Phase Kickback

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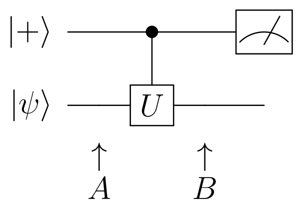


U - arbitrary

$|+\rangle \equiv |+x\rangle \equiv H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

A: $\frac{|0\rangle+|1\rangle}{\sqrt{2}}|\psi\rangle = \frac{|0\rangle|\psi\rangle+|1\rangle|\psi\rangle}{\sqrt{2}}$

B: $\frac{|0\rangle|\psi\rangle+|1\rangle U|\psi\rangle}{\sqrt{2}}$

$|\psi\rangle$ - an eigenvector of U with eigenvalue $e^{i\phi}$

$U|\psi\rangle = e^{i\phi}|\psi\rangle$

B: $\frac{|0\rangle|\psi\rangle+|1\rangle e^{i\phi}|\psi\rangle}{\sqrt{2}} = \frac{|0\rangle+e^{i\phi}|1\rangle}{\sqrt{2}}|\psi\rangle$.

Requirements for phase kickback to occur:

- *The control qubit(s) must be in a superposition.*

  This is because otherwise applying controlled operation will just change global phase:

  $$|1\rangle|\psi\rangle \xrightarrow{\boxed{U}} |1\rangle U|\psi\rangle = |1\rangle e^{i\phi}|\psi\rangle \cong |1\rangle|\psi\rangle$$

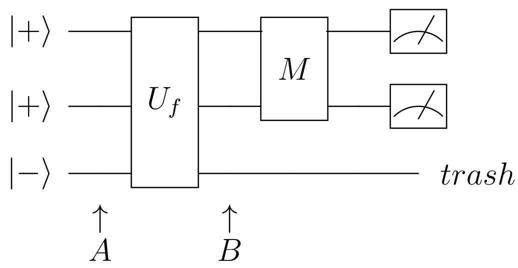
- $|\psi\rangle$ *must be an eigenvector of operator* U.

- *Arbitrary operator* U *must be applied in a controlled way.*

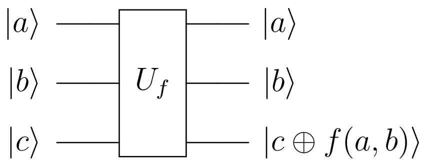
  If we apply U unconditionally, we would just change global phase of a state:

  $$U\frac{|0\rangle+|1\rangle}{\sqrt{2}}|\psi\rangle = e^{i\phi}\frac{|0\rangle+|1\rangle}{\sqrt{2}}|\psi\rangle \cong \frac{|0\rangle+|1\rangle}{\sqrt{2}}|\psi\rangle$$

f_{00}		f_{01}		f_{10}		f_{11}	
input	output	input	output	input	output	input	output
00	1	00	0	00	0	00	0
01	0	01	1	01	0	01	0
10	0	10	0	10	1	10	0
11	0	11	0	11	0	11	1



$|+\rangle \equiv |+x\rangle \equiv H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
$|-\rangle \equiv |-x\rangle \equiv H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$



A:
$|+\rangle|+\rangle|-\rangle$
$= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)|-\rangle$
$= \frac{1}{2}(|00\rangle|-\rangle + |01\rangle|-\rangle + |10\rangle|-\rangle + |11\rangle|-\rangle)$
$f = f_{00}$
B:

$$\begin{aligned} &\frac{1}{2}(|00\rangle U_{f_{00}}|-\rangle + |01\rangle|-\rangle + |10\rangle|-\rangle + |11\rangle|-\rangle) \\ =&\frac{1}{2}(|00\rangle(-1)|-\rangle + |01\rangle|-\rangle + |10\rangle|-\rangle + |11\rangle|-\rangle) \\ =&\frac{1}{2}(-|00\rangle + |01\rangle + |10\rangle + |11\rangle)|-\rangle \end{aligned}$$

$f = f_{01}$
B:

$$
\begin{aligned}
&\frac{1}{2}(|00\rangle|-\rangle + |01\rangle U_{f_{01}}|-\rangle + |10\rangle|-\rangle + |11\rangle|-\rangle) \\
&= \frac{1}{2}(|00\rangle|-\rangle + |01\rangle(-1)|-\rangle + |10\rangle|-\rangle + |11\rangle|-\rangle) \\
&= \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle)|-\rangle
\end{aligned}
$$

Phase kickback is used in many quantum algorithms, for instance:

- Deutsch
- Simon
- Shor
- Grover
- Phase estimation