Algoritmi e Strutture Dati

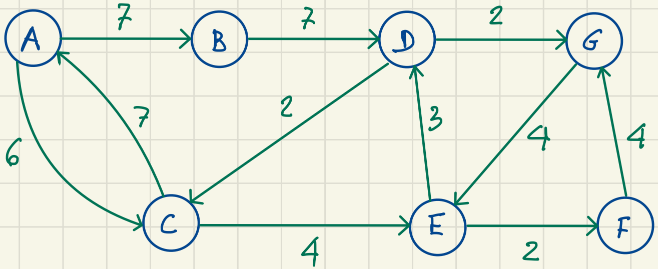
Lezione 25

25 novembre 2022

Cammini minimi

$G = (V, E)$ grafo orientato

$w: E \to \mathbb{R}$ funzione peso



$\pi = \langle v_0, v_1, \dots, v_k \rangle$ cammino da v_0 a v_k

$$w(\pi) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$$ peso del cammino

A B D G E $\underline{20}$

A C E $\underline{10}$

$A \to G$

A B D G 16

A C E F G 16

A C E D G 15

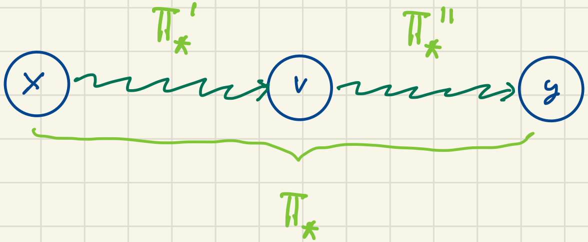
π_* è un CAMMINO MINIMO da x a y se

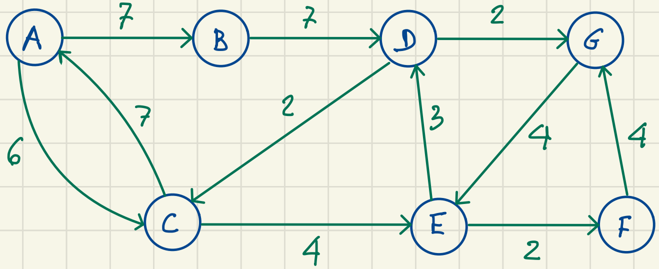
(1) π_* è un CAMMINO da x a y

(2) per ogni cammino π da x a y

$$w(\pi) \geq w(\pi_*)$$

CAMMINI MINIMI: proprietà



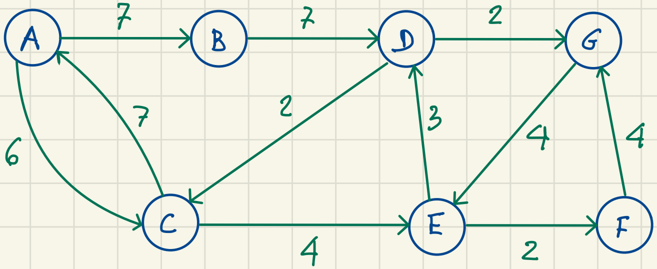


Se π_* è un cammino minimo da x a y e π_* passa per v:

- la parte di π_* da x a v è un cammino minimo da x a v
- " " di π_* da v a y " " " " da v a y

PRINCIPIO DI OTTIMALITA'

CAMMINI MINIMI: proprietà



- Se tutti i pesi sono positivi allora ogni cammino minimo semplice

$$\pi \qquad \underbrace{x \to \dots \to z}_{\pi'} \underbrace{\to \dots \to z}_{\pi''} \underbrace{\to \dots \to y}_{\pi'''}$$

$$w(\pi) = w(\pi') + \underbrace{w(\pi'')}_{>0} + w(\pi''')$$

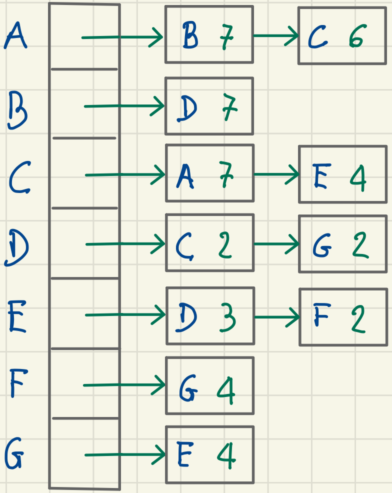
$$\overline{\pi} \qquad \underbrace{x \to \dots \to z}_{\pi'} \underbrace{\to \dots \to y}_{\pi'''}$$

