = \{0\}$

b. $\bigcup_{i \in I} A_i = \{x \mid x \in [-1,1]\}$; $\bigcap_{i \in I} A_i = \{0\}$

72. a. AIDS $\cup$ ALZHEIMERS = {genetics 0.7, virus 0.8, nutrition 0.3, bacteria 0.4, environment 0.4}

b) AIDS $\cap$ ALZHEIMERS = {genetics 0.2, virus 0.4, nutrition 0.1, bacteria 0.3, environment 0.3}

c) AIDS' = {genetics 0.8, virus 0.2, nutrition 0.9, bacteria 0.6, environment 0.7}

73. P(1) is true - every member of T is greater than 1, otherwise 1 would be the smallest member of T. Assume that P(k) is true, i.e., every member of T is greater than k. Consider P(k + 1), that every member of T is greater than k + 1. If P(k + 1) is not true, then there is some member of T $\leq$ k + 1. By the inductive hypothesis, every member of T is greater than k, therefore some member of T equals k + 1, and this is the smallest member of T. This is a contradiction, because we assumed T has no smallest member. Therefore P(k + 1) is true. By the first principle of induction, P(n) is true for all n, and T must be empty. This contradicts the fact that T is a non-empty set.

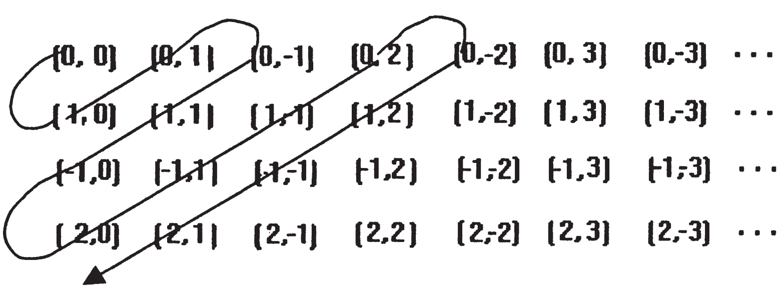
*74. If T is a non-empty set, then by the principle of well-ordering, T has a smallest member t_0. Then $P(t_0)$ is not true, so by statement 1', $t_0 \neq 1$. Also P(r) is true for all r, $1 \leq r \leq t_0 - 1$. This contradicts the implication in 2', so T is the empty set and therefore P(n) is true for all positive integers n.

*75. An enumeration of the set is 1, 3, 5, 7, 9, 11, ...

76. An enumeration of Z is 0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5, ...

*77. An enumeration of the set is a, aa, aaa, aaaa, ...

78. An enumeration of the set is shown by the arrow through the array



79. a. n = 0.249999....

100n = 24.9999....

99n = 100n – n = 24.9999.... – 0.2499999 = 24.75

n = 24.75/99 = 0.25

b. m = 0.250000...

100m = 25.0000...

99m = 100m – m = 25.0000... - 0.250000.... = 24.75

m = 24.75/99 = 0.25

c. n = m

80. Assume that the set has an enumeration

$$\begin{array}{l} z_{11}, z_{12}, z_{13}, z_{14}, \ldots \\ z_{21}, z_{22}, z_{23}, z_{24}, \ldots \\ z_{31}, z_{32}, z_{33}, z_{34}, \ldots \\ \vdots \end{array}$$

Now construct an infinite sequence Z of positive integers with $Z = z_1, z_2, z_3, \ldots$ such that $z_i \neq z_{ii}$ for all i. Then Z differs from every sequence in the enumeration, yet is a member of the set. This is a contradiction, so the set is uncountable.

81. Assume that the set has an enumeration

$$\begin{array}{l} s_{11}s_{12}s_{13}s_{14} \ldots \\ s_{21}s_{22}s_{23}s_{24} \ldots \\ s_{31}s_{32}s_{33}s_{34} \ldots \\ \vdots \end{array}$$

where each s_{ii} is either a or b. Now construct an infinite string $s = s_1s_2s_3s_4 \ldots$ such that $s_i = a$ if $s_{ii} = b$, and $s_i = b$ if $s_{ii} = a$. Then s differs from every string in the enumeration, yet is a member of the set. This is a contradiction, so the set is uncountable.

82. Let A be a countable set. Then A is finite or countably infinite. If A is finite and $B \subseteq A$, then B is finite, hence countable. If A is countably infinite, let a_1, a_2, a_3, be an enumeration of A. Using this same list but eliminating elements in A - B gives an enumeration of B.

*83. Let A and B be denunmerable sets with enumerations
$A = a_1, a_2, a_3, \ldots$ and $B = b_1, b_2, b_3, \ldots$
Then use the list $a_1, b_1, a_2, b_2, a_3, b_3, \ldots$ and eliminate any duplicates. This will be an enumeration of $A \cup B$, which is therefore denumerable.

84. $B = \{S | S \text{ is a set and } S \notin S\}$. (Perhaps you think that no set S can be an element of itself, in which case B is empty. But we can still talk about set B.) Then either $B \in B$ or $B \notin B$. If $B \in B$, then B has the property of all members of B, namely $B \notin B$. Hence both $B \in B$ and $B \notin B$ are true. If $B \notin B$, then B has the property characterizing members of B, hence $B \in B$. Therefore both $B \notin B$ and $B \in B$ are true.

EXERCISES 3.2

*1. $5 \cdot 3 \cdot 2 = 30$

*2. $4 \cdot 2 \cdot 2 = 16$

3. $4 \cdot 8 \cdot 6 = 92$

4. $4^{20} \cdot 5^{10}$

5. $26^3 \cdot 10^2$

6. $52^3 \cdot 10^2$

*7. $45 \cdot 13 = 585$

8. $3 \cdot 2\,(A - B - D) + 2 \cdot 4\,(A - C - D) = 14$

*9. 10^9

10. No - the number of different codes is $10 \cdot 10 = 100$, so not every apartment has its own code.

*11. $26 \cdot 26 \cdot 26 \cdot 1 \cdot 1 = 17{,}576$

12. $2 \cdot 4 \cdot 4 = 32$

13. $2 \cdot 2 \cdot 2 \cdot 2$ (fill in the 4 rows of the truth table with T or F)

*14. (3, 1)

B R	R B	R B	B R	B R	R B	R B	B R	
R	B	B	R	R	B	B	R	...

The cycles affect the (bottom) 2 elements of the stack, then the bottom 1 element of the stack. The stack number is 21.

(1, 3)

B R	B R	R B	R B	B R	B R	R B	R B	
R	R	B	B	R	R	B	B	...

The cycles affect the bottom 1 element of the stack, then the (bottom) 2 elementS of the stack. The stack number is 12.

(2, 2)

B R	R B	B R	R B	B R	R B	B R	R B	
R	B	R	B	R	B	R	B	...

The cycles always affect the (bottom) 2 elements of the stack. The stack number is 22.

(1, 1)

R	R	R	R	R	R	R	R	
R	R	R	R	R	R	R	R	...

The cycles always affect the (bottom) 1 element of the stack. The stack number is 11.

15. a. (1, 5)
 b.

R	R	G	G	B	B	R	R	G	G	B	B	
B	B	R	R	G	G	B	B	R	R	R	R	
G	G	B	B	R	R	G	G	B	B	G	G	
G	G	B	B	R	R	G	G	B	B	R	R	...

The cycles affect the bottom 1 element of the stack, then the (bottom) 3 elements of the stack. The stack number is 13.

 c. Using the stack number 221, we can recreate the stacks

B	R	B	B	R	B	B	R	B	B	...
R	B	R	R	B	R	R	B	R	R	

and then the juggling pattern, which is

B	R	B	B	R	B	B	R	B	B	...
R	B	R	R	B	R	R	B	R	R	
R	B	R	R	B	R	R	B	R	R	...
2	3	1	2	3	1	2	3	1	2	...

or (2, 3, 1).

