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PARES DE TRANSFORMADA Z

1) $x[n] = \delta[n]$

UTILIZANDO A DEFINIÇÃO DA TRANSF. Z UNILATERAL

$$X[z] = \sum_{n=0}^{+\infty} x[n] . z^{-n} = \sum_{n=0}^{+\infty} \underbrace{\delta[n] . z^{-n}}_{\underbrace{\delta[n]\, z^{-0}}_{\delta[n]}} = \sum_{n=0}^{+\infty} \delta[n] = 1$$

PORTANTO,

$$\boxed{\delta[n] \longleftrightarrow 1 \qquad \forall\ z}$$

2) $x[n] = \delta[n-n_0]$

$$X[z] = \sum_{n=0}^{+\infty} x[n] . z^{-n} = \sum_{n=0}^{+\infty} \delta[n-n_0]\, z^{-n} = \sum_{n=0}^{+\infty} \delta[n-n_0] . z^{-n_0}$$

$$= z^{-n_0} \sum_{n=0}^{+\infty} \delta[n-n_0] = z^{-n_0}$$

PORTANTO,

$$\boxed{\delta[n-n_0] \longleftrightarrow z^{-n_0} \qquad \forall\ z}$$

3) $x[n] = u[n]$

$$X[z] = \sum_{n=0}^{+\infty} x[n].z^{-n} = \sum_{n=0}^{+\infty} \underbrace{u[n]}_{=1}.z^{-n} = \sum_{n=0}^{+\infty} z^{-n} = \sum_{n=0}^{+\infty} \left(\frac{1}{z}\right)^n$$

NO CASO, TEMOS QUE

$$\sum_{k=m}^{n} \alpha^k = \frac{\alpha^{n+1} - \alpha^m}{\alpha - 1}, \quad \alpha \neq 1$$

$$\sum_{k=0}^{+\infty} \alpha^k = \frac{1}{1-\alpha}, \quad |\alpha| < 1$$

CONSIDERANDO $|1/z| < 1$, OU SEJA, $|z| > 1$

$$X[z] = \sum_{n=0}^{+\infty} \left(\frac{1}{z}\right)^n = \frac{1}{1 - 1/z} = \frac{z}{z-1}$$

PORTANTO,

$$\boxed{u[n] \longleftrightarrow \frac{z}{z-1}, \quad \text{COM } |z| > 1}$$

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4) $x[n] = n.u[n]$

$$X[z] = \sum_{n=0}^{+\infty} x[n].z^{-n} = \sum_{n=0}^{+\infty} n.\underbrace{u[n]}_{\substack{1,\ n \geq 0 \\ 0,\ n < 0}}.z^{-n} = \sum_{n=0}^{+\infty} n.z^{-n}$$

$$= \sum_{n=0}^{+\infty} n\left(\frac{1}{z}\right)^n$$

NO CASO, TEMOS QUE

$$\sum_{n=0}^{K} n.\alpha^n = \frac{\alpha + [K(\alpha-1)-1].\alpha^{K+1}}{(\alpha-1)^2}, \quad \alpha \neq 1$$

$$\sum_{n=0}^{+\infty} n\,\alpha^n = \frac{\alpha}{(\alpha-1)^2}, \quad |\alpha|<1$$

CONSIDERANDO $|1/z|<1$, OU SEJA, $|z|>1$

$$X[z] = \sum_{n=0}^{+\infty} n\left(\frac{1}{z}\right)^n = \frac{1/z}{\left(\frac{1}{z}-1\right)^2} = \frac{1}{z}\cdot\frac{1}{\frac{(1-z)^2}{z^2}} = \frac{z}{(z-1)^2}$$

PORTANTO,

$$\boxed{n.u[n] \longleftrightarrow \frac{z}{(z-1)^2}, \quad \text{COM } |z|>1}$$