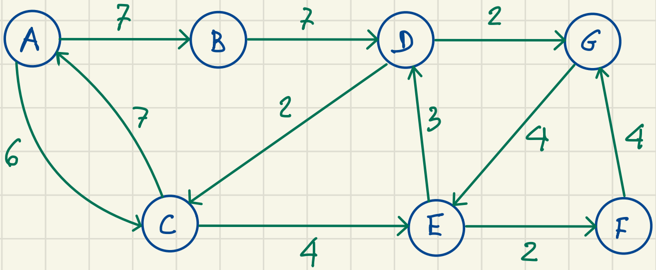
$$w(\overline{\pi}) = w(\pi') + w(\pi''') < w(\pi)$$

- Se ci sono pesi negativi, ma NON ci sono cicli di peso negativo allora se ∃ cammino da x a y allora ∃ cammino minimo da x a y che è semplice

GRAFI PESATI: alcune rappresentazioni

LISTA DI ADIACENZA CON PESI

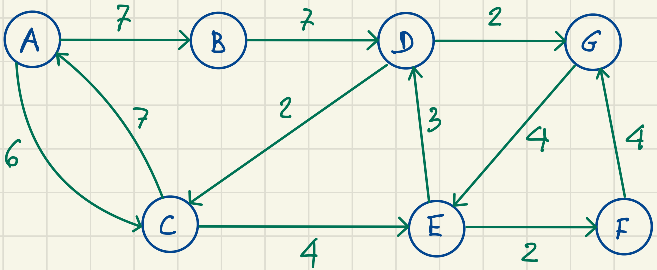




GRAFI PESATI: alcune rappresentazioni

MATRICE DEI PESI

	A	B	C	D	E	F	G
A	∞	7	6	∞	∞	∞	∞
B	∞	∞	∞	7	∞	∞	∞
C	7	∞	∞	∞	4	∞	∞
D	∞	∞	2	∞	∞	∞	2
E	∞	∞	∞	3	∞	2	∞
F	∞	∞	∞	∞	∞	∞	4
G	∞	∞	∞	∞	4	∞	∞



PROBLEMI "CAMMINI MINIMI"

- Trovare il cammino minimo tra due vertici
- Trovare i cammini minimi tra un vertice s e tutti gli altri
- Trovare i cammini minimi tra ogni coppia di vertici

CAMMINI MINIMI TRA TUTTE LE COPPIE DI VERTICI:

L'ALGORITMO DI Floyd & Warshall

Supponiamo $V = \{v_1, v_2, \dots v_n\}$

d_{ij} = peso del cammino minimo da v_i a v_j
(lunghezza)

Problema P: determinare d_{ij} $i = 1, \dots, n$, $j = 1, \dots, n$

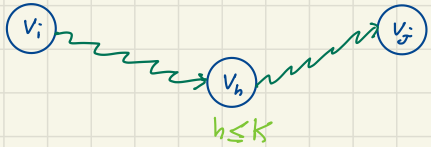
Problema $\mathcal{P}$: determinare d_{ij} $(i=1,\dots,n,\quad j=1,\dots,n)$

PROGRAMMAZIONE
DINAMICA

Per $k=0,\dots,n$ considero il problema vincolato:

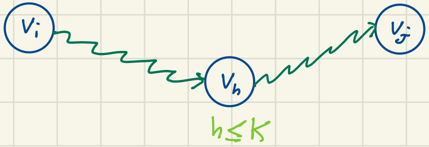
Problema $\mathcal{P}(k)$: determinare $d_{ij}^{(k)}$ $(i=1,\dots,n,\quad j=1,\dots,n)$

$d_{ij}^{(k)}$ = peso del cammino minimo da v_i a v_j
che nei passi intermedi visita
esclusivamente vertici di indice $\leq k$



Allora $\mathcal{P} = \mathcal{P}(n)$

Come risolvere $\mathcal{P}(k)$?