16. a. $3^2 = 9$
 b. 11, 12, 13, 21, 22, 23, 31, 32, 33
 c. From Exercise 14, the stack numbers 11, 12, 21, and 22 represent all the juggling patterns of length 2 using 2 balls. The remaining 5 stack numbers must involve 3 balls. Using them, the stacks and then the juggling patterns can be recreated:

stack number 13:

R	R	G	G	B	B	R	R	G	G	B	B	
B	B	R	R	G	G	B	B	R	R	R	R	
G	G	B	B	R	R	G	G	B	B	G	G	
G	G	B	B	R	R	G	G	B	B	R	R	...

pattern (1, 5) [see Exercise 15a and 15b) stack number 23:

R	R	B	B	R	R	B	B	R	R	B	B	
B	G	R	G	B	G	R	G	B	G	R	G	
G	B	G	R	G	B	G	R	G	B	G	R	
G	B	G	R	G	B	G	R	G	B	G	R	…

pattern (2, 4)

stack number 31:

R	G	G	B	B	R	R	G	G	B	B	R	
B	R	R	G	G	B	B	R	R	G	G	B	
G	B	B	R	R	G	G	B	B	R	R	G	
G	B	B	R	R	G	G	B	B	R	R	G	…

pattern (5, 1)

stack number 32:

R	G	G	R	R	G	G	R	R	G	G	R	
B	R	B	G	B	R	B	G	B	R	B	G	
G	B	R	B	G	B	R	B	G	B	R	B	
G	B	R	B	G	B	R	B	G	B	R	B	…

pattern (4, 2)

stack number 33

R	G	B	R	G	B	R	G	B	R	G	B	
B	R	G	B	R	G	B	R	G	B	R	G	
G	B	R	G	B	R	G	B	R	G	B	R	
G	B	R	G	B	R	G	B	R	G	B	R	…

pattern (3, 3)

17. $4^3 = 64$

*18. $26 + 26 \cdot 10 = 286$

19. $4 \cdot 3 \cdot 2 = 24$

20. $17 \cdot 16 + 24 \cdot 23 = 824$

*21. $5 \cdot 3 \cdot 4 \cdot 3 = 180$

22. $5 \cdot 4 \cdot 3 + 3 \cdot 4 \cdot 3 = 96$

23. $10 \cdot 7 \cdot 3 \cdot 2 \cdot 2 \cdot 2 = 1680$

24. $10 \cdot 7 \cdot 1 \cdot 2 \cdot 2 \cdot 2 + 10 \cdot 7 \cdot 2 \cdot 2 \cdot 2 \cdot 1 = 1120$

*25. $9\cdot10\cdot26\cdot10\cdot10+9\cdot10\cdot26\cdot10\cdot10\cdot10+9\cdot10\cdot26\cdot10\cdot10\cdot10\cdot10=25{,}974{,}000$

26. $2\cdot2\cdot2\cdot2$ (hamburger alone) + $2\cdot2\cdot2$ (fish sandwich alone) + 5 (beverage alone) + $2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2$ (hamburger and fish) + $2\cdot2\cdot2\cdot2\cdot5$ (hamburger and beverage) + $2\cdot2\cdot2\cdot5$ (fish and beverage) + $2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot5$ (all three) = 917

*27. $5\cdot3=15$

28. $2\cdot11\cdot6=132$

29. $9\cdot10\cdot2 = 180$ (2 choices, 0 or 5, for the third digit)

*30. 900 - 180 = 720

31. $9\cdot5\cdot3$ (middle digit even, third digit 0, 4, or 8) + $9\cdot5\cdot2$ (middle digit odd, third digit 2 or 6) = 135 + 90 = 225

32. 225 (numbers divisible by 4) + $9\cdot5\cdot1$ (middle digit odd, third digit 0) + $9\cdot10\cdot1$ (third digit 5) = 225 + 45 + 90 = 360

33. $9\cdot5\cdot1$ (middle digit even, third digit 0) = 45

34. 900 - 360 = 540

*35. $2^8=256$

36. $2^6=64$

*37. $1\cdot2^7$ (begin with 0) + $1\cdot2^6\cdot1$ (begin with 1, end with 0) = $2^7+2^6=192$

38. $2\cdot1\cdot2^6=2^7=128$

39. $1\cdot1\cdot1\cdot2^5=32$

40. 8 (one for each digit at which the 0 occurs)

41. $1\cdot1\cdot2^6$ (begin with 10) + $1\cdot1\cdot1\cdot2^5$ (begin with 110) + $1\cdot1\cdot1\cdot2^5$ (begin with 010) + $1\cdot1\cdot1\cdot2^5$ (begin with 000) = $2^6+3\cdot2^5=160$

42. $2\cdot2\cdot2\cdot2\cdot1\cdot1\cdot1\cdot1=2^4=16$

*43. 8 (This is the same problem as Exercise 40.)

44. 256 - 8 (contain exactly one 0) - 1 (contain no 0s) = 247

45. $6\cdot6=36$

*46. $6 \cdot 1 = 6$

47. $1 \cdot 1 = 1$

48. $2 + 2 + 2$ (two ways to get each of 6 & 1, 5 & 2, 4 & 3) + 2 (two ways to get 6 & 5) = 8

49. $5 \cdot 5 = 25$

50. $4 \cdot 5 \cdot 3 \cdot 3 \cdot 2 \cdot 2 = 720$

*51. $4 \cdot 1 \cdot 3 \cdot 3 \cdot 2 \cdot 2 = 144$

52. $1 \cdot 5 \cdot 3 \cdot 3 \cdot 1 \cdot 1 = 45$

53. $5 \cdot 3 \cdot 3 \cdot 2 \cdot 2 = 180$

54. $4 \cdot 3 \cdot 3 \cdot 3 \cdot 2 = 216$

*55. $52 \cdot 52 = 2704$

56. $4 \cdot 4 = 16$

57. $12 \cdot 12 = 144$

*58. $4 \cdot 4 = 16$ ways to get 2 of one kind; there are 13 distinct "kinds", so by the Addition Principle, the answer is $16 + 16 + \ldots + 16 = 13 \cdot 16 = 208$

59. $4 \cdot 48$ (flower king, bird nonking) + $4 \cdot 48$ (bird king, flower nonking) = 384

60. Face value of 5 can occur in 4 disjoint ways:

flower face value	bird face value
1	4
2	3
3	2
4	1

Each has $4 \cdot 4$ ways of occurring, so the total is $4 \cdot 16 = 64$

61. Face value of less than 5 can occur in the disjoint ways shown below:

flower face value	bird face value
1	1, 2, or 3
2	1 or 2
3	1

Each has $4 \cdot 4$ ways of occurring, so the total is $6 \cdot 16 = 96$

62. $40 \cdot 40 = 1600$

*63. $12 \cdot 52$ (flower face card, any bird card) + $40 \cdot 12$ (flower non-face card, bird face card) = 1104

<u>or</u>

$52 \cdot 52$ (total number of hands - Exercise 55) - $40 \cdot 40$ (hands with no face cards - Exercise 62) = 1104

64. 2704 - $48 \cdot 48$ (total number of hands - hands with no kings) = 400

65. 00111000 11001001 00100001 00001010

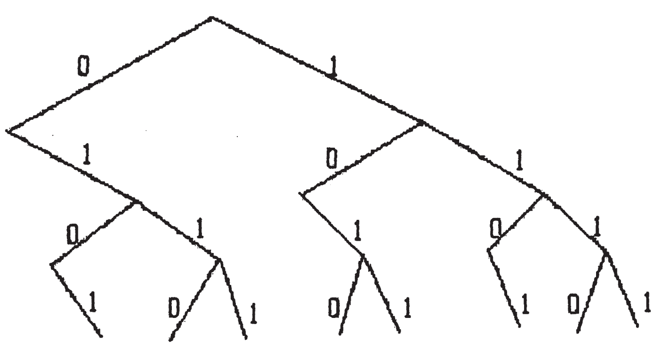
66. 0111000

*67. Using the formula
Total IP addresses = (# of A netids)(# of A hostids) + (# of B netids)(# of B hostids) + (# of C netids)(# of C hostids)
and eliminating the special cases gives $(2^7 - 1)(2^{24} - 2) + (2^{14})(2^{16} - 2) + (2^{21})(2^8 - 2)$ = (127)(16,777,214)+ (16,384)(65,534) + (2,097,152)(254) = 2,130,706,178 + 1,073,709,056 + 532,676,608 = 3,737,091,842

