Time Series Checking with Fuzzy Interval Temporal Logics*

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Abstract. Model checking is a very well-known problem, with many practical applications. A possible declination of such a problem in the interval logic setting is the so-called finite model checking, that consists of verifying an interval temporal logic formula, typically of Halpern and Shoham's logic of Allen's relations HS, on a fully represented finite interval model. A multivariate time series is a collection of temporally ordered sets of values, and they allow to describe a variety of situations, such as the medical history of an hospitalized patient or the sensor values during a plane flight. In this paper we argue how the recently introduced fuzzy generalization of interval temporal logic is a suitable language in which interesting properties of a multivariate time series can be expressed and checked, and we define, solve, and discuss the complexity of the multivariate time series fuzzy interval logic checking.

Keywords: Model checking · Time series · Fuzzy logic.

1 Introduction

A temporal variable is a variable whose value changes over time. A *time series* is a set of temporal variables. They can be *univariate*, if only one temporal variable is involved, or *multivariate*, otherwise. Each variable of a multivariate

* The authors acknowledge the partial support by the Italian INDAM GNCS project *Strategic Reasoning and Automated Synthesis of Multi-Agent Systems*

time series is an ordered collection of n real values, instead of a single value. Multivariate time (or temporal) series emerge in many application contexts: the temporal history of some hospitalized patients can be described by the time series of the values of his/her temperature, blood pressure, and oxygenation; the pronunciation of a word in sign language can be described by the time series of the relative and absolute positions of the ten fingers w.r.t. some reference point; different sport activities can be distinguished by the time series of some relevant physical quantities; all active sensors during a flight give time-changing values that form a time series. A repository of time series extracted from real data can be found in [3].

In its original formulation, *model checking* (*MC*) is the problem of verifying if a given formula is satisfied by a given model [12]. Usually, the model is the abstract representation of a system, in which the relevant properties become propositional letters, where the formula is written in a temporal logic and represents an interesting property. The prevailing adopted ontology for both the model and the logic is point-based: systems are represented in such a way that each state is a vertex on a Kripke model, atomic properties are descriptions of states, and the underlying logic a point-based temporal logic, often LTL or CTL [32,33]. As interval-based temporal logics emerged as a possible alternative to point-based ones, the concept of *interval temporal logic model checking* (*IMC*) emerged with them. Halpern and Shoham's interval temporal logic HS [16], which features one modality for each Allen relation [2], is the most representative interval-based temporal logic, and its model checking problem is the one that received the most attention. The problem of model checking HS formulas has been formulated in two ways. On the one side, model checking for full HS, interpreted over finite Kripke structures according to a state-based semantics has been studied in [22,28]. The authors showed that, under the homegeneity assumption, which constrains a proposition letter to hold over an interval if and only if it holds over each component state, the problem is non-elementarily decidable (EXPSPACE-hardness has been later shown in [5]). Since then, the attention was brought to HS fragments, which are often computationally much better [5,6,23,24,25]. Also, the model checking problem for some HS fragments extended with epistemic operators has been investigated in [18,19]. The semantic assumptions for these epistemic HS fragments differ from those of [22,28] in two important respects, making it difficult to compare the two families of logics: formulas are interpreted over the unwinding of the Kripke structure (computation-tree-based semantics) and interval labeling takes into account only the endpoints of intervals. The latter assumption has been later relaxed by making it possible to use regular expressions to define the labeling of proposition letters over intervals in terms of the component states [20]. On the other side, the problem of checking *finite, linear*, and *fully represented* interval models (FIMC problem) against HS formulas was formulated in [26], and its *infinite, periodical* generalization was presented in [27] for a fragment of HS. These interval model checking strategies share the following elements: first, the object to be checked is abstracted in some way, and, second, the underlying temporal logic is a *crisp* logic. We define

HS	Allen's relations
$\langle A \rangle$	$[x, y]R_A[x', y'] \Leftrightarrow y = x'$
$\langle L \rangle$	$[x, y]R_L[x', y'] \Leftrightarrow y < x'$
$\langle B \rangle$	$[x, y]R_B[x', y'] \Leftrightarrow x = x', y' < y$
$\langle E \rangle$	$[x, y]R_E[x', y'] \Leftrightarrow y = y', x < x'$
$\langle D \rangle$	$[x, y]R_D[x', y'] \Leftrightarrow x < x', y' < y$
$\langle O \rangle$	$[x, y]R_O[x', y'] \Leftrightarrow x < x' < y < y'$