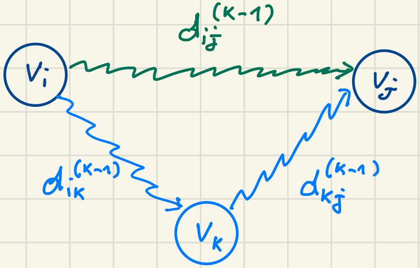


$\boxed{k=0}$

$$d_{ij}^{(0)} = \begin{cases} \omega(v_i, v_j) & \text{se } (v_i, v_j) \in E \text{ e } v_i \neq v_j \\ 0 & \text{se } v_i = v_j \\ \infty & \text{altrimenti} \end{cases}$$

$\boxed{k>0}$

$$d_{ij}^{(k)} = \min \left\{ d_{ij}^{(k-1)} , \; d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \right\}$$



$d_{ik}^{(k-1)} + d_{kj}^{(k-1)} < d_{ij}^{(k-1)}$?

```
ALGORITMO FloydWarshall (Grafo G) → Matrice
    Siano $D_0[1..n, 1..n], \dots, D_n[1..n, 1..n]$ matrici
    FOR i ← 1 TO n DO
        FOR j ← 1 TO n DO
            IF i = j THEN $D_0[i,j] \leftarrow 0$
            ELSE IF $(v_i, v_j) \in E$ THEN $D_0[i,j] \leftarrow \omega(i,j)$
            ELSE $D_0[i,j] \leftarrow \infty$
    FOR k ← 1 TO n DO
        FOR i ← 1 TO n DO
            FOR j ← 1 TO n DO
                IF $D_{k-1}[i,k] + D_{k-1}[k,j] < D_{k-1}[i,j]$ THEN
                    $D_k[i,j] \leftarrow D_{k-1}[i,k] + D_{k-1}[k,j]$
                ELSE
                    $D_k[i,j] \leftarrow D_{k-1}[i,j]$
    RETURN $D_n$
```

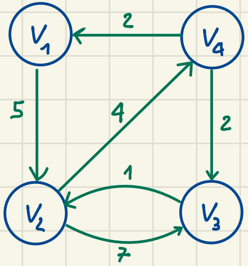
$$d_{ij}^{(0)} = \begin{cases} \omega(v_i, v_j) & \text{se } (v_i, v_j) \in E \text{ e } v_i \neq v_j \\ 0 & \text{se } v_i = v_j \\ \infty & \text{altrimenti} \end{cases}$$

$$d_{ij}^{(k)} = \min \left\{ d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \right\}$$

Tempo $O(n^3)$

Spazio $O(n^3)$

Esempio



$$M = \begin{pmatrix} \infty & 5 & \infty & \infty \\ \infty & \infty & 7 & 4 \\ \infty & 1 & \infty & \infty \\ 2 & \infty & 2 & \infty \end{pmatrix}$$

matrice dei pesi

$$D_0 = \begin{pmatrix} 0 & 5 & \infty & \infty \\ \infty & 0 & 7 & 4 \\ \infty & 1 & 0 & \infty \\ 2 & \infty & 2 & 0 \end{pmatrix}$$

$$D_1 = \begin{pmatrix} 0 & 5 & \infty & \infty \\ \infty & 0 & 7 & 4 \\ \infty & 1 & 0 & \infty \\ 2 & 7 & 2 & 0 \end{pmatrix}$$

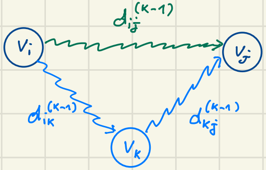
$v_4 \rightarrow v_1 \rightarrow v_2$

$$D_2 = \left(\qquad \right)$$

$$D_3 = \left(\qquad \right)$$

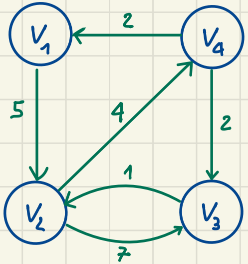
$$D_4 = \left(\qquad \right)$$

$$d_{ij}^{(0)} = \begin{cases} \omega(v_i, v_j) & \text{se } (v_i, v_j) \in E \text{ e } v_i \neq v_j \\ 0 & \text{se } v_i = v_j \\ \infty & \text{altrimenti} \end{cases}$$



$$d_{ij}^{(k)} = \min\left\{ d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \right\}$$