68. $A(1) = 1{,}300{,}000$
$A(n) = 2A(n-1)$ for $n \geq 2$
This is a linear first-order recurrence relation with constant coefficients; the solution is
$A(n) = 2^{n-1}A(1) = 2^{n-1}(1{,}300{,}000)$

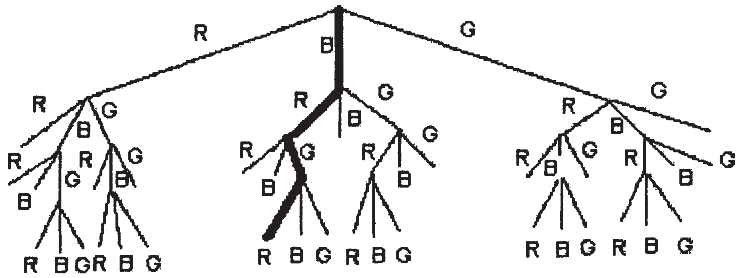
$3{,}737{,}091{,}842 = 2^{n-1}(1{,}300{,}000)$
$2^{n-1} = 3{,}737{,}091{,}842/1{,}300{,}000$
$n \cong \log_2(3737/1.3) + 1 \cong 12.49$ years, so in about 2006.

69.



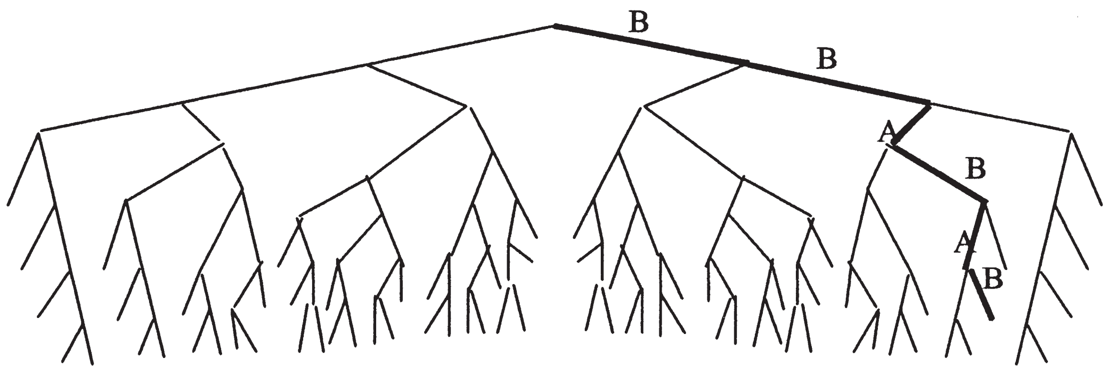
8 outcomes

*70.



33 ways; the outcome that is highlighted is BRGR.

71. A win = left branch, B win = right branch. There are 70 ways; the outcome labeled here would be BBABAB



72. For m = 2, the result follows from the Multiplication Principle.
Assume that for m = k, there are $n_1 \cdots n_k$ possible outcomes for the sequence of events 1 to k.
Let m = k + 1. Then the sequence of events 1 to k + 1 consists of the sequence of events 1 to k followed by event k + 1. The sequence of events 1 to k has $n_1 \cdots n_k$ possible outcomes by the inductive hypothesis. The sequence 1 to k followed by event k + 1then has $(n_1 \cdots n_k)n_{k+1}$ outcomes by the Multiplication Principle, which equals $n_1 \cdots n_{k+1}$.

73. For m = 2, the result follows from the Addition Principle.
Assume that for m = k there are $n_1 + \ldots + n_k$ possible outcomes from choosing one of these k events.
Let m = k + 1. Then choosing one of the k + 1 disjoint events can be thought of as choosing one of the k disjoint events from 1 to k or choosing event k + 1. The k disjoint events have $n_1 + \ldots + n_k$ possible outcomes by the inductive hypothesis, so by the Addition Principle, there are $(n_1 + \ldots + n_k) + n_{k+1} = n_1 + \ldots + n_{k+1}$ possible outcomes.

74. a.
P(1) = 1 (trivial case)
P(2) = 1 (only one way to multiply 2 terms)

For n > 2, let the last multiplication occur at position k, $1 \le k \le n - 1$. The product is then split into two products of k and (n – k) factors, respectively, which can be parenthesized in P(k) and P(n – k) ways, respectively. By the Multiplication Principle, there are P(k)P(n – k) ways to parenthesize for a fixed k. Each value for k gives a different set of parentheses, so by the Addition Principle,

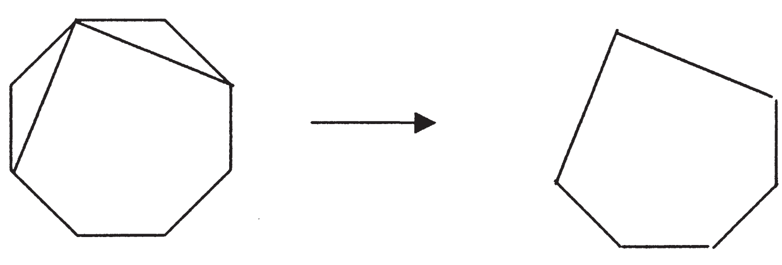
$$P(n) = P(1)P(n-1) + P(2)P(n-2) + \cdots + P(n-1)P(1)$$
$$= \sum_{k=1}^{n-1} P(k)P(n-k)$$

b. The proof will use the Second Principle of Induction.
P(1) = 1 = C(0)
P(2) = 1 = C(1)

Assume that P(r) = C(r – 1) for $1 \le r \le m$ and consider P(m + 1):

$$P(m+1) = \sum_{k=1}^{m} P(k)P(m+1-k) \quad \text{by the definition of P from part (a)}$$
$$= \sum_{k=1}^{m} C(k-1)C(m-k) \quad \text{inductive hypothesis}$$
$$= C(m) \quad \text{definition of C(m) from Exercise 33 of Section 2.4.}$$

75. *a. A pair of straight lines that shaves off two corners reduces the triangulation problem to a polygon with 2 less sides (see figures).



If n is odd, we want to do this process k times to reduce the polygon to a single triangle, so that

(n + 2) – 2k = 3
2k = n – 1

If n is even, we want to do this process k times to reduce the polygon to a 4-sided figure, so that

(n + 2) – 2k = 4
2k = n – 2

One final line triangulates the 4-sided figure, making the total number of lines needed to be

$$n - 2 + 1 = n - 1$$

b. $T(0) = 1$ (trivial case)
$T(1) = 1$ (number of ways to triangulate a 3-sided polygon)

For a fixed k, the (k + 1)-sided polygon can be triangulated in $T(k - 1)$ ways and the (n - k + 2)-sided polygon can be triangulated $T(n - k)$ ways, giving, by the Multiplication Principle, $T(k - 1)T(n - k)$ triangulations. Each value for k gives a different set of triangulations, so by the Addition Principle,

$$T(n) = T(0)T(n-1) + T(1)T(n-2) + \dots T(n-1)T(0)$$
$$= \sum_{k=1}^{n} T(k-1)T(n-k)$$

c. The recurrence relations for T(n) and C(n) are identical.

