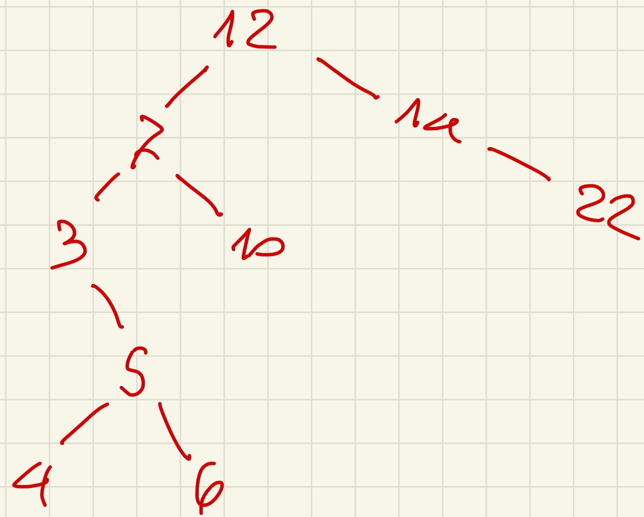
Algoritmi e Strutture Dati

Lezione 28

2 dicembre 2022

Esempio 12 7 10 14 3 5 22 4 6

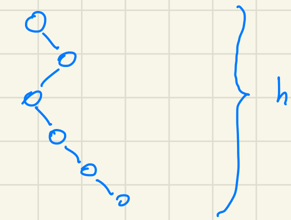


costo: inserimento, ricerca, cancellazione — Θ(altezza dell'albero)

Alberi binari: $\underbrace{\text{n° nodi}}_{n}$ vs $\underbrace{\text{altezza}}_{h}$

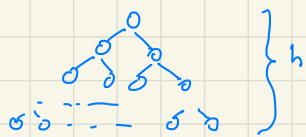
- Numero **minimo** di nodi per alberi di altezza h

$\underline{h+1}$



- Numero **massimo** di nodi per alberi di altezza h

$2^{h+1}-1$



albero completo di altezza h

$$\Rightarrow \quad h+1 \le n \le 2^{h+1}$$

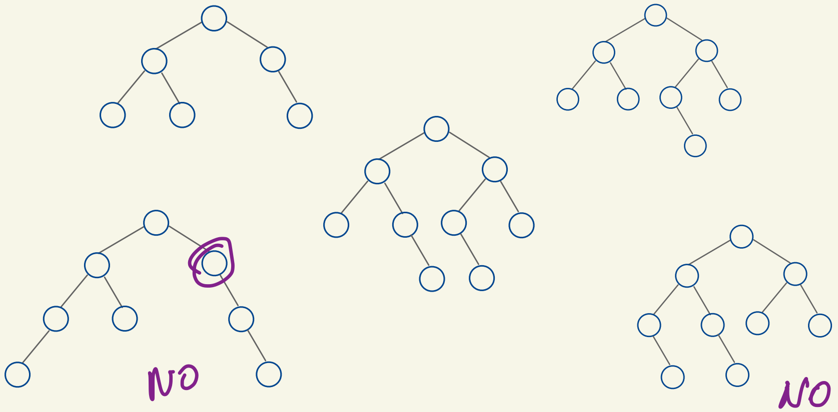
$$\Rightarrow \quad \log_2(n+1)-1 \le h \le n-1$$

Alberi bilanciati

Alberi perfettamente bilanciati

Definizione

Un albero binario è detto *perfettamente bilanciato* quando *per ogni nodo* la differenza in valore assoluto tra i numeri di nodi presenti nei suoi sottoalberi sinistro e destro è al massimo 1



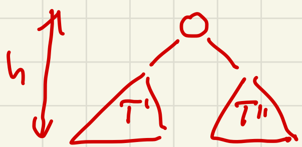
Alberi perfettamente bilanciati: $\underbrace{\text{n° nodi}}_{n}$ vs $\underbrace{\text{altezza}}_{h}$