Graphical representation

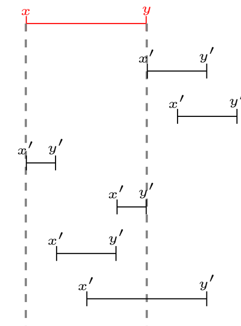

Fig. 1: Allen's interval relations and HS modalities.

multivariate time series model checking as the problem of checking if an interval temporal logic formula is satisfied by a finite multivariate time series. Time series are represented without abstraction, and we capture the intrinsic uncertainty carried by real-world data by checking formulas of the *fuzzy* generalization of HS (FHS) [13], effectively introducing the *fuzzy interval logic multivariate time series checking problem* (*TFIMC*). On the one hand, fuzzy model checking has received very little attention in the literature, having been attempted in [14,31] for fuzzy generalizations of CTL; however, these frameworks are not directly comparable with the one we present here. On the other hand, our approach could be associated with the concept of probabilistic model checking (PMC) on Markov models, which has a large recent history in the literature (see [17], and references within); however: *(i)* fuzzy logics generalize probabilistic logics by having both non-crisp accessibility relations and atomic properties, *(ii)* probabilistic model checking is point-based, and *(iii)* Markov models are, as Kripke models, abstractions of the underlying systems. The TFIMC problem is the direct generalization of the FIMC problem. The major obstacle in solving the latter lies in the representation of the model, which may be exponentially succinct w.r.t. the size of the input (i.e., the pair model+formula). In [26] it has been proved that FIMC is polynomial, but the necessary pre-process of the model causes a complexity blow-up. Such an obstacle does not present itself in solving TFIMC, as time series cannot be succinctly represented, and, as consequence, its complexity is lower than FIMC; however, the problem itself is not easier (in fact, is slightly more difficult), and such a difference is hidden, so to say, in the representation of the model.

2 Preliminaries: Fuzzy Interval Temporal Logic

Crisp interval temporal logic. Let $\mathbb{D} = \langle D, \leq \rangle$ be a linearly ordered set. An *interval* over $\mathbb{D}$ is an ordered pair $[x, y]$, where $x, y \in D$ and $x < y$. While in the original approach to interval temporal logic intervals with coincident endpoints were included in the semantics, in the recent literature they tend to be excluded except, for instance, in [4] where a two-sorted approach has been studied. If we exclude the identity relation, there are 12 different relations between two intervals in a linear order, often called *Allen's relations* [2]: the six relations R_A (adjacent to, or meets), R_L (later than), R_B (begins), R_E (ends), R_D (during), and R_O (overlaps), depicted in Figure 1, together with their inverses $R_{\overline{X}} = (R_X)^{-1}$, for each $X \in \{A, L, B, E, D, O\}$. We interpret interval structures as Kripke structures, with Allen's relations playing the role of the accessibility relations. Thus, we associate a universal modality $[X]$ and an existential modality $\langle X \rangle$ with each Allen's relation R_X. For each $X \in \{A, L, B, E, D, O\}$, the *inverse* of the modalities $[X]$ and $\langle X \rangle$ are the modalities $[\overline{X}]$ and $\langle \overline{X} \rangle$, corresponding to the inverse relation $R_{\overline{X}}$ of R_X. Halpern and Shoham's logic, denoted HS [16], is a multi-modal logic with formulæ built from a finite, non-empty set AP of atomic propositions (also referred to as propositional letters), the classical propositional connectives, and a modal operator for each Allen's relation, as follows:

$$\varphi ::= \bot \mid p \mid \neg\psi \mid \psi \vee \xi \mid \langle X \rangle \psi.$$

In the above grammar, $p \in AP$ and $X \in \{A, L, B, E, D, O, \overline{A}, \overline{L}, \overline{B}, \overline{E}, \overline{D}, \overline{O}\}$. The other propositional connectives and constants (e.g., $\rightarrow$, and $\top$), as well as the dual modalities (e.g., $[A]\varphi \equiv \neg\langle A \rangle \neg\varphi$), can be defined in the standard way. Given a formula of HS, its *inverse* formula is obtained by substituting every operator $\langle X \rangle$ with its inverse one $\langle \overline{X} \rangle$, and the other way around, for $X \in \{A, L, B, E, D, O\}$, while its *symmetric* is obtained by substituting every operator $\langle X \rangle$ with its inverse one $\langle \overline{X} \rangle$, and the other way around, for $X \in \{A, L, O\}$, and every $\langle B \rangle$ (resp., $\langle \bar{B} \rangle$) with $\langle E \rangle$ (resp., $\langle \bar{E} \rangle$), and the other way around.