Esempio



$$M = \begin{pmatrix} \infty & 5 & \infty & \infty \\ \infty & \infty & 7 & 4 \\ \infty & 1 & \infty & \infty \\ 2 & \infty & 2 & \infty \end{pmatrix}$$

matrice dei pesi

$$D_0 = \begin{pmatrix} 0 & 5 & \infty & \infty \\ \infty & 0 & 7 & 4 \\ \infty & 1 & 0 & \infty \\ 2 & \infty & 2 & 0 \end{pmatrix}$$

$$D_1 = \begin{pmatrix} 0 & 5 & \infty & \infty \\ \infty & 0 & 7 & 4 \\ \infty & 1 & 0 & \infty \\ 2 & 7 & 2 & 0 \end{pmatrix}$$

$v_1 \to v_2 \to v_3$

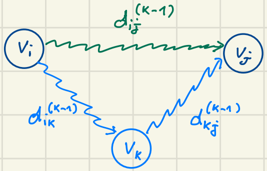
$$D_2 = \begin{pmatrix} 0 & 5 & 12 & 9 \\ \infty & 0 & 7 & 4 \\ \infty & 1 & 0 & 5 \\ 2 & 7 & 2 & 0 \end{pmatrix}$$

$$D_3 = \begin{pmatrix} 0 & 5 & 12 & 9 \\ \infty & 0 & 7 & 4 \\ \infty & 1 & 0 & 5 \\ 2 & 3 & 2 & 0 \end{pmatrix}$$

$V_1 \to V_2 \rightsquigarrow V_4 \to v_3$

$$D_4 = \begin{pmatrix} 0 & 5 & 11 & 9 \\ 6 & 0 & 6 & 4 \\ 7 & 1 & 0 & 5 \\ 2 & 3 & 2 & 0 \end{pmatrix}$$

$$d_{ij}^{(0)} = \begin{cases} w(v_i, v_j) & \text{se } (v_i, v_j) \in E \text{ e } v_i \neq v_j \\ 0 & \text{se } v_i = v_j \\ \infty & \text{altrimenti} \end{cases}$$



$$d_{ij}^{(k)} = \min\{ d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \}$$

$$d_{ij}^{(k)} = \min \left\{ d_{ij}^{(k-1)} , d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \right\}$$

- per $j = k$:

$$d_{ik}^{(k)} = \min \left\{ d_{ik}^{(k-1)} , d_{ik}^{(k-1)} + \overset{\overset{0}{\shortparallel}}{d_{kk}^{(k-1)}} \right\} = d_{ik}^{(k-1)}$$

- per $i = k$:

$$d_{kj}^{(k)} = \min \left\{ d_{kj}^{(k-1)} , d_{kk}^{(k-1)} + d_{kj}^{(k-1)} \right\} = d_{kj}^{(k-1)}$$

dunque

$$D_{k-1}[i,k] + D_{k-1}[k,j] = D_k[i,k] + D_k[k,j]$$

ALGORITMO FloydWarshall (Grafo G) → Matrice

```
Sia D[1..n, 1..n] una matrice
FOR i ← 1 TO n DO
    FOR j ← 1 TO n DO
        IF i = j THEN D[i,j] ← 0
        ELSE IF $(v_i, v_j) \in E$ THEN D[i,j] ← $\omega(i,j)$
        ELSE D[i,j] ← $\infty$
```

$$d_{ij}^{(0)} = \begin{cases} \omega(v_i, v_j) & \text{se } (v_i, v_j) \in E \text{ e } v_i \neq v_j \\ 0 & \text{se } v_i = v_j \\ \infty & \text{altrimenti} \end{cases}$$

$D_0[i,j]$

```
FOR K ← 1 TO n DO
    FOR i ← 1 TO n DO
        FOR j ← 1 TO n DO
            IF D[i,K] + D[K,j] < D[i,j] THEN
                D[i,j] ← D[i,K] + D[K,j]
RETURN D
```

$$d_{ij}^{(K)} = \min\left\{ d_{ij}^{(K-1)},\ d_{iK}^{(K-1)} + d_{Kj}^{(K-1)} \right\}$$

```
IF $D_{K-1}[i,K] + D_{K-1}[K,j] < D_{K-1}[i,j]$ THEN
    $D_K[i,j] \leftarrow D_{K-1}[i,K] + D_{K-1}[K,j]$
ELSE
    $D_K[i,j] \leftarrow D_{K-1}[i,j]$
```

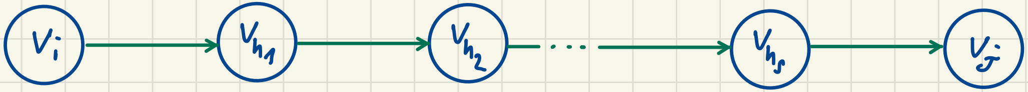
D_n

Spazio $O(n^2)$

Come ricavare il cammino minimo tra v_i e v_j?