• $n \geq 2^h$

Induzione su h

$h = 0$ — radice — $n = 1 = 2^0$ ok

$h > 0$



Almeno uno tra T' e T'' ha altezza $h-1$

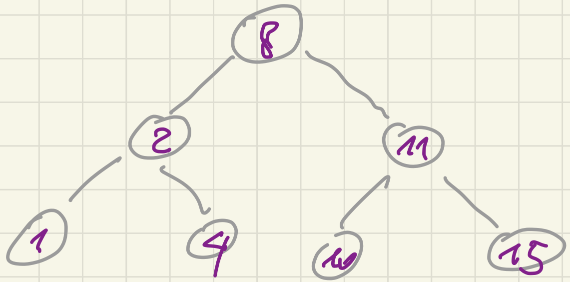
→ ip. ind.: ha almeno 2^{h-1} nodi

L'altro sottoalbero (per def) ha almeno $2^{h-1} - 1$ nodi

$$\#\text{tot. nodi} \geq \underset{\text{radice}}{\cancel{1}} + 2^{h-1} + 2^{h-1} \cancel{-1} = 2^h \qquad \square$$

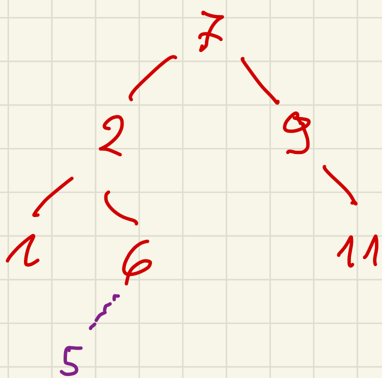
• $n \leq 2^{h+1} - 1$ $\Longrightarrow$ $h = \lfloor \log_2 n \rfloor - 1$