The semantics of HS is given in terms of *interval models* of the type:

$$M = \langle \mathbb{I}(\mathbb{D}), V \rangle,$$

where $\mathbb{D}$ is a linear order, $\mathbb{I}(\mathbb{D})$ is the set of all intervals over $\mathbb{D}$, and V is a *valuation function* $V : AP \mapsto 2^{\mathbb{I}(\mathbb{D})}$, which assigns to each atomic proposition $p \in AP$ the set of intervals $V(p)$ on which p holds. In this work, we are interested in finite structures and thus we restrict our attention to linear orders over finite domains. A finite domain of length n will be denoted $[n]$. The *truth* of a formula φ on a given interval $[x, y]$ in an interval model M is defined by structural induction on formulæ as follows:

$$\begin{array}{lll}
M, [x, y] \Vdash p & \text{if} & [x, y] \in V(p), \text{ for } p \in AP; \\
M, [x, y] \Vdash \neg\psi & \text{if} & M, [x, y] \not\Vdash \psi; \\
M, [x, y] \Vdash \psi \vee \xi & \text{if} & M, [x, y] \Vdash \psi \text{ or } M, [x, y] \Vdash \xi; \\
M, [x, y] \Vdash \langle X \rangle \psi & \text{if} & M, [z, t] \Vdash \psi \text{ for a } [z, t] \text{ s.t. } [x, y] R_X [z, t], \\
 & & \text{for } X \in \{A, L, B, E, D, O, \overline{A}, \overline{L}, \overline{B}, \overline{E}, \overline{D}, \overline{O}\}.
\end{array}$$

In the recent literature, several computational problems related to the logic HS have been studied, including the satisfiability problem, analyzed for the full logic in the original work by Halpern and Shoham [16], in which the authors prove that it is undecidable when the logic is interpreted in virtually all interesting classes of linearly ordered sets, and for various fragments (with different computational behaviours) in, among others, [1,8,9,10,21,29,30], the model checking problem, in [19,22,26], and, more recently, different knowledge extraction problems, in [7,11].

Fuzzy interval temporal logic. A formula of a fuzzy modal logic is evaluated in a Heyting Algebra. A *Heyting Algebra* is a structure:

$$\mathcal{H} = (H, \wedge, \vee, \rightarrow, 0, 1),$$

that is, a bounded distributive lattice with (non-empty) domain H, with internal operations $\wedge$ (*meet*[6]) and $\vee$ (*join*), both commutative, associative, and connected by the absorption law, in which a partial order can be defined as:

$$\alpha \preceq \beta \Leftrightarrow \alpha \wedge \beta = \alpha \Leftrightarrow \alpha \vee \beta = \beta.$$

The symbols 0 and 1 denote, respectively, least and the greatest elements of $\mathcal{H}$. In a Heyting algebra, the *relative pseudo-complement* of α w.r.t. β, usually denoted as $\alpha \rightarrow \beta$, and called *Heyting implication*, can be defined, and exists for every α and β [15]. A Heyting algebra is said to be *complete* if for every subset $S \subseteq H$, both its least upper bound $\bigvee S$ and its greatest lower bound $\bigwedge S$ exist, and it is said to be a *chain* if $\preceq$ is total. Typical realizations of Heyting algebras include the two-element Boolean algebra, the closed interval $[0, 1]$ in $\mathbb{R}$, and any finite linear chain. Given a complete Heyting chain $\mathcal{H}$ with domain H, the fuzzy generalization of HS is defined starting with a domain D enriched with two functions:

$$\widetilde{<}, \widetilde{=} : D \times D \rightarrow H,$$

and defining the structure:

$$\widetilde{\mathbb{D}} = \langle D, \widetilde{<}, \widetilde{=} \rangle$$

as a *fuzzy strictly linearly ordered set* if it holds, for every x, y, and z:

1. $\widetilde{=}(x, y) = 1 \Leftrightarrow x = y$ (*reflexivity of* $\widetilde{=}$);
2. $\widetilde{=}(x, y) = \widetilde{=}(y, x)$ (*symmetry of* $\widetilde{=}$);
3. $\widetilde{<}(x, x) = 0$ (*irreflexivity of* $\widetilde{<}$);
4. $\widetilde{<}(x, z) \succeq \widetilde{<}(x, y) \wedge \widetilde{<}(y, z)$ (*transitivity of* $\widetilde{<}$);
5. $\widetilde{<}(x, y) \succ 0 \,\&\, \widetilde{<}(y, z) \succ 0 \Rightarrow \widetilde{<}(x, z) \succ 0$ (*transfer of* $\widetilde{<}$);
6. $\widetilde{<}(x, y) = 0 \,\&\, \widetilde{<}(y, x) = 0 \Rightarrow \widetilde{=}(y, x) = 1$ (*weak totality*);
7. $\widetilde{=}(x, y) \succ 0 \Rightarrow \widetilde{<}(x, y) \prec 1$ (*non-contradiction of* $\widetilde{<}$ *over* $\widetilde{=}$).

[6] This is the classical nomenclature in lattice theory, and it should not be confused with Allen's relation *meets*, used in this paper.

Given a set of propositional letters AP and a complete Heyting algebra $\mathcal{H}$, a well-formed *fuzzy interval temporal logic* (FHS, for short) formula is obtained by the following grammar:

$$\varphi ::= \alpha \mid p \mid \psi \vee \xi \mid \psi \wedge \xi \mid \psi \to \xi \mid \langle X \rangle \psi \mid [X]\psi,$$

where $\alpha \in H$, $p \in AP$, and, as in the crisp case, $X \in \{A, L, B, E, D, O, \overline{A}, \overline{L}, \overline{B}, \overline{E}, \overline{D}, \overline{O}\}$. We use $\neg\varphi$ to denote the formula $\varphi \to 0$.

Given a fuzzy strictly linearly ordered set, we can now define the set of fuzzy strict intervals in $\mathbb{D}$:

$$\mathbb{I}(\widetilde{\mathbb{D}}) = \{[x, y] \mid \widetilde{<}(x, y) \succ 0\}.$$

Generalizing classical Boolean evaluation, propositional letters are directly evaluated in the underlying algebra, by defining a valuation function $\widetilde{V} : AP \times \mathbb{I}(\widetilde{\mathbb{D}}) \to H$ that generalizes the crisp function V. Similarly, Allen's relations now have a fuzzy behaviour, which is obtained by generalizing the original, crisp definition, and substituting every $=$ with $\widetilde{=}$ and every $<$ with $\widetilde{<}$:

$$\begin{aligned}
\widetilde{R}_A([x, y], [z, t]) &= \widetilde{=}(y, z);\\
\widetilde{R}_L([x, y], [z, t]) &= \widetilde{<}(y, z);\\
\widetilde{R}_B([x, y], [z, t]) &= \widetilde{=}(x, z) \wedge \widetilde{<}(t, y);\\
\widetilde{R}_E([x, y], [z, t]) &= \widetilde{<}(x, z) \wedge \widetilde{=}(y, t);\\
\widetilde{R}_D([x, y], [z, t]) &= \widetilde{<}(x, z) \wedge \widetilde{<}(t, y);\\
\widetilde{R}_O([x, y], [z, t]) &= \widetilde{<}(x, z) \wedge \widetilde{<}(z, y) \wedge \widetilde{<}(y, t).
\end{aligned}$$

Now, we say that an *$\mathcal{H}$-valued interval model* (or *fuzzy interval model*) is a tuple of the type:

$$\widetilde{M} = \langle \mathbb{I}(\widetilde{\mathbb{D}}), \widetilde{V} \rangle,$$

where $\widetilde{\mathbb{D}}$ is a fuzzy strictly linearly ordered set that respects the properties 1-7, and $\widetilde{V}$ is a fuzzy valuation function. We interpret an FHS formula in a fuzzy interval model $\widetilde{M}$ and an interval $[x, y]$ by extending the valuation $\widetilde{V}$ of propositional letters as follows, where $X \in \{A, L, B, E, D, O, \overline{A}, \overline{L}, \overline{B}, \overline{E}, \overline{D}, \overline{O}\}$ and $[z, t]$ varies in $\mathbb{I}(\widetilde{\mathbb{D}})$:

$$\begin{aligned}
\widetilde{V}(\alpha, [x, y]) &= \alpha;\\
\widetilde{V}(\psi \wedge \xi, [x, y]) &= \widetilde{V}(\psi, [x, y]) \wedge \widetilde{V}(\xi, [x, y]);\\
\widetilde{V}(\psi \vee \xi, [x, y]) &= \widetilde{V}(\psi, [x, y]) \vee \widetilde{V}(\xi, [x, y]);\\
\widetilde{V}(\psi \to \xi, [x, y]) &= \widetilde{V}(\psi, [x, y]) \to \widetilde{V}(\xi, [x, y]);\\
\widetilde{V}(\langle X \rangle \psi, [x, y]) &= \bigvee_{[z,t]} \{\widetilde{R}_X([x, y], [z, t]) \wedge \widetilde{V}(\psi, [z, t])\};\\
\widetilde{V}([X]\psi, [x, y]) &= \bigwedge_{[z,t]} \{\widetilde{R}_X([x, y], [z, t]) \to \widetilde{V}(\psi, [z, t])\}.
\end{aligned}$$

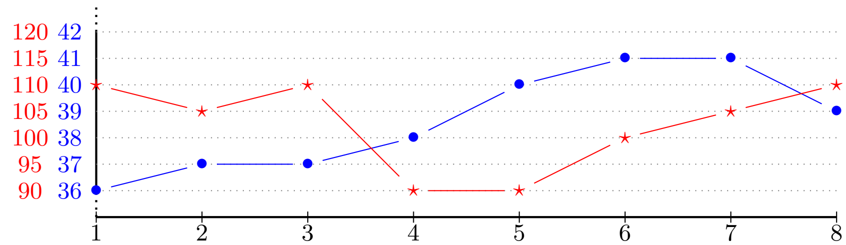

Fig. 2: A multivariate time series with two variables.

We say that a formula of FHS φ is α-*satisfied* at an interval $[x, y]$ in a fuzzy interval model $\widetilde{M}$, denoted $\widetilde{M}, [x, y] \Vdash_\alpha \varphi$, if $\widetilde{V}(\varphi, [x, y]) \succeq \alpha$. The formula φ is α-*satisfiable* if and only if there exists a fuzzy interval model and an interval in that model where it is α-satisfied. A formula is *satisfiable* if it is α-satisfiable for some $\alpha \in H$, $\alpha \neq 0$. A formula is α-*valid* if it is α-satisfied at every interval in every model, and *valid* if it is 1-valid. Observe that since a Heyting algebra, in general, does not encompass classical negation, and since our definition of satisfiability is graded, instead of absolute, then the usual duality of satisfiability and validity does not hold anymore.

Fuzzy HS was introduced in [13], where its satisfiability problem, along with certain expressive power problems, were studied.

3 Fuzzy Time Series Checking

Multivariate time series. A temporal variable is a variable whose value changes over time. A *time series* is a set of temporal variables. They can be *univariate*, if only one temporal variable is involved, or *multivariate*, otherwise. In data science, a time series defined over a set of temporal variables (or temporal attributes) $\mathcal{A} = \{A_1, \ldots, A_m\}$ is usually represented as $n \times m$ matrix:

$$T = \begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,m} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,m} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n,1} & a_{n,2} & \ldots & a_{n,m}. \end{pmatrix}$$

We denote the *domain* of an attribute A, that is, the set in which A_i takes values, by $dom(A)$. We assume that each variable A_i that forms a multivariate series T has n values, that and there are no missing values; so, the *length* of a multivariate time series n is well-defined. By $A(t)$, we denote the value of variable A at point t. An example of time series with $m = 2$ and $n = 8$ can be found in Figure 2; in this example, the variable A_1 represents the evolution of a patient's temperature during the observed period, while the variable A_2 represents his/her blood pressure during the same period.