EXERCISES 3.3

1. Let A = those who speak English
B = those who speak French
Then $|A \cup B| = 42$, $|A| = 35$, $|B| = 18$.
$|A \cap B| = |A| + |B| - |A \cup B| = 35 + 18 - 42 = 11$

*2. Let A = guests who drink coffee
B = guests who drink tea
Then $|A| = 13$, $|B| = 10$, and $|A \cap B| = 4$.
$|A \cup B| = |A| + |B| - |A \cap B| = 13 + 10 - 4 = 19$

3. Yes - 5

4. 18

*5. Let A = breath set, B = gingivitis set, C = plaque set
a. $|A \cap B \cap C| = 2$
b. $|A - C| = |A| - |A \cap C| = 12 - 6 = 6$

6. Let A = CS 120 set, B = CS 180 set, C = CS 215 set. Then
$|A \cup B \cup C| = 32 + 27 + 35 - 7 - 16 - 3 + 2 = 70$. Therefore 83 - 70 = 13 students are not eligible to enroll.

7. Let A = checking account set, B = regular savings set, C = money market savings set.
   a. $|A \cap C| = 93$.
   b. $|A - (B \cup C)| = |A| - |A \cap (B \cup C)|$ by Example 29
   $= |A| - |(A \cap B) \cup (A \cap C)|$
   $= |A| - (|A \cap B| + |A \cap C|)$ by Example 28 because $(A \cap B)$ and $(A \cap C)$ are disjoint
   $= 189 - (69 + 93) = 27$

*8. Let A = auto set, B = bike set, C = motorcycle set
   a. $|B - (A \cup C)| = |B| - |B \cap (A \cup C)|$ by Example 32
   $= |B| - |(B \cap A) \cup (B \cap C)|$
   $= |B| - (|B \cap A| + |B \cap C| - |B \cap A \cap C|)$
   $= 97 - (53 + 7 - 2) = 39$

   b. $|A \cup B \cup C| = 97 + 83 + 28 - 53 - 14 - 7 + 2 = 136$, so $150 - 136 = 14$ do not own any of the three.

9. From the Principle of Inclusion and Exclusion,
   $87 = 68 + 34 + 30 - 19 - 11 - 23 + |A \cap B \cap C|$, so $|A \cap B \cap C| = 8$.

10. No. Letting A, B, and C be the odor, lather, and ingredients sets, the Principle of Inclusion and Exclusion says that the union of the three sets would contain 491 people, yet only 450 were surveyed.

11. $|A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D|$
    $+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$

12. The terms in the expansion are equivalent to all of the subsets of $\{A_1, \ldots, A_n\}$ except for the empty set. Therefore there are $2^n - 1$ terms.

13. 5 (Use the Pigeonhole Principle, where suits are bins, cards are items)

*14. No - there are 13 different denominations (bins), so 12 cards could all be different.

15. 51; there are 50 potential pairs (bins)

16. 3; there are two genders (bins).

17. 367

18. Yes - there are 12 bins, so more than 24 items means that at least one bin has more than 2 items.

*19. There are 3 pairs - 1 and 6, 2 and 5, 3 and 4 - that add up to 7. Each element in the set belongs to one of these pairs. Apply the Pigeonhole Principle, where the pairs are the bins, and the numbers are the items.

20. 6

21. This follows from the Pigeonhole Principle, where the n possible remainders (the numbers 0 through n - 1) are the bins.

EXERCISES 3.4

1. *a. 42 b. 6720 c. 360 d. $\dfrac{n!}{[n-(n-1)]!} = \dfrac{n!}{1!} = n!$

2. $9! = 362{,}880$

3. $14! \sim 87{,}178{,}291{,}000$

4. $8! = 40{,}320$; $7!\cdot 3 = 15{,}120$ (there are only 3 choices for the last character)

*5. 5! (total permutations)
3! (arrangement of the 3 R's for each distinguished permutation)
$= 5\cdot 4 = 20$

6. Seat one person in one chair anywhere in the circle – the position doesn't matter. Choose seats for the remaining 5 people from the remaining 5 positions relative to the first person, which gives 5! = 120 arrangements.

*7. $P(15, 3) = \dfrac{15!}{12!} = 15\cdot 14\cdot 13 = 2730$

8. a. $(26)^3$
b. $P(26, 3) = 26\cdot 25\cdot 24 = 15600$

9. 19!

*10. $(2!)(11!)(8!) = 2(39{,}916{,}800)(40{,}320)$

men first or women first

arrange women

11. $11!\cdot C(12, 8)\cdot 8!$

select 8 of 12 slots between the men to fill with women

arrange men

12. Seat one person in one chair anywhere in the circle – the position doesn't matter. Choose seats for the remaining 18 people from the remaining 18 positions relative to the first person, which gives 18!

13. Seat one man in one chair anywhere in the circle – the position doesn't matter. Choose seats for the remaining 10 men in positions relative to the first man, which gives 10!. Each woman must be seated to a man's right, giving 11 locations for women of which 8 must be chosen – C(11,8). Then arrange the 8 women in these 8 chosen locations, giving 8! arrangements of women. The answer is thus 10!·C(11,8)·8!

14. *a. 120 b. 36 c.28 d. $\frac{n!}{(n-1)!1!} = n$

15. $C(n, n\text{-}1) = \frac{n!}{(n-1)!1!} = n$. The number of ways to select n - 1 objects from n objects is the number of ways to exclude 1 object, C(n, 1).

*16. $C(300, 25) = \frac{300!}{25!275!}$

17. C(18, 11)

*18. C(17, 5)·C(23, 7) = (6188)(245,157)

19. C(21, 4)·C(11, 3)

*20. C(7, 1) · C(14, 1) · C(4, 1) · C(5, 1) · C(2, 1) · C(3, 1) = 7·14·4·5·2·3 = 11,760

21. C(14, 2)·C(21, 4) (2 from manufacturing, 4 from the others)

22. C(3, 1)·C(30, 5) (1 from marketing, 5 from non-accounting and non-marketing)

*23. all committees - (none or 1 from manufacturing) = C(35, 6) - [C(21, 6) + C(14, 1)·C(21, 5)]

24. 1·1·1·1·C(48, 1) = 48 (one way to choose the four queens, a pool of 48 left for the 5^{th} card)

*25. C(13, 3)·C(13, 2)

26. C(13, 5)

27. 4·C(13, 2)·C(13, 1)·C(13, 1)·C(13, 1) (pick the suit that has 2 cards, select the 2 cards, then select the 1 from each remaining suit)

28. C(12, 5) (select all 5 cards from the 12 face cards)

29. 13·C(4,2)·C(12,3)·C(4,1)·C(4,1)·C(4,1) = 1,098,240 (select the kind for the pair, select the pair of that kind, select the three remaining kinds, select 1 card from each of those kinds)

*30. $C(13,2) \cdot C(4,2) \cdot C(4,2) \cdot C(44,1) = 123{,}552$ (select the two kinds for the pairs, select the pair of one kind, select the pair of the other kind, select the fifth card from the unused kinds)

31. $13 \cdot C(4,3) \cdot C(12,2) \cdot C(4,1) \cdot C(4,1) = 54{,}912$ (select the kind, select the three of that kind, select two remaining kinds, select the fourth and fifth card from the unused kinds)

32. $10 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 10 \cdot (4)^5 = 10{,}240$ (there are 10 possible starting points for the sequence, then a choice of 1 card from each of the four suits for each of the five cards)

*33. $4 \cdot C(13, 5) = 5{,}148$ (select the suit, then the 5 cards of that suit)

34. $13 \cdot C(4,3) \cdot 12 \cdot C(4,2) = 3{,}744$ (select a kind, select three of that kind, select the second kind, select a pair of that kind)

35. $13 \cdot C(4,4) \cdot C(48,1) = 624$ (select the kind, select the 4 cards of that kind, select the remaining card)

36. $4 \cdot 10 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 40$ (select the suit, select the starting point, select the 5 cards)

37. $4 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 4$ (select the suit, select the 5 cards)

*38. $C(48, 14)$

39. $C(32, 14)$

40. $C(16, 8) \cdot C(32, 6)$

41. $C(43, 14)$

*42. $C(32, 2) \cdot C(16, 12)$ (Choose the two processors from B, then the remaining 12 modules are assigned to cluster A.)