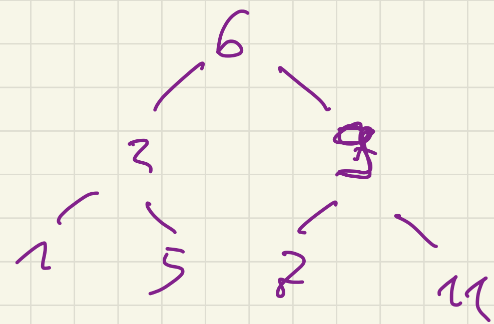
Altezza logaritmica nel n° nodi

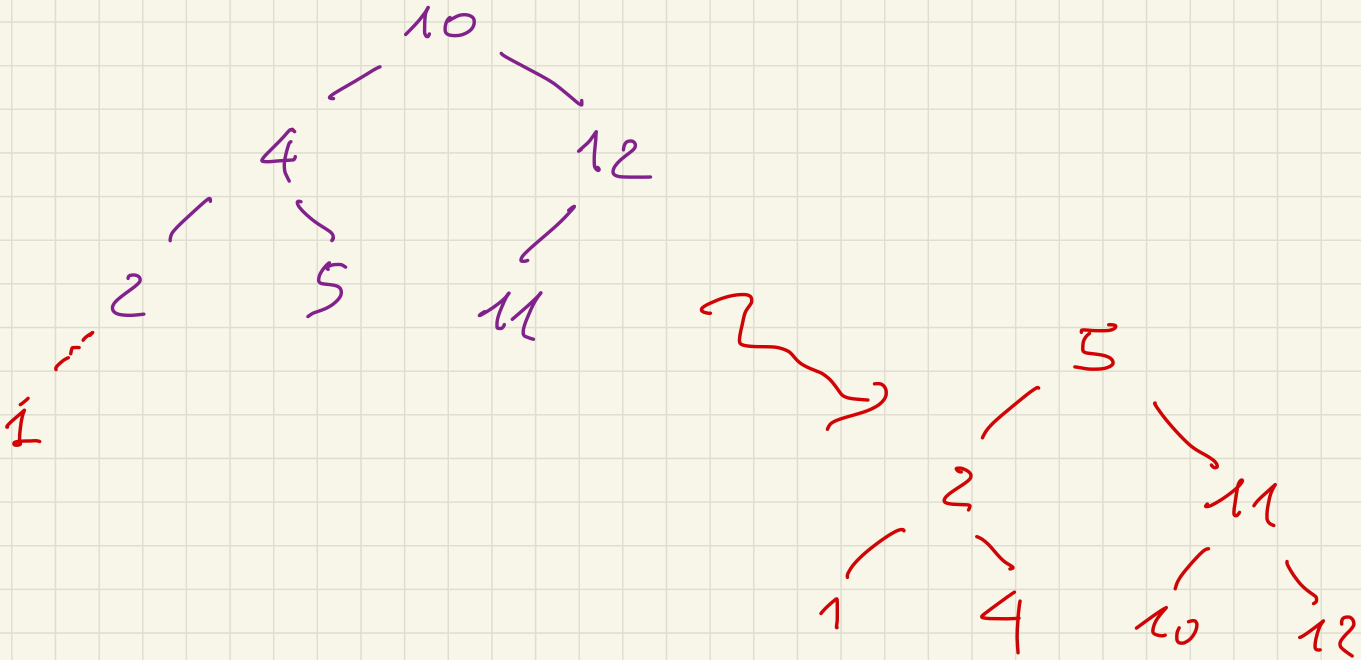


1 2 4 8 10 11 15

1 2 6 7 9 11







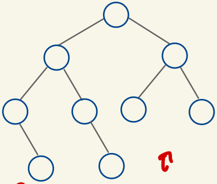
Sposto tutte le chiavi

$\Rightarrow$ costo almeno $\Omega(n)$

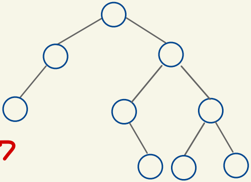
Alberi bilanciati in altezza

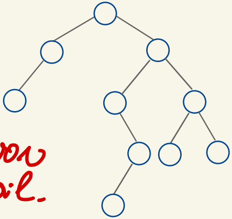
Definizione

Un albero binario è detto *bilanciato* (in altezza) o AVL[1] quando *per ogni nodo* la differenza in valore assoluto tra le altezze dei suoi sottoalberi sinistro e destro è al massimo 1

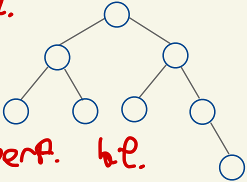


bilanciato ma non perf. bil.

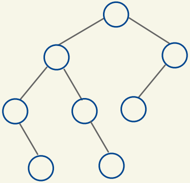




non bil.



perf. bil.



bil. non perf.

perf bilanc.
⇓
bilanciato

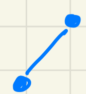
[1] Adelson-Velsky and Landis, 1962

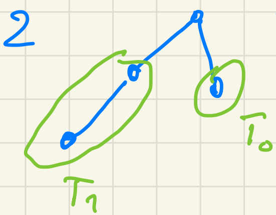
Alberi bilanciati in altezza: $\underbrace{\text{n° nodi}}_{n}$ vs $\underbrace{\text{altezza}}_{h}$