Problem definition. Let T be a time series of length n defined over a set of variables $\mathcal{A}$. We define a set of *decisions* $\mathcal{S}$ as:

$$\mathcal{S} = \{A \bowtie a \mid A \in \mathcal{A} \text{ and } a \in dom(A)\},$$

where $\bowtie \in \{\leq, <, =, >, \geq\}$, and we let $AP = \mathcal{S}$. Now, consider a time series T with n distinct points, and the finite domain $\widetilde{[n+1]}$ obtained by adding a point 0 at the beginning of the series with undefined values A_i for each i. After having fixed two concrete relations $\tilde{\cong}$ and $\tilde{<}$, we can form the set of intervals $\mathbb{I}(\widetilde{[n+1]})$, and define a function f to give truth values to decisions:

$$f : \mathcal{S} \times \mathbb{I}(\widetilde{[n+1]}) \to H.$$

By means of f, we allow a decision to be $\mathcal{H}$-valuated on an interval; in the most general case one may imagine different functions for different variables, although, for the sake of simplicity of exposition, here we assume that all variables are evaluated by the same function. Interpreting a formula of FHS on a time series T simply consists of defining a $\mathcal{H}$-valued interval model:

$$\widetilde{T} = \langle \mathbb{I}(\widetilde{[n+1]}), \widetilde{V}, f \rangle,$$

and of imposing that decisions are evaluated through f:

$$\widetilde{V}(A_i \bowtie a, [x, y]) = f(A_i \bowtie a, [x, y]).$$

Since the variables A_i are all undefined on 0, we may assume that f is undefined on the interval $[0, 1]$ as well. Given a time series T, an algebra $\mathcal{H}$, a value $\alpha \in H$, and a formula φ of FHS, the *fuzzy interval logic multivariate time series checking problem* (*TFIMC*) is the problem of establishing if it holds:

$$\widetilde{T}, [0, 1] \Vdash_\alpha \varphi.$$

In a way, we can say that the function f implicitly defines the domain of the algebra on which $\widetilde{T}$ is defined.

Concretizing a model: an example. Given a time series T, there are many ways to produce a concrete temporal model; some of them are more intuitive than others. In this example, we assume $\mathcal{H}$ to be the closed interval $[0, 1]$ in $\mathbb{R}$ equipped with the relation *minimum* (as meets) and *maximum* (as join), and we describe a concretization of f. The function f can be thought as a black-box function representing the domain-expert knowledge. In the example in Figure 2, for instance, one has to decide *when* temperature can be considered, say, *over 38 degrees*. One possible way is to define f, for a generic variable A, relation $\bowtie$, value a, and interval $[x, y]$, as the ratio between the number of points $x \leq t \leq y$ that satisfy $A \bowtie a$ and $y - x + 1$. Similarly, we have to describe a concretization of $\tilde{\cong}$ and $\tilde{<}$. Among the many ways that exist to do so, we fix a positive parameter $h \in \mathbb{N}$ (which can be thought of as an *horizon*), and define two parametric versions for $\tilde{\cong}$ and $\tilde{<}$ as follows: $\tilde{\cong}_h(x, y)$ is always 0, except when $|x - y| \leq h$,

Algorithm 1 Fuzzy interval logic multivariate time series checking algorithm.

function $\textsc{Check}(\widetilde{T}, \varphi)$
    $L = \emptyset$
    for $\psi \in sub(\varphi)$ in increasing length order **do**
        if $\psi = \gamma$ **then**
            for $[x, y] \in \mathbb{I}(\widetilde{[n+1]})$ **do**
                $L(\psi, [x, y]) = \gamma$
        if $\psi = A \bowtie a$ **then**
            for $[x, y] \in \mathbb{I}(\widetilde{[n+1]})$ **do**
                $L(\psi, [x, y]) = f(A \bowtie a, [x, y])$
        if $\psi = \tau \circ \xi$ **then**
            for $[x, y] \in \mathbb{I}(\widetilde{[n+1]})$ **do**
                $L(\psi, [x, y]) = L(\tau, [x, y]) \bullet L(\xi, [x, y])$
        if $\psi = \langle X \rangle \tau$ **then**
            for $[x, y] \in \mathbb{I}(\widetilde{[n+1]})$ **do**
                $z = 0$
                for $[z, t] \in \mathbb{I}(\widetilde{[n+1]})$ **do**
                    $z \leftarrow z \vee (R_{\widetilde{X}}([x, y], [z, y]) \wedge L(\tau, [z, t]))$
                $L(\psi, [x, y]) = z$
        if $\psi = [X] \tau$ **then**
            for $[x, y] \in \mathbb{I}(\widetilde{[n+1]})$ **do**
                $z = 1$
                for $[z, t] \in \mathbb{I}(\widetilde{[n+1]})$ **do**
                    $z \leftarrow z \wedge (R_{\widetilde{X}}([x, y], [z, y]) \wedge L(\tau, [z, t]))$
                $L(\psi, [x, y]) = z$