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5) $x[n] = n^2.u[n]$

$$X[z] = \sum_{n=0}^{+\infty} x[n].z^{-n} = \sum_{n=0}^{+\infty} n^2.\underbrace{u[n]}_{\substack{=1,\ n\geq 0\\ 0,\ n<0}}\,z^{-n} = \sum_{n=0}^{+\infty} n^2\,z^{-n}$$

$$= \sum_{n=0}^{+\infty} n^2\left(\frac{1}{z}\right)^n$$

NO CASO, TEMOS QUE

$$\sum_{n=0}^{K} n^2 \cdot \alpha^n = \frac{\alpha\left[(1+\alpha)(1-\alpha^K) - 2K(1-\alpha)\alpha^K - K^2(1-\alpha)^2\alpha^K\right]}{(1-\alpha)^3}, \quad \alpha \neq 1$$

$$\sum_{n=0}^{+\infty} n^2 \cdot \alpha^n = \frac{\alpha(1+\alpha)}{(1-\alpha)^3}, \quad |\alpha| < 1$$

CONSIDERANDO, $\left|\frac{1}{Z}\right| < 1$, OU SEJA, $|Z| > 1$

$$X[Z] = \sum_{n=0}^{+\infty} n^2 \left(\frac{1}{Z}\right)^n = \frac{\frac{1}{Z}\left(1+\frac{1}{Z}\right)}{\left(1-\frac{1}{Z}\right)^3} = \frac{\frac{1}{Z}\left(\frac{Z+1}{Z}\right)}{\frac{(Z-1)^3}{Z^3}} = \frac{Z(Z+1)}{(Z-1)^3}$$

PORTANTO,

$$\boxed{n^2 \cdot u[n] \longleftrightarrow \frac{Z(Z+1)}{(Z-1)^3}, \quad \text{COM } |Z| < 1}$$

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6) $x[n] = \alpha^n \cdot u[n]$

$$X[Z] = \sum_{n=0}^{+\infty} x[n] \cdot Z^{-n} = \sum_{n=0}^{+\infty} \alpha^n \cdot \underbrace{u[n]}_{\substack{=1,\ n \geq 0 \\ 0,\ n<0}} \cdot Z^{-n} = \sum_{n=0}^{+\infty} \left(\frac{\alpha}{Z}\right)^n$$

NO CASO, TEMOS QUE

$$\sum_{n=0}^{+\infty} r^n = \frac{1}{1-r}, \quad |r| < 1$$

CONSIDERANDO $\left|\frac{\alpha}{z}\right|<1$, OU SEJA, $|z|>|\alpha|$

$$X[z] = \sum_{n=0}^{+\infty}\left(\frac{\alpha}{z}\right)^n = \frac{1}{1-\frac{\alpha}{z}} = \frac{z}{z-\alpha}$$

PORTANTO,

$$\boxed{\alpha^n . u[n] \longleftrightarrow \frac{z}{z-\alpha}, \quad \text{COM } |z|>|\alpha|}$$

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7) $x[n] = \alpha^{n-1} . u[n-1]$

$$X[z] = \sum_{n=0}^{+\infty} x[n] . z^{-n} = \sum_{n=0}^{+\infty} \alpha^{n-1} . \underbrace{u[n-1]}_{\substack{=1,\ n\geq 1 \\ 0,\ n<1}} z^{-n} = \sum_{n=1}^{+\infty} \alpha^{n-1} . z^{-n}$$

$$= \alpha^{-1} . \sum_{n=1}^{+\infty}\left(\frac{\alpha}{z}\right)^n = \frac{1}{\alpha} . \sum_{n=1}^{+\infty}\left(\frac{\alpha}{z}\right)^n$$