n° minimo di nodi albero AVL di altezza h

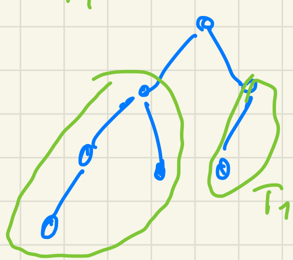
altezza — n° min di nodi

$h=0$  T_0 — $n_0 = 1$

$h=1$  T_1 — $n_1 = 2$

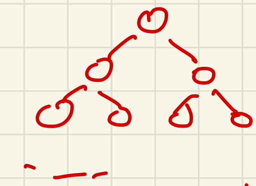
$h=2$ 

T_2 — $n_2 = 4$

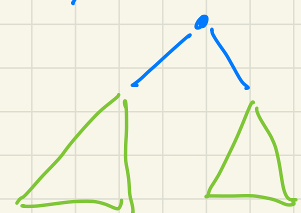
$h=3$ 

T_3

n° max nodi di un albero AVL



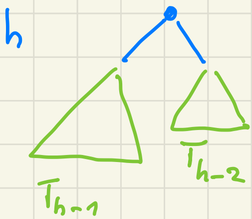
$2^{h+1}-1$

h 

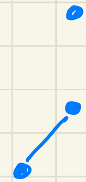
T_h

ALBERO di FIBONACCI DI ALTEZZA h

ALBERO AVL di altezza h con minimo n° nodi



T_h



T_0 $\quad n_0 = 1$

T_1 $\quad n_1 = 2$

$$n_h = \begin{cases} 1 & \text{se } h=0 \\ 2 & \text{se } h=1 \\ 1 + n_{h-1} + n_{h-2} & \text{altrimenti} \end{cases}$$

$$\begin{cases} F_1 = F_2 = 1 \\ F_k = F_{k-1} + F_{k-2} & k>2 \end{cases}$$

$$n_h = F_{h+3} - 1$$

Dim ind. su h

$h=0 \quad n_0 = 1 \quad F_3 - 1 = 2 - 1 = 1$ ok

$h=1 \quad n_1 = 2 \quad F_4 - 1 = 3 - 1 = 2$ ok

$h>1 \quad n_h = 1 + n_{h-1} + n_{h-2} \underset{\text{ip. ind.}}{=} \cancel{1} + \underbrace{F_{h+2} - \cancel{1} + F_{h+1}}_{F_{h+3}} - 1$

$= F_{h+3} - 1 \quad \square$

$$F_n \approx \frac{\phi^n}{\sqrt{5}} \qquad \phi = 1.618\ldots$$

$$n_h \approx \frac{\phi^{h+3}}{\sqrt{5}} \qquad \sqrt{5}\, n_h \approx \phi^{h+3} \qquad h+3 \approx \log_\phi(\sqrt{5}\, n_h)$$

$$h+3 \approx \log_\phi \sqrt{5} + \log_\phi n_h \qquad h = \Theta(\log n)$$

$$h < 1.44 \log_2 n$$

ALBERO DI FIBONACCI DI ALTEZZA h CON n NODI

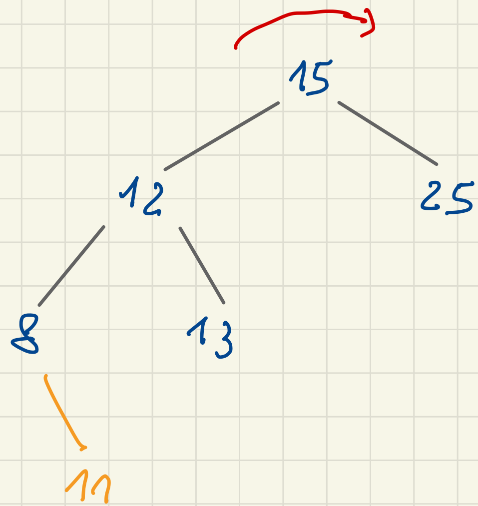
$$h = \Theta(\log n)$$

↓

alberi AVL di altezza h con minimo n° nodi

$\Longrightarrow$ Ogni albero AVL ha altezza h con $h = \Theta(\log n)$

Inserimento in alberi AVL



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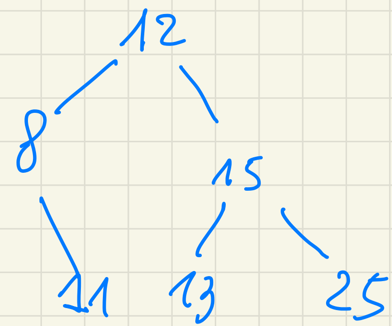
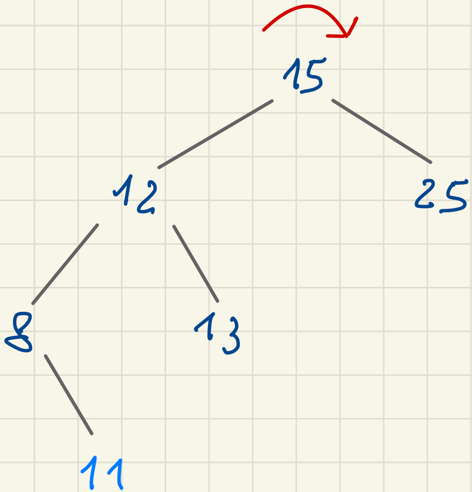
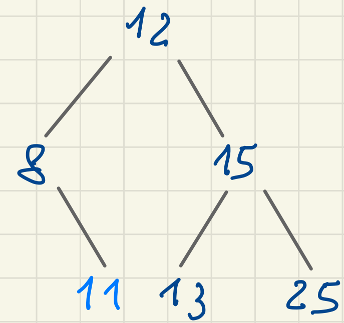