in which case it is $\frac{h-|x-y|}{h}$, and $\widetilde{<}_h(x,y)$ is always 1 (it is 0, when $x < y$), except when $y - x \leq h$, in which case it is $\frac{y-x}{h}$. It is immediate to see that they satisfy axioms 1-7. Moreover, if $h = 1$ our definition immediately reduces to the crisp definitions of $=$ and $<$. Consider, again, the time series from Figure 2 and its corresponding model $\widetilde{T}$ obtained with the above fuzzy equality and ordering relations, and in which we assume, for the purpose of this example, $h = 4$. A possibly interesting property to be evaluated on this time series is *starting from day two, it is never true that blood pressure is low while temperature is high.* We can translate such a property as *starting from day two, there is no interval in which blood pressure is low (i.e., less than or equal 100) and temperature is high (i.e., greater than or equal to 38).* The existence of such an interval is translated to FHS as:

$$\langle L \rangle (A_2 \leq 100 \wedge A_1 \geq 38).$$

Among all witnesses of such a condition in $\widetilde{T}$, we find the interval $[4, 5]$: $\widetilde{<}(1, 4) = \frac{3}{4}$, $f(A_1 \geq 38, [4, 5]) = \frac{2}{2}$, and $f(A_2 \leq 100, [4, 5]) = \frac{2}{2}$, so that $\widetilde{V}(\langle L \rangle (A_2 \leq 100 \wedge A_1 \geq 38), [0, 1]) \succeq 0.75$.

Algorithm. We are ready to formalize the fuzzy time series checking algorithm. Let $\widetilde{T} = \langle \mathbb{I}(\widetilde{[n+1]}), \widetilde{V}, f \rangle$ be a model based on some complete algebra $\mathcal{H}$, φ a formula of FHS, and $\alpha \in H$. Algorithm 1 is the adaptation of Emerson and Clarke's classical CTL algorithm to the interval, fuzzy case, and it returns `true` if and only if the value of φ on the (auxiliary) interval $[0, 1]$ is greater of equal α in $\widetilde{T}$. In Algorithm 1, we use the symbol $\circ \in \{\vee, \wedge, \rightarrow\}$ to denote a logical symbol, and the symbol $\bullet$ to denote its algebraic corresponding one. Unlike the crisp case, every sub-formula must be checked on every interval, because, in the general case, any two intervals $[x, y]$ and $[z, t]$ may be related by any relation $R_{\widetilde{X}}$. The auxiliary data structure L can be thought of as an hash table indexed by three elements, namely ψ, x, y, that is, a sub-formula, and two points. Accessing L may be considered to have constant time complexity. Formulas, classically represented as binary trees, can be pre-processed in order to identify repeating sub-formulas, so that the main cycle of Algorithm 1 can be implemented in an efficient way.

Complexity. Let $\widetilde{T} = \langle \mathbb{I}(\widetilde{[n+1]}), \widetilde{V}, f \rangle$ be a model based on m temporal attributes, each with n distinct points, and let k be the length of the input formula φ. We can assume that $m, k = o(n)$, that is, that there are much less temporal variables and much less sub-formulas than there are distinct points. Thus, we can express the size of the input as $O(n)$. Also, we can assume that both the join ($\vee$) and meet ($\wedge$) operations take constant time in n, and that each call to f takes time $O(n)$ (in the worst case scenario, in fact, each call to f requires exploring an interval with n points). The most external cycle is executed $O(n)$ times. In the worst-case scenario, during each execution ψ is a modal formula. Since there are $O(n^2)$ intervals in $\widetilde{T}$, the complexity of the modal case is $O(n^4)$. Therefore, the entire algorithm runs in $O(n^5)$.

Theorem 1. *The TFIMC problem is deterministic polynomial.*

4 Conclusions

In this paper we first defined, and then solved, the multivariate time series fuzzy interval logic checking problem. Despite its simplicity, the interest in this problems lies in the fact that multivariate time series are ubiquitous in certain areas of data science and learning, but they have never before been linked to the classical model checking problem. Yet, we believe that many interesting properties can be formulated, and therefore checked, on a time series, and that the recently introduced fuzzy interval temporal logic FHS is a suitable language to do so. As future work, our intention is to design and test an efficient implementation of our algorithm.

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