[NO CASO, TEMOS QUE

$$\sum_{n=m}^{k} r^n = \frac{r^{k+1}-r^m}{r-1}, \quad r \neq 1$$

$$\sum_{n=m}^{+\infty} r^n = \frac{r^m}{1-r}, \quad |r|<1$$
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CONSIDERANDO $\left|\frac{\alpha}{z}\right|<1$, OU SEJA, $|z|>|\alpha|$

$$X[z] = \frac{1}{z} . \frac{(\alpha/z)}{1-\alpha/z} = \frac{1}{z-\alpha}$$

PORTANTO,

$$\boxed{\alpha^{n-1} . u[n-1] \longleftrightarrow \frac{1}{z-\alpha} \quad \text{COM } |z|>|\alpha|}$$

8) $x[n] = n \cdot a^n\, u[n]$

$$X[z] = \sum_{n=0}^{+\infty} x[n] \cdot z^{-n} = \sum_{n=0}^{+\infty} n \cdot a^n \underbrace{u[n]}_{\substack{=1,\ n \geq 0 \\ 0,\ n<0}} \cdot z^{-n} = \sum_{n=0}^{+\infty} n \left(\frac{a}{z}\right)^n$$

NO CASO, TEMOS QUE

$$\sum_{n=0}^{+\infty} n \cdot r^n = \frac{r}{(r-1)^2}, \quad |r|<1$$

CONSIDERANDO $\left|\frac{a}{z}\right|<1$, OU SEJA, $|z|>|a|$

$$X[z] = \sum_{n=0}^{+\infty} n \left(\frac{a}{z}\right)^n = \frac{a/z}{\left(\frac{a}{z}-1\right)^2} = \frac{\frac{a}{z}}{\frac{(a-z)^2}{z^2}} = \frac{a \cdot z}{(z-a)^2}$$

PORTANTO,

$$\boxed{n \cdot a^n \cdot u[n] \longleftrightarrow \frac{a \cdot z}{(z-a)^2} \quad \text{COM } |z|>|a|}$$

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9) $x[n] = n^2 \cdot a^n \cdot u[n]$

$$X[z] = \sum_{n=0}^{+\infty} x[n] \cdot z^{-n} = \sum_{n=0}^{+\infty} n^2 \cdot a^n \cdot \underbrace{u[n]}_{\substack{=1,\ n \geq 0 \\ 0,\ n<0}}\, z^{-n} = \sum_{n=0}^{+\infty} n^2 \left(\frac{a}{z}\right)^n$$

NO CASO,

$$\sum_{n=0}^{+\infty} n^2 \cdot r^n = \frac{r(1+r)}{(1-r)^3}, \quad |r|<1$$

CONSIDERANDO $\left|\frac{\alpha}{z}\right|<1$, OU SEJA, $|z|>|\alpha|$

$$X[z] = \sum_{n=0}^{+\infty} n^2 \left(\frac{\alpha}{z}\right)^n = \frac{\frac{\alpha}{z}\left(1+\frac{\alpha}{z}\right)}{\left(1-\frac{\alpha}{z}\right)^3} = \frac{\frac{\alpha}{z}\left(\frac{z+\alpha}{z}\right)}{\frac{(z-\alpha)^3}{z^3}} = \frac{\alpha z(z+\alpha)}{(z-\alpha)^3}$$

PORTANTO,

$$\boxed{n^2 \cdot \alpha^n \cdot u[n] \longleftrightarrow \frac{\alpha z(z+\alpha)}{(z-\alpha)^3} \quad \text{COM } |z|>|\alpha|}$$

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10) $x[n] = |\alpha|^n \cdot \cos(\beta n)\, u[n]$

$$= \frac{|\alpha|^n}{2} \cdot e^{j\beta n}\, u[n] + \frac{|\alpha|^n}{2} \cdot e^{-j\beta n}\, u[n]$$