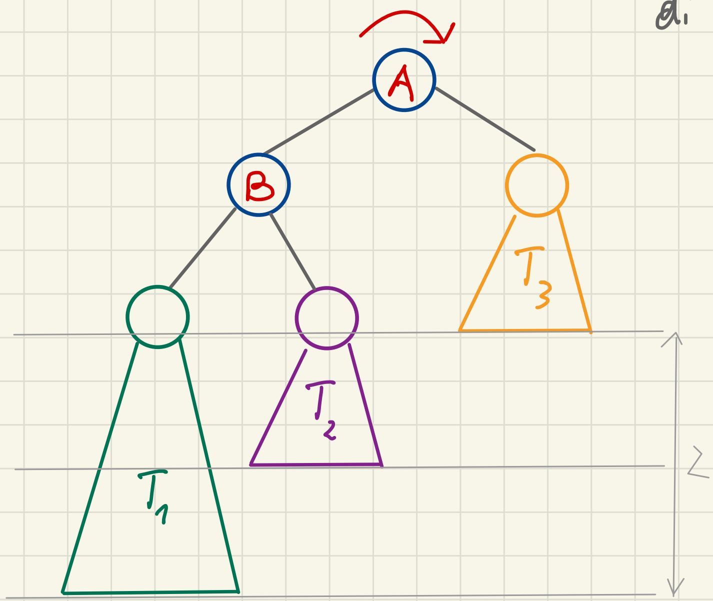
ROTAZIONE
vs DESTRA

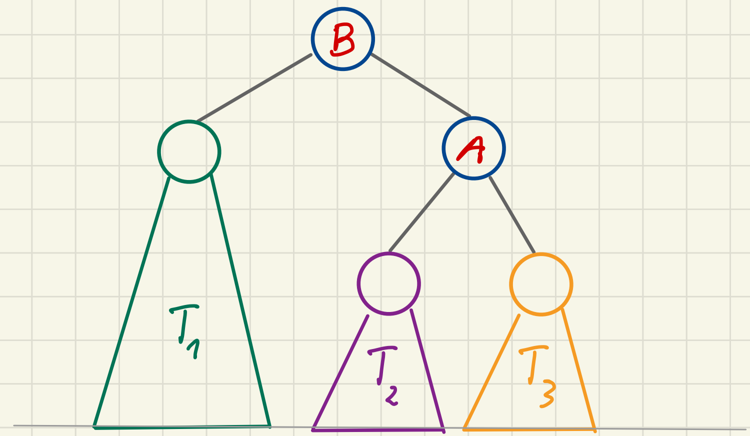


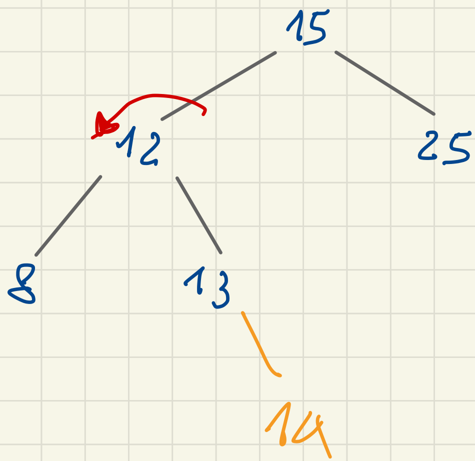


SBILANCIAMENTO a SX nel sottoalbero SX

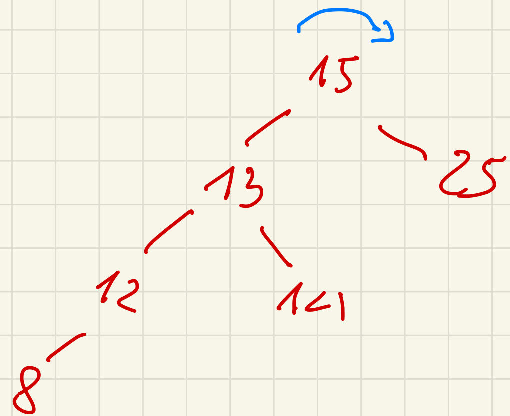
ROTAZIONE a DX
di perno A