*43. $C(12, 4) = 495$

44. $C(5, 2) \cdot C(7, 2) = (10)(21) = 210$

45. $C(5, 4) + C(7, 4) = 5 + 35 = 40$

46. $C(7, 3) \cdot C(5, 1) + C(7, 4) = 210$

47. $C(60, 2)$

*48. $C(60, 1) + C(60, 2)$

49. $C(59, 7)$

50. C(2, 1)·C(58, 6)

*51. C(12, 3) = 220

52. all committees - no independent = C(12, 3) - C(8, 3) = 164

*53. no Democrats + no Republicans - all independents (so as not to count twice)
= C(7, 3) + C(9, 3) - C(4, 3) = 115

54. all committees - (those without both) = 220 - 115 (from Exercises 47 and 49) = 105

*55. C(14, 6) = 3003

56. all groups - (those without both types) = 3003 - (no bores + no interesting)
= 3003 - [C(8, 6) + C(8, 6)] = 3003 - 56 = 2947

57. all - both = C(14, 6) - C(12, 4) = 2508
or
neither + exactly one = C(12, 6) + C(2, 1)·C(12, 5)

58. both + neither = C(12, 4) + C(12, 6) = 1419

*59. C(25, 5) - C(23, 5) (all committees - those with neither)
(Not 1·C(24, 4) + 1·C(24, 4) - this number is too big, it counts some combinations more than once)

60. C(12, 5) - C(10, 5) (all combinations - those without American history or English literature) or C(10,4) + C(10,4) +C(10,3) (history, no English + English, no history + both history and English)
(Not 1·C(11, 4) + 1·C(11, 4) - this number is too big, it counts some combinations more than once)

61. $\underbrace{C(4, 4)}_{\text{choose the 4 aces}}\cdot\underbrace{(36)}_{\text{39 non-clubs - 3 aces}} = 36$

(Not C(4, 4)·C(12, 1) - this number is too small; one of the aces is already the single club allowed, so the last card can be any non-club, non-ace)

62. $$\underbrace{C(3,3)}_{\text{choose 3 non-heart jacks}} \cdot \underbrace{C(12,2)}_{\text{choose 2 non-jack hearts}} + \overbrace{1}^{\text{jack of hearts}} \cdot \overbrace{C(3,2)}^{\text{2 other jacks}} \cdot [\overbrace{C(12,1)}^{\text{1 other heart}} \cdot \underbrace{36}_{\text{39 non-hearts - 3 jacks}}]$$

(Not $C(4,3) \cdot C(13,2)$ - this number is too small, one of the jacks could be a heart)

*63. a. $\dfrac{8!}{3!2!}$

b. $\dfrac{7!}{3!2!}$

64. a. $\dfrac{12!}{4!2!2!}$

b. $\dfrac{11!}{4!2!}$

65. $\dfrac{12!}{5!3!4!}$

66. 5 distinct characters would result in $5! = 5 \cdot 4 \cdot 3 \cdot 2$ code words. Since there are only 10 code words, this number has been divided by $4 \cdot 3$, which can only be written as $2! \cdot 3!$. Therefore there are two copies of one character and three copies of another.

*67. C(7, 5)

68. C(15, 12)

69. C(81, 48)

70. C(26, 12)

*71. a. C(8, 6) (choose 6 from 3 with repetitions)

b. C(7, 6) (choose 6 from 2 with repetitions)

c. C(5, 3) (3 of the 6 items are fixed, choose the remaining 3 from among 3, with repetitions)

72. a. C(10, 8)
b. C(7, 5) (3 of the 8 objects are fixed, choose the remaining 5 from among 3, with repetitions)
c. C(9, 8) (choose 8 from 2 with repetitions)
d. C(8, 6) (2 of the 8 objects are fixed, choose the remaining 6 from among 3, with repetitions)
e. C(9, 8) + C(8, 7) + C(7, 6) (zero chocolate chip cookies used - choose 8 from 2 with repetitions) + (one chocolate chip cookie used - choose remaining 7 from among 2 with repetitions) + (two chocolate chip cookies used - choose remaining 6 from among 2 with repetitions)

*73. a. C(16, 10)
b. C(9, 3) (7 of the 10 assignments are fixed, choose the remaining 3 from among 7, with repetitions)

74. a. C(10, 8)
b. C(8, 7) (1 assignment is fixed, choose the remaining 7 from among 2, with repetitions)

*75. C(13,10)

76. C(6, 4) (3 assignments are fixed, choose the remaining 4 from among 3, with repetitions)

77.

$$\frac{n!}{(n-1)!}+\frac{n!}{(n-2)!}=\frac{n!(n-2)!+n!(n-1)!}{(n-1)!(n-2)!}=\frac{n![(n-2)!+(n-1)!]}{(n-1)!(n-2)!}$$

$$=\frac{n[(n-2)!(1+(n-1))]}{(n-2)!}=n[1+(n-1)]=n\cdot n=n^2$$

78. $$C(n,r)=\frac{n!}{r!(n-r)!}=\frac{n!}{(n-r)!(n-(n-r))!}=C(n,n-r)$$

Whenever r objects are chosen from n, n - r objects are not chosen. Therefore the number of ways to choose r objects out of n is the same as the number of ways to choose n - r objects out of n.

79. C(r, 2) + C(n - r, 2) + r(n - r) =

$$\frac{r!}{(r-2)!2!}+\frac{(n-r)!}{(n-r-2)!2!}+r(n-r)$$

$$=\frac{r(r-1)}{2}+\frac{(n-r)(n-r-1)}{2}+r(n-r)$$

$$=\frac{r(r-1)+(n-r)(n-r-1)+2r(n-r)}{2}$$

$$=\frac{n^2-n}{2}=\frac{n!}{(n-2)!2!}$$

= C(n, 2)

*80. Consider selecting r elements from a set of n and putting those in bucket A, then selecting k of those r to put in bucket B. The left side multiplies the number of outcomes from those two sequential tasks.

Alternatively, we can select k elements from n and place them in bucket B, then r – k elements from the remaining n – k and place them in bucket A. The right side multiplies the number of outcomes from these two sequential tasks.

81. The union of the two sets is a set of size n + m, and there are C(n + m, r) ways to select r objects from it. Keeping the two sets separate, another way to choose r elements from the union is to choose k objects from one and r – k from the other, where $0 \le k \le r$. For a fixed k, using the Multiplication Principle gives C(n, k)C(m, r - k) outcomes. By the Addition Principle, all the ways to choose r objects is

$$\sum_{k=0}^{r} C(n,k)C(m,r-k)$$

Therefore both C(n + m, r) and $\sum_{k=0}^{r} C(n,k)C(m,r-k)$ represent the number of ways to choose r objects from the union of the two sets, and are thus equal in value.

82. $C(2) = \frac{1}{3}C(4,2) = \frac{1}{3}\frac{4!}{2!\cdot 2!} = \frac{4\cdot 3}{3\cdot 2} = 2$

$$C(3) = \frac{1}{4}C(6,3) = \frac{1}{4}\frac{6!}{3!\cdot 3!} = \frac{6\cdot 5\cdot 4}{4\cdot 2\cdot 3} = 5 \qquad C(4) = \frac{1}{5}C(8,4) = \frac{1}{5}\frac{8!}{4!\cdot 4!} = \frac{8\cdot 7\cdot 6\cdot 5}{5\cdot 2\cdot 3\cdot 4} = 14$$

83. a. A path sequence has length 2n and is composed of n R's and n D's in any order. There are C(2n, n) ways to select the positions for the n R's.
b. C(2n, n) = (n + 1)C(n)

EXERCISES 3.5

*1. |{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}| = 8
2. 3/8
3. 1/8
*4. 2/8 = 1/4
5. 0
*6. $6\cdot 6 = 36$
7. 1/36
8. 6/36 = 1/6

*9.