Matrice ausiliaria P :

$P[i,j]$ = massimo indice dei vertici intermedi sul cammino minimo da v_i a v_j



$$P[i,j] = \max\{h_1, h_2, \dots, h_s\}$$

Dai valori contenuti alla fine in P si può ricavare il cammino

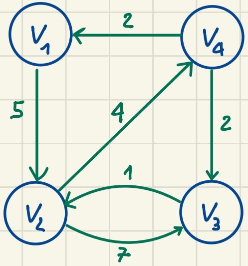
```
ALGORITMO FloydWarshall (Grafo G) → Matrice
    Siano D[1..n, 1..n] una matrice
    FOR i ← 1 TO n DO
        FOR j ← 1 TO n DO
            IF i = j THEN D[i,j] ← 0
            ELSE IF $(v_i, v_j) \in E$ THEN D[i,j] ← ω(i,j)
            ELSE D[i,j] ← ∞
            P[i,j] ← 0
```

$$d_{ij}^{(0)} = \begin{cases} \omega(v_i, v_j) & \text{se } (v_i, v_j) \in E \text{ e } v_i \neq v_j \\ 0 & \text{se } v_i = v_j \\ \infty & \text{altrimenti} \end{cases}$$

```
    FOR K ← 1 TO n DO
        FOR i ← 1 TO n DO
            FOR j ← 1 TO n DO
                IF D[i,K] + D[K,j] < D[i,j] THEN
                    D[i,j] ← D[i,K] + D[K,j]
                    P[i,j] ← K
    RETURN D , P
```

$$d_{ij}^{(K)} = \min\left\{ d_{ij}^{(K-1)}, d_{iK}^{(K-1)} + d_{Kj}^{(K-1)} \right\}$$

Esempio



$$M = \begin{pmatrix} \infty & 5 & \infty & \infty \\ \infty & \infty & 7 & 4 \\ \infty & 1 & \infty & \infty \\ 2 & \infty & 2 & \infty \end{pmatrix}$$

matrice dei pesi

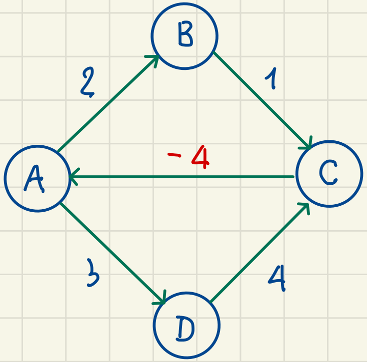
$$D = \begin{pmatrix} 0 & 5 & 11 & 9 \\ 6 & 0 & 6 & 4 \\ 7 & 1 & 0 & 5 \\ 2 & 3 & 2 & 0 \end{pmatrix}$$

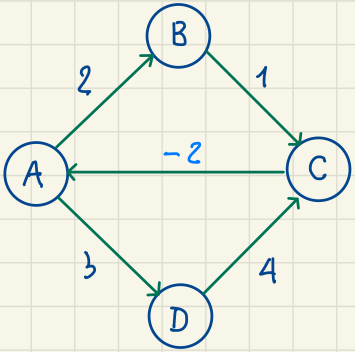
matrice delle distanze

$$P = \begin{pmatrix} 0 & \boxed{0} & \boxed{4} & \boxed{2} \\ 4 & 0 & 4 & \boxed{0} \\ 4 & 0 & 0 & 2 \\ 0 & 3 & \boxed{0} & 0 \end{pmatrix}$$

$$v_1 \rightarrow v_2 \longrightarrow v_4 \longrightarrow v_3$$

PESI NEGATIVI ?





Floyd & Warshall: corretto anche con pesi negativi purché non ci siano cicli negativi