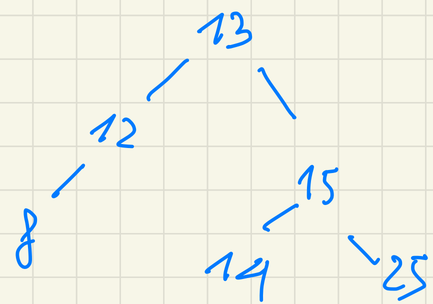


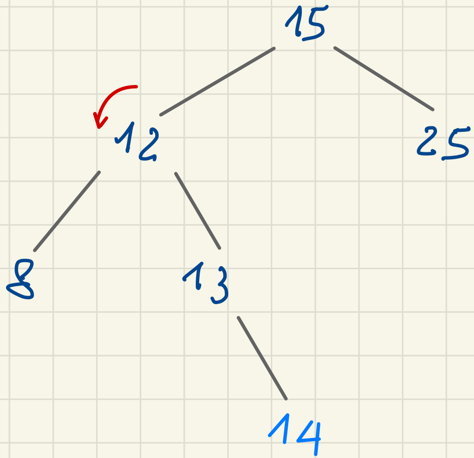


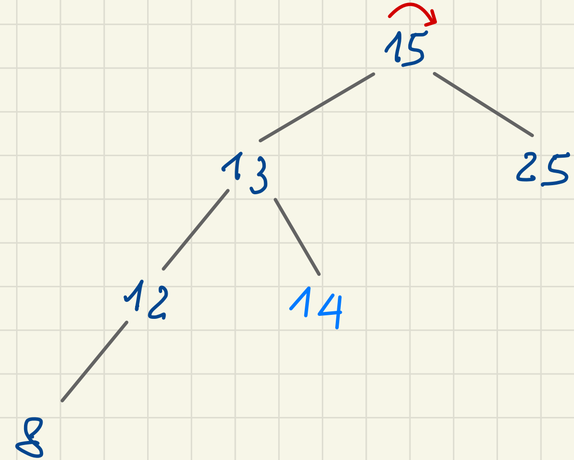


inserire 14






15
12
25
8
13
14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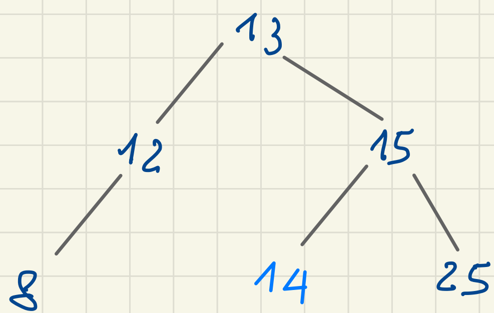
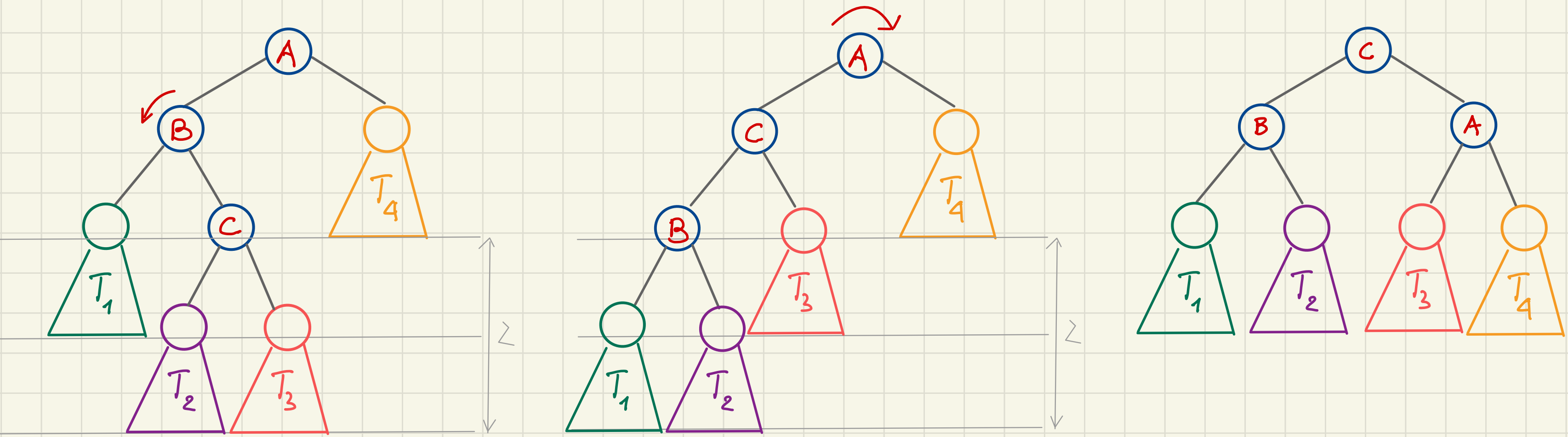