A	B
1	6
2	5
3	4
4	3
5	2
6	1

Probability is 6/36 = 1/6

10. (6 + 6 – 1)/36 = 11/36 (using Principle of Inclusion and Exclusion)

11.

A	B
5	6
6	5
6	6

Probability is 3/36 = 1/12.

12. An odd sum can happen by (odd, even) or (even, odd). Probability = $(3\cdot3 + 3\cdot3)/36 = 18/36 = 1/2$. (Intuitively, half the outcomes would be an odd number.)

*13. C(52, 2) = 1326.

14. C(13, 2) / C(52, 2) = 78/1326 = 1/17

*15. C(39, 2) / C(52, 2) = 741/1326 = 19/34

16. $C(13, 1)\cdot C(39, 1)/C(52, 2) = 13\cdot39 / 1326 = 13/34$ (one spade, one non-spade)

17. $(C(13, 2) + 13\cdot39) / C(52, 2) = 585/1326 = 15/34$ (both spades or one spade, one non-spade)
or
(C(52, 2) - C(39, 2)) / C(52, 2) = 585/1326 =15/34 (all hands minus both non-spades)

18. C(12, 2) / C(52, 2) = 66/1326 = 11/221

19. C(3, 2) / C(52, 2) = 3/1326 = 1/442

*20. (C(12, 2) + C(13, 2) – C(3, 2)) / C(52, 2) = 141/1326 = 47/442 (both face cards + both spades – both spade face cards)

*21 The size of the sample space is $10\cdot10\cdot10 = 1000$
a. 1/1000 (only one winning 3-digit number)
b. $1\cdot3!/1000$ (all permutations of the 3-digit number drawn) = $6/1000 \cong 1/167$
c. (3!/2!)/1000 (all distinct permutations of the 3-digit number drawn) = $3/1000 \cong 1/333$

22. The size of the sample space is $10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$
   a. 1/10,000 (only one winning 4-digit number)
   b. $1 \cdot 4!/10{,}000$ (all permutations of the 4-digit number drawn) = $24/10{,}000 \cong 1/417$
   c. $(4!/2!)/10{,}000$ (all distinct permutations of the 4-digit number drawn) = $12/10{,}000 \cong 1/833$
   d. $(4!/2! \cdot 2!)/10{,}000$ (all distinct permutations of the 4-digit number drawn) = $6/10{,}000 \cong 1/1667$
   e. $(4!/3!)/10{,}000$ (all distinct permutations of the 4-digit number drawn) = $4/10{,}000 \cong 1/2500$

23. The size of the sample space is $36 \cdot 35 \cdot 34 \cdot 33 \cdot 32 = 45{,}239{,}040$
   a. $5!/45{,}239{,}040 = 1/376992$. $5!$ = all permutations of the one winning set of 5 numbers.
   b. $31 \cdot 5 \cdot 5!/45{,}239{,}040 \cong 1/2432$. If 4 of the numbers must equal those actually drawn, there are 31 choices for the 5th number (it must be different from the other 4, and it cannot equal the 5th number actually drawn). There are 5 "wild card" positions for the 5th number. And for each set of numbers, there are 5! permutations.
   c. $C(31, 2) \cdot C(5, 2) \cdot 5! /45{,}239{,}040 \cong 1/81$. If 3 of the numbers must equal those actually drawn and the remaining two cannot equal the remaining 2 actually drawn, there is a pool of 31 allowable values for the remaining two numbers. There are 2 positions out of 5 that can be "wild card" positions for the two non-matching numbers. And for each set of numbers, there are 5! permutations.

24. The size of the sample space is $49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 42 = 9{,}610{,}695{,}360$
   a. $5!/ 9{,}655{,}934{,}400 = 1/80{,}089{,}128$. $5!$ = exactly one winning set of 6 numbers but all permutations of the group of 5.
   b. $C(44, 5) \cdot 5! \cdot 1/9{,}655{,}934{,}400 \cong 1/74$. The 5 numbers must be chosen from the 44 numbers not drawn in the group of 5; each set of numbers has 5! permutations; the 1 is the single choice for the Powerball number.

For Exercises 25-33, the size of the sample space is $C(52, 5) = 2{,}598{,}960$

25. $13 \cdot C(4, 2) \cdot C(12, 3) \cdot C(4, 1) \cdot C(4, 1) \cdot C(4, 1) / C(52, 5) = 1{,}098{,}240/2{,}598{,}960 \cong 0.423$

*26. $C(13, 2) \cdot C(4, 2) \cdot C(4, 2) \cdot C(44, 1) / C(52, 5) = 123{,}552/2{,}598{,}960 \cong 0.048$

27. $13 \cdot C(4, 3) \cdot C(12, 2) \cdot C(4, 1) \cdot C(4, 1) / C(52, 5) = 54{,}912/2{,}598{,}960 \cong 0.021$

28. $10 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 / C(52, 5) = 10{,}240/2{,}598{,}960 \cong 0.004$

*29. $4 \cdot C(13, 5) / C(52, 5) = 5{,}148/2{,}598{,}960 \cong 0.002$

30. $13 \cdot C(4, 3) \cdot 12 \cdot C(4, 2) / C(52, 5) = 3{,}744/2{,}598{,}960 \cong 0.0014$

31. $13 \cdot C(4, 4) \cdot C(48, 1) / C(52, 5) = 624/2{,}598{,}960 \cong 0.0002$

32. $4 \cdot 10 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \,/\, C(52, 5) = 40/2{,}598{,}960 \cong 0.000015$
33. $4 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \,/\, C(52, 5) = 4/2{,}598{,}960 \cong 0.0000015$

34. a. The probability of a straight flush, 0.0000015, is much less than the probability of a full house, 0.0014, and so is a better hand.
   b. The probability of four of a kind, 0.0002, is much less than the probability of a straight, 0.004, and so is a better hand.

*35. 365^n (each of the n persons has one of 365 possible birthdays)

36. $|E| = (365)(364)(363)\ldots(365 - n + 1)$ (each of the n persons has a different birthday), so $P(E) = (365)(364)(363)\ldots(365 - n + 1)/(365)^n$

37. $B = E'$, so from Exercise 36, $P(B) = 1 - P(E) = 1 - (365)(364)(363)\ldots(365 - n + 1)/(365)^n$

38. When $n = 22$, using the expression for P(B) from Exercise 37 gives $P(B) \cong 0.475695$, When $n = 23$, $P(B) \cong 0.507297$,

39. a. $P(E_2') = 1 - P(E_2) = 1 - 0.45 = 0.55$
   b. $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = 0.37 + 0.45 - 0.14 = 0.68$
   c. $P((E_1 \cup E_2)') = 1 - P(E_1 \cup E_2) = 1 - 0.68 = 0.32$

*40. a. $P(E_1) = p(1) + p(3) + p(5) = 0.6$
   b. $P(E_2) = p(3) + p(6) = 0.25$
   c. $P(E_3) = p(4) + p(5) + p(6) = 0.65$
   d. $P(E_2 \cap E_3) = p(6) = 0.15$
   e. $P(E_1 \cup E_3) = P(E_1) + P(E_3) - P(E_1 \cap E_3) = 0.6 + 0.65 - p(5) = 0.95$
   or
   $P(E_1 \cup E_3) = p(1) + p(3) + p(4) + p(5) + p(6) = 0.95$

41. a.