UTILIZANDO A DEFINIÇÃO DA TRANSF Z UNILATERAL

$$X[z] = \sum_{n=0}^{+\infty} x[n] \cdot z^{-n} = \frac{1}{2} \cdot \sum_{n=0}^{+\infty} |\alpha|^n \left(\frac{e^{j\beta}}{z}\right)^n + \frac{1}{2} \cdot \sum_{n=0}^{+\infty} |\alpha|^n \left(\frac{e^{-j\beta}}{z}\right)^n$$

$$= \frac{1}{2} \cdot \sum_{n=0}^{+\infty} \left(\frac{|\alpha| \cdot e^{j\beta}}{z}\right)^n + \frac{1}{2} \cdot \sum_{n=0}^{+\infty} \left(\frac{|\alpha| \cdot e^{-j\beta}}{z}\right)^n$$

NO CASO,

$$\sum_{n=0}^{+\infty} r^n = \frac{1}{1-r} \quad \text{COM } |r|<1$$

CONSIDERANDO $|z| > |\alpha|$,

$$X[z] = \frac{1}{2}\cdot\frac{1}{1 - \frac{|\alpha|.e^{j\beta}}{z}} + \frac{1}{2}\cdot\frac{1}{1 - \frac{|\alpha|.e^{-j\beta}}{z}}$$

$$= \frac{1}{2}\cdot\left[\frac{z}{z-|\alpha|.e^{j\beta}} + \frac{z}{z-|\alpha|.e^{-j\beta}}\right]$$

$$= \frac{z}{2}\left[\frac{z-|\alpha|\,e^{-j\beta} + z-|\alpha|\,e^{j\beta}}{(z-|\alpha|.e^{j\beta})(z-|\alpha|.e^{-j\beta})}\right]$$

$$= \frac{z}{2}\cdot\left[\frac{2z-2|\alpha|.\cos\beta}{z^2-2|\alpha|.\cos\beta z+|\alpha|^2}\right]$$

PORTANTO,

$$\boxed{|\alpha|^n.\cos(\beta n)\,u[n] \longleftrightarrow \frac{z(z-|\alpha|.\cos\beta)}{z^2-2|\alpha|.\cos\beta z+|\alpha|^2}, \quad \text{com } |z|>|\alpha|}$$

II) $x[n] = |\alpha|^n\,\mathrm{sen}(\beta n)\,u[n] = \frac{|\alpha|^n}{2}\,e^{j\beta n}\,u[n] - \frac{|\alpha|^n}{2}\,e^{-j\beta n}\,u[n]$

$$X[z] = \frac{1}{2}\sum_{n=0}^{+\infty}\left(\frac{|\alpha|.e^{j\beta}}{z}\right)^n - \frac{1}{2}\cdot\sum_{n=0}^{+\infty}\left(\frac{|\alpha|.e^{-j\beta}}{2}\right)^n$$

$$\left[\begin{array}{l}\text{NO CASO,}\\ \displaystyle\sum_{n=0}^{+\infty} r^n = \frac{1}{1-r},\ |r|<1\end{array}\right]$$

CONSIDERANDO $|z| > |\alpha|$,

$$X[z] = \frac{1}{2} \cdot \frac{1}{1 - \frac{|\alpha| . e^{j\beta}}{z}} - \frac{1}{2} \frac{1}{1 - \frac{|\alpha| . e^{-j\beta}}{z}}$$

$$= \frac{1}{2} \left[\frac{z}{z - |\alpha| . e^{j\beta}} - \frac{z}{z - |\alpha| . e^{-j\beta}} \right]$$

$$= \frac{z}{2} \frac{(z - |\alpha| . e^{-j\beta} - z + |\alpha| e^{j\beta})}{(z - |\alpha| . e^{j\beta})(z - |\alpha| . e^{-j\beta})}$$

$$= \frac{z\, |\alpha| . \text{sen}\, \beta}{z^2 - 2|\alpha| . \cos\beta . z + |\alpha|^2}$$

PORTANTO,

$$\boxed{|\alpha|^n . \text{sen}(\beta n)\, u[n] \longleftrightarrow \frac{z|\alpha| . \text{sen}\, \beta}{z^2 - 2|\alpha| . \cos\beta . z + |\alpha|^2}, \text{ COM } |z| > |\alpha|}$$