13
12
15
8
14
25

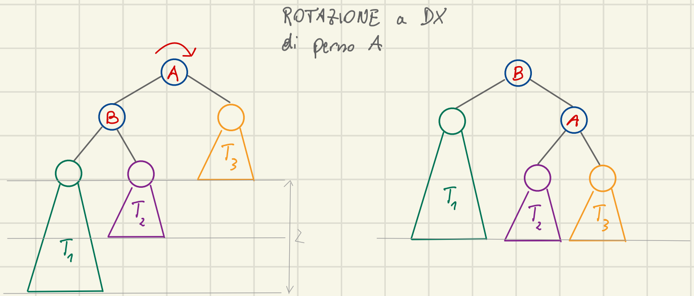
SBILANCIAMENTO a DX nel sottoalbero SX

1. ROTAZIONE a SX
di perno B

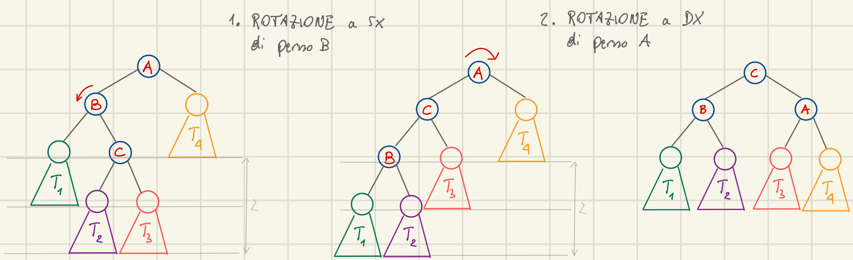
2. ROTAZIONE a DX
di perno A



SBILANCIAMENTO a SX nel sottoalbero SX



SBILANCIAMENTO a DX nel sottoalbero SX



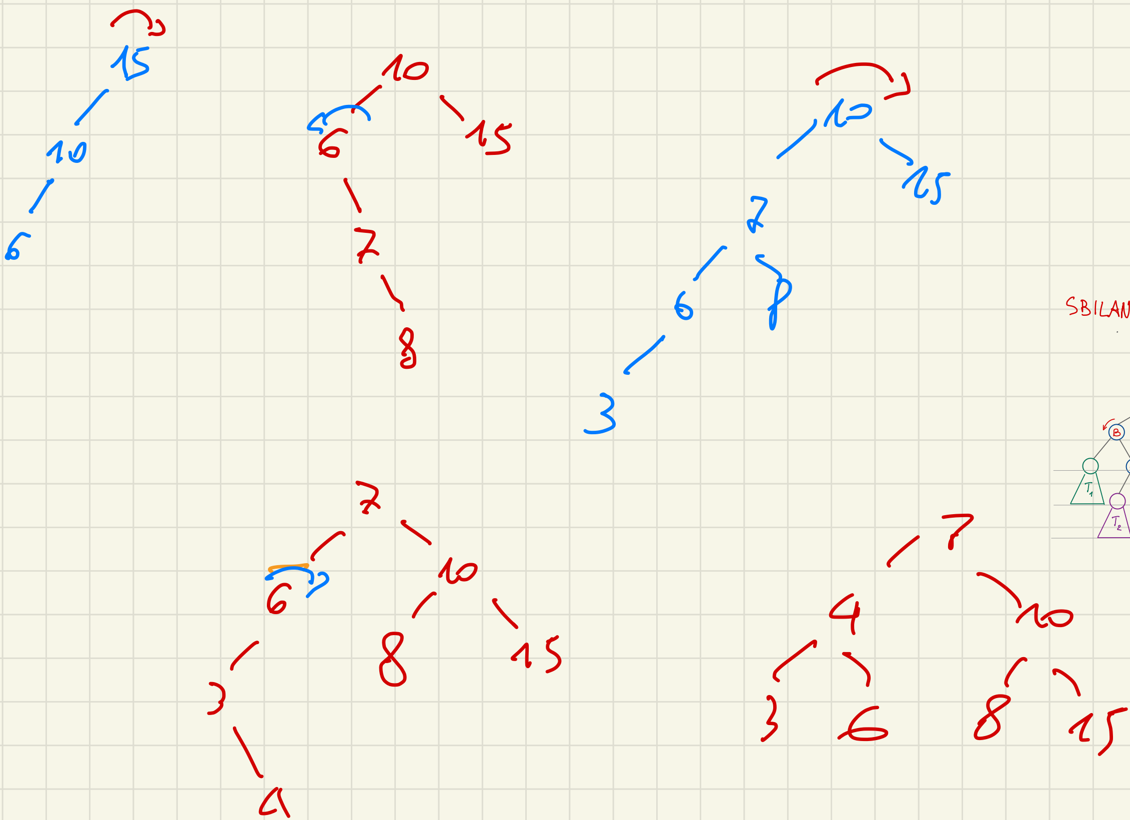
- Schemi simmetrici per sbilanciamenti nel sottoalbero dx

- Il ribilanciamento può essere effettuato in tempo costante

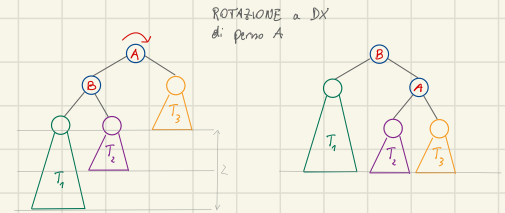
Esempio

Disegnare l'albero AVL che si ottiene a partire da un albero vuoto inserendo uno dopo l'altro, nell'ordine indicato, i seguenti numeri:

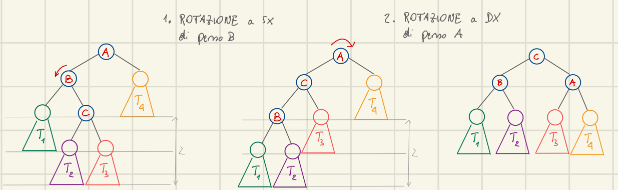
15 10 6 7 8 3 4



SBILANCIAMENTO a SX nel sottoalbero SX



SBILANCIAMENTO a DX nel sottoalbero SX



ALBERI AVL

n chiavi	Tempo	Perf. Sbilanciato
Ricerca	$\Theta(\log n)$	$\Theta(\log n)$
Inserimento	$\Theta(\log n)$	$\Theta(n)$
Cancellazione	$\Theta(\log n)$	$\Theta(n)$