D	R	I
8/13	4/13	1/13

   b. 8/13
   c. 9/13

42. a. $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.08}{0.34} \cong 0.24$
   b. $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.17 + 0.34 - 0.08 = 0.43$
   c. $P(A \cup B)' = 1 - P(A \cup B) = 1 - 0.43 = 0.57$

For Exercises 43-49, the sample space is {GGG, GGB, GBG, GBB, BGG, BGB, BBG, BBB}

43. 4/8 = 1/2

*44. 3/8

45. 1/8

46. P(at least one girl) = P(no girls)' = 1 – 1/8 = 7/8

47. 1/8

*48. P(three girls | first two girls) = P(three girls)/P(first two girls) = (1/8) / (2/8) = 1/2

49. P(at least 1 boy and at least 1 girl | at least 1 boy) = (6/8) / (7/8) = 6/7

50. a. $P(E_i \mid F) = \dfrac{P(E_i \cap F)}{P(F)}$ (1)

$P(F \mid E_i) = \dfrac{P(F \cap E_i)}{P(E_i)}$ or $P(F \cap E_i) = P(F \mid E_i)\,P(E_i)$ (2)

$P(F \cap E_i) = P(E_i \cap F)$ so substitute from Equation (2) into Equation (1), giving

$$P(E_i \mid F) = \frac{P(F \mid E_i)\,P(E_i)}{P(F)}$$

b. The events $E_i, 1 \le i \le n$, are all disjoint , therefore the events $F \cap E_i, 1 \le i \le n$, are all disjoint.

$$F = F \cap S = F \cap (E_1 \cup E_2 \cup \ldots \cup E_n) = (F \cap E_1) \cup (F \cap E_2) \cup \ldots \cup (F \cap E_n)$$

Because the probability of the union of disjoint events is the sum of the probabilities of each event, $P(F) = \sum_{k=1}^{n} P(F \cap E_k)$

c. From Equation (2) of part (a),

$$P(F \cap E_i) = P(F \mid E_i)\,P(E_i)$$

Substituting into the result of part (b) gives

$$P(F) = \sum_{k=1}^{n} P(F \mid E_k)P(E_k)$$

d. Substituting the result of part (c) into the result of part (a) gives

$$P(E_i \mid F) = \frac{P(F \mid E_i)P(E_i)}{\sum_{k=1}^{n} P(F \mid E_k)P(E_k)}$$

51. From the given information, $P(E_1) = 0.62$ and $P(E_2) = 0.38$; also $P(F \mid E_1) = 0.014$ and $P(F \mid E_2) = 0.029$. Using Bayes' Theorem,

$$P(E_1 \mid F) = \frac{P(F \mid E_1)P(E_1)}{\sum_{k=1}^{2} P(F \mid E_k)P(E_k)} = \frac{(0.014)(0.62)}{(0.014)(0.62) + (0.029)(0.38)} \cong 0.44$$

$$P(E_2 \mid F) = \frac{P(F \mid E_2)P(E_2)}{\sum_{k=1}^{2} P(F \mid E_k)P(E_k)} = \frac{(0.029)(0.38)}{(0.014)(0.62) + (0.029)(0.38)} \cong 0.56$$

The patient is more likely to be on medication Y.

*52. a.

X	2	3	4	5	6	7	8	9	10	11	12
p	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

b. $E(X) = 2(1/36) + 3(2/36) + 4(3/36) + 5(4/36) + 6(5/36) + 7(6/36) + 8(5/36) + 9(4/36) + 10(3/36) + 11(2/36) + 12(1/36) = 252/36 = 7$.

53. a. The sample space is {red, green, blue}. The random variable is the prize amount, and the probability is computed from the number of each color in the box.

x_i	red	green	blue
$X(x_i)$	3	6	10
$p(x_i)$	43/78	27/78	8/78

$E(X) = 3(43/78) + 6(27/78) + 10(8/78) = 371/78 = 4.75$

b. The expected value of the prize money is \$0.25 less than the cost of the game card, so after 100 games, the casino could expect a profit of 100(0.25) = \$25.00.

54. Letting G = good (virus-free) file and B = bad file,

x_i	G	BG	BBG	BBBG
$X(x_i)$	1	2	3	4
$p(x_i)$	$\left(\frac{9}{12}\right)$	$\left(\frac{3}{12}\right)\left(\frac{9}{11}\right)$	$\left(\frac{3}{12}\right)\left(\frac{2}{11}\right)\left(\frac{9}{10}\right)$	$\left(\frac{3}{12}\right)\left(\frac{2}{11}\right)\left(\frac{1}{10}\right)\left(\frac{9}{9}\right)$

$E(X) = 1(9/12) + 2(27/132) + 3(54/1320) + 4(6/1320)$
$= 1(990/1320) + 2(270/1320) + 3(54/1320) + 4(6/1320) = 1716/1320 = 1.3$

*55. The sample space of input values has n + 1 elements (n elements from the list, or not in the list), all equally likely.

x_i	L_1	L_2	...	L_n	L_{n+1} (not in list)
$X(x_i)$	1	2	...	n	n
$p(x_i)$	1/(n + 1)	1/(n + 1)	...	1/(n + 1)	1/(n + 1)

Then $E(X) = \sum_{i=1}^{n+1} X(x_i)p(x_i) =$

$$\sum_{i=1}^{n} i\left(\frac{1}{n+1}\right) + n\left(\frac{1}{n+1}\right) = \frac{1}{n+1}\sum_{i=1}^{n} i + \frac{n}{n+1} = \frac{1}{n+1}(1+2+\ldots+n) + \frac{n}{n+1} = \frac{1}{n+1}\frac{n(n+1)}{2} + \frac{n}{n+1}$$
$$= \frac{n^2+3n}{2(n+1)}$$

56. The sample space of input values has n + 1 elements (n elements from the list, or not in the list). The probability that the target is not in the list is 0.8. The probability that the target is in the list is 0.2, divided equally among the n positions in the list.

x_i	L_1	L_2	...	L_n	L_{n+1} (not in list)
$X(x_i)$	1	2	...	n	n
$p(x_i)$	0.2/n	0.2/n	...	0.2/n	0.8

Then $E(X) = \sum_{i=1}^{n+1} X(x_i)p(x_i) =$

$$\sum_{i=1}^{n} i\left(\frac{0.2}{n}\right) + n(0.8) = \frac{0.2}{n}\sum_{i=1}^{n} i + 0.8n = \frac{0.2}{n}(1+2+\ldots+n) + 0.8n = \frac{0.2}{n}\frac{n(n+1)}{2} + 0.8n$$
$$= 0.1(n+1) + 0.8n = 0.9n + 0.1$$

EXERCISES 3.6

1. *a. $(a + b)^5 = C(5, 0)a^5 + C(5, 1)a^4b + C(5, 2)a^3b^2 + C(5, 3)a^2b^3 + C(5, 4)ab^4 + C(5, 5)b^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

   b. $(x + y)^6 = C(6, 0)x^6 + C(6, 1)x^5y + C(6, 2)x^4y^2 + C(6, 3)x^3y^3 + C(6, 4)x^2y^4 + C(6, 5)xy^5 + C(6,6)y^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

   *c. $(a + 2)^5 = C(5, 0)a^5 + C(5, 1)a^4(2) + C(5, 2)a^3(2)^2 + C(5, 3)a^2(2)^3 + C(5, 4)a(2)^4 + C(5, 5)(2)^5 = a^5 + 10a^4 + 40a^3 + 80a^2 + 80a + 32$

d. $(a - 4)^4 = C(4, 0)a^4 + C(4, 1)a^3(-4)^2 + C(4, 2)a^2(-4) + C(4, 3)a(-4)^3 + C(4, 4)(-4)^4$
$= a^4 - 16a^3 + 96a^2 - 256a + 256$

*e. $(2x + 3y)^3 = C(3, 0)(2x)^3 + C(3, 1)(2x)^2(3y) + C(3, 2)(2x)(3y)^2 + C(3, 3)(3y)^3$
$= 8x^3 + 36x^2y + 54xy^2 + 27y^3$

f. $(3x - 1)^5 = C(5, 0)(3x)^5 + C(5, 1)(3x)^4(-1) + C(5, 2)(3x)^3(-1)^2 + C(5, 3)(3x)^2(-1)^3$
$+ C(5, 4)(3x)(-1)^4 + C(5, 5)(-1)^5 = 243x^5 - 405x^4 + 270x^3 - 90x^2 + 15x - 1$

g. $(2p - 3q)^4 = C(4, 0)(2p)^4 + C(4, 1)(2p)^3(-3q) + C(4, 2)(2p)^2(-3q)^2 + C(4, 3)(2p)(-3q)^3$
$+ C(4, 4)(-3q)^4 = 16p^4 - 96p^3q + 216p^2q^2 - 216pq^3 + 81q^4$

h. $(3x + \frac{1}{2})^5 = C(5, 0)(3x)^5 + C(5, 1)(3x)^4(\frac{1}{2}) + C(5, 2)(3x)^3(\frac{1}{2})^2 + C(5, 3)(3x)^2(\frac{1}{2})^3$
$+ C(5, 4)(3x)(\frac{1}{2})^4 + C(5, 5)(\frac{1}{2})^5 = 243x^5 + \frac{405}{2}x^4 + \frac{132}{2}x^3 + \frac{45}{4}x^2 + \frac{15}{16}x + \frac{1}{32}$

2. $120a^7b^3$

3. $924x^6y^6$

*4. $C(9, 5)(2x)^4(-3)^5 = -489{,}888x^4$

5. $15{,}120a^3b^4$

*6. $C(8, 8)(-3y)^8 = 6561y^8$

7. $729x^6$

*8. $C(5, 2)(4x)^3(-2y)^2 = 2560x^3y^2$

9. $-1701x^5$

10. $(a + b + c)^3 = ((a + b) + c)^3 = C(3, 0)(a + b)^3 + C(3, 1)(a + b)^2c + C(3, 2)(a + b)c^2 +$
$C(3, 3)c^3 = [C(3, 0)a^3 + C(3, 1)a^2b + C(3, 2)ab^2 + C(3, 3)b^3]$
$+ 3(a^2 + 2ab + b^2)c + 3(a + b)c^2 + c^3$
$= a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$

11. $(1 + 0.1)^5 = C(5, 0)(1) + C(5, 1)(0.1) + C(5, 2)(0.1)^2 + C(5, 3)(0.1)^3 + C(5, 4)(0.1)^4 +$
$C(5, 5)(0.1)^5 = 1 + 0.5 + 10(0.01) + 10(0.001) + 5(0.0001) + 0.00001$
$= 1 + 0.5 + 0.1 + 0.01 + 0.0005 + 0.00001 = 1.61051$

*12. $C(8, 1)(2x - y)^7 5^1 = C(8, 1)(5)[C(7, 4)(2x)^3(-y)^4] = C(8, 1)C(7,4)(2)^3(5)x^3y^4 =$ $11{,}200x^3y^4$

13. $C(9, 2)(x + y)^7(2z)^2 = C(9, 2)[C(7, 2)x^5y^2](2z)^2 = 3024x^5y^2z^2$

14. $C(n + 2, r) = C(n + 1, r - 1) + C(n + 1, r)$ (Pascal's formula)
$= C(n, r - 2) + C(n, r - 1) + C(n, r - 1) + C(n, r)$ (Pascal's formula again)
$= C(n, r) + 2C(n, r - 1) + C(n, r - 2)$

15. Basis: $n = k$: $C(k, k) = C(k+1, k+1)$ true because both equal 1
Assume $C(k, k) + C(k+1, k) + \ldots + C(n-1, k) = C(n, k+1)$
Then $C(k, k) + \ldots + C(n-1, k) + C(n, k) = C(n, k+1) + C(n, k)$
$= C(n, k+1) + [C(n+1, k+1) - C(n, k+1)]$ by Pascal's formula
$= C(n+1, k+1)$

16. From the Binomial Theorem with $a = 1$, $b = (-1)$:
$C(n, 0) - C(n,1) + C(n,2) - \ldots + (-1)^n C(n,n) = (1 + (-1))^n = 0^n = 0$

*17. From the Binomial Theorem with $a = 1$, $b = 2$:
$C(n, 0) + C(n, 1)2 + C(n, 2)2^2 + \ldots + C(n, n)2^n = (1 + 2)^n = 3^n$

18. a. From the Binomial Theorem with $a = 1$, $b = 2^{-1}$,

$$C(n, 0) + C(n, 1)2^{-1} + C(n, 2)2^{-2} + \ldots + C(n, n)2^{-n} = (1 + 2^{-1})^n$$

so, multiplying by 2^n,

$$C(n, 0)2^n + C(n, 1)2^{n-1} + C(n, 2)2^{n-2} + \ldots + C(n, n)2^{n-n} = 2^n(1 + 2^{-1})^n$$

or

$$C(n, 0)2^n + C(n, 1)2^{n-1} + C(n, 2)2^{n-2} + \ldots + C(n, n)$$
$$= 2^n\left(1+\frac{1}{2}\right)^n = 2^n\left(\frac{2+1}{2}\right)^n = 2^n\left(\frac{3^n}{2^n}\right) = 3^n$$

b. From the symmetry of Pascal's triangle (or from Exercise 74 of Section 3.4), $C(n, r) = C(n, n - r)$. Thus the coefficients in the sum of Exercise 17 can be "reversed" - $C(n, 0)$ gets replaced with $C(n, n)$ and so forth, giving the desired result.

19. a. $C(n, 0) + C(n, 1)x + C(n, 2)x^2 + C(n, 3)x^3 + \ldots + C(n, n)x^n$

b. Differentiating both sides of the equation
$(1 + x)^n = C(n, 0) + C(n, 1)x + C(n, 2)x^2 + \ldots + C(n, n)x^n$
gives
$n(1 + x)^{n-1} = C(n, 1) + 2C(n, 2)x + 3C(n, 3)x^2 + \ldots + nC(n, n)x^{n-1}$

c. follows from (b) with $x = 1$

d. follows from (b) with $x = -1$

20. $(1 + x)^n = C(n, 0) + C(n, 1)x + C(n, 2)x^2 + C(n, 3)x^3 + \ldots + C(n, n)x^n$

Integrating both sides with respect to x,

$$\frac{(1+x)^{n+1}}{n+1} = C(n,0)x + \frac{C(n,1)x^2}{2} + \frac{C(n,2)x^3}{3} + \ldots + \frac{C(n,n)x^{n+1}}{n+1} + C$$

To evaluate the constant of integration C, let $x = 0$:

$$\frac{1}{n+1} = 0 + 0 + \ldots + 0 + C$$

so

$$\frac{(1+x)^{n+1}}{n+1} = C(n,0)x + \frac{C(n,1)x^2}{2} + \frac{C(n,2)x^3}{3} + \ldots + \frac{C(n,n)x^{n+1}}{n+1} + \frac{1}{n+1}$$

or

$$\frac{(1+x)^{n+1} - 1}{n+1} = C(n,0)x + \frac{C(n,1)x^2}{2} + \frac{C(n,2)x^3}{3} + \ldots + \frac{C(n,n)x^{n+1}}{n+1}$$

Let $x = 1$. Then

$$\frac{2^{n+1} - 1}{n+1} = C(n,0) + \frac{1}{2}C(n,1) + \frac{1}{3}C(n,2) + \ldots + \frac{1}{n+1}C(n,n)$$

Let $x = -1$. Then

$$\frac{-1}{n+1} = -C(n,0) + \frac{1}{2}C(n,1) - \frac{1}{3}C(n,2) + \ldots + \frac{(-1)^{n+1}}{n+1}C(n,n)$$

or

$$\frac{1}{n+1} = C(n,0) - \frac{1}{2}C(n,1) + \frac{1}{3}C(n,2) + \ldots + \frac{(-1)^n}{n+1}C(n,n)$$

21. a. Out of all the intersections of m sets, $1 \le m \le k$, we want the ones that pick all m elements from the k elements in B. There are C(k,m) ways to do this.
   b. $C(k, 1) - C(k, 2) + C(k, 3) - \ldots + (-1)^{k+1}C(k, k)$
   c. From Exercise 16,

   $$C(k, 0) - C(k, 1) + C(k, 2) - \ldots + (-1)^k C(k, k) = 0$$

   or

   $$C(k, 1) - C(k,2) + \ldots (-1)^{k+1}C(k, k) = C(k,0)$$

   but $C(k,0) = 1$